s={s1,s2}={0,1} ; Xik={0,1} iid ; $x_i = \{x_{i1}, x_{i2}, \dots x_{in}\}$

$$H(x_1, x_2 \dots x_n) = -\sum_{x_{i1}, x_{i2}, \dots x_{in}} P(x_{i1}, x_{i2} \dots x_{in}) \, ld \, P(x_{i1}, x_{i2}, \dots x_{in}) =$$

$$= -\sum_{x_{i1}, x_{i2}, \dots x_{in}} P(x_{i1}) P(x_{i2}) \dots P(x_{in}) \left[ld \, P(x_{i1}) + ld \, P(x_{i2}) \dots ld \, P(x_{in}) \right]$$

$$P(x_i) = P(x_{i1}, x_{i2}, \dots x_{in}) = \prod_{k=1}^{n} P(x_{ik})$$

$$H(x_1, x_2, \dots x_n) = -\sum_{x_{i1}} ld \, P(x_{i1}) \underbrace{\sum_{x_{i2}, x_{i3}, \dots x_{in}} P(x_{i1}, \dots x_{in})}_{P(x_{i1})} -$$

$$- \sum_{x_{i2}} ld \, P(x_{i2}) \underbrace{\sum_{x_{i1}, x_{i3}, \dots x_{in}} P(x_{i1}, \dots x_{in})}_{P(x_{i2})} \dots - \sum_{x_{in}} ld \, P(x_{in}) \underbrace{\sum_{x_{i1}, \dots x_{in-1}} P(x_{i1}, \dots x_{in})}_{P(x_{in-1})}$$

$$= H(x_{i1}) + H(x_{i2}) + \dots + H(x_{in}) =$$ / $x_{i1}, x_{i2}, \dots x_{in}$ $\underline{\underline{iid}}$ /

$$= n \cdot H(X_i) = n \cdot H(X)$$

$x_i \Rightarrow$ BINARY VARIABLE.

$\boxed{\text{VAŽNOST IZVEDUVANJETO VO P2.20 E NAJVAŽNO!!!}}$ ✓

→ OVAJ DOKAZ MOŽE DA SE KORISTI ZA DOKAŽUVANJE NA ENTROPIJATA NA PROŠIREN IZVOR NAMESTO DOKAZOT ŠTO SE KORISTI VO SKRIPTATA OD PREDAVANJA (PP.150)

- AKO SE ODI SO NOMENKLATURATA OD SKRIPTATA

$$S = \{S_1, S_2, \dots S_2\} \qquad W_i = \{S_{i1}, S_{i2}, \dots S_{in}\}$$

$$H(S_1, S_2, \dots S_n) = -\sum_{s_{i1} \in S} \sum_{s_{i2} \in S} \dots \sum_{s_{in} \in S} P(s_{i1}, s_{i2}, \dots s_{in}) \, ld \, P(s_{i1}, s_{i2}, \dots s_{in}) =$$

$$= \sum_{s_{i1}} \sum_{s_{i2}} \dots \sum_{s_{in}} P(s_{i1}, s_{i2}, \dots s_{in}) \left[ld \, P(s_{i1}) + ld \, P(s_{i2}) \dots + ld \, P(s_{in}) \right] =$$

$$= \sum_{s_{i1}} ld \, P(s_{i1}) \underbrace{\sum_{s_{i2}, s_{i3} \dots s_{in}} P(s_{i1}, s_{i2}, \dots s_{in})}_{P(s_{i1})} + \sum_{s_{i2}} ld \, P(s_{i2}) \underbrace{\sum_{s_{i1}, s_{i3} \dots s_{in}} P(s_{i1}, s_{i2}, \dots s_{in})}_{P(s_{i2})} + \dots$$

$$= \sum_{s_{i1} \in S} P(s_{i1}) \, ld \, P(s_{i1}) + \sum_{s_{i2} \in S} P(s_{i2}) \, ld \, P(s_{i2}) + \dots = n \cdot H(S)$$

1

$$H(x, g(x)) = H(x) + \underbrace{H(g(x)|x)}_{0} = H(x)$$
$$H(x, g(x)) = H(g(x)) + \underbrace{H(x|g(x))}_{\neq 0} \geq H(g(x)) \Rightarrow$$

$\boxed{H(x) \geq H(g(x))}$ ($)

(b) TREBA ĆE DA SE DOKAŽE:

$$H(x_1, x_2, \ldots x_4) \geq H(Z_1, Z_2, \ldots Z_k) \rightarrow \text{TRUE!!!}$$

$$Z_1, Z_2, \ldots Z_k = f(x_1, x_2, \ldots x_4) \Rightarrow (\$) \Rightarrow$$

(c) FIND GOOD MAP "f" ON SEQUENCES OF LENGTH 4

- 0000 = 0
+ 0001 = 1
+ 0010 = 1
+ 0011 = 2
- 0100 = 1
+ 0101 = 2
+ 0110 = 2
+ 0111 = 3
- 1000 = 1
+ 1001 = 2
- 1010 = 2
+ 1011 = 3
- 1100 = 2
- 1101 = 3
- 1110 = 3
- 1111 = 4

$$H(x_1, x_2, x_3, x_4) = 4\,H(p) = 4 \cdot H(p)$$

$$H(p) = -p\,ld\,p - (1-p)\,ld(1-p)$$

VKUPNO IMA:

1.) SEKVENCI ZA SO 0 "1"-CI = 1
2.) SEKVENCI ZA SO 1 "1"-CA = 4
3.) SEKVENCI ZA SO 2 "1"-CI = 6
4.) SEKVENCI ZA SO 3 "1"-CI = 4
5.) SEKVENCI ZA SO 4 "1"-CI = 1

0
4
12
12
4
32

- TREBA DA SE PODELAT VO 2 GRUPI SO IST BROJ NA EDINICI:

GRUPA 1 $(Z_1 = 0)$ (+)

8 NIZI

4 × 2 EDINICI = 8
2 × 1 EDINICA = 2
2 × 3 = 6
~~0000~~

16 NULI + 16 EDINICI

GRUPA 2 $(Z_1 = 1)$ (−)

8 NIZI

2 + 2 EDINICI = 4
2 × 1 EDINICA = 2
2 × 3 = 6
1 × 4 = 4
0000

16 NULI + EDINICI 16

$$H(x_1, x_2, x_3, x_4 | Z_1 = 0) = -P(0001)\,ld\,P(0001) - P(0010)\,ld\,P(0010) -$$
$$- P(0011)\,ld\,P(0011) - P(0101)\,ld\,P(0100) - P(0111)\,ld\,P(0111) - P(1001)\,ld\,P(1001)$$
$$- P(1011)\,ld\,P(1011) = -4(1-p)^2 p^2\,ld(1-p)^2 p^2 - 2(1-p)p^3\,ld(1-p)p^3 - 2(1-p)^3 p\,ld(1-p)^3 p$$

$P(0110)\,ld\,P(0110) -$

$$H(x_1, x_2, x_3, x_4 | Z_2 = 1) = -P(000)\,ld\,P(000) - P(0100)\,ld\,P(0100) - P(1000)\,ld\,P(1000)$$
$$- P(1010)\,ld\,P(1010) - P(1100)\,ld\,P(1100) - P(1101)\,ld\,P(1101) - P(1110)\,ld\,P(1110) -$$
$$- P(1111)\,ld\,P(1111) = -(1-p)^4\,ld(1-p)^4 - 2(1-p)^3 p\,ld(1-p)^3 p - 2(1-p)^2 p^2\,ld(1-p)^2 p^2 -$$
$$- 2p^3(1-p)\,ld\,p^3(1-p) - p^4\,ld\,p^4$$

$H(x_1, x_2, x_3, x_4 | z_1) = H(x_1, x_2, x_3, x_4, z_1) - \underbrace{H(z_1)}_{1} = 1 - H(x_1, x_2, x_3, x_4, z_1)$

- EDITION 1 SOLUTION: USE THE FACT THAT ALL SEQUENCES WITH SAME NUMBER OF ONES ARE EQUALLY LIKELY. E.G.:

SEQUENCE WITH ONE „1": (e.g. 0001; 0010...): $p\cdot(1-p)^3$
SEQUENCE WITH TWO „1"-s: (e.g. 0011; 0101...): $p^2(1-p)^2$
SEQUENCE WITH THREE „1": (e.g. 0111, 1101 ...) $p^3(1-p)$

FOR EXAMPLE 0001, 0010, 0100, 1000 ARE EQUALLY LIKELY AND CAN BE USED TO GENERATE 2 PURE RANDOM BITS. AN EXAMPLE OF MAPPING TO GENERATE RANDOM BITS IS:

0000 → Λ

x4 { 0001 → 00   0010 → 01   0100 → 10   1000 → 11

x6 { 0011 → 00   0110 → 01   1100 → 10   1001 → 11
     0101 → 0   1010 → 1

x4 { 1110 → 11   1101 → 10   1011 → 01   0111 → 00

1111 → Λ ✓

⊛

$$\boxed{E[K] = 4\cdot p\cdot q^3\cdot 2 + \underline{4\cdot p^2\cdot q^2\cdot 2} + \underline{2\,p^2 q^2\cdot 1} + 4\cdot p^3\cdot q\cdot 2 = 8\cdot p q^3 + 10 p^2 q^2 + 8 p^3 q}$$

FOR $p = \frac{1}{2} \Rightarrow E[K] = 1.625$

- CONSIDER SEQUENCES WITH „K" ONES. THERE ARE $\binom{4}{k}$ SUCH SEQUENCES, WHICH ARE ALL EQUALLY LIKELY. IF $\binom{4}{k}$ WERE A POWER OF 2, THEN WE COULD GENERATE $ld\binom{4}{k}$ PURE RANDOM BITS FROM SUCH A SET. HOWEVER IN GENERAL $\binom{4}{k}$ IS NOT A POWER OF TWO AND BEST WE CAN DO IS DEVIDE THE SET OF $\binom{4}{k}$ ELEMENTS INTO SUBSET OF SIZES WHICH ARE POWERS OF 2

- LARGEST SET HAS A SIZE: $2^{\lfloor ld\binom{4}{k}\rfloor}$

e.g. $\binom{4}{2} = 6 \qquad 2^{\lfloor ld 6\rfloor} = 2^{\lfloor 2.5\rfloor} = 2^2 = 4$

AND CAN BE USED TO GENERATE $\lfloor \log\binom{4}{k}\rfloor$ RANDOM BITS.

LET $l = \lfloor \log\binom{4}{k}\rfloor$ THAN AT LEAST HALF OF THE ELEMENTS BELONG TO A SET OF SIZE 2^l AND WOULD GENERATE l RANDOM BITS, AT LEAST FORTH OF THE ELEMENTS BELONG TO A ~~SET OF~~ SIZE 2^{l-1} AND WOULD GENERATE $l-1$ RANDOM BITS AND ETC.

- ON THE AVERAGE THE NUMBER OF BITS GENERATED IS:

$$E[K \mid k\ 1\text{'s in sequence}] = \frac{1}{2}l + \frac{1}{4}(l-1) + \frac{1}{8}(l-2) \cdots + \frac{1}{2^l}\cdot 1$$

$$= \left[\frac{l}{2} + \frac{l}{4} + \frac{l}{8} + \cdots + \frac{l}{2^l}\right] - \left[\frac{1}{4} + \frac{2}{8} + \frac{3}{16} \cdots + \frac{l-1}{2^l}\right]$$

$$= l\left[\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \frac{1}{2^l}\right] - \frac{1}{2}\left[\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots + \frac{l-1}{2^{l-1}}\right] =$$

$$S = 1 + q + q^2 + \cdots q^n \quad \Big|\ *q \qquad S = \frac{1-q^{n+1}}{1-q}$$

$$qS = q + q^2 + q^3 + \quad q^{n+1}$$

$$q = \frac{1}{2} \Rightarrow \frac{1 - \frac{1}{2}^{n+1}}{1-\frac{1}{2}}$$

$$S(1-q) = 1 - q^{n+1}$$

$$S = \frac{\frac{2^{n+1}-1}{2^{n+1}}}{\frac{1}{2}} = \frac{2^{n+1}-1}{2^n} = \left(2 - \frac{1}{2^n}\right)$$

$$= \frac{l}{2}\cdot\left(2 - \frac{1}{2^{l-1}}\right) - \frac{1}{4}\left[1 + 1 + \frac{3}{4} + \cdots \frac{l-1}{2^{l-2}}\right] =$$

$$\frac{1}{4}(l-1) = \frac{1}{2^n}(l-1) = \frac{1}{2^n}\left(l - \underline{n+1}\right)$$

$$\frac{1}{2^l}(l - l + 1) = \frac{1}{2^l}$$

$$= \frac{2 \cdot l}{2} - \frac{l}{2^l} - \frac{1}{4}\left[2 + \frac{3}{4} + \cdots \frac{l-1}{2^{l-2}}\right] =$$

$$= l - \frac{1}{4}\left[2 + \frac{3}{4} + \cdots \frac{l-1}{2^{l-2}} + \frac{l}{2^{l-2}}\right] > l - 1$$

$$\frac{1}{4}(l-1) + \frac{1}{8}(l-2) + \frac{1}{2^4}(l-3) + \frac{1}{2^5}(l-4) + \cdots + \frac{1}{2^l}(l-1)$$

$$S = \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \cdots \frac{l-1}{2^l} = \frac{1}{4}\left(1 + 1 + \frac{3}{4} + \frac{4}{8} + \cdots + \frac{l-1}{2^{l-2}}\right)$$

$$S = \sum_{l=2} \frac{l-1}{2^l} = 1 \ \text{MAPLE} \qquad \lim_{l\to\infty} \frac{l}{2^l} \to 0$$

$$S = \sum_{l=2}^{\infty} \frac{l-1}{2^l} = \sum_{l=1}^{\infty} \frac{l-1}{2^l} = 1 + \sum_{l=0}^{\infty} \frac{l-1}{2^l} =$$

$$= 1 + \underbrace{\sum_{l=0}^{\infty} \frac{l}{2^l}}_{S_1} - \underbrace{\sum_{l=0}^{\infty} \frac{1}{2^l}}_{S_2}$$

$$S_2 = \frac{1}{1-\frac{1}{2}} = 2$$

3291909

$$S_1 = \sum_{l=0}^{\infty} \frac{l}{2^l} \qquad S_1(x) = \sum_{l=0}^{\infty} l \cdot x^l$$

$$\int x^l dx = \frac{x^{l+1}}{l+1} \qquad S_1(x) = x \sum_{l=0}^{\infty} l \cdot x^{l-1}$$

$$\frac{S_1(x)}{x} = \sum_{l=0}^{\infty} l \cdot x^{l-1}$$

$$\int \frac{S_1(x)}{x} dx = \sum_{l=0}^{\infty} \cancel{l} \frac{x^l}{\cancel{l}} = \sum_{l=0}^{\infty} x^l = \frac{1}{1-x}$$

$$\underline{0 \leq x \leq 1}$$

$$\int \frac{S_1(x)}{x} dx = \frac{1}{1-x} \qquad \frac{S_1(x)}{x} = \frac{1}{(1-x)^2}$$

$$S_1(x) = \frac{x}{(1-x)^2} \qquad x = \frac{1}{2} \Rightarrow S_1\left(\frac{1}{2}\right) = \frac{\frac{1}{2}}{\left(\frac{1}{2}\right)^2} = 2$$

$$\boxed{S = S_1 - S_2 + 1 = 2 - 2 + 1 = 1}$$

$$E[k] \geq l - \underbrace{\frac{l}{2^l}}_{\to 0 \text{ (for } l \to \infty)} - 1 \geq l - 1$$

• HENCE THE EXPECTED NUMBER OF PURE RANDOM BITS PRODUCED BY THIS ALGORITHM IS: (VIDI N12.3 ⊛)

$$E[k] \geq \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \left\lfloor \log_2 \binom{n}{k} - 1 \right\rfloor \geq \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \left[ld \binom{n}{k} - 2 \right]$$

$$= \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} ld \binom{n}{k} - 2 \underbrace{\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}}_{1} = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} ld \binom{n}{k} - 2$$

UKUPNA VEROVATNOĆA MORA DA BUDE = 1
DOPOLNITELNI DOKAZ NEMA

$$\geq \sum_{n(p-\varepsilon)\leq k\leq n(p+\varepsilon)} \binom{n}{k} p^k q^{n-k} \log\binom{n}{k} - 2 \geq$$

NOW FOR SUFFICIENTLY LARGE n, THE PROBABILITY THAT THE NUMBER OF 1'S IN SEQUENCE IS CLOSE TO $n \cdot p$ IS NEAR "1" (BY THE WEEK LOW OF LARGE NUMBERS) FOR SUCH SEQUENCES $\frac{k}{n}$ IS CLOSE TO p AND HANCE THERE EXIST δ SUCH THAT

$$\binom{n}{k} \geq 2^{n\left(H\left(\frac{k}{n}\right)-\delta\right)} \geq 2^{n(H(p)-2\delta)}$$

USING STIRLING'S APPROXIMATION FOR THE BINOMIAL COEFFICIENT AND THE CONTINUITY OF THE ENTROPY FUNCTION. IF WE ASSUME THAT "n" IS LARGE ENOUGH SO THAT THE PROBABILITY THAT ! $n(p-\varepsilon)\leq k \leq n(p+\varepsilon)$ IS GREATER THAN $1-\varepsilon$, THAN WE SEE THAT

$$E[k] \geq (1-\varepsilon)n(H(p)-2\delta)-2$$

$$\geq \sum_{n(p-\varepsilon)\leq k\leq n(p+\varepsilon)} \binom{n}{k} p^k q^{n-k} \left[n(H(p)-2\delta)\right] - 2 = n(H(p)-2\delta) \underbrace{\sum_{n(p-\varepsilon)\leq k\leq n(p+\varepsilon)} \binom{n}{k} p^k q^{n-k}}_{\geq 1-\varepsilon} - 2$$

$$\geq n(H(p)-2\delta)(1-\varepsilon)-2$$ hence $\underline{E[k] \geq (1-\varepsilon)\cdot n(H(p)-2\delta)-2}$

WHICH IS VERY GOOD SINCE $nH(p)$ IS AN UPPER BOUND ON THE NUMBER OF PURE RANDOM BITS THAT CAN BE PRODUCED FROM THE BENT COIN SEQUENCE.

$\boxed{\text{2.18 WORLD SERIES}}$ THE WORLD SERIES IS SEVEN-GAMES SERIES THAT TERMINATES AS SOON AS EITHER TEAM WINS FOUR GAMES. LET X BE THE RANDOM VARIABLE THAT REPRESENTS THE OUTCOME OF THE WORLD SERIES BETWEEN THE TEAMS A & B. POSSIBLE VALUES OF X ARE AAAA, BABABAB AND BBBAAAA. LET Y BE THE NUMBER OF GAMES PLAYED WHICH RANGES FROM 4 TO 7. ASSUMING THAT A&B ARE EQUALY MATCHED AND THAT THE GAMES ARE INDEPENDENT, CALCULATE: $H(X)$, $H(Y)$, $H(Y|X)$, AND $H(X|Y)$.

$X \in \{ \underset{x_1}{AAAA}, \underset{x_2}{BABABAB}, \underset{x_3}{BBBAAAA} \}$ $\quad Y \in \{ \underset{y_1}{4}, \underset{y_2}{5}, \underset{y_3}{6}, \underset{y_4}{7} \}$

NUMBER OF GAMES PLAYED

$P_X = \{ \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \}$?

$H(X) = \frac{1}{3}(ld\,3)\cdot 3 = \underline{ld\,3}$

$P(Y=4 \mid X=x_1)$

$P(Y \mid X=x_1) = ?$ $\quad P(Y \mid X=x_1) = \{1, 0, 0, 0\}$

$P(Y \mid X=x_2) = \{0, 0, 0, 1\}$ $\quad P(Y=7 \mid X=x_2) = 1$

$P(Y \mid X=x_3) = \{0, 0, 0, 1\}$ $\quad P(Y=7 \mid X=x_3) = 1$

$H(Y \mid X=x_1) = ld\,1 = \emptyset$ $\quad H(Y \mid X=x_2) = \emptyset$ $\quad H(Y \mid X=x_3) = \emptyset$

$H(Y \mid X) = \sum_{x \in X} P(x)\cdot H(Y \mid X=x) = P(x_1)\cdot H(Y \mid x_1) + P(x_2)\, H(Y \mid x_2) + P(x_3)\cdot H(Y \mid x_3) = \underline{\emptyset}$

AKO GO ZNAEŠ X ODMA ZNAEŠ KOLKO NATPREVARI SE IGRANI

LOGIČNO SLEDI

$P(X \mid Y=y_1) = \{1, 0, 0\}$ $\quad P(X \mid Y=y_2) = \{0, 0, 0, 0\}$

$P(x=x_1 \mid Y=y_1)$ $\quad P(X \mid Y=y_3) = \{0, 0, 0, 0\}$

$P(X \mid Y=y_4) = \{0, \frac{1}{2}, \frac{1}{2}\}$

$H(X \mid Y) = P(y_1)\cdot \underbrace{H(X \mid y_1)}_{\emptyset} + \cancel{P(y_2)\cdot H(X \mid y_2)} + \cancel{P(y_3)\, H(X \mid y_3)}$
$+ P(y_4)\, H(X \mid y_4) = \frac{35}{56}\left[0 + \frac{1}{2} + \frac{1}{2}\right] = \underline{\frac{35}{56}}$

$P_Y = \left\{ \frac{1}{56}, \frac{5}{56}, \frac{15}{56}, \frac{35}{56} \right\}$ ZOŠTO E OVA VAKA VIDI NA PP.9 INAKU TOA SE KOEFICIENTITE DOLU

$\binom{7}{4} = 35$ $\quad \binom{6}{4} = 15$; $\binom{5}{4} = 5$; $\binom{4}{4} = 1$

$\sum_{k=4}^{4} \binom{4}{k} \frac{1}{2}^{k} \cdot \frac{1}{2}^{4-k} = 1 \cdot \frac{1}{16} = \frac{1}{16}$

$\boxed{\begin{matrix} n=7 \\ k=4 \end{matrix}}$ $\binom{7}{4} = 35$

$35 \cdot \frac{1}{16} \cdot \frac{1}{8} = \frac{35}{128}$

$\cancel{\sum_{k=4}^{5} \binom{4}{k} \frac{1}{2}^{k} \cdot \frac{1}{2}^{4-k} = \frac{1}{16}} + 5 \cdot \frac{1}{2}\cdot\frac{1}{16} = \frac{5}{32}$

$\cancel{2 \times 10 = 20}$

$\cancel{\sum_{k=4}^{6} \binom{4}{k}}$ $\boxed{\begin{matrix} n=6 \\ k=4 \end{matrix}}$ $\binom{6}{4} = 15$ $\quad 15 \cdot \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^2 = \frac{15}{64}$

1 000
- 001
- 010
+ 011
- 100
+ 101
+ 110
1 111

KOMBINACII SO 2 EDINICI

$$\binom{3}{2} = \frac{3!}{2!\cdot 1!} = \frac{6}{2} = 3$$

IMAŠ ISTO TOČNO KOMBINACII SO 2 NULI

VKUPNO KOMBINACII $2^3 = 8$

VEROJATNOSTA DA PADNAT 2 EDINICI $\frac{3}{8}$

- PRIMEROV E OD 3 UTAKMICI DA POBEDI „1"

VEROJATNOSTA E : $\frac{3}{6} = \frac{1}{2}$

- VEROJATNOSTA OD TRI UTAKMICI DA POBEDI „0" E

$\frac{3}{6} = \frac{1}{2}$

- NE SE BROJAT „1"

$Y \in [2,3]$

- MEČOT SE ZAVRŠIL ZA 2 UTAKMICI

001 ILI

REZULTATOT OD MEČOVITE MOŽE DA BIDE:

	00	010	011	100	101	11
WIN	0	0	1	0	1	1

VEROJATNOSTA DA SE ZAVRŠI UTAKMICATA VO 2 MEČA E: $\frac{2}{5}$

VEROJATNOSTA DA SE ZAVRŠI VO 3 MEČA E $\frac{3}{5}$

4-RI UTAKMICI

0000 1001
+ 0001 1010
+ 0010 1011 -
0011 1100
+ 0100 1101 -
0101 1110 -
0110 1111
- 0111
+ 1000

3 POBEDI

000	0111
0010	1011
0100	1101
1000	1110
$\frac{4}{8}$	$\frac{4}{8}$

35
21
$\boxed{56}$

5 UTAKMICI 3 POBEDI

7 UTAKMICI 4 POBEDI

$\binom{4}{4} = 1 \quad \binom{5}{4} = 5 \quad \binom{6}{4} = 15 \quad \binom{7}{4} = 35$

$\boxed{35+15+5+1 = 56}$

$P_4 = \frac{1}{56}$ $P_5 = \frac{5}{56}$ $P_6 = \frac{15}{56}$ $P_7 = \frac{35}{56}$

$$\boxed{P_Y = \left\{ \frac{1}{56}, \frac{5}{56}, \frac{15}{56}, \frac{35}{56} \right\}} \quad (1)$$

$Y \in \{4, 5, 6, 7\}$

$$H(Y) = \frac{1}{56} \, ld\, 56 + \frac{5}{56} \cdot ld \frac{56}{5} + \frac{15}{56} \, ld \frac{56}{15} + \frac{35}{56} \, ld \frac{56}{35}$$

$\doteq 1.35$

POSLO SUM REZONIRAL NO IMAM GREŠKA VO MOETOT VO MATLAB. POSLEDNATA UTAKMICA MORA DA POBEDI TIMOT „1"

- NAPRAVIV SIMULACIJA VO MATLAB OD KADE OVA GOLE MI STANA APSOLUTNO JASNO!!!
- IMENO SE POKAŽA DEKA VKUPNIOT BROJ NA KOMBINACII KADE TURNIROT SE ZAVRŠUVA SO 4-RI POBEDENI UTAKMICI OD VKUPNO 4 ILI 5 ILI 6 ILI 7 UTAKMICI E 56 KOMBINACII. AKO GI ZEMEŠ VO PREDVID KOMBINACIITE KOGA POBEDUVA DRUGIOT TEAM TOGAŠ IMAŠ 112 KOMBINACII NO BROJOT NA OUTCOMES POGOLEM JA SE DOBIVA ISTATA E DVOJNO VEROJATNOST ZA P_Y KAKO VO (1).

JAS SUM ZEMAL POLLE 1111 DA MOŽE Ø

EDITION 1 SOLUTION:

- THERE ARE 2 (AAAA, BBBB) WORLD SERIES WITH 4 GAMES. EACH HAPPENS WITH PROBABILITY $\left(\frac{1}{2}\right)^4$
- THERE ARE $8 = 2\binom{4}{3}$ WORLD SERIES WITH 5 GAMES. EACH HAPPENS WITH PROBABILITY $\left(\frac{1}{2}\right)^5$

- ODJASNUVANJE ZOŠO ZEMA $\binom{4}{3}$ E TOA ŠTO POSLEDNATA UTAKMICA TREBA DA BIDE POBEDA T.E.:

$\underbrace{????}A$ ILI $????B$

OVDE TREBA DA IM 3xA PA ZATOA $\binom{4}{3}$

- THERE ARE $20 = 2 \cdot \binom{5}{3} = 2 \cdot 10 = 20$ WORLD SERIES WITH 6 GAMES. EACH HAPPENS WITH PROBABILITY $\left(\frac{1}{2}\right)^6$
- THERE ARE: $40 = 2 \cdot \binom{6}{3} = 2 \cdot 20 = 40$ WORLD SERIES WITH 7 GAMES. EACH HAPPENS WITH PROBABILITY $\left(\frac{1}{2}\right)^7$

• $Y \in \{4,5,6,7\}$ $\boxed{P_Y = \left\{2\cdot\left(\frac{1}{2}\right)^4, 8\cdot\left(\frac{1}{2}\right)^5, 20\cdot\left(\frac{1}{2}\right)^6, 40\cdot\left(\frac{1}{2}\right)^7\right\}}$ MMV

$P_Y = \left\{\frac{1}{8}, \frac{1}{4}, \frac{20}{64}, \frac{40}{128}\right\} = \left\{\frac{1}{8}, \frac{1}{4}, \frac{5}{16}, \frac{5}{16}\right\}$

$\boxed{\frac{2+4+10}{16} = \frac{16}{16} = 1}$

$H(Y) = \frac{1}{8}\cdot 3 + \frac{1}{4}\cdot 2 + \left(\frac{5}{16}\, ld \frac{16}{5}\right)\cdot 2 = \frac{3}{8} + \frac{1}{2} + \frac{5}{8}\, ld\frac{16}{5} =$

$= \underline{1.92379} \doteq 1.924$

• $H(X) = ?$ IMA VKUPNO 2 NASTANI SO 4 POSLEDOVATELNI POBEDI, 8 KOMBINACII KADE OD 5 NATPREVARI SE POSTIGNATI 4 POBEDI, 20 KOMBINACII ZA 4 POBEDI OD 6 NATPREVARI I 40 KOMBINACII ZA 4 POBEDI OD 7 NATPREVARI. ZNAČI X MOŽE DA BIDE EDNA OD SITE TIE 70 KOMBINACII. BILO KOJA OD TIE KOMBINACII MOŽE DA SE SLUČI SO VEROJATNOST:

1.) 4 OD 4	$\frac{1}{16}$	3.) 4 OD 6	$1/64$
2.) 4 OD 5	$1/32$	4.) 4 OD 7	$1/128$

$H(X) = 2\cdot\frac{1}{16}\, ld 16 + 8\cdot\frac{1}{32}\, ld 32 + 20\,\frac{1}{64}\, ld 64 + 40\,\frac{1}{128}\, ld 128 =$ MMV

$= \frac{1}{8}\cdot 4 + \frac{1}{4}\cdot 5 + 20\cdot\frac{3}{32} + \frac{40\cdot 7}{128} = \frac{1}{2} + \frac{5}{4} + \frac{15}{8} + \frac{35}{16}$

$= \frac{8+20+30+35}{16} = \frac{28+65}{16} = \frac{93}{16} = 5.8125$

• $H(Y|X) = ?$ $H(Y/X) = \sum_{x\in X} p(x)\, H(Y|X=x)$

$H(Y|X=x) = 0$ — NEMA NEIZVESNOST. AKO GO ZNAEŠ X SIGURNO GO ZNAEŠ Y

• $H(X|Y) = ?$ $\Rightarrow H(X|Y) = \sum_{y\in Y} p(y)\, H(X|Y=y)$

$H(X|Y=4) = 2\cdot\frac{1}{16}\cdot ld 16 = 2\cdot\frac{1}{4} = 1/2$

$H(X|Y=5) = 8\cdot\frac{1}{32}\cdot ld 32 = \frac{1}{4}\cdot 5 = \frac{5}{4}$ $\boxed{H(X|Y=6) = 20\cdot\frac{1}{64}\cdot 6 = \frac{15}{8}}$

$H(X|Y=7) = 40 \frac{1}{128} \cdot 7 = \frac{35}{16}$

$H(X|Y) = p(Y=4) \cdot \frac{1}{2} + p(Y=5) \cdot \frac{5}{4} + p(Y=6) \frac{15}{8} + p(Y=7) \cdot \frac{35}{16}$

$H(X|Y) = \frac{1}{8} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{5}{4} + \frac{5}{16} \cdot \frac{15}{8} + \frac{5}{16} \cdot \frac{35}{16} =$

$= \frac{1}{16} + \frac{5}{16} + \frac{75}{128} + \frac{175}{256} = \frac{6 \cdot 16 + 150 + 175}{256} = \frac{96 + 325}{256}$

$= \frac{421}{256} = \underline{1.64453}$?

$H(X|Y) = H(X,Y) - H(Y) = H(X) + \cancel{H(Y|X)}^{\to 0} - H(Y) =$

$= H(X) - H(Y) = 5.8125 - 1.92379 \doteq \underline{3,889}$

? $\Rightarrow$ ТРЕБА ДА Е ЕДНАКВО НА 3.889. ЗОШТО НЕ Е ЕДНАКВО ?

$\boxed{\text{2.19 Infinite Entropy}}$ THIS PROBLEM SHOWS THAT ENTROPY OF A DISCRETE RANDOM VARIABLE CAN BE INFINITE. LET $A = \sum_{n=2}^{\infty} (n \log^2 n)^{-1}$. [IT IS EASY TO SHOW THAT A IS FINITE BY BOUNDING THE INFINITE SUM BY INTEGRAL OF $(x \log^2 x)^{-1}$]. SHOW THAT THE INTEGER-VALUED RANDOM VARIABLE X DEFINED BY $Pr(X=n) = (A\, n \log^2 n)^{-1}$ for $n = 2, 3, \ldots$ HAS $H(X) = +\infty$

- APROXIMATION OF SUM BY DEFINITE INTEGRALS

$$\int_{s=a-1}^{b} f(s)\,ds \le \sum_{i=a}^{b} f(i) \le \int_{s=a}^{b+1} f(s)\,ds \quad \text{FOR INCREASING FUNCTION.}$$

$$\int_{s=a}^{b+1} f(s)\,ds \le \sum_{i=a}^{b} f(i) \le \int_{s=a-1}^{b} f(s)\,ds \quad \text{for DECREASING FUNCTION.}$$

NMV

$$\int_{1+\varepsilon}^{\infty} \frac{1}{x \log^2 x} dx = \frac{1}{\ln(1+\varepsilon)} \qquad \varepsilon > 0$$

$$\boxed{\lim_{\varepsilon \to 0} \frac{1}{\ln(1+\varepsilon)} = +\infty}$$

$$\Pr(x=n) = (A\, n\, ld^2 n)^{-1} \qquad H(X) = +\infty$$
$$n = 2, 3, \dots$$
$$A = \sum_{n=2}^{\infty} \left(n\, ld^2(n)\right)^{-1}$$

$$H(X) = \sum_{n=2}^{\infty} p(x)\, ld \frac{1}{p(x)} = -\sum_{n=2}^{\infty} p(x)\, ld\, p(x) =$$
$$= -\sum_{n=2}^{\infty} (A\, n\, ld^2 n)^{-1}\, ld\, (A\, n\, ld^2 n)^{-1} = -\frac{1}{A}\sum_{n=2}^{\infty} \frac{1}{n\, ld^2 n}\left[ld \frac{1}{A} + ld \frac{1}{n\, ld^2 n}\right] = +\underbrace{\frac{ld A}{A}\sum_{n=2}^{\infty}\frac{1}{n\, ld^2 n}}_{S1} + \underbrace{\frac{1}{A}\sum_{n=2}^{\infty}\frac{1}{n\, ld^2 n}\, ld(n\, ld^2 n)}_{S2}$$

- POZORNUVANJE log sum inequality

$a_1, a_2, \dots a_n$ & $b_1, b_2, \dots b_n$ } NONNEGATIVE

$$\sum_{i=1}^{n} a_i\, ld \frac{a_i}{b_i} \geq \left(\sum_{i=1}^{n} a_i\right) ld \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}$$

WITH EQUALITY if $\frac{a_i}{b_i} = \text{const}$

PROOF: $f(t) = t\, ld\, t$ is STRICTLY CONVEX

BY JENSEN'S INEQUALITY $E[f(x)] \geq f(E[x])$

$$\sum \alpha_i f(t_i) \geq f\left(\sum \alpha_i t_i\right) \qquad \alpha_i \geq 0 \quad \sum_i \alpha_i = 1$$

$$\alpha_i = \frac{b_i}{\sum_{j=1}^{n} b_j} \quad \& \quad t_i = \frac{a_i}{b_i}$$

$$\sum \frac{b_i}{\sum b_j}\, ld \frac{a_i}{b_i} \geq ld\left[\sum \frac{\cancel{b_i}}{\sum b_j} \cdot \frac{a_i}{\cancel{b_i}}\right] = ld \sum \frac{a_i}{\sum b_j}$$

$$= \left| \sum_i \alpha_i = 1 \right| = \sum \alpha_i\, ld \sum \frac{a_i}{\sum b_j} = \sum \alpha_i\, ld \frac{\sum a_i}{\sum b_i}$$

$$D(p\|q) = \sum p(x)\, ld \frac{p(x)}{q(x)} \geq \underbrace{\left(\sum p(x)\right)}_{1} ld \frac{\sum p(x)}{\sum q(x)} = 1\, ld 1 = 0$$

$$\underline{D(p\|q) \geq 0}$$

$$H(x) = -\sum_{n=2}^{\infty} p(x)\, ld \left(A \cdot n\, ld^2 n\right)^{-1} \doteq -\sum_{n=2}^{\infty} p(x)\, ld \frac{\left(\sum_{n=2}^{\infty} (n\, ld^2(n))\right)^{-1}}{n\, ld^2 n}$$

$$= -\sum_{n=2}^{\infty} p(x)\, ld \frac{1}{n\, ld^2 n \sum_{n=2}^{\infty} (n\, ld^2(n))^{-1}}$$

$$S_1 = \frac{ld A}{A} \underbrace{\sum_{n=2}^{\infty} \frac{1}{n\, ld^2 n}}_{A} = \frac{ld A}{A} \cdot A = \underline{\underline{ld A}}$$

$$S_2 = \frac{1}{A} \sum_{n=2}^{\infty} \frac{1}{n\, ld^2 n} ld(n\, ld^2 n) \doteq \underline{\underline{\frac{1}{A} \cdot H(n\, ld^2 n)}}$$

$$H(n\, ld^2 n) = \sum_{n=1}^{\infty} \frac{1}{n\, ld^2 n} ld(n\, ld^2 n) = \underbrace{\frac{1}{1 \cdot 0} ld 0}_{\infty} + S_2$$

$$S_2 = E[ld\, Y] \qquad \boxed{Y = n\, ld^2 n}$$

$$E[ld\, Y] \geq ld\{E[Y]\} = \sum_{n=2}^{\infty} \frac{1}{n\, ld^2 n} (\cancel{n}\, ld^2 n) = 1$$

$$\boxed{H(x) \geq ld A + 1}$$

$$ld A = ld\left[\sum_{n=2}^{\infty} \frac{1}{n\, ld^2(n)}\right] = \left(\sum \frac{1}{n\, ld^2 n}\right) ld \frac{\sum \frac{1}{n\, ld^2(n)}}{\sum_{1} b_i}$$

$$\leq \sum_{n=2}^{\infty} \frac{1}{n\, ld^2 n}\, ld \frac{1}{n\, ld^2 n}$$

$$\Pr(x=n) = \frac{1}{A\, n\, ld^2 n} \qquad n = 2, 3, \ldots$$

$$\sum_{n=2}^{\infty} \frac{1}{A\, n\, ld^2 n} = 1 \qquad \frac{1}{A} \sum_{n=2}^{\infty} \frac{1}{n\, ld^2 n} = 1$$

$$\boxed{A = \sum_{n=2}^{\infty} \frac{1}{n\, ld^2 n}}$$

· Vo MAPLE GRAFIČKI SE POKAZUVA DEKA

$A \doteq 1$

$$x \in \{2, 3, 4, 5, \ldots\}$$

$$P_x \doteq \left\{ \frac{1}{2}, \frac{1}{3}\cdot\frac{1}{ld^2 3}, \frac{1}{16}, \frac{1}{5}\cdot\frac{1}{ld^2 5}, \ldots \right\}$$

$$x = x_8 = 8 \qquad P_x = \frac{1}{72} = \frac{1}{8 \cdot ld 8}$$

$$P_Y = \left\{ \frac{1}{2}, \frac{1}{16}, \frac{1}{72}, \ldots \right\} = \frac{1}{2^n \cdot n^2} \qquad n = 1, \ldots$$

$$H(Y) = \sum_{n=1}^{\infty} \frac{1}{2^n \cdot n^2} \cdot ld(2^n \cdot n^2) = \sum_{n=1}^{\infty} \frac{1}{2^n \cdot n^2} \cdot (n \cdot \cancel{ld 2}^{\,1} + 2\, ld\, n)$$

$$H(Y) = \underbrace{\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n}}_{\ln 2} + \underbrace{\sum_{n=1}^{\infty} \frac{1}{2^{n-1} \cdot n^2}\, ld\, n}_{0.1943}$$

$$H(x) = \sum_{n=2}^{\infty} \frac{1}{A\, n\, ld^2 n} \cdot ld(A \cdot n\, ld^2 n) = \frac{1}{A} \sum_{n=2}^{\infty} \frac{1}{n\, ld^2 n} \left(ld A + ld\, ld^2 n^{\sqrt{n}} \right)$$

$$= \frac{1}{A} \cdot ld A \underbrace{\sum_{n=2}^{\infty} \frac{1}{n\, ld^2 n}} + \frac{1}{A} \sum_{n=2}^{\infty} \frac{ld\left(ld^2 n^{\sqrt{n}}\right)}{n\, ld^2 n}$$

$$= ld A + \frac{1}{A} \sum_{n=2}^{\infty} \frac{2\, ld\, ld\, n^{\sqrt{n}}}{n\, ld^2 n}$$

$$= \operatorname{ld} A + \frac{2}{A}\sum_{n=2}^{\infty} \frac{\operatorname{ld}(\sqrt{n}\cdot \operatorname{ld} n)}{n \operatorname{ld}^2 n} = \operatorname{ld} A + \frac{2}{A}\sum_{n=2}^{\infty} \frac{\frac{1}{2}\operatorname{ld} n + \operatorname{ld}(\operatorname{ld} n)}{n \operatorname{ld}^2 n}$$

$$= \operatorname{ld} A + \frac{2}{A}\cdot\frac{1}{2}\sum_{n=2}^{\infty} \frac{1}{n \operatorname{ld} n} + \frac{2}{A}\sum_{n=2}^{\infty} \frac{\operatorname{ld}\left[\operatorname{ld}\left(2^{\operatorname{ld} n}\right)\right]}{n \operatorname{ld}^2 n}$$

$$x = e^{\ln x} \,/\, \ln \qquad \ln x = \ln x \quad \Leftarrow$$

$$x = 2^{\operatorname{ld} x} \,/\, \operatorname{ld} \qquad \operatorname{ld} x = \operatorname{ld} x$$

$$H(X) = \frac{1}{A}\sum_{n=2}^{\infty} \frac{1}{n \operatorname{ld}^2 n}\left(\operatorname{ld} A n + \operatorname{ld} \operatorname{ld}^2 n\right) = \operatorname{ld} A + \underbrace{\frac{1}{A}\sum_{n=2}^{\infty} \frac{1}{n \operatorname{ld} n}}_{\to \infty \text{ (MANJE)}} +$$

$$+ \underbrace{\frac{2}{A}\sum_{n=2}^{\infty} \frac{\operatorname{ld}(\operatorname{ld} n)}{n \operatorname{ld}^2 n}}_{\to \infty \text{ (MANJE)}}$$

$$\int_2^{\infty} \frac{dx}{x \operatorname{ld} x} = \int_2^{\infty} \frac{1}{\operatorname{ld} x}\, d(\ln x) = \operatorname{ld} 2 \int_2^{\infty} \frac{1}{\ln x}\, d(\ln x)$$

$$= \operatorname{ld} 2 \, \ln(\ln x)\Big|_2^{\infty} = \operatorname{ld} 2 \left[\underbrace{\ln(\ln \infty)}_{\infty} - \ln(\ln 2)\right] = \infty$$

$$\sum_{n=2}^{\infty} \frac{1}{n \operatorname{ld} n} > \int_2^{\infty} \frac{dx}{x \ln x} = \infty \Rightarrow \sum_{n=2}^{\infty} \frac{1}{n \operatorname{ld} n} \to \infty$$

VIDI PP.11

$$\Rightarrow H(X) = \operatorname{ld} A + \frac{1}{A}\sum_{n=2}^{\infty} \frac{1}{n \operatorname{ld} n} + \frac{2}{A}\sum_{n=2}^{\infty} \frac{\operatorname{ld}(\operatorname{ld} n)}{n \operatorname{ld}^2 n} \to \infty$$

2.20 RUN-LENGTH CODING! Let $X_1, X_2, \ldots X_n$ be (possibly dependant) binary random variables. Suppose that one calculates the run lengths $R = (R_1, R_2, \ldots)$ of this sequence (in order as they occur). For example, the sequence $X = 0001100100$ yields run lengths $R = (3,2,2,1,2)$. Compare $H(X_1, X_2, \ldots, X_n)$, $H(R)$, and $H(X_n, R)$. Show all equalities and inequalities, and bound all differences.

$$H(X) = H(X, f(X)) = H(X) + \underbrace{H(f(X)|X)}_{\emptyset}$$

FANO'S INEQUALITY REVISITED

$g(Y) = \hat{X} \qquad X \to Y \to \hat{X}$

$$\boxed{P_e = P_r\{\hat{X} \neq X\}}$$

$$\boxed{H(P_e) + P_e \log|X| \geq H(X|\hat{X}) \geq H(X|Y)}$$

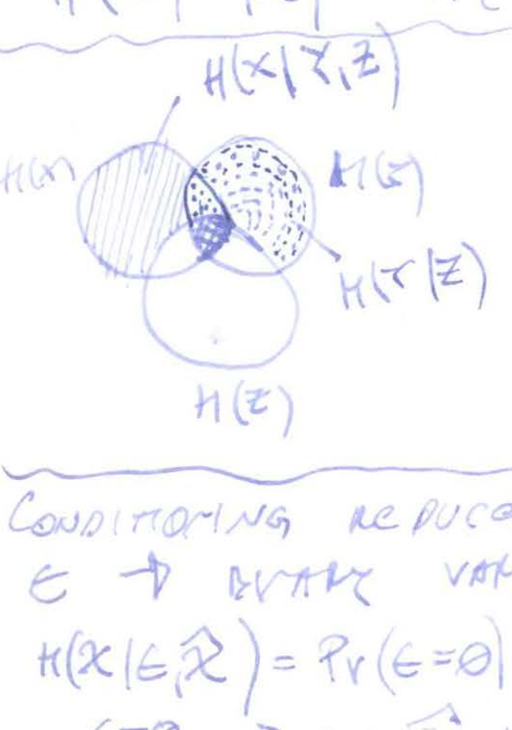
can be weakened to: $1 + P_e \operatorname{ld}|X| \geq H(X|Y)$

$$P_e \geq \frac{H(X|Y) - 1}{\operatorname{ld}|X|}$$

Proof:

$$E = \begin{cases} 1 & \text{if } \hat{X} \neq X \\ 0 & \text{if } \hat{X} = X \end{cases}$$

$$H(E, X|\hat{X}) = \overbrace{H(E|\hat{X})}^{\leq H(P_e)} + \overbrace{H(X|\hat{X}, E)}^{\leq P_e \operatorname{ld}|X|} = H(X|\hat{X}) + \underbrace{H(E|X, \hat{X})}_{\emptyset} \quad (*)$$



$$H(X, Y|Z) = H(X|Y, Z) + H(Y|Z)$$

$$H(X|Y, Z) = H(X|Z) - I(X;Y|Z)$$

$$I(X;Y|Z) = I(X;Y,Z) - I(X;Z)$$

$$I(X;Y,Z) = I(X;Z) + I(X;Y|Z)$$

Conditioning reduces entropy: $H(E|\hat{X}) \leq H(E) = H(P_e)$

$E \to$ binary variable $P_e = P_r\{\hat{X} \neq X\} = P_r\{E = 1\}$

$$H(X|E, \hat{X}) = P_r(E=0) \cdot H(X|\hat{X}, E=0) + P_r(E=1) \cdot H(X|\hat{X}, E=1)$$

$E = 0 \Rightarrow X = \hat{X} \quad H(X|\hat{X}, E=0) = \emptyset$

$$H(X|E,\hat{X}) = \underbrace{Pr(E=0)}_{1-Pe}\cdot 0 + Pe\, H(X|\hat{X}, E=1) \leq Pe\, H(X) \leq \underline{Pe\, ld|X|}$$

$\Rightarrow$ /(*)/ $H(X|\hat{X}) \leq H(Pe) + Pe\, ld|X|$

- DATAPROCESSING INEQUALITY:

$X \to Y \to \hat{X}$ IS MARKOV CHAIN

$$I(X;Y) \geq I(X;\hat{X}) \qquad \cancel{H(X)} - H(X|Y) \geq \cancel{H(X)} - H(X|\hat{X})$$

$$H(X|Y) \leq H(X|\hat{X}) \Rightarrow \boxed{H(Pe) + Pe\, ld|X| \geq H(X|Y)}$$

COROLLARY: FOR ANY TWO RANDOM VARIABLES X & Y WITH $p = Pr(X \neq Y)$

$$\boxed{H(p) + p \log|X| \geq H(X|Y)}$$

PROOF: LET $\hat{X} = Y$ IN FANO'S INEQUALITY.

- FOR ANY RANDOM VARIABLES X & Y, IF ESTIMATOR $g(Y)$ TAKES VALUES IN THE SET „X"

$$H(Pe) + Pe \log(|X|-1) \geq H(X|Y)$$

$$H(X,E|\hat{X}) = H(X|\hat{X}) + \underbrace{H(E|X,\hat{X})}_{} = H(E|\hat{X}) + H(X|\hat{X},E)$$

$$H(X|\hat{X},E) = \underbrace{Pr(E=0)}_{(1-p)}\cdot \underbrace{H(X|\hat{X},E=0)}_{0} + Pr(E=1)\underline{H(X|\hat{X},E=1)}$$

$$H(X|\hat{X},E=1) \leq H(X) \leq ld(|X|-1)$$

ONLY ERRORED OUTCOMES

- PROBABILITY OF $X=$ AND RELATED INEQUALITIES. WERE X, X' ARE I.I.D.

$$Pr(X=X') = \sum_x p^2(x)$$

$$X \in \{x_1, x_2, \dots x_n\} \qquad X' \in \{x_1, x_2, \dots x_n\}$$

$$P_X = P_{X'} = \{p_1, p_2, \dots p_n\}$$

$$P(X'=X) = p_1 \cdot p_1 + p_2 \cdot p_2 + \dots + p_n \cdot p_n = \sum_{i=1}^{n} p^2(x_i)$$

$Pr(X=x_1)$ $P(X'=x_1)$ $Pr(X=x_n)$ $P(X'=x_n)$

LEMMA 2.10.1: IF X & X' ARE I.I.D WITH ENTROPY $H(X)$ THEN $P_r(X=X') \geq 2^{-H(X)}$ WITH EQUALITY IF AND ONLY IF X HAS UNIFORM DISTRIBUTION.

PROOF: SUPPOSE THAT $X \sim p(x)$. BY JENSEN'S INEQUALITY WE HAVE:

$$2^{E[\operatorname{ld} p(x)]} \leq E\left[2^{\operatorname{ld} p(x)}\right]$$

$$2^{-H(X)} = 2^{\sum p(x) \operatorname{ld} p(x)} \leq \sum p(x)\, 2^{\operatorname{ld} p(x)} = \sum p^2(x)$$

$$\Rightarrow \boxed{\sum p^2(x) \geq 2^{-H(x)}} \quad \text{PROVED!!!}$$

COROLLARY: LET X, X' BE INDEPENDENT WITH $X \sim p(x)$, $X' \sim r(x)$ $\quad x, x' \in \mathcal{X}$ THEN

$$P_r(X = X') \geq 2^{-H(p)-D(p\|r)}$$

$$P_r(X = X') \geq 2^{-H(r)-D(r\|p)}$$

Proof:

$$2^{-H(p)-D(p\|r)} = 2^{-\sum p(x)\operatorname{ld}\frac{1}{p(x)} - \sum p(x)\operatorname{ld}\frac{p(x)}{r(x)}}$$

$$= 2^{\sum p(x)\operatorname{ld} p(x) + \sum p(x)\operatorname{ld}\frac{r(x)}{p(x)}} = 2^{\sum p(x)\operatorname{ld} r(x)} \leq$$

$$\leq \sum p(x)\, 2^{\operatorname{ld} r(x)} = \sum p(x)\cdot r(x) = \underline{P_r(X = X')}$$

- $X_1, X_2, \dots X_i \dots X_n$ POSIBLY DEPENDANT BINARY RANDOM VARIABLES

$R = (R_1, R_2, \dots)$ - RUN LENGHT EG. $\underline{R = (3,2,2,1,2)}$

$H(X_1, X_2, \dots X_n)$ VS. $H(R)$ VS. $H(X_n, R)$

$$H(X_1, X_2, \dots X_n) = \sum_{i=1}^{n} H(X_i | X_{i-1}, \dots X_1)$$

E.G
$X = \{00011001000\}$

$P_r(X_1 = 0) = 1-p$, $P_r(X_1 = 1) = p$

$$P_R = \left\{(1-p)^3, p^2, (1-p)^2, p, (1-p)^3\right\}$$

$\boxed{p = \frac{1}{2}}$

$$P_R = \left\{\left(\frac{1}{2}\right)^{r_1}, \left(\frac{1}{2}\right)^{r_2}, \dots \left(\frac{1}{2}\right)^{r_n}\right\}$$

$$\sum_{i=1}^{n} r_i = n$$

$$Pr(R=3) = Pr(x_1=0)\cdot Pr(x_2=0)\cdot Pr(x_3=0) = (1-p)^3$$

$$H(R) = -\sum_{i=1}^{n} P(R_i)\, ld\, P(R_i)$$

ПРЕСМЕТКА ЗА КАКО ГОЛЕМИ СЕ ...

$$X_i = \{x_{i1}, x_{i2}, \dots x_{in}\} \qquad |X| = 2^n$$

$$P_{x_i} = \left(\frac{1}{2}\right)^n \qquad \sum_{i=1}^{2^n} P_i = \frac{1}{2^n}\cdot 2^n = 1$$

$$x_i \in \{0,1\} = \{s_1, s_2\}\ ;\ z=2\ ;\ i=1,2,\dots n$$

$$H(x_1,x_2,\dots x_n) = -\sum_{x_1\in X_1}\sum_{x_2\in X_2}\dots\sum_{x_n\in X_n} P(x_1,x_2,\dots x_n)\, ld\, P(x_1,\dots x_n)$$

$$= -\sum_{x_1\in X_1}\sum_{x_2\in X_2}\dots\sum_{x_n\in X_n} P(x_1,x_2,\dots x_n)\left[ld\,P(x_1) + ld\,P(x_2)\dots ld\,P(x_n)\right]$$

$$= \sum_{x_1\in X_1} ld\,P(x_1)\sum_{x_2,x_3\dots x_n} P(x_1,x_2,\dots x_n) + \sum_{x_2} ld\,P(x_2)\sum_{x_1,x_3\dots x_n} P(x_1,x_2,\dots x_n)$$

$$\dots + \sum_{x_n} ld\,P(x_n)\sum_{x_1,x_2\dots x_{n-1}} P(x_1,x_2,\dots x_n) = \sum_{x_1\in X_1} P(x_1)\, ld\,P(x_1) +$$

$$+ \sum_{x_2\in X_2} P(x_2)\, ld\,P(x_2) + \dots + \sum_{x_n\in X_n} P(x_n)\, ld\,P(x_n) = n\cdot H(p)$$

$$\underline{H(x_1,x_2,x_3\dots x_n) = n\cdot H(p)}$$ АКО $x_1,x_2\dots x_n$ ARE INDEPENDANT

$$Pr(x_i=1) = p \qquad Pr(x_i=0) = 1-p$$

$H(R) = ?$ $\quad \underline{\underline{2^n \text{ - КОМБИНАЦИИ}}}$ $\quad P(R) = \dfrac{1}{2^n}$ АКО СЕ РАВНОМЕРНО ВЕРОЈАТНИ

$$H(R) = -\sum P(R)\, ld\,P(R) = +2^n\cdot\frac{1}{2^n}\, ld\, 2^n = \underline{\underline{n}}$$

$$H(x_1,x_2\dots x_n) = n\cdot H(p) = H(R)\cdot H(p)$$

$$H(p) = -p\, ld\, p - (1-p)\, ld(1-p) \qquad \text{FOR } p=\tfrac{1}{2} \Rightarrow$$

$$H(x_1,x_2,\dots x_n) = H(R)$$

$H(x_n, R) = ?$

$$H(x_1, x_2, \dots x_n) = \sum_{i=1}^{n} H(x_i \mid x_{i-1}, \dots x_1) = -\sum_{x_1 \dots x_2} P(x_1, x_2 \dots x_n)\,(ld\,P(x_1 \dots x_n))$$

$$H(x_1, x_2, \dots x_n) \geq \sum_{i=1}^{n} H(x_i)$$

$$H(0,0,1) = H(2,1) \qquad H(0,0,1) = H(0) + H(0|0) + H(1|0,0)$$

$$H(2,1) = H(2) + H(1|2)$$

$$H(R_1, R_2, \dots R_n) = \underbrace{H(R_1)}_{H(x_1, x_2 \dots x_{R_1})} + \underbrace{H(R_2|R_1)}_{H(x_{R_1+1}, x_{R_1+2}, \dots x_{R_1+R_2} / x_1, x_2, \dots x_{R_1}) =} + \overbrace{H(R_3|R_1, R_2)}^{*} + \dots$$

$$* = H(x_{R_1+R_2+1}, \dots x_{R_1+R_2+R_3} / x_1, x_2, \dots x_{R_1+R_2})$$

$$H(x_1, x_2 / x_3 x_4) = H(x_1, x_2, x_3 x_4) - H(x_3 x_4)$$

$$= \cancel{H(x_1, x_2, \dots x_{R_1})} + \cancel{H(x_1, x_2, \dots x_{R_1+R_2})} - \cancel{H(x_1, x_2, \dots x_{R_1})} +$$
$$+ \cancel{H(x_1, x_2 \dots x_{R_3})} - \cancel{H(x_1, x_2, \dots x_{R_1+R_2})} + H(x_1, x_2, \dots x_{R_4}) -$$
$$- \cancel{H(x_1, x_2, \dots x_{R_3})} \dots + H(x_1, x_2, \dots x_n)$$

ZNAČI: $\boxed{H(x_1, x_2, \dots x_n) = H(R_1, R_2, R_3 \dots)}$

$$H(x_n, R) = H(R) + H(x_n|R) = H(x_n) + H(R|x_n)$$

$$H(x_n) = H(p) \qquad x_n = \begin{cases} 1 & \text{WITH } Pr = p \\ 0 & \text{WITH } Pr = 1-p \end{cases}$$

$$H(R|x_n) = p \cdot H(R/1) + (1-p) H(R/0) = H(R)$$

$$H(x_n, R) = H(x_n) + H(R) = H(p) + H(R).$$

Addition 1 SOLUTION: SINCE THE RUN LENGTHS ARE FUNCTIONS OF $x_1, x_2, \dots x_n \Rightarrow$ $H(R) \leq H(x)$. ANY x_i TOGETHER WITH RUN LENGHT DETERMINE THE ENTIRE SEQUENCE $x_1, x_2, \dots x_n$ hence:

$$H(x_i, R) = H(x_i) + \underbrace{H(R|x_i)}_{\emptyset \Rightarrow \text{АКО ГО ЗНАЕШ } x_i \text{ ГО ЗНАЕШ } R} = H(x_i)$$

САКА ДА КАЖЕ:

$$H(x_1, x_2 \dots x_n, R) = H(x_1, x_2, \dots x_n) + \underbrace{H(R|x_1, x_2, \dots x_n)}_{\emptyset}$$

$$\boxed{H(\underbrace{x_1, x_2, \dots x_n}_{x_i}, R) = H(x_1, x_2, \dots x_n)}$$

CONDITIONING REDUCES ENTROPY

$$H(x_i, R) = H(x_i) \qquad H(x_i) = H(x_i, R) = H(R) + H(x_i|R) \leq$$

$$\leq H(R) + H(x_i)$$

ОВА Е ЛЕЗОН, КАДЕ ЗЕМАМ ДЕКА $x_i \stackrel{def}{=} (x_1, x_2, \dots x_n)$

- Sekak vo rešenieto of Edition 1 pod x_i poprazni kanat: $x_i = \begin{cases} 1 & Pr(1) = p \\ 0 & Pr(0) = 1-p \end{cases}$

ДЕМЕК АКО $x_i = 1$, ЗНАЕШ ДЕКА ЗА ТОА НАСТАН ПОЛЗИ... НА КОНТОКУВАЊЕ ИЗНЕСУВА

$R = (3, 2, 2, 1, 2) \qquad x = (111, 00, 11, 0, 11)$

ЗНАЧИ ЗА МАКСИМУМ $(x_1, x_2, \dots x_n)$ ЗА ЗАПИС:

$$H(x_1, x_2, \dots x_n) = H(x_i, R) = H(R) + H(x_i|R)$$

$$\leq \left| \begin{matrix}\text{CONDITIONING} \\ \text{REDUCES entropy}\end{matrix} \right| \leq H(R) + H(x_i) \leq H(R) + 1$$

$$H(x_i) = -p\, ld\, p - (1-p)\, ld\, p \qquad H(x_i)\Big|_{max} = 1 \quad \text{ЗА } p = \tfrac{1}{2}$$

$\boxed{2.21}$ MARKOV INEQUALITY FOR PROBABILITIES.
LET $p(x)$ BE A PROBABILITY MASS FUNCTION
PROVE, FOR ALL $d \geq 0$, THAT

$$Pr\{p(X) \leq d\}\, ld \frac{1}{d} \leq H(X)$$

$$Pr(p(X) \leq d) = ? \qquad x \in \{x_1, x_2, \dots x_n\}$$

$$p(x) = \{p_1, p_2, \dots p_n\}$$

$$Pr(p(x) \leq d) = Pr(p_1 \leq d) \cdot Pr(p_2 \leq d) \cdots Pr(p_n \leq d)$$

$$H(x) = -\sum_{i=1}^{n} p_i\, ld\, p_i = -E[ld\, p_i] \geq -ld\, E[p_i]$$

$$E[ld\, p_i] \leq \left| \begin{matrix} ld\ e \\ \text{КОНКАВНА} \end{matrix} \right| \leq ld\, E[p_i]$$

$$H(\mathcal{X}) \geq -\mathrm{ld}\, E[p_i] \qquad E[p_i] = \sum_{i=1}^{n} p_i^2$$

$$H(\mathcal{X}) \geq -\mathrm{ld}\left\{\sum_{i=1}^{n} p_i^2\right\} = ?$$

$$\sum_{i=1}^{n} p_i^2 \leq 1$$

$$D(p\|q) = \sum p \,\mathrm{ld}\, \frac{p}{q} = -\sum p\, \mathrm{ld}\, \frac{1}{p} - \sum p\, \mathrm{ld}\, \frac{1}{q}$$

$q = \frac{1}{|\mathcal{X}|}$ } uniform

$x \in \{x_1, x_2, \dots x_n\}$

$p(x) \in \{p_1, p_2, \dots p_n\}$

$q(x) \in \{q_1, q_2, \dots q_n\}$

$$H(\mathcal{X}) = \sum_{i=1}^{|\mathcal{X}|} q(x)\, \mathrm{ld}\, \frac{1}{q(x)} = \sum_{i=1}^{|\mathcal{X}|} \frac{1}{|\mathcal{X}|} \cdot \mathrm{ld}|\mathcal{X}| = |\mathcal{X}| \cdot \frac{1}{|\mathcal{X}|} \cdot \mathrm{ld}|\mathcal{X}| = \mathrm{ld}|\mathcal{X}|$$

$$I(\mathcal{X}, \mathcal{Y}) = \sum_{\substack{x \in \mathcal{X} \\ y \in \mathcal{Y}}} p(x,y)\, \mathrm{ld}\, \frac{p(x,y)}{p(x) \cdot p(y)} =$$

$$= \underbrace{\sum_{x,y} p(x,y)\, \mathrm{ld}\, \frac{1}{p(x)}}_{H(\mathcal{X})} + \underbrace{\sum_{x,y} p(x,y)\, \mathrm{ld}\, \frac{1}{p(y)}}_{H(\mathcal{Y})} - \underbrace{\sum_{x,y} p(x,y)\, \mathrm{ld}\, \frac{1}{p(x,y)}}_{H(\mathcal{X}, \mathcal{Y})}$$

$$I(\mathcal{X}, \mathcal{Y}) = H(\mathcal{Y}) + \cancel{H(\mathcal{X})} - \cancel{H(\mathcal{X})} - H(\mathcal{Y}|\mathcal{X}) =$$
$$= H(\mathcal{Y}) - H(\mathcal{Y}|\mathcal{X})$$

$$I(\mathcal{X}, \mathcal{Y}) = D(p(x,y) \,\|\, p(x)\, p(y)) = \sum_{x,y} p(x,y)\, \mathrm{ld}\, \frac{p(x,y)}{p(x)\, p(y)}$$

$$\mathcal{Y} = \{p_1, p_2, \dots, p_n\}, \quad \boxed{p_1 < p_2 < p_3 < \dots < p_n} \; (*)$$

$$\Pr(\mathcal{Y} \leq 1) = \sum_{i=1}^{m} p_i(x) \qquad H(\mathcal{X}) = -\sum_{i=1}^{n} p(x)\, \mathrm{ld}\, p(x)$$

$$\Pr(\mathcal{Y} \leq 1) = \sum_{i=1}^{\lambda \leq 1} p_i(x) = 1 = P(p(x) \leq 1)\big|_{max}$$

$$H(X) = \sum_{x=1}^{n} p(x)\, ld \frac{1}{p(x)} = E\left[ld \frac{1}{p(x)} \right] = -E\left[ld\, p(x) \right] \geq$$

$$\geq - ld \left\{ E\left[p(x)\right] \right\} = - ld \left\{ \sum_{i=1}^{n} p_i^2 \right\}$$ POMNOŽENI KAKO VO ⊕

$$\sum_{i=1}^{n} p_i(x) = p_1 + p_2 + \cdots p_n = \frac{d}{2} \qquad p_n = d$$

$$\left(\sum_{i=1}^{n} p_i^2 \right) \leq \left(\sum_{i=1}^{n} p_i \right)^2 \quad / (ld)$$

$$ld \left(\sum_{i=1}^{n} p_i^2 \right) \leq ld \left(\sum_{i=1}^{n} p_i \right)^2 \qquad - ld \sum_{i=1}^{n} p_i^2 \geq - ld \left(\sum_{i=1}^{n} p_i \right)^2$$

$$H(X) \geq - ld \left[\sum_{i=1}^{n} p_i^2 \right] \geq - ld \left(\sum_{i=1}^{n} p_i \right)^2 = -2\, ld \sum_{i=1}^{n} p_i$$

$$\geq -2\, ld \frac{d}{2} = 2\, ld \frac{2}{d} = 2 + ld \frac{1}{d}$$

Edition 4 SOLUTION

$$P(p(x) < d) \cdot ld \frac{1}{d} = \sum_{x: p(x) < d} p(x) \cdot ld \frac{1}{d} \leq \sum_{x: p(x) < d} p(x)\, ld \frac{1}{p(x)}$$

$\leq$ / OVA VAŽI ZOŠTO $p(x) < d \quad ld \frac{1}{p(x)} > ld \frac{1}{d}$ / $\leq$

$$\sum_{x} p(x)\, ld \frac{1}{p(x)} = \underline{\underline{H(X)}}$$

ZA SITE VREDNOSTI NA X SUMATA E SIGURNO POGOLEMA OD ONA PRETHODNO KADE IMAŠ SAMO PODMNOŽESTVO OD „X"

$$\boxed{H(X) \geq P(p(x) < d) \cdot ld \frac{1}{d}}$$ → MARKOV INEQUALITY

PROBLEM 2.22 LOGICAL ORDER OF IDEAS. IDEAS HAVE BEEN DEVELOP IN ORDER OF NEED AND THEN GENERALIZED IF NECESSARY. REORDER FOLLOWING IDEAS, STRONGEST FIRST, IMPLICATIONS FOLLOWING:

(a) CHAIN RULE FOR $I(X_1, \dots X_n; Y)$, CHAIN RULE FOR $D(p(x_1, x_2, \dots x_n) \| q(x_1, x_2, \dots x_n))$, AND CHAIN RULE FOR $H(X_1, X_2, \dots X_n)$

(b) $D(f \| g) \geq 0$, JENSEN'S INEQUALITY, $I(X;Y) \geq 0$

$$D(p(x) \| q(x)) = \sum p(x) \, ld \frac{p(x)}{q(x)} \qquad \overbrace{q(x) = \frac{1}{|X|}}^{\text{UNIFORM}}$$

$$H(X) = \left| X \sim \text{Uniform} \right| = \sum \frac{1}{|X|} ld |X| = |X| \cdot \frac{1}{|X|} \cdot ld|X| = \underline{ld|X|}$$

$$D(p(x) \| q(x)) \geq 0 \qquad -\sum_x p(x) \, ld \frac{1}{p(x)} + \sum p(x) \, ld \frac{1}{q(x)} \geq 0$$

$$H(X) \leq \sum p(x) \, ld \frac{1}{q(x)} = ld|X| \cdot \sum p(x) = ld|X|$$

$|X|$

MAXIMUM ENTROPY RECALL

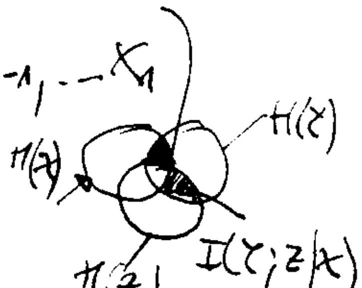
- CHAIN RULE FOR ENTROPY:

$$H(X_1, X_2, \dots X_n) = \sum_{i=1}^{n} H(X_i | X_{i-1}, \dots X_1)$$

- CHAIN RULE FOR MUTUAL ENTROPY

$$I(X_1, X_2, \dots X_n; Y) = \sum_{i=1}^{n} I(X_i; Y | X_{i-1}, \dots X_1)$$

$$I(X, Y; Z) = I(X; Z) + I(Y; Z | X)$$

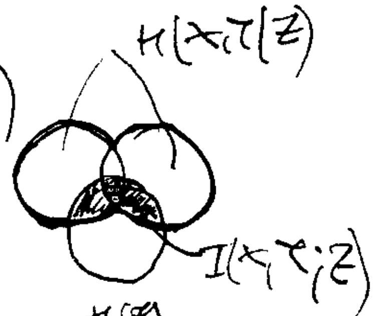


$$H(X, Y) = H(X) + H(Y|X)$$

$$H(X, Y, Z) = H(X) + H(Y|X) + H(Z|X, Y)$$

$$I(X, Y; Z) = H(X, Y) - H(X, Y | Z)$$

$$H(X, Y) - H(X, Y|Z) = H(X) - H(X|Z) + H(Y|X) - H(Y|X, Z)$$



$$D(p(x_1,x_2,\dots x_n)\|q(x_1,x_2,\dots x_n)) = \sum_{x_1 x_2\dots x_n} p(x_1,x_2\dots x_n)\, ld\, \frac{p}{q}$$

$$= -\underbrace{\sum_{x_1 x_2\dots x_n} p(x_1,x_2,\dots x_n)\, ld\, \frac{1}{p(x_1 x_2\dots x_n)}}_{H(x_1,x_2\dots x_n)} + \sum_{x_1 x_2\dots x_n} p(x_1\dots x_n)\, ld\, \frac{1}{q(x_1\dots}$$

IF $q_1(x_1,\dots x_n) = p(x_1)\cdot p(x_2)\cdots p(x_n) \Rightarrow$

$$D(p\|q) = -H(x_1,x_2,\dots x_n) + H_1(x_1) + H_2(x_2)\dots H_n(x_n)$$

(6) JENSEN'S INEQUALITY, $D(p\|q)\geq 0$, $I(X;Y)\geq 0$
(1) (2) (3)

(1) E NAJAKO ZOŠTO SE KORISTI ZA DOKAŽUVANJE NA (2), A (2) SE KORISTI ZA DOKAŽUVANJE NA (3)

(4) $H(x_1,x_2,\dots x_n)$ (3) ; $I(x_1\dots x_n;Y)$ (2)
$D(p(x_1\dots x_n)\|q(x_1\dots x_n))$ (1)

$\dots I(X;Y) = D(p(x,y)\|p(x)p(y)) \Rightarrow$ AKO JA ZNAEŠ RELATIVNATA ENTROPIJA TOGAŠ MOŽEŠ DA JA PRESMETAŠ TRANSINFORMACIJATA.
$H(X) = I(X;X) \Rightarrow$ AKO JA ZNAEŠ TRANSINFO ZA NAJDEŠ ENTROPIJATA.
AMAČINATA JE

PROBLEM 2.23 CONSIDER SEQUENCE OF "n" BINARY RANDOM VARIABLES $X_1, X_2, \dots, X_n$
EACH SEQUENCE WITH AN EVEN NUMBER OF 1'S HAS PROBABILITY $2^{-(n-1)}$, AND EACH SEQUENCE WITH AN ODD NUMBER OF 1'S HAS PROBABILITY 0.
FIND MUTUAL INFORMATIONS
$I(X_1,X_2)$, $I(X_2;X_3|X_1)\dots$, $I(X_{n-1};X_n|X_1,\dots X_{n-2})$

$x_1, x_2, \dots x_n$ EVEN NUMBER OF 1's $P = 2^{-(n-1)}$

ODD NUMBER OF 1's $P = 0$

0	000	e	100	o	4
1	001	o	101	e	5
2	010	o	110	e	6
3	011	e	111	o	7

ČETRI NASTANI SO VEROJATNOST

$P_3 = \frac{1}{2^2} = \frac{1}{4}$

00	e
01	o
10	o
11	e

$P_2 = \frac{1}{2}$ } DVA NASTANI $\quad P_4 = \frac{1}{8}$ } 8 NASTANI

0	e
1	o

$P_1 = \frac{1}{2}$ } EDEN NASTAN

$$\boxed{x_i = \begin{cases} 0 & 1-p \\ 1 & p \end{cases}}$$

$P(x_1) = \frac{1}{2}$ $\quad P(x_1, x_2) = \frac{1}{2}$ $\quad P(00) = \frac{1}{4}$, $P(11) = \frac{1}{4}$

$$P(x_1, x_2) = P(0,0) + P(1,1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$I(x_1, x_2) = H(x_1) - H(x_1|x_2) = H(x_2) - H(x_2|x_1)$$

$H(x_1) = 2 \cdot \frac{1}{2} \cdot ld\, 2 = 1$ $\quad H(x_1 x_2) = H(x_1) + H(x_2|x_1)$
$\quad = H(x_2) + H(x_1|x_2)$

$$H(x_1|x_2) = \underset{1}{P(x_2)} \cdot H(x_1|x_2 = 0) + P(x_2) \cdot H(x_1|x_2 = 1)$$

$$H(x_1|x_2 = 0) = -\sum_{x_1 = 0} P(x_1|x_2 = 0)\, ld\, P(x_1|x_2 = 0) = \frac{1}{2}\, ld\, 2 = \frac{1}{2}$$

$x_1 = 0$ } SAMO OVAA KOMBINACIJA ← 00 KREIRA

$$P(x_1 x_2) = P(x_1) P(x_1|x_2) = \frac{1}{2} \cdot \underbrace{P(0|1)}_{0} + \underbrace{\frac{1}{2}(1|1)}_{\neq 0} + \underbrace{\frac{1}{2}(0|0)}_{\neq 0} + \underbrace{\frac{1}{2}(1|0)}_{0}$$

$$\boxed{P(0,0) = \frac{1}{2} \quad P(1,1) = \frac{1}{2} = \frac{1}{2^{n-1}}}$$

$P(0) = \frac{1}{2^{n-1}} = \frac{1}{2^0} = 1$

$$\boxed{I(x_1, x_2) = H(x_1) + H(x_2) - H(x_1, x_2) = 1 + 1 - 1 = 1}$$

$$H(x_1 x_2) = -\sum_{x_1, x_2} P(x_1, x_2)\, ld\, P(x_1, x_2) =$$

$$= -P(0,0)\, ld\, P(0,0) - P(1,1)\, ld\, P(1,1) = \frac{1}{2} + \frac{1}{2} = 1$$

$p(x_1,x_2)$ $x_2 \backslash x_1$	0	1	$p(x_2)$
0	$\frac{1}{2}$	0	$\frac{1}{2}$
1	0	$\frac{1}{2}$	$\frac{1}{2}$
$p(x_1)$	$\frac{1}{2}$	$\frac{1}{2}$	

$$p(x_1) = \sum_{x_2} p(x_1, x_2)$$

- V0 GENERALEN SLUČAJ KADAR $q \neq \frac{1}{2}$

$x_2 \backslash x_1$	0	1	$p(x_2)$
0	$1-q$	0	$1-q$
1	0	q	q
$p(x_1)$	$1-q$	q	

$$\left.\begin{aligned} p(0,0) &= 1-q = \frac{1}{2} \\ p(1,1) &= q = \frac{1}{2} \end{aligned}\right\} q = \frac{1}{2}$$

$$H(x_1) = -q \operatorname{ld} q - (1-q) \operatorname{ld}(1-q)$$

$$H(x_2) = -q \operatorname{ld} q - (1-q) \operatorname{ld}(1-q)$$

$$I(x_1, x_2) = -2q \operatorname{ld} q - 2(1-q) \operatorname{ld}(1-q) - 1$$

- $H(x_1, x_2, x_3) = \sum_{x_1, x_2, x_3} p(x_1, x_2, x_3) \operatorname{ld}(x_1, x_2, x_3) =$

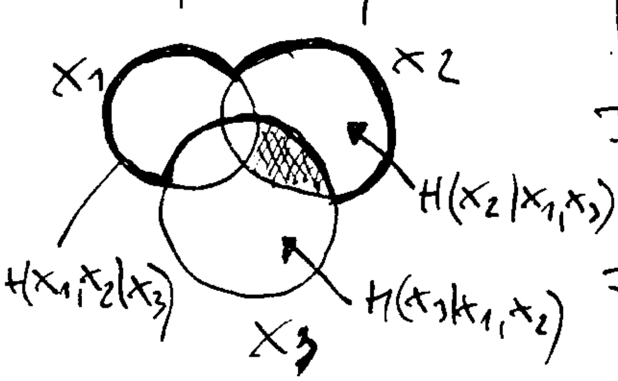
$$= -p(000) \operatorname{ld} p(000) - p(011) \operatorname{ld} p(011) - p(101) \operatorname{ld} p(101) -$$

$$- p(110) \operatorname{ld} p(110) = \frac{1}{4} \cdot 2 + \frac{1}{4} \cdot 2 + \frac{1}{4} \cdot 2 + \frac{1}{4} \cdot 2 =$$

$$= 1/2 + 1/2 + 1/2 + 1/2 = 2$$

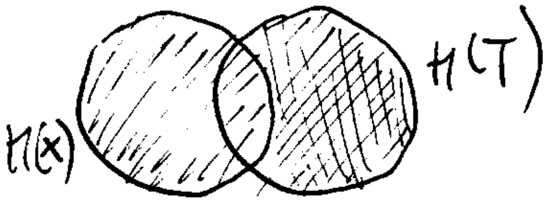
- $I(x_2; x_3 | x_1) = ?$

$$\boxed{I(x_2; x_3 | x_1) = I(x_1, x_2; x_3) - I(x_1; x_3)}$$

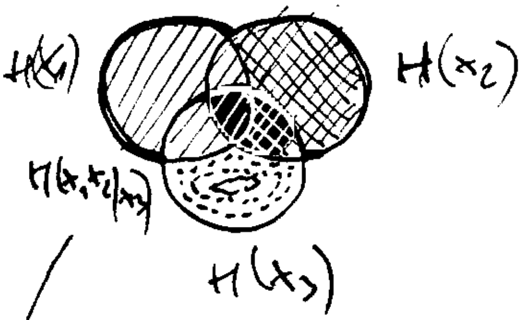


$$I(x_1, x_2; x_3) = \sum_{i=1}^{2} I(x_i; x_3 | x_{i-1} \ldots x_1)$$

$$= I(x_1; x_3) + I(x_2; x_3 | x_1)$$



$$\underline{H(X)+H(Y|X)=H(X,Y)}$$



$$H(X_1,X_2,X_3)=H(X_1)+H(X_2|X_1)+$$
$$+\ H(X_3|X_1,X_2)$$

$$H(X_1,X_2,\dots X_n)=\frac{\cancel{2^n}}{\cancel{2}}\cdot\frac{1}{\cancel{2^{n-1}}}\,ld\,2^{n-1}=$$
$$=(n-1)$$

$$\boxed{H(X_1,X_2,\dots X_n)=n-1}$$

$$I(X_2;X_3|X_1)=I(X_1,X_2;X_3)-I(X_1;X_3)$$

$$I(X_1,X_2;X_3)=I(X_2;X_3|X_1)+I(X_1;X_3)$$

$$=H(X_1,X_2,X_3)-H(X_1,X_2|X_3)-H(X_3|X_1,X_2)$$

$$=H(X_1,X_2,X_3)-H(X_1|X_3)-H(X_2|X_1,X_3)-H(X_3|X_1,X_2)$$

$$I(X_1;X_3)=H(X_1)-H(X_1|X_3)$$

$$I(X_2;X_3|X_1)=H(X_1,X_2,X_3)-H(X_1)-\underline{H(X_2|X_1,X_3)}-H(X_3|X_1X_2)$$

$$I(X_2;X_3|X_1)=H(X_2|X_1)-H(X_2|X_1X_3)$$

$$\boxed{\begin{aligned}I(X_2;X_3|X_1)&=H(\cancel{X_1,X_2,X_3})-\underbrace{H(X_1)-H(X_2|X_1)}-H(X_3|X_1X_2)-H(X_2|X_1X_3)+H(X_2|X_1)\\&=H(X_2|X_1)-H(X_2|X_1,X_3)^{\cancel{H(X_1,X_2,X_3)}}\end{aligned}}$$

→ <u>DOKAZ NA DEFINICIJATA 2.60 !!!</u>

– DA AMA ~~SCHAK~~ GO KORIST GRAFIKOT OD

$$H(X_2|X_1)=H(X_1,X_2)-H(X_1)$$

$$H(X_2|X_1,X_3)=H(X_1,X_2,X_3)-H(X_1,X_3)$$

$$I(X_2;X_3|X_1)=H(X_1,X_2)-H(X_1)-H(X_1,X_2,X_3)+H(X_1,X_3)$$

$I(x_2;x_3|x_1) = 2H_2 - H_1 - H_3$

$H_2 = H(x_1,x_2) = H(x_1,x_3) = H(x_2,x_3) \dots$

$H_3 = H(x_1,x_2,x_3) = H(x_2,x_3,x_4)$

$I(x_2;x_3|x_1) = 2\cdot 1 - 1 - 2 = -1$?

$I(x_2;x_3|x_1) = H(x_2|x_1) - H(x_2|x_1 x_3) = H(x_1 x_2) - H(x_1) - \underline{H(x_2|x_1 x_3)}$

→ $\emptyset$ AKO GI ZNAŠ x_1 & x_3 GO ZNAŠ x_2 1 MMV

$H(x_1,x_2,x_3) = H(x_1) + H(x_2|x_1) + H(x_3|x_1,x_2)$

$H(x_1,x_3,x_2) = H(x_1) + H(x_3|x_1) + \underline{H(x_2|x_1,x_3)}$

$H(x_2|x_1,x_3) = H(x_1,x_2,x_3) - H(x_1) - H(x_3|x_1)$

$I(x_2;x_3|x_1) = H(x_1 x_2) - \cancel{H(x_1)} - H(x_1,x_2,x_3) + \cancel{H(x_1)} + \underline{H(x_3|x_1)}$

$H(x_3|x_1) = H(x_1,x_3) - H(x_1)$

$H(x_1,x_3,x_2) = H(x_2) + H(x_1 x_3|x_2)$

000	⊕
001	
010	
011	⊕
100	
101	⊕
110	⊕
111	

$H(x_1,x_2,x_3) = -P(000)\,ld\,P(000) - P(011)\,ld\,P(011) - P(101)\,ld\,P(101) - P(110)\,ld\,P(110)$

$H(x_1,x_3) = P(\overset{13}{00})\,ld\,P(00) - P(01)\,ld(P(01)) - P(11)\,ld\,P(11) - P(10)\,ld\,P(10) = \left(\frac{1}{4}\cdot 2\right)\cdot 4 =$

$H(x_1,x_3) = H(x_1,x_2,x_3)$

$H(x_3|x_1) = P(x_1=0)\cdot H(x_3|x_1=0) + P(x_1=1)\,H(x_3|x_1=1)$

$H(x_3|x_1=0) = \sum_{x_3} P(x_3|x_1=0)\,ld\,P(x_3|x_1=0) =$

$= \underbrace{P(x_3=0|x_1=0)}_{=P(000)=\frac{1}{4}}\,ld\,P(x_3=0|x_1=0) + \underbrace{P(x_3=1|x_1=0)}_{P(011)=\frac{1}{4}}\,ld\,P(x_3=1|x_1=0)$

$= \frac{1}{4}\cdot 2 + \frac{1}{4}\cdot 2 = \frac{1}{2} + \frac{1}{2} = 1$

$H(x_2|x_1 x_3) = H(x_1,x_2,x_3) - H(x_1) - H(x_3,x_1) = 2 - 1 - 1 = \emptyset$

POKAZ NA 1

$I(X_2;X_3|X_1) = H(X_2|X_1) - \underbrace{H(X_3|X_1,X_2)}_{=\emptyset} = H(X_1,X_2) - H(X_1)$

AKO ZNAŠ X_1, X_2 ŽE ZNAŠ I X_3

$$\boxed{I(X_2;X_3|X_1) = (n-1) - 1 = n-2 = 2-2 = \emptyset}$$

$I(X_{n-1};X_n|X_1, \dots X_{n-2})$

- $I(X_3;X_4|X_1,X_2) = H(X_3|X_1,X_2) - \underbrace{H(X_3|X_1,X_2,X_3)}$

$= H(X_3|X_1,X_2) = H(X_1,X_2,X_3) - H(X_1,X_2) = 3-\cancel{1}-2+\cancel{1} = 3-2 = 1$

$\quad\quad 3-1 \quad\quad 2-1$

- $I(X_4;X_5|X_1,X_2,X_3) = H(X_4|X_1X_2X_3) - \underbrace{H(X_5|X_1X_2X_3X_4)}_{\emptyset} =$

$= H(X_1X_2X_3X_4) - H(X_1X_2X_3) = 4-\cancel{1}-3+\cancel{1} = 1$

- $I(X_{n-1};X_n|X_1, \dots X_{n-2}) = H(X_{n-1}|X_1 \dots X_{n-2}) -$

$- \underbrace{H(X_n|X_1, \dots X_{n-1})}_{\emptyset} = H(X_1 \dots X_{n-2}, X_{n-1}) -$

$- H(X_1 \dots X_{n-2})$

$= (n-1)-1-[(n-2)-1] = n-1-1-(n-2)+1 = (n-2)-(n-2)+1 = 1$

+ 0000	1000
0001	1001 +
0010	1010 +
+ 0011	1011
0100	1100 +
+ 0101	1101
+ 0110	1110
0111	1111 +

$H(X_1X_2X_3X_4) = \left(\frac{1}{8} \cdot 3\right) \cdot 8 = 3 = n-1$

$\boxed{\text{PROBLEM 2.24}}$ AVERAGE ENTROPY. Let $H(p) = -p\,ld\,p - (1-p)\,ld(1-p)$ be binary entropy function.

(a.) Evaluate $H(\frac{1}{4})$ using the fact that $\log_2 3 \doteq 1.584$. (Hint: You may wish to consider the experiment with 4 equally likely outcomes, one of which is more interesting than others.)

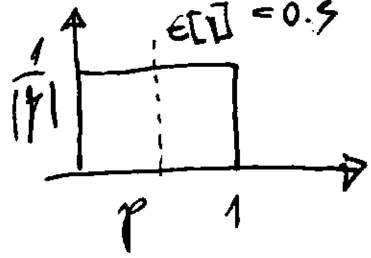
(b) CALCULATE AVERAGE ENTROPY $H(p)$ WHEN PROBABILITY IS CHOSEN UNIFORMLY IN RANGE $0 \leq p \leq 1$

(c) (OPTIONAL) CALCULATE THE AVERAGE ENTROPY $H(p_1, p_2, p_3)$ WHERE (p_1, p_2, p_3) IS UNIFORMELY DISTRIBUTED VECTOR. GENERALIZE TO DIMENSION "n".

(a) $H\left(\frac{1}{4}\right) = \frac{1}{4}\cdot 2 + \left(\frac{3}{4}\right)\cdot \mathrm{ld}\,\frac{4}{3} = \frac{1}{2} + \frac{3}{4}\cdot(2 - \mathrm{ld}\,3) = \frac{1}{2} + \frac{3}{2} - 1.189$

$H\left(\frac{1}{4}\right) = \frac{4}{2} - 1.189 = 2 - 1.189 = \underline{0.811}$

(b)



$p \in [p_1, p_2, \dots p_n]$

$|p| = n$

$$\overline{H(p)} = \sum_p P(p)\, H(p) = \sum_p \frac{1}{|p|} H(p) = \frac{1}{|p|} \sum_p H(p)$$

$H(p) \Rightarrow$ CONCAVE FUNCTION

$$\boxed{E[H(p)] \leq H(E[p]) = H\left(\tfrac{1}{2}\right) = 1}; \quad E[p] = \sum_{p_i} \frac{1}{|p|}\cdot p_i = \frac{1}{2}$$

E.G.: $p \in [0; 0.1; 0.2, 0.3, \dots 0.9, 1]$ $\quad |p| = \underline{\underline{11}}$

$0.1 + 0.2 + \dots + 1 = \frac{1}{10} + \frac{2}{10} + \dots + \frac{10}{10}$

$= \frac{1}{10}(1 + 2 + \dots + 10) = \frac{55}{10} = \underline{\underline{5{,}5}}$

$E[p] = \frac{1}{11}\sum_{i=1}^{11} p_i = \frac{5{,}5}{11} = 0.5$

$a_n = a_{n-1} + d = a_1 + (n-1)d$

$S = a_1 + a_2 + \dots + a_n = a_1 + a_1 + d + a_1 + 2d + \dots + a_1 + (n-1)d$

$= n\cdot a_1 + d\,\underbrace{[1 + 2 + \dots (n-1)]}_{= \frac{n(n-1)}{2}}$

$S = a_n + a_{n-1} + \dots + a_1$

$2S = (a_1 + a_n) + (a_2 + a_{n-1}) + \dots + (a_n + a_1)$

$(2a_1 + (n-1)d) + \underbrace{(a_1 + d + a_1 + (n-2)d)}_{2a_1 + (n-1)d} + \dots + 2a_1 + (n-1)d$

$2S = n\cdot[2a_1 + (n-1)d]$ $\qquad S = \frac{n}{2}[2a_1 + (n-1)d]$

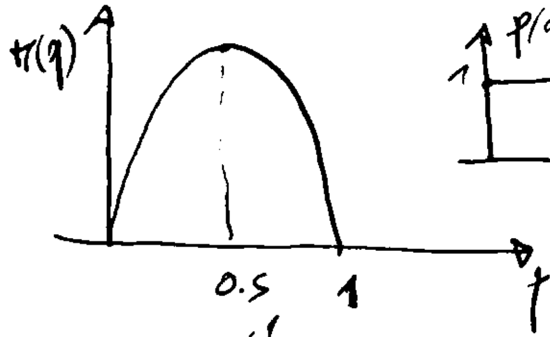
$$\boxed{S = n\cdot a_1 + \frac{n(n-1)}{2}d}$$

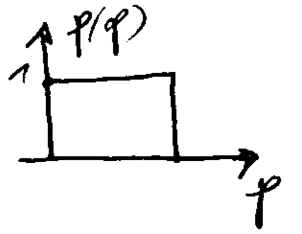
$a_1 = 1 \quad d = 1 \quad n = 10$

$S = 10 + \frac{10\cdot 9}{2}\cdot 1 = 10 + 45 = 55$

$$H(q) = -q\,ld\,q - (1-q)\,ld(1-q)$$

$$ld\,q = \frac{\ln q}{\ln 2} \qquad 7 = ld\,q, \quad 2^7 = q$$





$$\int_0^1 H(q)\cdot \cancel{p(q)}\,dq = \int_0^1 H(q)\,dq = -\int_0^1 q\,ld\,q + (1-q)\,ld(1-q)\,dq$$

$$= -\frac{1}{\ln 2}\int_0^1 q\ln q\,dq - \frac{1}{\ln 2}\int_0^1 (1-q)\ln(1-q)\,dq = \underline{\frac{1}{2\ln 2}} \doteq 0.72135$$

$$(\ln x)' = \frac{1}{x} \qquad \left(\ln\frac{1}{x}\right)' = \frac{1}{\frac{1}{x}}\cdot\left(\frac{1}{x}\right)' = x\cdot\frac{-1}{x^2} = -\frac{1}{x}$$

$$(\ln(\ln x))' = \left(\frac{1}{\ln x}\right)\cdot\frac{1}{x}$$

$$(\sin(x^2))' = 2x\cos(x^2) \qquad ?$$

$$\int_0^1 q\ln\frac{1}{q}\,dq = -\int_0^1 q^2\ln\frac{1}{q}\,d\left(\ln\frac{1}{q}\right)$$

$$\int q\ln q\,dq \quad \left|\begin{array}{l} u = \ln q \quad du = \frac{1}{q}\,dq \\ v = \int q\,dq = \frac{q^2}{2}\end{array}\right| =$$

$$= \ln q\cdot\frac{q^2}{2} - \int\frac{q^{\cancel{2}}}{2}\cdot\frac{1}{\cancel{q}}\,dq = \frac{q^2}{2}\ln q - \frac{1}{2}\int q\,dq =$$

$$= \frac{q^2}{2}\ln q - \frac{q^2}{4}$$

FUNKCIONALNA TRANSFORMACIJA NA SLUČAJNI PROM.:

$$h = -q\,ld\,q - (1-q)\,ld(1-q) \qquad P(q) = \begin{cases}1 & 0\le q\le 1\\ 0 & \text{otherwise}\end{cases}$$

$$P(h) = \left.\frac{P(q)}{\frac{dh}{dq}}\right|_{q=f(h)} \qquad \frac{dh}{dq} = ld(1-q) - ld\,q$$

$$P(h) = \frac{-p\,ld\,p - (1-p)\,ld(1-p)}{ld(1-p) - ld\,p}\Bigg|_{p=f(h)}$$

$$h = -p\,ld\,p - (1-p)\,ld(1-p) = \cancel{-p\,2^{-p\,ld\,p}}$$

$$p = 2^{ld(p\,ld\,p)} \,/\, ld \qquad \boxed{ld\,p = ld(ld\,p)} \qquad \cancel{\boxed{p = 2^{ld\,p}}}$$

$$x = e^{\ln x} \qquad \boxed{\ln x = \ln x} \qquad x = e^{\ln x}$$

$$h = ld\,p^{-p} + ld(1-p)^{-(1-p)} \qquad h = ld\left[p^{p}(1-p)^{(1-p)}\right]^{-1}$$

$$2^{h} = \left[p^{p}(1-p)^{1-p}\right]^{-1} \qquad 2^{h} = \frac{1}{p^{p}(1-p)^{(1-p)}}$$

- So koristenje na matlce:

$$P(p) = [\underbrace{\tfrac{1}{11};\tfrac{1}{11};\;---\;;\tfrac{1}{11}}_{N-1}]$$

$$p \in \{0;0{,}1;0{,}2;\;----,1\} \qquad \boxed{N = |p|}$$

$$H(p) \in \{0;0{,}469;0{,}72193;0{,}88129;0{,}97095;1\}$$

$$P(H) = \left[\tfrac{2}{11};\tfrac{2}{11};\tfrac{2}{11};\tfrac{2}{11};\tfrac{2}{11};\tfrac{1}{11}\right\}$$

$$E[H(p)] = \sum_{i=1}^{6} H(p)\,\boxed{P(H)} = 0{,}6442 \Big\}\ \text{matlce}$$

$$E[H(p)] = \sum_{p=1}^{11} H(p)\cdot P(H) = 0{,}68357 \Big\}\ \text{matlce}$$

$$E[H(p)] = \sum_{p=1}^{1001} H(p)\cdot P(H) = 0{,}72062$$

$$p = \frac{(h-1)}{N} \qquad E[H(p)] = \sum_{h=1}^{N}\frac{1}{N}\cdot\left[\frac{h-1}{N}\,ld\frac{N}{(h-1)} + \left(1-\frac{h-1}{N}\right)ld\frac{1}{1-\frac{h-1}{N}}\right.$$

$$= \sum_{h=1}^{N}\frac{1}{N}\left[\frac{h-1}{N}\,ld\frac{N}{h-1} + \left(\frac{N-h+1}{N}\right)ld\frac{N}{N-h+1}\right] = \frac{1}{N^2}\sum_{h=1}^{N}(h-1)\,ld\frac{N}{(h-1)} + (N-h+1)\,ld\frac{N}{N-h+1}$$

$p \in \left\{ \frac{n-1}{N} \, , \, n = 1 \ldots N+1 \right\} \qquad H \in \left\{ H(p) \quad n = 1 \ldots N+1 \right\}$

$P(p) = \left\{ \frac{1}{N+1} \, , \, n = 1 \ldots N+1 \right\} \qquad P(H) = \left\{ \frac{1}{N+1} \, , \, n = 1 \ldots N+1 \right\}$

$$E[H(p)] = \sum_{n=1}^{N+1} \frac{1}{N+1} \overbrace{\left[\frac{n-1}{N} \, ld \frac{N}{n-1} + \frac{N-n+1}{N} \, ld \frac{N}{N-n+1} \right]}^{H(p)}$$

$$\boxed{E[H(p)] = \frac{1}{N(N+1)} \sum_{n=1}^{N+1} \left[(n-1) \, ld \frac{N}{N-1} + (N-n+1) \, ld \frac{N}{N-n+1} \right]}$$

$$(n-1)\left[ld\,N - ld(N-1) \right] + (N-n+1)\left[ld\,N - ld(N-n+1) \right] =$$

$$= n\,\cancel{ld N} - \cancel{ld N} - (n-1)\,ld(N-1) + N\,ld\,N - \cancel{n\,ld N} + \cancel{ld N} - (N-n+1)\,ld(N-n+1)$$

$$= -(n-1)\,ld(N-1) + N\,ld\,N - (N-n+1)\,ld(N-n+1)$$

$$= N\,ld\,N - (n-1)\,ld(N-1) - (N-n+1)\,ld(N-n+1) =$$

$$= ld \frac{N^N}{(N-1)^{n-1}(N-n+1)^{N-n+1}}$$

(c) $H(p_1, p_2, p_3) = ?$ WHERE (p_1, p_2, p_3) IS UNIFORMLY DISTRIBUTED PROBABILITY VECTOR.

$$H(p) = -p \, ld\, p - (1-p)\, ld(1-p)$$

$$H(p_1, p_2) = H(p_1) + H(p_2 | p_1) = \underbrace{H(p_1) + H(p_2)}_{p_1 \,\&\, p_2} \quad \text{i.i.d INDEPENDENT}$$

$$H(p_1, p_2, p_3) = H(p_1) + H(p_2 | p_1) + H(p_3 | p_1 p_2) = \sum_{i=1}^{3} H(p_i)$$

$p_i \in \left\{ \frac{n-1}{N} \, , \, n = 1, \ldots N+1 \right\} \qquad H \in \left\{ H(p_i) \quad n = 1, \ldots N+1 \right\}$

$P(p_i) = \left\{ \frac{1}{N+1} \, , \, n = 1, \ldots N+1 \right\} \qquad P(H) = \left\{ \frac{1}{N+1} \, , \, n = 1, \ldots N+1 \right\}$

$$E[H(p_1, p_2, p_3)] = E[H(p_1)] + E[H(p_2)] + E[H(p_3)] =$$

$$= \frac{3}{N(N+1)} \sum_{n=1}^{N+1} \left[(n-1) \, ld \frac{N}{n-1} + (N-n+1) \, ld \frac{N}{N-n+1} \right]$$

$p_1, p_2 \,\&\, p_3$ ARE i.i.d

$$E[H(p_1, p_2, \ldots p_n)] = \frac{n}{N(N+1)} \sum_{i=1}^{N+1} \left[(i-1) \, ld \frac{N}{i-1} + (N-i+1) \, ld \frac{N}{N-i+1} \right]$$

• EDITION 1 SOLUTION:

$$H\left(\frac{1}{4}\right) + \frac{1}{4}\,ld\,3 = 2$$

$$\boxed{\left(\frac{1}{4}\,ld\,\cancel{4}^{2} + \frac{1}{4}\,ld\,\cancel{4}^{2}\right)\cdot 2 = 2}$$

$$p_1\begin{cases} 00 & P(00) = 1/4 \\ 01 & P(01) = 1/4 \end{cases}\qquad p_2\begin{cases} 10 & P(10) = 1/4 \\ 11 & P(11) = 1/4 \end{cases}$$

$$P_2 = \frac{3}{4} \qquad \Rightarrow P_2 = 1 - P_1$$

$$H = -P_1\,ld\,P_1 - P_2\,ld\,P_2 = \frac{1}{4}\,ld\,4 + \frac{3}{4}\,ld\,\frac{4}{3}$$

$$= \frac{1}{4}\,ld\,4 + \frac{3}{4}(ld\,4 - ld\,3) =$$

$$= \frac{1}{2} + \frac{3}{2} - \frac{3}{4}\,ld\,3 = 2 - \frac{3}{4}\,ld\,3 = 2 - 0.75\cdot 1.584 = 2 - 1.188 = 0.812$$

$$ld\,\frac{1}{3/4} = ld\,\frac{4}{3} = 2 - ld\,3$$

$$H\left(\frac{3}{4}\right) = \frac{3}{4}\,ld\,\frac{4}{3} + \frac{1}{4}\,ld\,4 = \frac{3}{4}\,ld\,4 - \frac{3}{4}\,ld\,3 + \frac{1}{4}\,ld\,4 = \underline{\underline{2 - \frac{3}{4}\,ld\,3}}$$

$$H\left(\frac{1}{4}\right) = \frac{1}{4}\,ld\,4 + \frac{3}{4}\,ld\,\frac{4}{3} \qquad H\left(\frac{1}{4}\right) + H\left(\frac{3}{4}\right) = 2$$

$$H\left(\frac{1}{4}\right) = 2 - H\left(\frac{3}{4}\right)$$

$$H\left(\frac{1}{4}\right) = 2 + \frac{3}{4}\,ld\,\frac{4}{3} \qquad H\left(\frac{1}{4}\right) = \cancel{2}\,\cancel{-2} + \frac{3}{4}\,ld\,3$$

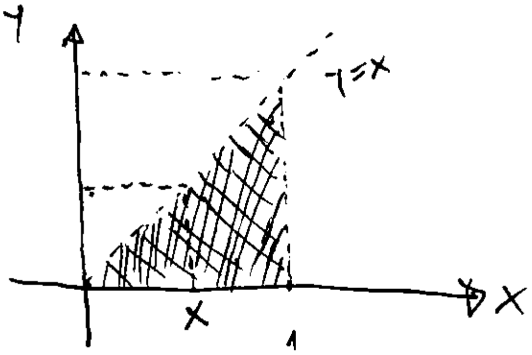
$$H\left(\frac{1}{4}\right) = \frac{3}{4}\,ld\,3 = \frac{3}{4}\cdot 1.584 = 0.75\cdot 1.584 = 1.188 \Big\}?$$

$$H\left(\frac{1}{4}\right) = \frac{1}{4}\,ld\,4 + \frac{3}{4}\cdot ld\,\frac{4}{3} = 1/2 + \frac{3}{4}\,ld\,4 - \frac{3}{4}\,ld\,3 = \underline{\underline{2 - \frac{3}{4}\,ld\,3}}$$

$$p = 1/4 \qquad (1-p) \qquad 3/2$$

(c) CHOOSING A UNIFORMLY DISTRIBUTED PROBABILITY VECTOR (p_1, p_2, p_3) IS EQUIVALENT TO CHOOSING POINT (p_1, p_2) FROM THE TRIANGLE $0 \leq p_1 \leq 1$, $p_1 \leq p_2 \leq 1$. THE PROBABILITY DENSITY FUNCTION HAS CONSTANT VALUE 2 BECAUSE THE AREA OF THE TRIANGLE IS 1/2. SO AVERAGE $H(p_1, p_2, p_3)$ IS: (nats)

$$-2\int_0^1\int_{p_1}^1 p_1 \ln p_1 + p_2 \ln p_2 + (1-p_1-p_2)\ln(1-p_1-p_2)\,dp_2\,dp_1$$

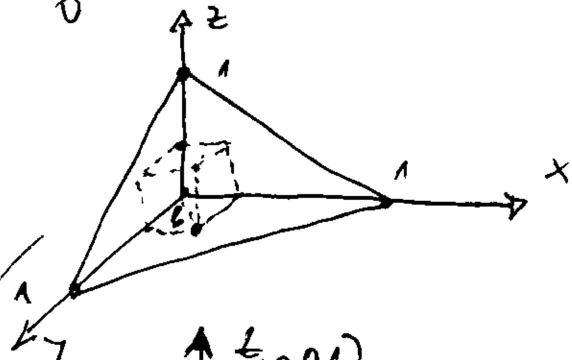


$0 \le x \le 1 \qquad x \le y \le 1$

$P = \int_0^1 f(x)\,dx \qquad \int g(y)\,dy$

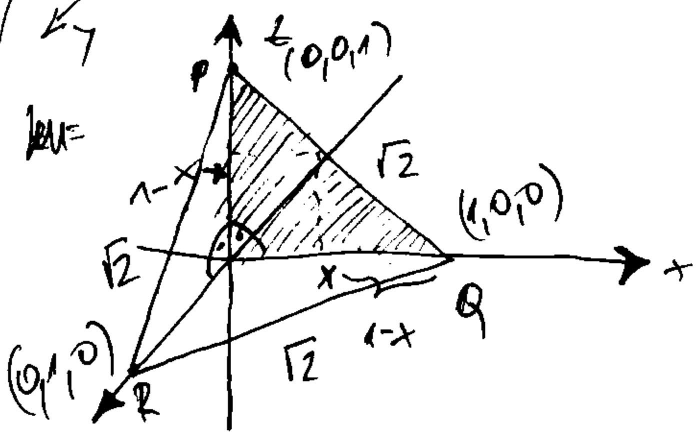
$P = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$

$P = \int_0^1 y\,dy = \frac{y^2}{2} = \frac{1}{2}$

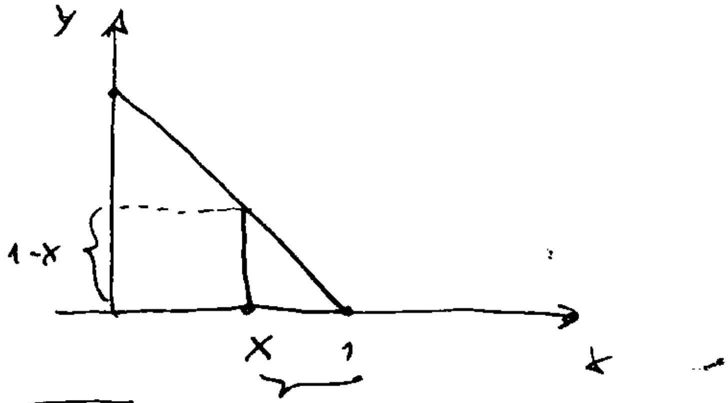


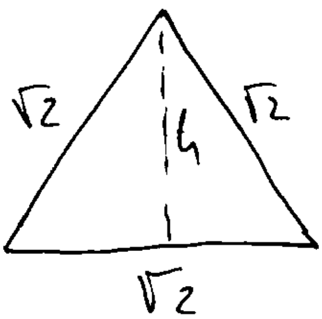
$P = \frac{1\cdot 1\cdot 1}{2} = \frac{1}{2}$

$I = \int_0^1 \int_x^1 f(x,y)\,dy\,dx$



$\boxed{z = -x + 1}$





$h = \sqrt{-\left(\frac{\sqrt{2}}{2}\right)^2 + (\sqrt{2})^2} = \sqrt{2 - \frac{1}{2}} = \sqrt{\frac{3}{2}}$ (*)

$P = \frac{a\cdot h}{2} = \frac{\sqrt{2}\cdot\sqrt{\frac{3}{2}}}{2} = \frac{\sqrt{3}}{2}$

$b = \sqrt{x^2 + (1-x)^2} = \sqrt{x^2 + 1 - 2x + x^2} = \sqrt{2x^2 - 2x + 1} = \sqrt{x^2 + y^2}$

$\boxed{\bar{n} \circ (\bar{r} - \bar{r}_0) = 0}$ $\qquad \vec{n}\cdot\vec{r} = \vec{n}\cdot\vec{r}_0$

RAVENKA NA RAMNINA

$n = \langle a, b, c\rangle \qquad r_0 = \langle x_0, y_0, z_0\rangle$

$r = \langle x, y, z\rangle$

$\langle a, b, c\rangle \circ \langle x - x_0, y - y_0, z - z_0\rangle = 0$

$a(x - x_0) + b(y - y_0) + c(y - y_0) = \emptyset$

$\overline{PQ} = \langle 0,0,1\rangle - \langle 1,0,0\rangle = \langle -1, 0, 1\rangle$

$\overline{PR} = \langle 0,0,1\rangle - \langle 0,1,0\rangle = \langle 0, -1, 1\rangle$

$$n = \overline{PQ} \times \overline{PR} = \begin{vmatrix} i & j & k \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{vmatrix} = 1\cdot i - 1\cdot j + 1\cdot k$$

$$\vec{n} = \langle 1, -1, 1 \rangle$$

a, b, c

$$x_0 = P = \langle 0, 0, 1 \rangle$$

$$\vec{n}\cdot(\vec{r} - \vec{r_0}) = 0 \qquad \langle 1, -1, 1 \rangle \cdot \langle (x-0), (y-0), (z-1) \rangle = 0$$

$$(x-0) + (y-0) + (z-1) = 0 \qquad \boxed{x + y + z - 1 = 0}$$

PODEDRAVNO VAZI $\quad \left(x - \frac{1}{3}\right) + \left(y - \frac{1}{3}\right) + \left(z - \frac{1}{3}\right)$

$$\boxed{z = 1 - x - y}$$

$$I = \int_0^1 \int_0^1 (1 - x - y)\, dy\, dx = \int_0^1 \left(y - xy - \frac{y^2}{2}\right)\Big|_0^1 dx =$$

$$= \int_0^1 \left(1 - x - \frac{1}{2}\right) dx = \int_0^1 \left(\frac{1}{2} - x\right) dx = \left(\frac{1}{2}x - \frac{x^2}{2}\right)\Big|_0^1$$

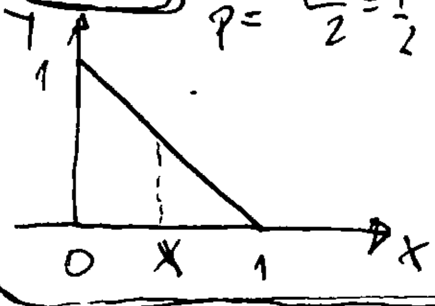
$$= \frac{1}{2} - \frac{1}{2} = 0 \quad ?$$

x − PO PRAVI
y − PO KRIVI

$$I = \int_0^1 \int_x^1 (1 - x - y)\, dy\, dx = \int_0^1 \left(y - xy - \frac{y^2}{2}\right)\Big|_x^1 dx =$$

$$= \int_0^1 \left[\frac{1}{2} - x - \left(x - x^2 - \frac{x^2}{2}\right)\right] dx = \int_0^1 \left(\frac{1}{2} - 2x + x^2 + \frac{x^2}{2}\right) dx$$

$$= \left(\frac{1}{2}x - 2\frac{x^2}{2} + \frac{x^3}{3}\cdot\frac{3}{2}\right)\Big|_0^1 = 1 - 1 + \frac{1}{2} = 0$$



$$P = \frac{1\cdot 1}{2} = \frac{1}{2}$$

$$\int_0^1 \int_0^{-x+1} dy\, dx = \int_0^1 y\Big|_0^{-x+1} dx = \int_0^1 (1 - x)\, dx$$

$$y = -x + 1$$

$$= \left(x - \frac{x^2}{2}\right)\Big|_0^1 = 1 - \frac{1}{2} = \frac{1}{2}$$

VOLUME

$$I = \int_0^1 \int_0^{1-x} (1-x-y)\,dy\,dx = \int_0^1 \left(y - xy - \frac{y^2}{2}\right)\Bigg|_0^{1-x} dx =$$

$$= \int_0^1 \left[1-x - x(1-x) - \frac{(1-x)^2}{2}\right] dx = \int_0^1 1 - x - x + x^2 - \frac{1}{2} + x - \frac{x^2}{2}\,dx$$

$$= \int_0^1 \frac{1}{2} - x + \frac{x^2}{2}\,dx = \left(\frac{x}{2} - \frac{x^2}{2} + \frac{x^3}{6}\right)\Bigg|_0^1 = \frac{1}{6} \qquad \boxed{V = \frac{1}{6}}$$

AREA

$$A = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)}\,dA$$

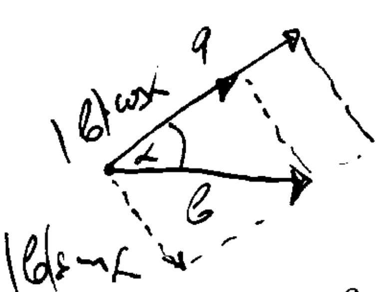
$$z = 1-x-y \qquad \frac{\partial z}{\partial x} = -1 \qquad \frac{\partial z}{\partial y} = -1$$

$$A = \iint_D \sqrt{1+1+1}\,dy\,dx = \sqrt{3}\int_0^1\int_0^{1-x} dy\,dx =$$

$$= \sqrt{3}\int_0^1 (1-x)\,dx = \sqrt{3}\left(x - \frac{x^2}{2}\right)\Bigg|_0^1 = \frac{\sqrt{3}}{2}$$

ODGOVARA SO ⊛ NA NAL. 36

$$A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij}$$

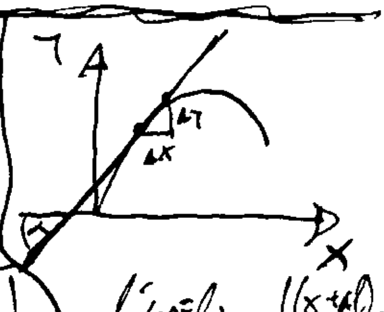


$$\vec{b} \cdot \vec{a} = |a| \cdot |b| \cdot \cos\alpha$$

$$\boxed{\vec{a} \times \vec{b} = |a| \cdot |b| \cdot \sin\alpha}$$

$$a = \Delta x \cdot \vec{i} + f_x(x_i, y_i) \cdot \Delta x\,\vec{k}$$

$$b = \Delta y \cdot \vec{j} + f_y(x_i, y_i)\,\Delta y\,\vec{k}$$



$$f'(x) = \lim \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$f' = \operatorname{tg}\alpha = \frac{\Delta y}{\Delta x}$$

$$a \times b = \begin{vmatrix} i & j & k \\ \Delta x & 0 & f_x(x_i, y_i)\Delta x \\ 0 & \Delta y & f_y(x_i, y_i)\Delta y \end{vmatrix} = -f_x \Delta x \Delta y \, \vec{i} - f_y \Delta x \Delta y \, \vec{j} + \Delta x \Delta y = \Delta A\left(-f_x \vec{i} - f_y \vec{j} + \vec{k}\right)$$

$$\Delta T_{ij} = |a \times b| = \sqrt{f_x^2(x_i y_i) + f_y(x_i y_i)^2 + 1}$$

$$A(S) = \lim_{n,m \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} \sqrt{1 + f_x^2(x_i, y_i) + f_y^2(x_i, y_i)}$$

$$\boxed{A(S) = \iint_D \sqrt{1 + f_x^2(x,y) + f_y^2(x,y)} \, dA}$$

$p_2 \backslash p_1$	0	1	$P(p_2)$
0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
$P(p_1)$	$\frac{1}{2}$	$\frac{1}{2}$	

$$P(p_{1i}, p_{2i}, p_{3i}) = \frac{1}{\underbrace{|(p_1, p_2, p_3)|}}$$

$P(p_1, p_2, p_3)$

2^8 комб

$p_1 \in \{0, 1\}$

$p_2 \in \{0, 1\}$

$p_3 \in \{0, 1\}$

$$P(p_1) = \sum_{p_2} \sum_{p_3} P(p_1, p_2, p_3)$$

$$\boxed{P(0) = 4 \cdot \frac{1}{8} = \frac{1}{2}}$$

$$P(1) = \sum_{p_2} \sum_{p_3} P(p_1, p_2, p_3) = 4 \cdot \frac{1}{8} = \frac{1}{2}$$

ДИМЕНЗИЈАТА ~~на~~ Т.Е. КАРДИНАЛНОСТА НА ВЕКТОРОТ.

$0 \leq p_1 \leq 1 \qquad p_1 \leq p_2 \leq 1$

$$-2 \int_0^1 \int_{p_1}^1 p_1 \ln p_1 + p_2 \ln p_2 + (1 - p_1 - p_2) \ln(1 - p_1 - p_2) \, dp_2 \, dp_1$$

$$= -2 \int_0^1 \left(p_2 \, p_1 \ln p_1 + \frac{p_2^2}{2} \ln p_2 - \frac{p_2^2}{4} + I \right) \Big|_{p_1}^1 dp_1$$

$$I = \int (1-\gamma_1-\gamma_2)\ln(1-\gamma_1-\gamma_2)\,d\gamma_2$$

$$u = \ln(1-\gamma_1-\gamma_2) \qquad du = \frac{-1}{1-\gamma_1-\gamma_2}\,d\gamma_2$$

$$v = \int (1-\gamma_1-\gamma_2)\,d\gamma_2 = \gamma_2 - \gamma_1\gamma_2 - \frac{\gamma_2^2}{2}$$

$$I = \left(\gamma_2-\gamma_1\gamma_2-\frac{\gamma_2^2}{2}\right)\ln(1-\gamma_1-\gamma_2) + \int\left(\gamma_2-\gamma_1\gamma_2-\frac{\gamma_2^2}{2}\right)\frac{d\gamma_2}{1-\gamma_1-\gamma_2}$$

$$= \left(\gamma_2-\gamma_1\gamma_2-\frac{\gamma_2^2}{2}\right)\ln(1-\gamma_1-\gamma_2) - \frac{1}{2}\left(\frac{\gamma_2}{2}+\gamma_1-1\right)\gamma_2 +$$

$$+ \ln(\gamma_1+\gamma_2-1)\left[-\frac{1}{2}+\gamma_1-\frac{\gamma_1^2}{2}\right]$$

$$= \gamma_2\left(1-\gamma_1-\frac{\gamma_2}{2}\right)\ln(1-\gamma_1-\gamma_2) + \left(1-\frac{2\gamma_1+\gamma_2}{2}\right)\frac{1}{2}$$

$$-\left(\frac{1}{2}-\gamma_1+\frac{\gamma_1^2}{2}\right)\ln(1-\gamma_1+\gamma_2)$$

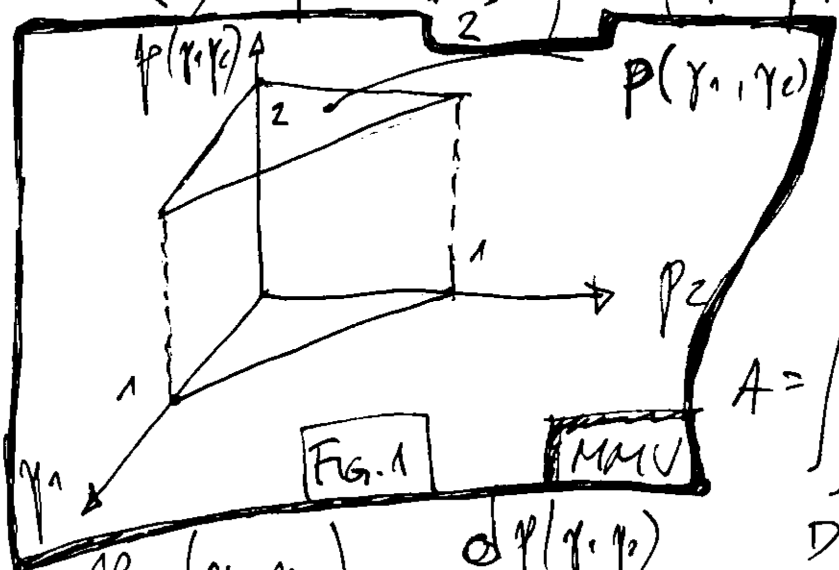


FIG. 1

$$\int_0^1 \gamma(\gamma_1,\gamma_2)\,d\gamma_1 = 2\cdot\gamma_1\Big|_0^1 =$$

$$A = \iint \sqrt{1+p_{\gamma_1}^2(\gamma_1\gamma_2)+p_{\gamma_2}^2(\gamma_1\gamma_2)}\,d\gamma_1 d\gamma_2$$

$$p_{\gamma_1}(\gamma_1\gamma_2) = \frac{\partial p(\gamma_1\gamma_2)}{\partial\gamma_1} = \frac{\partial}{\partial\gamma_1}(2) = 0$$

$$p_{\gamma_2}(\gamma_1,\gamma_2) = 0$$

$$A = \int_0^1\int_p^{1-\gamma_1}\sqrt{1+0+0}\,d\gamma_1 d\gamma_2 = \int_0^1\int_0^{1-\gamma_1} d\gamma_2\,d\gamma_1 =$$

$$= \int_0^1 (1-\gamma_1)\,d\gamma_1 = \gamma_1 - \frac{\gamma_1^2}{2}\Big|_0^1 = 1-\frac{1}{2} = \frac{1}{2}$$

$$\boxed{V=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(\gamma_1,\gamma_2)\, d\gamma_1 d\gamma_2 = 1} \Rightarrow$$

$$V = A\cdot 2 = \frac{1}{2}\cdot 2 = 1$$

SO OGLED NA TOA ŠTO SE RABOTI ZA UNIFORMNO RASPREDELEN VEKTOR NA VEROJATNOSTI VOLUMENOT NA FIG 1 TREBA = 1

$$V = \int_0^1\int_0^{1-\gamma_1} 2\cdot d\gamma_2\, d\gamma_1 = 2\int_0^1 (1-\gamma_1)\, d\gamma_1 = 2\left(\gamma_1 - \frac{\gamma_1^2}{2}\right)\Big|_0^1$$

$$= 2\cdot\left(1-\frac{1}{2}\right) = 1$$

$$E\left[H(\gamma_1,\gamma_2,\gamma_3)\right] = \iiint \underbrace{p(\gamma_1,\gamma_2,\gamma_3)}_{2} H(\gamma_1\,\gamma_2\,\gamma_3)\, d\gamma_1 d\gamma_2 d\gamma_3$$

$$H(\gamma_1\,\gamma_2\,\gamma_3) = -\left[\gamma_1 \operatorname{ld}\gamma_1 + \gamma_2 \operatorname{ld}\gamma_2 + (1-\gamma_1-\gamma_2)\operatorname{ld}(1-\gamma_1-\gamma_2)\right]$$

$\gamma_1,\gamma_2,\gamma_3$ SE STATISTIČKI NEZAVISNI $\Rightarrow$

$$H(\gamma_1\,\gamma_2\,\gamma_3) = H(\gamma_1) + H(\gamma_2) + H(\gamma_3)$$

$$\boxed{\gamma_3 = 1-\gamma_1-\gamma_2}$$

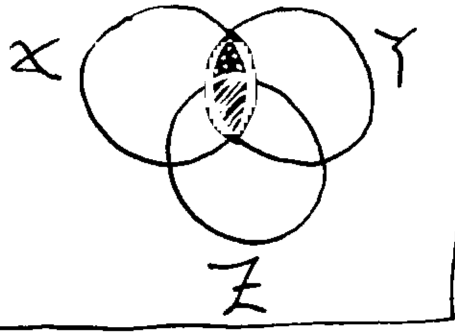
$$p(\gamma_1\,\gamma_2\,\gamma_3) = p_1\cdot\gamma_2\cdot\gamma_3$$

$$\Rightarrow H(\gamma_1\,\gamma_2\,\gamma_3) = -\gamma_1\operatorname{ld}\gamma_1 - \gamma_2\operatorname{ld}\gamma_2 - (1-\gamma_1-\gamma_2)\operatorname{ld}(1-\gamma_1-\gamma_2)$$

$$E\left[H(\gamma_1\,\gamma_2\,\gamma_3)\right] = -2\int_0^1\int_0^{1-\gamma_1} \gamma_1\operatorname{ld}\gamma_1 + \gamma_2\operatorname{ld}\gamma_2 + (1-\gamma_1-\gamma_2)\operatorname{ld}(1-\gamma_1-\gamma_2)\, d\gamma_2 d\gamma_1$$

$$= \frac{5}{6\ln 2}$$ MAPLE!!! MultiHopMimoChannel 3.18.24

PROBLEM 2.25 Venn diagrams. There isn't really a notion of mutual information to three random variables. Here is one attempt at a definition: Using Venn diagrams, we can see that the mutual information common to three random variables X, Y, Z can be defined by: $I(X;Y;Z) = I(X;Y) - I(X;Y|Z)$,



$$I(X;Y;Z) = I(X;Y) - I(X;Y|Z)$$

THIS QUANTITY IS SYMMETRIC IN X, Y, AND Z, DESPITE THE PRECEDING ASYMMETRIC DEFINITION

UNFORTUNATELY $I(X;Y;Z)$ IS NOT NECESSARILY NONNEGATIVE. FIND X, Y, AND Z SUCH THAT $I(X;Y;Z) < 0$, AND PROVE:

(a) $I(X;Y;Z) = H(X,Y,Z) - H(X) - H(Y) - H(Z) + I(X;Y) + I(Y;Z) + I(Z;X)$

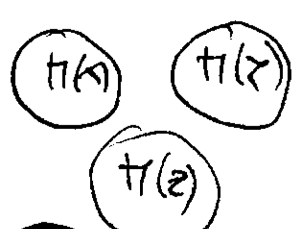
(b) $I(X;Y;Z) = H(X,Y,Z) - H(X,Y) - H(Y,Z) - H(Z,X) + H(X) + H(Y) + H(Z)$

THE FIRST IDENTITY CAN BE UNDERSTOOD USING THE VENN DIAGRAM ANALOGY FOR ENTROPY AND MUTUAL INFORMATION. SECOND FOLLOWS FROM THE FIRST.

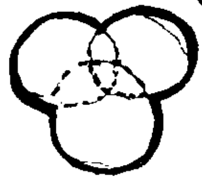
(b)
$$I(X;Y;Z) = H(X,Y,Z) - H(X) - H(Y) - H(Z) + H(X) - \underbrace{(H(X,Y) - H(Y))}_{H(X|Y)} + H(Y) - \underbrace{[H(Y,Z) - H(Z)]}_{H(Y|Z)} +$$
$$+ H(Z) - \underbrace{[H(X,Z) - H(X)]}_{H(Z|X)} = H(X,Y,Z) - \cancel{H(X)} - \cancel{H(Y)} - \cancel{H(Z)}$$
$$+ \cancel{H(X)} - H(X,Y) + \cancel{H(Y)} + H(Y) - H(Y,Z) + \cancel{H(Z)} + H(Z) - H(X,Z)$$
$$+ H(X) = H(X,Y,Z) - H(X,Y) - H(Y,Z) - H(X,Z) + H(X) + H(Y) + H(Z)$$

PROOFED!!!

(a) $H(X) + H(Y) + H(Z) =$



$H(X,Y,Z) =$



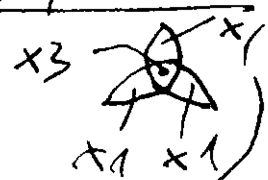
(−)

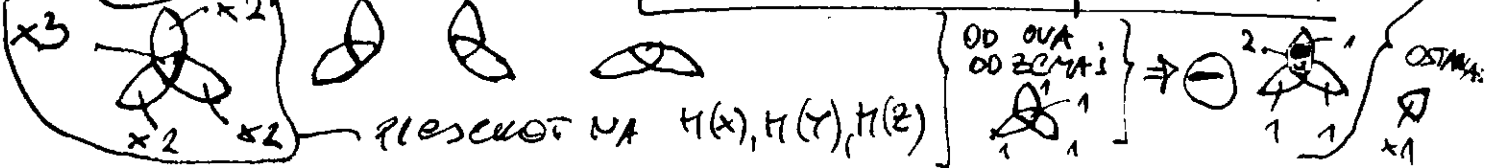
$I(X,Y) =$ ⬯

$I(Y,Z) =$ ⬯

$I(X,Z) =$ ⬯

+





PRESECI NA $H(X), H(Y), H(Z)$

OD OVA OD ZEMA ⇒ (−)

OSTANA:

So ova grafički go pokazav (a)

$$I(X;Y|Z) = -I(X;Z) + I(Y,Z;X)$$

$$I(Y,Z;X) = I(X;Z) + I(X;Y|Z)$$

⊙ Koga $I(X;Y;Z) < 0$?

- Će bide pomalo od nula ako na primer:

$I(X;Y|Z) > 0$ a $I(X;Y) = 0$

X, Y se statistički nezavisni no ako go vidiš „Z" X i Y stanuvaat megjusebno zavisni!

- $I(X;Y;Z) = I(X;Y) - I(X;Y|Z)$

$I(X;Y;Z) < 0$ IF $I(X;Y) \leq I(X;Y|Z)$

- $I(X;Y;Z) = I(X;Z) - I(X;Z|Y)$

$I(X;Y;Z) < 0$ IF $I(X;Z) \leq I(X;Z|Y)$

- $I(X;Y;Z) = I(Y;Z) - I(Y;Z|X)$

$I(X;Y;Z) < 0$ IF $I(Y;Z) \leq I(Y;Z|X)$

- Vo generalen slučaj ako dve od promenlivi X, Y, Z se statistički nezavisni togaš $I(X;Y;Z) < 0$

- Dokaz na (a) po analitički pat (editor i na ispitot da go vajat rešan)

$$I(X;Y;Z) = I(X;Y) - I(X;Y|Z) = I(X;Y) - [I(X;Z) + I(Y,Z;X)]$$

$$/I(Y,Z;X) = I(Z;X) + I(Y;X|Z) = I(X;Z) + I(X;Y|Z)/$$

$$I(X;Y;Z) = I(X;Y) + I(X;Z) - I(Y,Z;X)$$

$$I(Y,Z;X) = H(Y,Z) - H(Y,Z|X) = H(Y,Z) - [H(X,Y,Z) - H(X)]$$

$$I(X;Y;Z) = I(X;Y) + I(X;Z) - \underline{H(Y,Z)} + H(X,Y,Z) - H(X) =$$

$$= /H(Y,Z) = H(Y) + H(Z|Y) = \underline{H(Y) - I(Y;Z) + H(Z)}/ =$$

$$= I(X;Y) + I(X;Z) + I(Y;Z) - H(Y) - H(Z) + H(X,Y,Z) - H(X) =$$

$$= H(X,Y,Z) - H(X) - H(Y) - H(Z) + I(X;Y) + I(Y;Z) + I(X;Z)$$

$\boxed{2.26}$? ANOTHER PROOF OF NONNEGATIVITY OF RELATIVE ENTROPY. IN VIEW OF THE FUNDAMENTAL NATURE OF THE RESULT $D(p\|q) \geq 0$, WE WILL GIVE ANOTHER PROOF.

(a) SHOW THAT $\ln x \leq x-1$ for $0 \leq x \leq \infty$

(b) JUSTIFY THE FOLLOWING STEPS:

$$-D(p\|q) = \sum_x p(x) \ln \frac{q(x)}{p(x)} \leq \sum_x p(x)\left(\frac{q(x)}{p(x)} - 1\right) \leq 0$$

(c) WHAT ARE THE CONDITIONS FOR EQUALITY

(a) $\ln x \leq x - 1 \qquad \ln x - x + 1 \leq 0$ $\qquad$ $\ln x$ CONCAVE FUNCTION

$f(x) \leq 0 \qquad f(x) = \ln x - x + 1$

$f'(x) = \frac{1}{x} - 1 + 0 \qquad f''(x) = -\frac{1}{x^2} \leq 0 \quad \forall x$

$\Rightarrow f(x)$ e KONVEKSNA FUNKCIJA

• (a) MOŽE DA SE DOKAŽE NA DRUG NAČIN KAKO VO PROBLEM 2.13 SO KORISTENJE NA TAYLOROVA FORMULA: $\quad f'(x) = \frac{1}{x} \quad f''(x) = -\frac{1}{x^2}$

$$\ln(x) = \underbrace{f(1)}_{0} + \frac{f'(1)}{1}(x-1) + \frac{f''(1)}{2}(x-1)^2 =$$

$$= \left(\frac{1}{x_0}\right)\Big|_{x_0=1} \cdot (x-1) - \frac{1}{x_0^2}\Big|_{x_0=1}(x-1)^2 + \frac{2}{x_0^3}\Big|_{x_0=1}(x-1)^3 - \dots$$

$$= 1\cdot(x-1) - 1\cdot(x-1)^2 + 2(x-1)^3 - \dots$$

$$= \left| \left(-\frac{1}{x^2}\right)' = -\left(x^{-2}\right)' = -(-2)\,x^{-2-1} = \frac{2}{x^3} \right| =$$

$$\doteq (x-1) - (x-1)^2 \leq \underline{x-1} \qquad \boxed{\ln(x) \leq x-1}$$

$$-D(p\|q) = -\sum_{x\in A} p(x) \operatorname{ld} \frac{p(x)}{q(x)} = \sum_{x\in A} p(x) \operatorname{ld} \frac{q(x)}{p(x)}$$

$$\left| E[f(x)] \geq f[E[x]] \right|_{\text{CONVEX}} \leq \operatorname{ld} \sum p(x) \frac{q(x)}{p(x)} = \operatorname{ld} \sum_{x\in A} q(x) \leq$$

$$\leq \left| A \text{ IS SUBSET OF } \mathcal{X} \right| = \operatorname{ld} \sum_x q(x) \overset{x\in A}{=} \operatorname{ld} 1 = 0 \qquad -D(p\|q) \leq 0$$

$\boxed{D(p\|q) \geq 0}$

$$-D(p\|q) = \sum_x p(x)\, ld \frac{q(x)}{p(x)} \leq \left| \ln x \leq x - 1 \right|$$

$$\left| ld(x) = \frac{\ln(x)}{\ln 2} \right| \leq \frac{1}{\ln 2} \sum_x p(x) \cdot \left(\frac{q(x)}{p(x)} - 1 \right) =$$

$$= \frac{1}{\ln 2} \sum_x q(x) - p(x) \leq 0$$

$x \in [1, 2, 3, 4]$

$p(x) = \left[\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right] \qquad q(x) = \left[\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right]$

$q(x) - p(x) = \left[-\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{4}\right]$

$\sum q(x) - p(x) = -\frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{1}{4} \geq 0$

$$-D(p\|q) = \sum_x p(x)\, ld \frac{q(x)}{p(x)} \leq ld \underbrace{\sum_x \cancel{p(x)} \cdot \frac{q(x)}{\cancel{p(x)}}}_{\sum_x q(x) = 1} = ld\, 1 = 0$$

(c) $p(x) = q(x)$

$\boxed{2.27}$ GROUPING RULE FOR ENTROPY. LET $p = (p_1, p_2, \dots, p_m)$ BE A PROBABILITY DISTRIBUTION OF "m" ELEMENTS (I.E. $p_i \geq 0$ $\sum_{i=1}^m p_i = 1$). DEFINE A NEW DISTRIBUTION q ON "$m-1$" ELEMENTS AS $q_1 = p_1$, $q_2 = p_2$, … $q_{m-2} = p_{m-2}$, $q_{m-1} = p_{m-1} + p_m$. SHOW THAT

$$H(p) = H(q) + (p_{m-1} + p_m)\, H\left(\frac{p_{m-1}}{p_{m-1} + p_m}, \frac{p_m}{p_{m-1} + p_m} \right)$$

$$H(p) = \sum_{i=1}^{m} p_i\, ld\, p_i \qquad H(q) = \sum_{i=1}^{m-1} q_i\, ld\, q_i$$

$$H(p) = -\left[\sum_{i=1}^{m-2} p_i\, ld\, p_i + p_{m-1}\, ld\, p_{m-1} + p_m\, ld\, p_m \right] =$$

$$= -\Big[\underbrace{\sum_{i=1}^{m-2} p_i\, ld\, p_i + (p_{m-1} + p_m)\, ld(p_{m-1} + p_m)}_{-H(q)} - (p_{m-1} + p_m)\, ld\, (p_{m-1} + p_m)$$

$$+ p_{m-1}\, ld\, p_{m-1} + p_m\, ld\, p_m$$

$$H(p) = \underbrace{-\sum_{i=1}^{m-1} q_i \,\mathrm{ld}\, q_i}_{H(q)} + \underbrace{(p_{m-1}+p_m)\,\mathrm{ld}\,(p_{m-1}+p_m) - p_{m-1}\,\mathrm{ld}\, p_{m-1}}_{q_{m-1}\,\mathrm{ld}\, q_{m-1}} \quad - p_m \,\mathrm{ld}\, p_m$$

$$H(p) = H(q) + (p_{m-1}+p_m)\,\mathrm{ld}\,(p_{m-1}+p_m) - p_{m-1}\,\mathrm{ld}\,p_{m-1} - p_m\,\mathrm{ld}\,p_m$$

$$(*)\quad H\left(\frac{p_{m-1}}{p_{m-1}+p_m}, \frac{p_m}{p_{m-1}+p_m}\right) = -\frac{p_{m-1}}{p_{m-1}+p_m}\,\mathrm{ld}\,\frac{p_{m-1}}{p_{m-1}+p_m} - \frac{p_m}{p_{m-1}+p_m}\,\mathrm{ld}\,\frac{p_m}{p_{m-1}+p_m}$$

$$H(p) = H(q) + p_{m-1}\cdot\mathrm{ld}\,(p_{m-1}+p_m) + p_m\,\mathrm{ld}\,(p_{m-1}+p_m) - p_{m-1}\,\mathrm{ld}\,(p_{m-1}) - p_m\,\mathrm{ld}\,p_m$$

$$H(p) = H(q) + p_{m-1}\,\mathrm{ld}\left(\frac{p_{m-1}+p_m}{p_{m-1}}\right) + p_m\,\mathrm{ld}\,\frac{p_{m-1}+p_m}{p_m} =$$

$$= H(q) + \left[\frac{p_{m-1}}{p_{m-1}+p_m}\,\mathrm{ld}\,\frac{p_{m-1}+p_m}{p_{m-1}} + \frac{p_m}{p_{m-1}+p_m}\,\mathrm{ld}\,\frac{p_{m-1}+p_m}{p_m}\right]\cdot(p_{m-1}+p_m)$$

$$= H(q) + (p_{m-1}+p_m)\left[-\frac{p_{m-1}}{p_{m-1}+p_m}\,\mathrm{ld}\,\frac{p_{m-1}}{p_{m-1}+p_m} - \frac{p_m}{p_{m-1}+p_m}\,\mathrm{ld}\,\frac{p_m}{p_{m-1}+p_m}\right]$$

$$= H(q) + (p_{m-1}+p_m)\cdot H\left(\frac{p_{m-1}}{p_{m-1}+p_m}, \frac{p_m}{p_{m-1}+p_m}\right)$$

PROOFED!!!

PROBLEM 2.28 MIXING INCREASES ENTROPY. SHOW THAT THE ENTROPY OF THE PROBABILITY DISTRIBUTION $(p_1, p_2, \ldots, p_i, \ldots, p_j, \ldots p_m)$ IS LESS THEN THE ENTROPY OF THE DISTRIBUTION: $\left(p_1, p_2, \ldots, \frac{p_i+p_j}{2}, \ldots, \frac{p_i+p_j}{2}, \ldots p_m\right)$. SHOW THAT IN GENERAL ANY TRANSFER OF PROBABILITY THAT MAKES THE DISTRIBUTION MORE UNIFORM INCREASES ENTROPY.

$$D(p\|q) = +\sum_x p(x)\,\mathrm{ld}\,\frac{p(x)}{q(x)} \qquad q(x) = \frac{1}{|X|} \quad \forall x \in X$$

$$D(p\|q) = \sum_x p(x)\,\mathrm{ld}\,p(x) - \sum_x p(x)\cdot\mathrm{ld}\,\frac{1}{|X|} = -H(X) + \mathrm{ld}|X|$$

$$-\mathrm{ld}|X|$$

$$-H(X) + \mathrm{ld}|X| \geq 0$$

$$H(X) \leq \mathrm{ld}|X|$$

$$H(X) = \sum_{n=1}^{m} p_n \,\mathrm{ld}\, \frac{1}{p_n} \qquad H(Y) = \sum_{n=1}^{m} q_n \,\mathrm{ld}\, \frac{1}{q_n}$$

$$\boxed{H(x_i, x_j) = -p_i \,\mathrm{ld}\, p_i - p_j \,\mathrm{ld}\, p_j} \qquad H(y_i, y_j) = 2 \cdot \frac{p_i + p_j}{2} \,\mathrm{ld}\, \frac{2}{p_i + p_j}$$

$$p_i \,\mathrm{ld}\, \frac{1}{p_i} + p_j \,\mathrm{ld}\, \frac{1}{p_j} < (p_i + p_j) \,\mathrm{ld}\, \frac{2}{p_i + p_j} = p_i \,\mathrm{ld}\, \frac{2}{p_i + p_j}$$

$$+ p_j \,\mathrm{ld}\, \frac{2}{p_i + p_j} \; ; \quad p_i \,\mathrm{ld}\, \frac{1}{p_i} \cdot \frac{p_i + p_j}{2} + p_j \,\mathrm{ld}\, \frac{1}{p_j} \, \frac{p_i + p_j}{2} =$$

$$= p_i \,\mathrm{ld}\, \frac{p_i + p_j}{2 p_i} + p_j \,\mathrm{ld}\, \frac{p_i + p_j}{2 p_j} = p_i \,\mathrm{ld}\left(\frac{1}{2} + \frac{p_j}{2 p_i}\right) +$$

$$+ p_j \,\mathrm{ld}\left(\frac{p_i}{2 p_j} + \frac{1}{2}\right) < 0$$

$$p_i \,\mathrm{ld}\left(\frac{1}{2} + \frac{p_j}{2 p_i}\right) < p_j \,\mathrm{ld}\left(\frac{2 p_j}{p_i + p_j}\right) \quad ?$$

$$\mathrm{ld}\left(\frac{1}{2} + \frac{p_j}{2 p_i}\right)^{p_i} \cdot \left(\frac{1}{2} + \frac{p_i}{2 p_j}\right)^{p_j} < 0$$

$$\left(\frac{1}{2} + \frac{p_j}{2 p_i}\right)^{p_i} \cdot \left(\frac{1}{2} + \frac{p_i}{2 p_j}\right)^{p_j} < 1$$

$$\left(\frac{1}{2} + \frac{p_j}{2 p_i}\right)^{p_i} < \left(\frac{1}{2} + \frac{p_i}{2 p_j}\right)^{-p_j}$$

$$x \in \{x_i, x_j\} \qquad P = \{p_i, p_j\}$$

$$E\left[\mathrm{ld}\, \frac{1}{p(x)}\right] = E\left[-\mathrm{ld}\,[p(x)]\right] = -E\{\mathrm{ld}\,[p(x)]\}$$

$$= -\left[p_i \cdot \mathrm{ld}\, p_i + p_j \cdot \mathrm{ld}\, p_j\right] \qquad \ln \Rightarrow \text{CONCAVE FUNCTION}$$

$$E\{\mathrm{ld}\,[p(x)]\} < \mathrm{ld}\, E[p(x)]$$

$$E[p(x)] = p_i \cdot p_i + p_j \cdot p_j \qquad E[p(x)] = E\left[2^{\mathrm{ld}\, p(x)}\right]$$

$$E[p(x)] \geq 2^{E[\mathrm{ld}\, p(x)]} = 2^{H(x)}$$

• $p = \{p_i, p_j\}$ $\qquad q = \{q_i, q_j\} = \left\{\frac{p_i+p_j}{2}, \frac{p_i+p_j}{2}\right\}$

$$D(p\|q) = \sum p \, ld \frac{p}{q} = \underbrace{-\sum p \, ld \frac{1}{p}}_{-H(p)} - \sum p \, ld \frac{1}{q} \geq 0$$

$$-H(p) - ld \frac{2}{p_i+p_j} \sum p \geq 0 \qquad -H(p) \geq (p_i+p_j) \, ld \frac{2}{p_i+p_j}$$

$$H(p) \leq (p_i+p_j) \, ld \frac{2}{p_i+p_j} = H(q)$$

$$H(q) = \sum q \cdot ld \frac{1}{q} = \cancel{2} \cdot \frac{p_i+p_j}{\cancel{2}} \, ld \frac{2}{p_i+p_j} = (p_i+p_j) \, ld \frac{2}{p_i+p_j}$$

$$\Rightarrow H(p_i, p_j) \leq H(q_i, q_j) = H\left(\frac{p_i+p_j}{2}, \frac{p_i+p_j}{2}\right)$$

• GENERALIZED

$$p = \{p_{i1}, p_{i2}, \dots p_{in}\} \qquad q = \left\{\frac{\sum_{j=1}^{n} p_{ij}}{n}, \frac{\sum_{j=1}^{n} p_{ij}}{n} \dots \frac{\sum_{j=1}^{n} p_{ij}}{n}\right\}$$

$$D(p\|q) = \sum_{j=1}^{n} p_{ij} \, ld \frac{p_{ij}}{q_{ij}} = \underbrace{-\sum_{j=1}^{n} p_{ij} \, ld \frac{1}{p_{ij}}}_{-H(p)} -$$

$$-\sum_{j=1}^{n} p_{ij} \, ld \left(\frac{\sum p_{ij}}{n}\right) = -H(p) - \left(ld \frac{\sum p_{ij}}{n}\right) \sum_{j=1}^{n} p_{ij} =$$

$$= H(q) = -\cancel{n} \cdot \frac{\sum_{j=1}^{n} p_{ij}}{\cancel{n}} \, ld \frac{\sum_{j=1}^{n} p_{ij}}{n} = -\left[ld\left(\frac{\sum p_{ij}}{n}\right)\right] \sum_{j=1}^{n} p_{ij} =$$

$$= -H(p) + H(q) \geq 0 \Rightarrow \boxed{H(p) \leq H(q)}$$

GENERALIZED PROOF!!! ...

$$\sum a_i \, ld \frac{b_i}{c_i} \geq \left(\sum a_i\right) ld \frac{\sum b_i}{\sum c_i} \Big\} \text{ LOG-SUM INEQUALITY}$$

• EDITION 1 SOLUTION:

$$P_1 = (p_1, \dots, p_i, \dots, p_j, \dots p_n)$$
$$P_2 = \left(p_1, \dots, \frac{p_i+p_j}{2}, \dots, \frac{p_i+p_j}{2}, \dots p_n\right)$$

$$H(P_2) - H(P_1) = p_i \, ld \, p_i + p_j \, ld \, p_j - \cancel{2}\left(\frac{p_i+p_j}{\cancel{2}}\right) ld \frac{p_i+p_j}{2}$$

$$= \underbrace{p_i \, ld \, p_i + p_j \, ld \, p_j}_{} - (p_i+p_j) \, ld \frac{p_i+p_j}{2}$$

$$\sum_{n\in\{i,j\}} p_n \, ld \, p_n \geqslant \left(\sum_{n\in\{i,j\}} p_n\right) \cdot ld \sum_{n\in\{i,j\}} p_n$$

$$= p_i \, ld \, p_i + p_j \, ld \, p_j - (p_i+p_j) \, ld (p_i+p_j) + (p_i+p_j) \cdot ld \, 2$$

$$= p_i \, ld (2 p_i) + p_j \, ld (2 p_j) - (p_i+p_j) \, ld (p_i+p_j) =$$

$$= \sum_{n\in\{i,j\}} p_n \, ld (2 p_n) - \left(\sum_{n\in\{i,j\}} p_n\right) ld \sum_{n\in\{i,j\}} p_n \geqslant$$

$$\geqslant \left(\sum_{n\in\{i,j\}} p_n\right) ld \frac{\sum_{n\in\{i,j\}} p_n}{\frac{1}{2}} - \left(\sum_{n\in\{i,j\}} p_n\right) ld \sum_{n\in\{i,j\}} p_n =$$

$$= \cancel{\left(\sum_{n\in\{i,j\}} p_n\right) ld \sum_{n\in\{i,j\}} p_n} + \sum_{n\in\{i,j\}} p_n \, \cancelto{1}{ld 2} - \cancel{\left(\sum_{n\in\{i,j\}} p_n\right) ld \sum_{n\in\{i,j\}} p_n}$$

$$= \sum_{n\in\{i,j\}} p_n \geqslant \emptyset \Rightarrow \boxed{H(P_2) > H(P_1)}$$

• GENERALIZED:

$$P_1 = \{p_1, \dots, p_{i_1}, \dots p_{i_n}, \dots p_m\}$$
$$P_2 = \left\{p_1, \dots, \frac{\sum p_{ij}}{n}, \dots, \frac{\sum p_{ij}}{n}, \dots p_m\right\}$$

$$H(P_2) - H(P_1) = \sum_{j=1}^{n} p_{ij} \, ld \, p_{ij} - n \frac{\sum_{j=1}^{n} p_{ij}}{n} \, ld \frac{\sum_{j=1}^{n} p_{ij}}{n} \geqslant$$ log-sum inequal.

$$\left(\sum_{j=1}^{n} p_{ij}\right) ld \sum_{j=1}^{n} p_{ij} - \left(\sum_{j=1}^{n} p_{ij}\right) ld \sum_{j=1}^{n} p_{ij} + \sum_{j=1}^{n} p_{ij} \, ld \, n =$$

$$= \sum_{j=1}^{n} p_{ij} \, ld \, n \geqslant \emptyset \Rightarrow H(P_2) \geq H(P_1)$$

$\boxed{\text{PROBLEM 2.29}}$ INEQUALITIES. LET X, Y & Z ARE JOINT RANDOM VARIABLES. PROVE THE FOLLOWING INEQUALITIES AND FIND CONDITIONS FOR EQUALITY.

(a) $H(X,Y|Z) \geq H(X|Z)$ (b) $I(X,Y;Z) \geq I(X;Z)$

(c) $H(X,Y,Z) - H(X,Y) \leq H(X,Z) - H(X)$

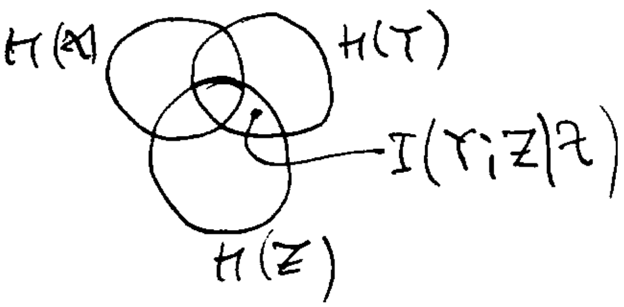
(d) $I(X;Z|Y) \geq I(Z;Y|X) - I(Z;Y) + I(X;Z)$

(a) $H(X,Y|Z) = H(X|Z) + \underbrace{H(Y|X,Z)}_{\geq 0} \Rightarrow H(X,Y|Z) \geq H(X|Z)$

CONDITION FOR EQUALITY: $\underline{Y = f(X,Z)} \Rightarrow H(Y|X,Z) = 0$

(b.) $I(X,Y;Z) \geq I(X;Z)$

$I(X,Y;Z) = I(X;Z) + \underbrace{I(Y;Z|X)}_{\geq 0}$



$I(Y;Z|X) = H(Y|X) - H(Y|X,Z)$

$I(Y;Z|X) = 0$ IF Y & Z ARE INDEPENDENT GIVEN "X"

$\underline{H(Y|X,Z) = H(Y|X)}$

$X \to Y \to Z$

DATA PROCESSING INEQUALITY:

$p(x,y,z) = p(x) \cdot p(y|x) \cdot p(z|y)$

$I(X;Y) \geq I(X;Z)$

$I(X;Y,Z) = I(Y,Z;X) = I(X;Y) + \overbrace{I(X;Z|Y)}^{0} =$

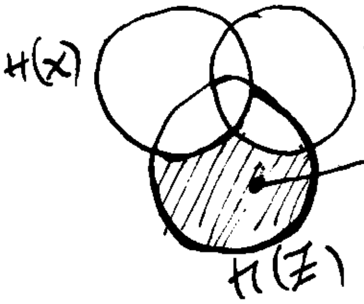
$= I(X;Z) + \underbrace{I(X;Y|Z)}_{\geq 0} \Rightarrow \underline{I(X;Y) \geq I(X;Z)}$

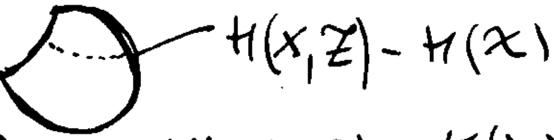
$I(X;Z|Y=y) = H(X|Y=y) - H(X|Z,Y=y) = H(X|Y=y) - H(X|Y=y) = 0$

(c) $H(X,Y,Z) - H(X,Y) = \underbrace{H(X) + H(Y|X)}_{H(X,Y)} + H(Z|X,Y) - \cancel{H(X,Y)}$

$= H(Z|X,Y) = H(X,Z) + H(Y|X,Z) - H(X,Y) = H(X,Z) + H(Y|X,Z) - H(X) - H(Y|X)$

$H(Y|X,Z) = H(X,Y,Z) - H(X,Z)$





(LHS)

$$H(X,Y,Z) - H(X,Y) = H(Z|X,Y)$$

$$H(X,Z) - H(X) = \cancel{H(X)} + H(Z|X) - \cancel{H(X)} = H(Z|X) \quad \text{(RHS)}$$

$$\boxed{H(Z|X,Y) \le H(Z|X)}$$ → TRUE SINCE CONDITIONING REDUCES ENTROPY !!

$$\left.\begin{aligned} H(X,Z|Y) &= H(\cancel{X}|Y) + H(X|Y,Z) \\ H(Y,Z|X) &= H(Z|X) + H(Y|X,Z) \end{aligned}\right\} ?$$

$$I(X,Y) = H(X) - H(X|Y) \ge 0 \qquad H(X) \ge H(X|Y)$$

$$I(Y;Z|X) = H(Z|X) - H(Z|X,Y) \ge 0$$

$$H(Z|X) \ge H(Z|X,Y)$$

CONDITION FOR EQUALITY: $I(Y;Z|X) = 0$

$\underbrace{Y \to X \to Z}_{\text{MARKOV CHAIN}}$ ⇒ Z & Y ARE STATISTICALLY INDEPENDENT GIVEN X, i.e.

$$\Pi_c = \begin{bmatrix} 1-\gamma & \gamma & 0 \\ 0 & 1-\gamma & \gamma \\ \gamma & 0 & 1-\gamma \end{bmatrix} \qquad P(Y_j) = \sum_{i=1}^{n} P(y_j|x_i) \cdot P(x_i) \quad j = 1,2,\dots,r$$

$$\boxed{Y = \Pi_c^T \cdot X}$$

$$\Pi_c = \begin{bmatrix} P(y_1|x_1) & P(y_2|x_1) & \dots & P(y_r|x_1) \\ P(y_1|x_2) & P(y_2|x_2) & \dots & P(y_r|x_2) \\ \dots & \dots & & \dots \\ P(y_1|x_n) & P(y_2|x_n) & \dots & P(y_r|x_n) \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_r \end{bmatrix} = \begin{bmatrix} P(y_1|x_1) & P(y_1|x_2) & \dots & P(y_1|x_n) \\ \dots & & & \\ \dots & & & \\ P(y_r|x_1) & P(y_r|x_2) & \dots & P(y_r|x_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$C = \upsilon(X,Y) \max_{P(x_i)} I(X;Y) = \upsilon(X,Y)\left[H(Y) - H(Y|x_i)\right]$$

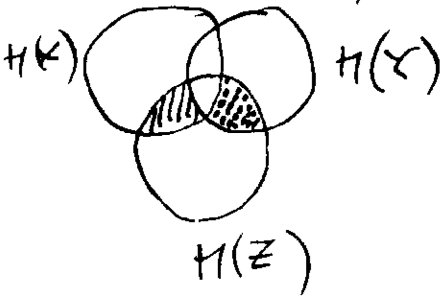
$\max H(Y) = \operatorname{ld} r$ } FOR UNIFORM DISTRIBUTION OF OUTPUT SYMBOLS

$\boxed{H(Y|X) = H(Y|x_i)}$ → SYMMETRICAL Π_c, UNIFORM $P(x)$

$$H(Y|x_i) = (1-p)\operatorname{ld}\frac{1}{1-p} + p \operatorname{ld}\frac{1}{p} = H(Y|X)$$

$$\boxed{C = \sigma(x,y)\left[\log r - (1-\gamma)\log\frac{1}{1-\gamma} - \gamma\log\frac{1}{\gamma}\right]}$$

(d) $I(X;Z|Y) \geq I(Z;Y|X) - I(Z;Y) + I(X;Z)$ ①



LHS $I(X;Z|Y) = I(X;Y,Z) - I(X;Y)$

RHS $I(Z;Y|X) = I(Y;X,Z) - I(X;Y)$

(*) $\boxed{I(X,Z;Y) = I(X;Y) + I(Z;Y|X) = I(Z;Y) + I(X;Y|Z)}$

$I(X;Y,Z) - I(X;Y) \geq I(Y;X,Z) - I(X;Y) - I(Z;Y) + I(X;Z)$

$I(X;Y,Z) \geq \underbrace{I(Z;Y) + I(X;Y|Z)}_{= I(Y,Z;X)} - I(Z;Y) + I(X;Z)$

□

ALTERNATIVELY IF WE START FROM RHS:

VO OVOJ SLUCAJ SO OVA SE POKAZUVA DEKA LSH = RSH !!!

$I(Z;Y|X) + I(X;Z) = I(X,Y;Z)$

$I(X,Y;Z) - I(Z;Y) = I(X,Y;Z) - I(Y;Z) = I(X;Z|Y)$

(*) $I(X,Z;Y) = I(X;Y) + I(Z;Y|X) = I(Z;Y) + I(X;Y|Z)$ THEN → 0

IF: $X \to Z \to Y$

$p(x,y,z) = p(x)\cdot p(y|x)\cdot p(z|y)$

BIDEJKI $I(X;Y|Z) \geq 0$!

REPLACE IN ①

$I(Z;Y|X) \geq I(Z;Y) - I(X;Y)$

$I(X;Z|Y) = \underbrace{I(X;Y,Z) - I(X;Y)}_{LHS} \geq \cancel{I(Z;Y)} - \cancel{I(X;Y)} - \cancel{I(Z;Y)} + I(X;Z)$

$I(X;Y,Z) \geq I(X;Z)$

$\cancel{I(X;Z)} + I(X;Y|Z) \geq \cancel{I(X;Z)}$

$I(X;Y|Z) \geq 0$

- CONDITION FOR EQUALITY IF $X \to Z \to Y$ i.e. X, Z, Y FORM A MARKOV CHAIN.

- VO EDITION 1 ZA IMAAT GREŠKA SO ZNAKOT KAKO VO ①. „AKO GO PREFRLIŠ $I(Z;Y)$ NA LHS“:

$I(X;Z|Y) + I(Z;Y) = I(X,Y;Z)$ "="

RHS: $I(Z;Y|X) + I(X;Z) = I(X,Y;Z)$

PROBLEM 2.30 MAXIMUM ENTROPY. FIND THE PROBABILITY MASS FUNCTION $p(x)$ THAT MAXIMIZES THE ENTROPY $H(X)$ OF NON-NEGATIVE INTEGER-VALUED RANDOM VARIABLE X SUBJECT TO CONSTRAINT

$$\boxed{E[X] = \sum_{n=0}^{\infty} n\,p(n) = A}$$

for fixed VALUE $A > 0$. EVALUATE THIS MAXIMUM $H(X)$.

$$S = \sum_{n=0}^{\infty} x^n \qquad 0 < x \leq 1 \qquad x = \frac{1}{2}$$

$$S = \sum_{n=0}^{n} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n}$$

$$= 1 + x^1 + x^2 + \dots + x^n$$

$$x \cdot S = x + x^2 + \dots + x^n + x^{n+1}$$

$$S - xS = 1 - x^{n+1} \qquad S = \frac{1 - x^{n+1}}{1 - x} = (\quad)$$

$n \to \infty$, $0 < x < 1$ $\Rightarrow$ $\boxed{S = \frac{1}{1-x}}$

$$\hat{S} = \sum_{n=0}^{\infty} n x^n = ? \qquad S = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\int \sum_{n=0}^{\infty} x^n \, dx = \int \frac{1}{(1-x)} \, dx \qquad \frac{x^{n+1}}{n+1} \; !$$

$$\left(\sum_{n=0}^{\infty} x^n \right)' = \left(\frac{1}{1-x} \right)' \Rightarrow \sum_{n=0}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$$

$n = m-1$
$n = m+1$
$m \triangleq n$

$$\sum_{n=0}^{\infty} n x^{n-1} = \sum_{m=1}^{\infty} n x^{n-1} + 0 = \sum_{n=0}^{\infty} (n+1) x^n =$$

$$= \sum_{n=0}^{\infty} n x^n + \sum_{n=0}^{\infty} x^n = \frac{1}{(1-x)^2} \; ; \quad S_2 = \frac{1}{(1-x)^2} - \frac{1}{1-x}$$

$$\boxed{S_2 = \frac{1 - x + x}{(1-x)^2} = \frac{+x}{(1-x)^2}}$$

$E[X] = \sum_{n=0}^{\infty} n\, p(n) = A$ $\quad (\ln x)' = \frac{1}{x}$ $\quad$ TAЊ 070243461

$H(X) = -\sum_{n=0}^{\infty} p(n)\,\mathrm{ld}\,p(n)$ $\quad \frac{d}{dp}\left[H(X)\right] = 0$

$\sum_{n=0}^{\infty}\left(p'\,\mathrm{ld}\,p(n) + \frac{p}{\ln 2}\cdot\frac{1}{p}\right) = 0$ $\quad \lim_{n\to\infty} n\left(p'\,\mathrm{ld}\,p + \frac{1}{\ln 2}\right) = 0$

$p'\cdot\frac{\ln p}{\cancel{\ln 2}} = -\frac{1}{\ln 2}$ $\quad \cancel{p'}\cdot\ln p = -1$

$p' = \frac{-1}{\ln p}$ $\quad \ln p^{-p'} = 1$ $\quad p^{-p'} = 2$

$\ln\left(\frac{1}{p}\right) = 1$ $\quad \frac{1}{p} = 2$ $\quad \boxed{p = \frac{1}{e}}$

$\frac{d}{dp}\left(E[X]\right) = \sum_{n=0}^{\infty} n\cdot p'(n) = 0$ $\quad \left[\sin(x^2)\right]' = \cos(x^2)\cdot 2x$

$E[X] = \frac{1}{2}\sum_{n=0}^{\infty} n\cdot\frac{1}{2^n} = \frac{1}{2}\cdot\frac{x}{(1-x)^2} = \left| x = \frac{1}{2}\right| = \frac{\frac{1}{2}}{\frac{1}{4}} = 2\cdot\frac{1}{2} = 1$

$E[X] = (q-1)\sum_{n=0}^{\infty} n\,\frac{1}{q^{n+1}} = \frac{q-1}{q}\,\frac{\frac{1}{q}}{\left(1-\frac{1}{q}\right)^2} = \frac{\frac{q-1}{q^2}}{\left(\frac{q-1}{q}\right)^2} = \frac{(q-1)\cancel{q^2}}{\cancel{q^2}(1-q)^2} = \frac{q-1}{(1-q)^2}$

$n \in \{0, 1, 2, 3, \dots\}$

$p(n) = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\right\}$ $\quad \sum_{n=0}^{\infty} p(n) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n} = \frac{1}{2}\,\frac{1}{1/2} = \frac{1}{2}\cdot 2 = 1$

$q = 3 \quad p(n) = \left\{\frac{1}{3}, \frac{1}{9}, \dots\right\} \Rightarrow \sum_{n=0}^{\infty} p(n) = \frac{1}{3}\sum_{n=0}^{\infty}\frac{1}{3^n} = \frac{1}{3}\,\frac{1}{\frac{2}{3}} = \frac{1}{2}$

VO OPŠT SLUČAJ $p(n)$ MOŽE DA SE DEFINIRA KAKO

$\boxed{p(n) = \frac{q-1}{q^{n+1}}}$ $\quad \sum_{n=0}^{\infty} p(n) = 1$ ZA $\forall q > 1$

$$E[X] = \sum_{n=0}^{\infty} n \frac{q-1}{q^{n+1}} = \frac{q-1}{q} \sum_{n=0}^{\infty} \frac{n}{q^n} = \frac{q-1}{q} \cdot \frac{\frac{1}{q}}{\left(1-\frac{1}{q}\right)^2} = \frac{q-1}{q^2} \frac{q^2}{(q-1)^2}$$

$$\boxed{E[X] = \frac{1}{q-1}}$$

$$\boxed{\begin{array}{ll} \text{If } q=2 & \Rightarrow E[X]=1 \\ \text{If } q=3 & \Rightarrow E[X]=1/2 \end{array}}$$

$$H(X) = -\sum_{n=0}^{\infty} \frac{q-1}{q^{n+1}} \ln \frac{q-1}{q^{n+1}} \qquad \frac{dH(X)}{dq} = 0$$

$$-\frac{d}{dq}\left(\frac{q-1}{q^{n+1}} \ln \frac{q-1}{q^{n+1}} \right) = \frac{\left(1 + \ln \frac{q-1}{q^{n+1}}\right)\left((q-1)n - 1\right)}{q^{n+2}}$$

(a) $(q-1)n - 1 = 0 \qquad q-1 = \frac{1}{n} \qquad \boxed{q = \frac{1}{n} + 1}$

(b) $\ln \frac{q-1}{q^{n+1}} = -1 \qquad \frac{q-1}{q^{n+1}} = \frac{1}{e} \qquad (q-1) \cdot e = q^{n+1}$

$$\boxed{q^{n+1} - eq + e = 0}$$

$$\sum_{n=0}^{\infty} \frac{1}{q^{n+2}} \left[n(q-1)-1\right] \cdot \left[\ln\left(\frac{q-1}{q^{n+1}}\right)+1\right] = -\frac{\ln q}{(q-1)^2} = 0$$

$$\Rightarrow \boxed{q = e} \qquad \Rightarrow \boxed{p(n) = \frac{e-1}{e^{n+1}}} \ ?$$

$$p(n) = \left(\frac{1}{p}-1\right) \cdot p^{n+1} = \frac{1-p}{\cancel{p}} p^{n+\cancel{1}} = (1-p)p^n$$

$$\sum_{n=0}^{\infty} p(n) = \sum_{n=0}^{\infty} (1-p)p^n = (1-p)\sum_{n=0}^{\infty} p^n = (1-p)\frac{1}{1-p} = 1$$

$$E[X] = \sum_{n=0}^{\infty} n(1-p)p^n = (1-p)\sum_{n=0}^{\infty} np^n = (1-p)\frac{p}{(1-p)^2}$$

$$\boxed{E[X] = \frac{p}{1-p}} \qquad p = \frac{1}{2} \Rightarrow \frac{\frac{1}{2}}{\frac{1}{2}} = 1; \quad p = \frac{1}{3}; \quad \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

$$\cdot\, p^{n-1}\left[n-(n+1)p\right]\cdot\left[\ln\left(p^n(1-p)\right)+1\right]$$

$$\frac{d}{dp}\left(H(p)\right) = \sum_{n=0}^{\infty} p^{n-1}\left[n-(n+1)p\right]\left[\ln\left(p^n(1-p)\right)+1\right] = -\frac{\ln p}{(p-1)^2}$$

$$H(p) = -\frac{p\ln(1-p) - \ln(1-p) - p\ln p}{\ln 2\,(p-1)}$$ SO KONSTENTA NA MALE

$$H(p) = -\frac{(p-1)\ln(1-p) - p\ln p}{\ln 2\,(p-1)} = \frac{p\ln p + (1-p)\ln(1-p)}{\ln 2\,(p-1)}$$

$$\boxed{H(p) = \frac{(1-p)\ln(1-p) + p\ln p}{\ln 2\,(p-1)}}$$

$\boxed{p=\frac{1}{2}}$ $\quad p(n) = (1-p)p^n = \frac{1}{2}\cdot\left(\frac{1}{2}\right)^n$

$$H(X) = -\sum_{n=0}^{\infty}\frac{1}{2}\left(\frac{1}{2}\right)^n \operatorname{ld}\frac{1}{2}\left(\frac{1}{2}\right)^n = +\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n \operatorname{ld} 2^{n+1}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(n+1)\left(\frac{1}{2}\right)^n = \frac{1}{2}\left[\sum_{n=0}^{\infty} n\frac{1}{2^n} + \sum_{n=0}^{\infty}\frac{1}{2^n}\right]$$

$$= \frac{1}{2}\left[\frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} + \frac{1}{\frac{1}{2}}\right] = \frac{1}{2}\left[2+2\right] = \frac{4}{2} = 2$$

SUFFICIEN STATISTICS (RECALL)

- FAMILY OF PROBABILITY MASS FUNCTIONS $\{f_\theta(x)\}$
- X – SAMPLE FROM A DISTRIBUTION IN THIS FAMILY
- $T(X)$ ANY STATIC (FUNCTION OF THE SAMPLE)

THAN: $\theta \to X \to T(X)$

FROM DATA PROCESSING INEQUALITY FOR ANY θ

$$I(\theta; T(X)) \leq I(\theta; X)$$

- $T(X)$ IS SUFFICIENT STATISTIC RELATIVE TO THE FAMILY $\{f_\theta(x)\}$ IF X & θ ARE INDEPENDENT GIVEN $T(X)$

i.e.: $\boxed{\theta \to T(X) \to X}$

- $I(\theta; X) = I(\theta; T(X))$ IF $T(X)$ IS SUFFICIENT STATISTIC

EXAMPLES OF SUFFICIENT STATISTIC:

1 $x_1, x_2, \dots x_n$ $x_i \in \{0,1\}$ INDEPENDENT AND IDENTICALLY DISTRIBUTED

SEQUENCE OF COIN TOSSES WITH UNKNOWN PARAM.

$\theta = P_r(x_i = 1)$

GIVEN "n", THE NUMBER OF 1'S IS A SUFFICIENT STATISTIC FOR θ. $T(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} x_i$

$$P_r\left\{ (x_1, x_2, \dots x_n) = (x_1, x_2, \dots x_n) \,\middle|\, \sum_{i=1}^{n} x_i = k \right\} =$$

$$= \begin{cases} \frac{1}{\binom{n}{k}} & \text{if } \sum x_i = k \\ 0 & \text{OTHERWISE} \end{cases}$$

$$\left.\theta \to \sum x_i \to (x_1, x_2, \dots x_n)\right\}$$ FORMS MARKOV CHAIN AND T IS SUFFICIENT STATISTIC FOR θ

$$I(\theta; T(X)) = H(T(X)) - H(T(X) \mid \theta)$$

- EXAMPLE:

A 0 1 1
B 1 0 1
C 1 1 0

$$\binom{n}{k} = \frac{3!}{2! \cdot (3-2)!} = \frac{6}{2} = 3$$

$\boxed{n = 3} \Longrightarrow$ GIVEN n = NUMBER OF ONES IS SUFFICIENT STATISTIC OF θ

$$P(X_1 = 1) = \underbrace{\tfrac{1}{3}}_{P(B)} \cdot 1 + \underbrace{\tfrac{1}{3}}_{P(C)} \cdot 1 = \tfrac{1}{3} + \tfrac{1}{3} = \tfrac{2}{3}$$

$$P(X_2 = 1) = P(A) \cdot 1 + P(C) \cdot 1 = \tfrac{1}{3} + \tfrac{1}{3} = \tfrac{2}{3}$$

$$P(X_3 = 1) = P(A) \cdot 1 + P(B) \cdot 1 = \tfrac{1}{3} + \tfrac{1}{3} = \tfrac{2}{3}$$

- ZNAČI AKO GA ZNAEŠ DOLŽINATA NA SEKVENCATA I BROJOT NA EDINICI VO NEA TOGAŠ GI ZNAEŠ I VEROJATNOSTITE DA x_i $(i = 1 \dots 3)$ BIDE EDNAKOV NA 1. AKO GO ZNAEŠ $T(X)$ ISTO, TAKA GA ZNAEŠ VEROJATNOSTA NA POJAVUVANJE NA SEKVENCATA: x_a, x_b, x_c

② IF X IS NORMALLY DISTRIBUTED WITH MEAN θ AND VARIANCE 1:

$$f_\theta = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\theta)^2}{2}} = N(\theta,1)$$

AND $x_1, x_2, \ldots, x_n$ ARE DRAWN INDEPENDENTLY ACCORDING TO THIS DISTRIBUTION, A SUFFICIENT STATISTIC FOR θ IS THE SAMPLE MEAN $\bar{X}_n = \frac{1}{n}\sum x_i$. IT CAN BE VERIFIED THAT THE CONDITIONAL DISTRIBUTION OF $x_1, x_2, \ldots, x_n$, CONDITIONED ON $\bar{X}_n$ & n DOES NOT DEPEND ON θ.

(GIVEN STATISTIC $T = \bar{X}_n$ THE DISTRIBUTION OF DATA IS INDEPENDENT OF THE PARAMETER θ) MMV

$$\theta \to \bar{X}_n \to x_1, x_2, \ldots, x_n$$

$$P(x_1, x_2, \ldots x_n \mid \bar{x}_n, n) = P(x_1, x_2, \ldots x_n)$$

$$P(X,Y,Z) = P(X)\cdot P(Y|X)\cdot P(Z|Y) \qquad X \to Y \to Z$$

$$P(Y,Z|X) = \frac{P(X,Y,Z)}{P(X)} = \frac{P(X,Y)\cdot P(Z|X,Y)}{P(X)} = \frac{P(X,Y)P(Z|Y)}{P(X)}$$

$$P(Y,Z|X) = P(Y|X)\,P(Z|Y) \qquad P(X)\cdot P(Y,Z|X) = P(X,Y,Z) = P(X)\cdot P(Y|X)\cdot P(Z|Y)$$

$$P(X,Z|Y) \stackrel{?}{=} \frac{P(X,Y,Z)}{P(Y)} = \frac{P(X,Y)\cdot P(Z|X,Y)}{P(Y)} = P(X|Y)\cdot P(Z|Y)$$

③ If $f_\theta = \text{Uniform}(\theta, \theta+1)$, A SUFFICIENT STATISTIC OF θ IS:

$$T(x_1, x_2, \ldots x_n) = (\max\{x_1, x_2, \ldots x_n\}, \min\{x_1, x_2, \ldots x_n\})$$

AGAIN, ONE CAN SHOW THAT THE DISTRIBUTION OF DATA IS INDEPENDENT OF THE PARAMETER GIVEN STATISTIC T.

• СЕ НАВРАЌАМ НА ПРИМЕР 1 ЗА ДА ЈА ДОКАЖАМ НЕЗАВИСНОСТА НА СЕКВЕНЦАТА НА ПАЃЕЊЕ ПАРИЧКИ ОД ПАРАМЕТАРОТ $\theta = P_r(x_i = 1)$ АКО СЕ ЗНАЕ БРОЈОТ НА ЕДИНИЦИ ВО СЕКВЕНЦАТА

$$T(X) = \sum_{i=1}^{n} x_i$$

• НОВА $\theta = \frac{1}{2}$ (СИМУЛАЦИЈА ВО MAPLE)

0 0 0 1 0	$T(X) = 1$
0 1 1 1 1	$T(X) = 4$
0 0 1 1 0	$T(X) = 2$
1 0 1 0 0	$T(X) = 2$
1 0 0 0 0	$T(X) = 1$

$$\theta \to T(X) \to X$$

ТРЕБА ДА СЕ ДОКАЖЕ ДЕКА

$$\boxed{P(X \mid T(X), \theta) = P(X \mid T(X))}$$ MMV

ПРИ ДАДЕНО $T(X)$ X НЕ ЗАВИСИ ОД θ !!! МНОГУ ЛОГИЧНО !!

$P(x|T(x)) = ?$ $\quad P(x|T(x)=1) = \frac{1}{\binom{4}{k}} = \frac{1}{\binom{5}{1}} = \frac{1}{5}$

$P(x|T(x)=2) = \frac{1}{\binom{5}{2}} = \frac{1}{\frac{5!}{3!\cdot 2!}} = \frac{1}{10}$

$P(x|T(x)=3) = \frac{1}{\binom{5}{3}} = \frac{1}{10}$ $\quad P(x|T(x)=4) = \frac{1}{\binom{5}{4}} = \frac{1}{5}$

$P(x|T(x)=5) = 1$

- OVA E FAKT! AKO GO ZNAEŠ T(x) SE EDNO TI E ~~ZAŠTO ŠTO NAJVEROJ S KOJA BILA VREDNOSTA NA "θ"~~ ZAŠTO ŠTO NAJVEROJ... (POJAVUVANJE NA "1" VO NIZATA) VEĆE SE SLUČIL A T(x) ODKAŽUVA KOLKU PATI SE SLUČIL.

$$\boxed{P(x, T(x))} = \sum_{\forall t} \left[P(T(x)=t) \cdot P(x|T(x)=t) \right]$$

ZA DA JA ODREDIŠ P(x,T) ČE TI TREBA DA GO ZNAEŠ P(T(x)) NO TOA NE JA MENUVA KONSTATACIJATA ZA SUFICIENT STATISTIC.

$$P(y|x) = \frac{P(x,y)}{P(x)}$$

$y \backslash x$	1	2	$P(y)$
1	1/4	1/4	1/2
2	1/4	1/4	1/2
$P(x)$	1/2	1/2	

$y \backslash x$	1	2
1	1/2	1/2
2	1/2	1/2

$$\left.\begin{aligned} P(1,1) &= P(1)\cdot P(1|1) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} \\ P(2,1) &= P(2)\cdot P(1|2) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} \end{aligned}\right\}$$

TREBA DEFINI VO x DA GO OPIŠUVA 10 PEDICI ZAŠTO TOA E BITNO ZA SAME ZA

- SE NAVRAĆAM NA PRIMER (2)

$\bar{x}_n$ e VSUŠNOST EDNAKVO NA θ $(\bar{x}_n \approx \theta)$ ZA LOGIČNO MI IZGLEDA DEKA USLOVNATA RASPREDELBA NA $x_1, x_2, \ldots x_n$ AKO JI ZNAEŠ $\bar{x}_n$ ... DA NE ZAVISI OD "θ"

- THE MINIMAL SUFFICIENT STATISTIC IS A SUFFICIENT STATISTIC THAT IS FUNCTION OF ALL OTHER STATISTICS.

$$\theta \to T(x) \to U(x) \to x$$

- MINIMAL SUFFICIENT STATISTIC MAXIMALLY COMPRESS THE INFORM. FOR ABOUT θ IN THE SAMPLE. OTHER SUFFICIENT STATISTIC MAY CONTAIN ADDITIONAL IRRELEVANT INFORMATION.

• ~~EDITION~~ **EDITION 1 SOLUTION:** [FACENGT IKI 2002]

$$D(p\|q)=\sum_{i=1}^{n} p_i \,ld \frac{p_i}{q_i} = +\sum_{i=1}^{n} p_i\,ld\,p_i - \sum_{i=1}^{n} p_i\,ld\,q_i \geq 0$$

$$\sum p_i\,ld\,p_i \geq \sum p_i\,ld\,q_i \qquad -\sum_{i=1}^{n} p_i\,ld\,p_i \leq -\sum p_i\,ld\,q_i$$

• IF $\boxed{q_i=\frac{1}{n}}$ $\boxed{H(X)\leq ld\,n}$ - [070300003] MANE

$$q_i=\alpha(\beta)^i; \quad -\sum_{i=0}^{\infty} p_i\,ld\,p_i \leq -\sum_{i=0}^{\infty} p_i\,ld\,q_i = -\sum_{i=0}^{\infty} p_i\,ld(\alpha\beta^i)$$

$$= -\sum_{i=0}^{\infty} p_i\,ld(\alpha) - \underbrace{\sum_{i=0}^{\infty} p_i\cdot i}_{=A}\,ld\,\beta = -ld\,\alpha - A\cdot ld\,\beta$$

$$\boxed{-\sum_{i=0}^{\infty} p_i\,ld\,p_i \leq -ld\,\alpha - A\,ld\,\beta}$$

$$\sum_{i=0}^{\infty}\alpha\cdot\beta^i = 1 = \alpha\cdot\frac{1}{1-\beta}$$

• $$\sum_{i=0}^{\infty} i\cdot q_i = \sum_{i=0}^{\infty} i\cdot\alpha\cdot\beta^i = A = \alpha\cdot\frac{\beta}{(1-\beta)^2}$$

$$\alpha\cdot\frac{\beta}{(1-\beta)^2} = \frac{\alpha}{(1-\beta)}\frac{\beta}{(1-\beta)} = \frac{\beta}{1-\beta} = A$$

$$A=\frac{\beta}{1-\beta} \qquad A-A\cdot\beta=\beta \qquad \beta(A+1)=A \qquad \boxed{\beta=\frac{A}{1+A}}$$

$$\alpha\frac{\beta}{(1-\beta)^2} = \alpha\frac{\frac{A}{1+A}}{\left(1-\frac{A}{1+A}\right)^2} = \alpha\frac{\frac{A}{1+A}}{\left(\frac{1+A-A}{1+A}\right)^2} = \alpha\frac{\frac{A}{1+A}}{\left(\frac{1}{1+A}\right)^2} = A(1+A)$$

$$\alpha\cdot\cancel{A}(1+A)=\cancel{A} \qquad \boxed{\alpha=\frac{1}{1+A}}$$

• ZNAČI DISTRIBUCIJATA ŠTO ZA MAKSIMIZIRA ENTROPIJATA:

$$\boxed{p_i=\frac{1}{1+A}\left(\frac{A}{1+A}\right)^i}$$

• MAKSIMALNATA ENTROPIJA E:

$$H(p)\Big|_{max} = -ld\alpha - A\, ld\beta = -ld\frac{1}{1+A} - A\, ld\frac{A}{1+A} =$$

$$= -ld\frac{1}{1+A} - A\, ld\frac{1}{1+A} - A\, ld\, A = \underline{(1+A)\, ld(1+A) - A\, ld\, A}$$

• GENERALIZIRANA FORMA NA DISTRIBUCIJA:

$$p_i = \alpha\, \beta^i$$

IA SE PRETHODNO DOBIVA ILI SO KORISTENJE NA LAGRANGE MULTIPLIKATORI UATE:

$$F(p_i, \lambda_1, \lambda_2) = -\sum_{i=0}^{\infty} p_i\, ld(p_i) + \lambda_1 \underbrace{\left(\sum_{i=0}^{\infty} p_i - 1\right)}_{\text{CONSTRAINT (1)}} + \lambda_2 \underbrace{\left(\sum_{i=0}^{\infty} i\, p_i - A\right)}_{\text{CONSTRAINT (2)}}$$

WIKIPEDIA (MMV)

• LAGRANGE MULTIPLIERS PROVIDES STRATEGY FOR FINDING THE MAXIMA & MINIMA OF FUNCTION SUBJECT TO CONSTRAINTS.

MAXIMIZE $f(x,y)$

SUBJECT TO $g(x,y) = c$

- WE INTRODUCE NEW VARIABLE (λ) CALLED LAGRANGE MULTIPLIER AND STUDY THE LAGRANGE FUNCTION DEFINED

$$\Lambda(x,y,\lambda) = f(x,y) + \lambda(g(x,y) - c)$$

IF $f(x,y)$ IS A MAXIMUM FOR ORIGINAL CONSTRAINT PROBLEM, THEN THERE EXIST λ SUCH (x,y,λ) IS STATIONARY POINT FOR THE LAGRANGE FUNCTION (STATIONARY POINTS ARE THOSE POINTS WHERE THE PARTIAL DERIVATIVES OF Λ ARE ZERO)

- VO SLUČAJOT: $f(x,y) = H(p_i) = -\sum_{i=0}^{\infty} p_i\, ld(p_i)$

$$g_1(x,y) = \sum_{\forall x} p(x) = \sum_{i=1}^{\infty} p_i = 1 \qquad g_2(x,y) = \sum_{i=1}^{\infty} i\, p_i = A$$

$$\to F(p_i, \lambda_1, \lambda_2) = -\sum_{i=0}^{\infty} p_i\, ld(p_i) + \lambda_1\left(\sum_{i=0}^{\infty} p_i - 1\right) + \lambda_2\left(\sum_{i=0}^{\infty} i\, p_i - A\right)$$

$$\frac{dF}{dp_i} = 0 \Rightarrow \frac{-1}{\ln 2}\sum_{i=0}^{\infty}\left(\ln(p_i) + \cancel{p_i}\cdot\frac{1}{\cancel{p_i}}\right) + \lambda_1(\infty) + \lambda_2\left(\sum_{i=0}^{\infty} i\cdot 1 - A\right) =$$

?

$$\frac{dF}{d\lambda_1} = 0 \quad \sum_{i=0}^{\infty} p_i = 1 \qquad \frac{dF}{d\lambda_2} = 0 \Rightarrow \sum_{i=0}^{\infty} i\, p_i = A$$

$$F(p_i,\lambda_1,\lambda_2) = -\sum_{i=0}^{\infty} p_i \, ld\, p_i + \lambda_1\left(\sum_{i=0}^{\infty} p_i - 1\right) + \lambda_2\left(\sum_{i=0}^{\infty} i\, p_i - A\right)$$

$$= \sum_{i=0}^{\infty}\left(-p_i \, ld\, p_i + \lambda_1 p_i + \lambda_2 i\, p_i\right) - \lambda_1 - A\cdot\lambda_2$$

$$\frac{d\, F(p_i,\lambda_1,\lambda_2)}{d p_i} = \sum_{i=0}^{\infty}\left(- ld\, p_i - p_i \frac{1}{p_i \ln 2} + \lambda_1 + i\lambda_2\right) = 0$$

$$\sum_{i=0}^{\infty}\left(-ld\, p_i - \frac{\ln e}{\ln 2} + \lambda_1 + i\lambda_2\right) = \sum_{i=0}^{\infty} -\left(ld\, p_i + ld\, e\right) + \lambda_1 + i\lambda_2 = 0$$

$$\sum_{i=0}^{\infty}\left(- ld\left(e\cdot p_i\right) + \lambda_1 + i\lambda_2\right) = 0$$

$$\lambda_1 + i\lambda_2 = ld(e p_i) \qquad e p_i = 2^{\lambda_1 + i\lambda_2}$$

$$p_i = \frac{1}{e}\cdot 2^{\lambda_1}\cdot 2^{i\lambda_2} = \frac{2^{\lambda_1}}{e}\cdot\left(2^{\lambda_2}\right)^i = \alpha\cdot\beta^i$$

$$\alpha = \frac{2^{\lambda_1}}{e}, \quad \beta = 2^{\lambda_2}$$

~~PETAN~~ SPIPORSKY

070200516

PAPER REVISION | ELSEVIER REVISION

$$I(\gamma) = \left(\frac{\gamma}{\bar\gamma}\right)^{n+2k} e^{-\frac{(b+1)}{\bar\gamma}\gamma} = \Big/ \gamma = \frac{x^2}{d} \Big/ = \left(\frac{x^2}{d\bar\gamma}\right)^{n+2k} e^{-\frac{b+1}{\bar\gamma}\cdot\frac{x^2}{d}}$$

$$I(\gamma) = \frac{1}{(d\bar\gamma)^{n+2k}} \left(x^{2n+4k} e^{-\frac{b+1}{\bar\gamma}\frac{x^2}{d}} \right) A$$

$$\int \frac{A\, e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} x^{2n+4k} e^{-\frac{b+1}{\bar\gamma}\frac{x^2}{d} - \frac{x^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int x^{2n+4k} \cdot e^{-\left(\frac{b+1}{\bar\gamma d} + \frac{1}{2}\right)x^2} dx = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} x^{2n+4k} e^{-px^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} x^{2(n+2k)} e^{-px^2} dx = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} x^{2m} \cdot e^{-px^2} = \frac{(2m-1)!!}{2(2p)^m}\sqrt{\frac{\pi}{p}}$$

$$\boxed{m = n+2k}$$

$$= \frac{(2n+4k-1)!!}{2\sqrt{2}(2\gamma)^{n}\sqrt{\gamma}} = \frac{(2n+4k-1)!!}{2^{n+1}\sqrt{2}\cdot \gamma^{n+\frac{1}{2}}} = \frac{(2n+4k-1)!!}{2^{n+1}\sqrt{2}\left(\frac{2b+2+\bar{\gamma} d}{2\bar{\gamma} d}\right)}$$

$$= \frac{(2n+4k-1)!!}{2^{n+1}\sqrt{2}\,\frac{(2b+2+\bar{\gamma} d)^{n+\frac{1}{2}}}{2^{n+\frac{1}{2}}(\bar{\gamma} d)^{n+\frac{1}{2}}}} = \frac{(2n+4k-1)!!\,(\bar{\gamma} d)^{n+\frac{1}{2}}}{2\,(2b+2+\bar{\gamma} d)^{n+\frac{1}{2}}}$$

$$= \frac{(2n+4k-1)!!\,(\bar{\gamma} d)^{n+2k+\frac{1}{2}}}{2\,(2b+2+\bar{\gamma} d)^{n+2k+\frac{1}{2}}} =$$

$$(2m-1)!! = \frac{2^m}{\sqrt{\pi}}\cdot\Gamma\left(m+\frac{1}{2}\right) \qquad \underbrace{(2n+4k-1)!! = \frac{2^{n+2k}}{\sqrt{\pi}}\,\Gamma\left(n+2k+\tfrac{1}{2}\right)}$$

→ (8.339.2) GRADSHTEYN

$$= \frac{\frac{2^{n+2k}}{\sqrt{\pi}}\,\Gamma\left(n+2k+\frac{1}{2}\right)(\bar{\gamma} d)^{n+2k+\frac{1}{2}}}{2\,(2b+2+\bar{\gamma} d)^{n+2k+\frac{1}{2}}} =$$

$$= \frac{2^{n+2k-1}\,\Gamma\left(n+2k+\frac{1}{2}\right)\cdot(\bar{\gamma} d)^{n+2k+\frac{1}{2}}}{\sqrt{\pi}\,(2b+2+\bar{\gamma} d)^{n+2k+\frac{1}{2}}} \qquad \boxed{3.461.2}$$

SO OVA SE DOKAZUVA (12.3) VO MultilogMIMOb.m

$$P_{eas} = \frac{1}{2\cdot 2^{d}\, m!}\lim_{\gamma\to 0}\frac{d^{m} F_{\gamma}(\gamma)}{d\gamma^{m}}\cdot\prod_{i=1}^{m}(2\cdot i-1)$$

$$F(\gamma) = 1 + \frac{F'(0)}{1!}\cdot(\gamma-0) + \frac{F''(0)}{2!}\cdot(\gamma-0)^2$$

$$F(\gamma) = 1 + \frac{1}{\Gamma^2(m)}\sum_{k=0}^{m-1}\sum_{n=0}^{m-1} A\cdot\left(\frac{g}{g_m}\right)^{n+2k}\cdot e^{-\frac{(c+1)g}{g_m}}$$

$$\frac{d}{dg}\left[\left(\frac{g}{g_m}\right)^{n+2k}\cdot e^{-\frac{(c+1)g}{g_m}}\right] = \frac{(n+2k)}{g_m}\left(\frac{g}{g_m}\right)^{n+2k-1} e^{-\frac{(c+1)g}{g_m}} + \left(\frac{g}{g_m}\right)^{n+2k}\cdot(-1)\left(\frac{c+1}{g_m}\right)e^{-\frac{(c+1)g}{g_m}}$$

$$P_{eas} = \frac{1}{2 \cdot \bar{d}^n \cdot n!} \cdot \lim_{g \to 0} \left\{ \frac{d^n}{dg^n} \left[CDF(\gamma) \right] \right\} \cdot \prod_{i=1}^{n} (2i-1)$$

$$\prod_{i=1}^{n} (2i-1) = (2n-1)!! = \frac{2^n}{\sqrt{\pi}} \cdot \Gamma\left(n + \frac{1}{2}\right)$$

$$P_{eas} = \frac{1}{2 \cdot \bar{d}^n \cdot n!} \cdot \frac{2^n}{\sqrt{\pi}} \Gamma\left(n + \frac{1}{2}\right) \cdot \underbrace{\lim_{\gamma \to 0} \frac{d^n}{d\gamma^n} \left[F_P(\gamma) \right]}$$

$$\frac{d}{d\gamma}\left[F_P(\gamma)\right] = 0 + \frac{1}{\Gamma^2(n)} \sum_{k=0}^{n-1} \sum_{k=0}^{n-1} A \cdot \left[\frac{(2k+n)}{\gamma} - \frac{(c+1)}{\bar{\gamma}} \right] \left(\frac{\gamma}{\bar{\gamma}} \right)^{n+2k} \cdot e^{-\frac{(c+1)\gamma}{\bar{\gamma}}}$$

$$= \frac{1}{\Gamma^2(n)} \sum_{k=0}^{n-1} \sum_{k=0}^{n-1} A \left[\frac{(2k+n)}{\bar{\gamma}} - \frac{(c+1)\gamma}{\bar{\gamma}^2} \right] \left(\frac{\gamma}{\bar{\gamma}} \right)^{n+2k-1} \cdot e^{-\frac{(c+1)\gamma}{\bar{\gamma}}}$$

$$P_1 = +2\left(k + \frac{n}{2}\right)\bar{\gamma} - (c+1)\gamma$$

$$P_2 = 4\left(k + \frac{n}{2} - \frac{1}{2}\right)\left(k + \frac{n}{2}\right)\bar{\gamma}^2 - 4\left(k + \frac{n}{2}\right)(c+1)\bar{\gamma}\gamma + (c+1)^2\gamma^2$$

$$P_3 = 8\left(k + \frac{n}{2}\right)\left(k + \frac{n}{2} - \frac{1}{2}\right)\left(k + \frac{n}{2} - 1\right)\bar{\gamma}^3 - 12\left(k + \frac{n}{2}\right)\left(k + \frac{n}{2} - \frac{1}{2}\right)(c+1)\gamma\bar{\gamma}^2 + 6\left(k + \frac{n}{2}\right)(c+1)^2\gamma^2\bar{\gamma} - (c+1)^3\gamma^3$$

$$P_i = \sum_{i=0}^{N} \frac{\Gamma(N-i-2k-n)(-1)^{N-i}}{\Gamma(-2k-n)} \gamma^i \cdot \bar{\gamma}^{N-i} (c+1)^i$$

$$(n)_k = n \cdot (n+1) \cdot (n+2) \cdots (n+k-1) \qquad (n)_k = \frac{\Gamma(n+k)}{\Gamma(n)}$$

$$\prod_{i=1}^{N} \left(n - \frac{i-1}{2}\right) = \left(n - \frac{1}{2}\right) \cdot (n-1)$$

FOR $N = 3$

MMV

$$P_1 = (2k+n)\bar{\gamma} - (c+1)\gamma$$

$$P_2 = (2k+n-1)(2k+n)\bar{\gamma}^2 - 2(2k+n)(c+1)\bar{\gamma}\gamma + (c+1)^2\gamma^2$$

$$P_3 = (2k+n)(2k+n-1)(2k+n-2)\bar{\gamma}^3 - 3(2k+n)(2k+n-1)(c+1)\gamma\bar{\gamma}^2 + 3(2k+n)(c+1)^2\bar{\gamma}\gamma^2 - (c+1)^3\gamma^3$$

$$\boxed{\prod_{j=0}^{N-i-1}(2k+n-j) = \frac{\Gamma(N-i-2k-n)(-1)^{N-i}}{\Gamma(-2k-n)}}$$

19.65 \$

$$\prod_{i=1}^{N}(m-i) = \frac{(-1)^N\,\Gamma(N+1-m)}{\Gamma(1-m)} \quad ②$$

$$(m-1)(m-2)\cdots(m-N) = (-1)^N(1-m)(2-m)\cdots(N-m)$$

$$= (1-m)(1-m+1)(1-m+2)\cdots(1-m+N-1)\cdot(-1)^N =$$

$$= \underline{\underline{(1-m)_N}}\,(-1)^N = \circledast$$

$$\text{pochhammer}\,(n)_k = n\cdot(n+1)\cdots(n+k-1) = \underline{\underline{\frac{\Gamma(n+k)}{\Gamma(n)}}}$$

$$\circledast = \underline{\frac{\Gamma(1-m+N)}{\Gamma(1-m)}}\cdot(-1)^N$$

$N-1-m =$
$N-1+1-1-m =$
$N-2+1-m$

$$\prod_{i=0}^{N-1}(2k+n-i) = ? \qquad \prod_{i=0}^{N-1}(m-i) = m\cdot(m-1)\cdots(m-N+1)$$

$$= m\cdot(1-m)\cdots(N-1-m)\cdot(-1)^{N-1} = (-1)^{N-1}\, m\,(1-m)(1-m+1)\cdots(1-m+N\ldots$$

$$=(-1)^{N-1}\cdot m\,(1-m)_{N-1} = (-1)^{N-1}\frac{\Gamma(\cancel{1}-m+N-\cancel{1})}{\Gamma(1-m)}\cdot m = \frac{(-1)^{N-1}\cdot m\cdot\Gamma(N\ldots}{\Gamma(1-m)}$$

$$\Gamma(1-m) = \Gamma[-(m-1)] = (-1)^{m-2}\,1\cdot2\cdot3\cdots(m-2)$$

$$\Gamma(z-n) = \frac{(-1)^n\Gamma(z)}{(1-z)_n} \qquad (1-z)_n = \frac{(-1)^n\Gamma(z)}{\Gamma(z-n)}$$

$$(1-z)_n = (1-z)\cdot(1-z+1)\cdot(1-z+2)\cdots(1-z+n-1) =$$

$$=(-1)^n(z-1)(z-2)\cdot(z-3)(z-4)\cdots(z-n) = (-1)^n\prod_{i=1}^{n}(z-i)$$

$$\prod_{i=1}^{n}(z-i) = \frac{(1-z)_n}{(-1)^n} = \frac{\Gamma(1-z+n)}{(-1)^n\cdot\Gamma(1-z)}\cdot\frac{(-1)^n}{(-1)^n} = \frac{(-1)^n\Gamma(1-z+\ldots}{\Gamma(1-z)}$$

ISTO KAKO ①

$$(-z)_n = (-z)\cdot(-z+1)\cdot(-z+2)\cdots(-z+n-1) = (-1)^n\cdot(z)(z-1)(z-2)\cdot(z\ldots$$

$$= (-1)^n\prod_{i=0}^{n-1}(z-i);\quad \boxed{\prod_{i=0}^{n-1}(z-i) = \frac{(-z)_n}{(-1)^n} = \frac{\Gamma(n-z)\,(-1)^n}{\Gamma(-z)}}\ ③$$

$$\frac{d^N}{d\gamma^N}\left[\left(\frac{\gamma}{\bar\gamma}\right)^{n+2k} e^{-\frac{(c+1)\gamma}{\bar\gamma}}\right] = \frac{(-1)^N e^{-\frac{\gamma(c+1)}{\bar\gamma}}}{(\gamma\bar\gamma)^N} \sum_{i=0}^{N}\binom{N}{i}\left(\frac{\gamma}{\bar\gamma}\right)^{2k+n}\frac{\Gamma(N-i-2k-n)}{\Gamma(-2k-n)}\gamma^{i}\bar\gamma^{N-i}(c+1)^{i}$$

$$\frac{d^N}{d\gamma^N}\left[\left(\frac{\gamma}{\bar\gamma}\right)^{n+2k} e^{-\frac{(c+1)\gamma}{\bar\gamma}}\right] = (-1)^N e^{-\frac{\gamma(c+1)}{\bar\gamma}} \sum_{i=0}^{N}\binom{N}{i}\left(\frac{\gamma}{\bar\gamma}\right)^{2k+n}\frac{\Gamma(N-i-2k-n)}{\Gamma(-2k-n)}\gamma^{i-N}\bar\gamma^{-i}(c+1)^{i}$$

$$= (-1)^N e^{-\frac{\gamma(c+1)}{\bar\gamma}} \sum_{i=0}^{N}\binom{N}{i}\frac{\Gamma(N-i-2k-n)}{\Gamma(-2k-n)}\frac{\gamma^{i}}{\gamma^N\bar\gamma^{i}}(c+1)^{i}\left(\frac{\gamma}{\bar\gamma}\right)^{2k+n}$$

$$= (-1)^N e^{-\frac{\gamma(c+1)}{\bar\gamma}} \sum_{i=0}^{N}\binom{N}{i}\frac{\Gamma(N-i-2k-n)}{\Gamma(-2k-n)}\frac{\gamma^{i+2k+n}}{\gamma^N}\frac{(c+1)^{i}}{\bar\gamma^{i+2k+n}}$$

$$\boxed{dF_1 = \frac{n e^{\frac{-\gamma(c+1)}{\bar\gamma}}}{\gamma^n}\sum_{k=0}^{n-1}\sum_{n=0}^{n-1}\sum_{i=0}^{n}\frac{(-1)^{k+n}\Gamma(n-k)\Gamma(2k)\Gamma(n-i-2k-n)}{\Gamma(k+1)\Gamma(n+1)\Gamma(n-n)\Gamma(2k-n+n+1)\Gamma(i+1)\Gamma(n-i+1)\Gamma(-2k-n)}\cdot\frac{(c+1)^{n+i}c^{k}\cdot\gamma^{i+2k+n}}{\bar\gamma^{i+2k+n}}}$$

070275543 VLADO DIMCEV (CAMCE)

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)} \qquad \Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^n}(2n-1)!! = \frac{\sqrt{\pi}}{2^n}\cdot\frac{(2n)!}{2^n\cdot n!}$$

$$\Gamma\left(k+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,(2k)!}{2^{2k}\cdot k!} = \frac{\sqrt{\pi}\,\Gamma(2k+1)}{2^{2k}\,\Gamma(k+1)}$$

$$\sin(2\pi k + n\pi) = \sin((2k+n)\pi)$$

$$\lim_{\gamma\to 0}\frac{d^n F_1(\gamma)}{d\gamma^n} = \frac{1+c^n}{\bar\gamma^n} \qquad \prod_{i=1}^{n}(2i-1) = (2n-1)!!$$

$$\boxed{(2n-1)!! = \frac{2^n}{\sqrt{\pi}}\Gamma\left(n+\frac{1}{2}\right)}$$

$$P_{eas} = \frac{1}{2\,d^n\,n!} = \underbrace{\frac{1+c^n}{\bar\gamma^n}}_{\lim[F_1(\gamma)]^{(n)}}\cdot\frac{2^n}{\sqrt{\pi}}\Gamma\left(n+\frac{1}{2}\right) = \frac{1}{2\,n!}\frac{(c^n+1)\,2^n}{d^n\bar\gamma^n\sqrt{\pi}}\Gamma\left(n+\frac{1}{2}\right)$$

$$P_{eas} = \frac{(c^n+1)\, 2^{n-1}\, \Gamma(n+1/2)}{\gamma^n \bar{\gamma}^n \cdot \sqrt{\pi}\ \Gamma(n+1)}$$

$$P_{eas} = \frac{\Gamma(2n+1)\cdot (c^n+1)}{2^{n+1}\cdot \Gamma^2(n+1)\, d^n \cdot \bar{\gamma}^n}$$

$$\boxed{\frac{\Gamma(2n+1)}{2^{n+1}\Gamma^2(n+1)} = \frac{2^{n-1}\cdot\Gamma(n+\frac{1}{2})}{\Gamma(n+1)\sqrt{\pi}}}$$

$\Delta\Pi$

$\sqrt{\pi}$

$$\Gamma\left(k+\frac{1}{2}\right) = \frac{1}{2^n}\,\Gamma\left(\frac{1}{2}\right)\prod_{k=1}^{n}(1+k2-2) = \frac{1}{2^n}\Gamma\left(\frac{1}{2}\right)\prod_{k=1}^{n}(2k-1)$$

$$\boxed{\Gamma\left(k+\frac{1}{2}\right) = \frac{1}{2^k}\sqrt{\pi}\,(2k-1)!!}$$

$$1\cdot 3\cdot 5\cdot 7\cdot \cdots (2k-1) =$$

$$= 1\cdot(2\cdot 2-1)(2\cdot 3-1)(2\cdot 4-1)\cdots(2k-1) =$$

$$= 1\cdot 2^{k-1}\left(2-\frac{1}{2}\right)\left(3-\frac{1}{2}\right)\cdots\left(k-\frac{1}{2}\right)$$

$$\underbrace{\frac{3}{2}\quad\frac{5}{2}}_{15/4}$$

NEMA POTREBA OD KOMLICIRANJE!!! OVI DIREKTNO SO KAKO NA NZZ.

$\Delta *$

$$\prod_{j=1}^{n}(2j-1) = (2n-1)!! = \frac{(2n)!}{2^n\cdot n!}$$

$$\frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}}\cdot 2^n = (2n-1)!! = 1\cdot 3\cdot 5\cdot \cdots (2n-1)$$

$$\frac{\Gamma(n+1+n)}{\Gamma(n+1)} = (n+1)_n$$

$$\left.\begin{aligned}\Gamma(2n+1) &= 1\cdot 2\cdot \ldots (2n)\\ \Gamma(n+1) &= 1\cdot 2\cdot \ldots n\end{aligned}\right\}$$

$$\frac{\Gamma(2n+1)}{2^{n+1}\Gamma(n+1)\Gamma(n+1)} = \frac{(n+1)_n}{2^{n+1}\Gamma(n+1)}$$

$$\boxed{(2n-1)!! = \frac{(2n)!}{2^n\cdot n!}}$$

$*\Delta$ MMW FROM WOLFRAM DOUBLE FACTOR.

$$\frac{2^{n-1}\Gamma(n+\frac{1}{2})}{\Gamma(n+1)\sqrt{\pi}} = \frac{\cancel{2^{n-1}}\,\frac{1}{\cancel{2^n}}\cdot\cancel{\sqrt{\pi}}\,(2n-1)!!}{\Gamma(n+1)\cancel{\sqrt{\pi}}} = \frac{(2n-1)!!}{2\Gamma(n+1)} = \frac{(2n)!}{2^n n!\cdot 2\Gamma(n+1)}$$

$$= \frac{(2n)!}{2^{n+1}n!\cdot\Gamma(n+1)} = \frac{\Gamma(2n+1)}{2^{n+1}\,\Gamma^2(n+1)}$$

DOKAZANO!!! $\Delta\Pi$

$$\Gamma(2k-m+n+1) = \Gamma(2k+m-n+1) = \Gamma(2k+n+1-m) \Rightarrow$$

$$(2k+n)_{1-m} = \frac{\Gamma(2k+n+1-m)}{\Gamma(2k+n)}$$

$$\boxed{{}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}}$$

$$\boxed{{}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}}$$

. MAPLE DEFINITION OF GENERALIZED HYPERGEOM. FCN

$$pFq(n,d,z) = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p}(n_i)_k}{\prod_{j=1}^{q}(d_j)_k} \frac{z^k}{k!}$$

$$\boxed{{}_2F_1(n,d,z) = \sum_{k=0}^{\infty} \frac{(n_1)_k (n_2)_k}{(d)_k} \frac{z^k}{k!}}$$

$$\frac{\Gamma(m-2k-n-i)}{\Gamma(m-2k-n)} = (m-2k-n)_{-i}$$

$$(n)_{-k} = n\cdot(n-1)(n-2)\cdots(n-k+1) = (-1)^{k-1} n(1-n)(2-n)\cdots(k-1-n)$$

$$= (-1)^k(-n)[(-n)+1][(-n)+2]\cdots[(-n)+k-1] =$$

$$= (-1)^k \cdot (-n)_k$$

OVA NE E DOBRO !!!
VIDI GO IZLAZOT NA K112.69

$$F(\alpha,\beta;\gamma;1) = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}$$

$$\text{pochammer}(4,-3) = 4\cdot(4-1)(4-2)(4-3) = 4\cdot3\cdot2\cdot1 = 24$$

• (From Mupad) IF $\boxed{n<0}$

$$\boxed{\text{pochhammer}(x,n) = \frac{1}{\text{pochhammer}(x+n,-n)}} \quad \boxed{\text{MMV}}$$

$$\frac{(m+1-i)_i}{(m-2k-n-i)_i} = \frac{(m-i+1)_i}{(m-i-2k-n)_i} = \frac{(\overbrace{m-i-2k-n}^{x}+\overbrace{2k+n}^{b})_i}{(\underbrace{m-i-2k-n}_{x})_i}$$

$$= \frac{(x+b)_i}{(x)_i} = \frac{(x+b)\cdot(x+b+1)\cdots(x+b+i-1)}{x(x+1)(x+2)\dots\dots(x+i-1)} =$$

$$= \frac{(x)_b\cdot(x+b)_i}{(x)_b\cdot(x)_i} = \frac{x(x+1)\cdots(x+b-1)\cdot(x+b)(x+b+1)\cdots(x+b+i-1)}{} = \frac{(x)_{b+i}}{(x)_b\,(x)_i}$$

20500999

$$\frac{(m+1-i)_i}{(m-2k-n-i)_i} = \frac{(m-i-2k-n)_{2k+n+i}}{(m-i-2k-n)_{2k+n}\cdot(m-i-2k-n)_i}$$

PORCELANOSA, FON, FRANCUSKI MARKET,

$$\frac{(m+1-i)_i}{(m-2k-n-i)_i} = \left|\begin{matrix} m=3 \\ i=1..3\end{matrix}\right| = \frac{(3+1-i)_{\overset{=1}{i}}}{(3-i-b)_i} = \frac{(3+1-1)_1}{(2-b)} = \frac{(3)_1}{2-b}$$

$\boxed{i=2}$ $$\frac{(3+1-2)_2}{(3-2-b)_2} = \frac{(2)_2}{(3-l)_2} = \frac{2\cdot 3}{(3-l)\,(4-l)}$$

$\boxed{i=3}$ $$\frac{(3+1-3)_3}{(3-3-b)_3} = \frac{(1)_3}{(-b)_3} = \frac{1\cdot 2\cdot 3}{(-b)(-b+1)(-b+2)}.$$

$(-m)_i$ $\quad i=1 \quad (-m)_1 = -3$ $\qquad \boxed{i=2}$ $(-m)_2 = (-3)\cdot(-3+1) = (-1)^i\cdot 3\cdot 2$

$\boxed{i=3}$ $(-3)_3 = (-3)\cdot(-3+1)(-3+2) = (-3)(-2)(-1) = (-1)^3\cdot 3\cdot 2\cdot 1$

$$\boxed{(m+1-i)_i = \frac{(-m)_i}{(-1)^i}} \quad \boxed{\text{MMV}} \qquad (\underbrace{m-2k-n-1}_{"m"}+1-i)_i = \frac{(-m+2k+n+1)_i}{(-1)^i}$$

$$\frac{\cancel{\Gamma(m-2k-n)}}{\cancel{\Gamma(-2k-n)}} = (m)_{(-2k-n)} = \frac{1}{(m+n)_{-n}} = \frac{1}{(m-2k-n)_{2k+n}}$$

$$= \frac{1}{\frac{\Gamma(m-2k-n+2k+n)}{\Gamma(m-2k-n)}} = \frac{\Gamma(m-2k-n)}{\Gamma(m)}$$

$n<0$?

$$\frac{\Gamma(m-2k-n)}{\Gamma(-2k-n)} = (-2k-n)_m$$

$$(-n)_m = (-n)(-n+1)(-n+2)\cdots(-n+m-1) =$$

$$= (-1)^m (n)\cdot(n-1)\cdot(n-2)\cdots(n-m+1)\,\frac{(n-m)\cdot(n-m-1)\cdots 1}{(n-m)(n-m-1)\cdots 1}$$

$\Gamma(n+1)$, $\Gamma(n-m+1)$

$$\boxed{(-n)_m = \frac{(-1)^m\Gamma(n+1)}{\Gamma(n-m+1)}}$$

MMV

MNOGO JAKO IZVEDOVANJE !!!

$$\frac{\Gamma(2k)}{\Gamma(k+1)} = \frac{\cancel{1\cdot 2\cdot 3\cdots k}(k+1)\cdots(2k-1)}{\cancel{1\cdot 2\cdot 3\cdots k}} = (k+1)_{k-1}$$

$$(m+1-i)_i = \frac{(-m)_i}{(-1)^i} = \frac{\Gamma(m+1)}{\Gamma(m+1-i)}$$

$$\frac{\Gamma(m-k)}{k\,\Gamma(k)} = \frac{(-k)_m}{k} = \frac{(-1)^m\cdot\Gamma(k+1)}{k\;\Gamma(k-m+1)} = \frac{(-1)^m\Gamma(k)}{\Gamma(k-m+1)}$$

$$\frac{\Gamma(2k+n+1)}{\Gamma(n+1)} = (n+1)_{2k} \qquad \Gamma(2k) = 1\cdot 2\cdots(2k-1)$$

$$\boxed{\Gamma(2k) = \frac{\Gamma(2k+1)}{2k}} \qquad \frac{\Gamma(m-i-2k-n)}{\Gamma(-2k-n)} = (-2k-n)_{(m-i)} =$$

$$= \frac{(-1)^{m-i}\Gamma(2k+n+1)}{\Gamma(2k+n+1-m+i)}$$

70

$$\frac{\Gamma(2k+n+1-m)}{\Gamma(2k+n+1)} = (2k+n+1)_{(-m)} = \frac{1}{(2k+n+1-m)_m}$$

$$= \frac{1}{\frac{\Gamma(2k+n+1-m+m)}{\Gamma(2k+n+1-m)}} = \frac{\Gamma(2k+n+1-m)}{\Gamma(2k+n+1)}$$

$$\frac{\Gamma(2k+n+1-m)}{\Gamma(2k)} = (2k)_{n+1-m}$$

$$\frac{(2k)_{n+1}}{(2k)_{n+1-m}} = \frac{\cancel{(2k)}\cdot\cancel{(2k+1)}\cdots(2k+n)}{\cancel{(2k)}\cdot\cancel{(2k+1)}\cdots(2k+n-m)}$$

$$= (2k+n-m+1)(2k+n-m+2)\cdots(2k+n) =$$

$$= (2k+n-m+1)_m = (2k+n-m+1)\cdots\underbrace{(2k+n-m+1+m-1)}_{2k+n}$$

$$(2k+n-m+1)_m = \frac{\Gamma(2k+n+1)}{\Gamma(2k+n-m+1)}$$

$$\frac{(2k+n-m+1,\, m)}{(2k,\, n+1-m+i)} = \frac{(2k+n+1-m,\, m)}{(2k,\, n+i+1-m)} = \frac{\sqrt{(2k+n+1-m)}\,(2k+n+1-m+1)(2k+n+1-m+\ldots}{(2k)(2k+1)(2k+2)\cdots}$$

$$\frac{\Gamma(n-k)}{\Gamma(k+1)} = \frac{\Gamma(n-k)}{k\,\Gamma(k)} =$$

$$\Gamma(n-k) = \frac{1\cdot 2\cdot 3\cdots(n-k-1)\;(n-k)(n-k+1)\cdots(n-k+k)}{(n-k)_k}$$

$$\Gamma(n-k) = \frac{\Gamma(n)}{(n-k)_k} \qquad (n-k)_k = \frac{\Gamma(n)}{\Gamma(n-k)}$$

$$\frac{(2k+n-m+1, m)}{(2k, n+1-m+i)} = \frac{(2k+n-m+1)(2k+n-m+2)\cdots(2k+n)}{(2k)\cdot(2k+1)(2k+2)\cdots(2k+n+i-m)}$$

$$= \frac{\Gamma(2k+n-m)}{}$$

$$\boxed{(a-m)_m = ?} \qquad (a-m)_m = \frac{\Gamma(a)}{\Gamma(a-m)}$$

$$(2k+n+1-m, m) = \frac{\Gamma(2k+n+1)}{\Gamma(2k+n+1-m)}$$

$$(m-k)_k = (m-k)(m-k+1)\cdots(m-\cancel{k}+k-1)$$
$$= (m-k)(m-k+1)\cdots(m-1)$$

$\boxed{k=0}$ $\quad (m-k)_k = 1$

$\boxed{k=1}$ $\quad (m-k)_k = (m-1)_1 = (m-1)\cdot(m-1+\cancel{1-1})$

$\boxed{k=2}$ $\quad (m-2)_2 = (m-2)\cdot(m-2+2-1) = (m-2)(m-1)$

$\boxed{k=3}$ $\quad (m-3)_3 = (m-3)\cdot(m-3+1)(m-3+2) =$
$$= (m-3)(m-2)(m-1)$$

$$(m-k)_k = \frac{\Gamma(m)}{\Gamma(m-k)} = \left|\begin{matrix} m=4 \\ k=2 \end{matrix}\right| = \frac{\cancel{1\cdot 2}\cdot 3\cdot 4}{\cancel{1\cdot 2}}$$

$$(2k, n+1+i-m) = \frac{\Gamma(2k+n+1+i-m)}{\Gamma(2k)}$$

No. $$\frac{\Gamma(2k)}{\Gamma(k+1)} = \frac{(2k-1)!}{k!} = \frac{\cancel{1\cdot 2\cdot 3\cdots k}(k+1)\cdots(2k-1)}{\cancel{k!}}$$

$$= (k+1)_{k-1}$$

$$\boxed{(-n)_m = \frac{(-1)^m\,\Gamma(n+1)}{\Gamma(n-m+1)}}$$

$$(-1)^m(-n)_m = \frac{\Gamma(n+1)}{\Gamma(n-m+1)} = \frac{\Gamma(n+1)}{\Gamma(n+1-m)}$$

$$(-1)^n\,\frac{\Gamma(n+1-m)}{\Gamma(n+1)} = \frac{1}{(-n)_m}$$

$$\neq 1$$

$$\frac{(-1)^{k+n+m-i}}{\text{pochammer}(2k,\ n+1-m+i)} = \frac{(-1)^{-2m}}{(-1)^{k+n+m-i}(2k)_{n+1-m+i}\cdot(-1)^{-2m}} =$$

$$= \frac{1}{(-1)^{k+n-m-i}(2k)_{n+1-m+i}}\cdot\frac{(-1)^{2i}}{(-1)^{2i}} = \frac{1}{(-1)^{k+n-m+i}(2k)_{n+1-m+i}}$$

$$= \frac{1}{(-1)^{k-1}(-1)^{n+1-m+i}(2k)_{n+1-m+i}} = \frac{1}{(-1)^{k-1}\,\Gamma}$$

$$(m+1-i)_i = \frac{(-m)_i}{(-1)^i}$$

$$(2k+n+1-m,\ m) = \frac{(-2k-n)_m}{(-1)^m} = (-1)^m(-2k-n)_m$$

$$\boxed{\frac{\Gamma(2k+n+1)}{\Gamma(2k+n-m+1)} = (-1)^m\,(-2k-n)_m}$$

$$\boxed{\frac{\Gamma(2k)}{\Gamma(2k+n-m+1)}} = \frac{1}{\frac{\Gamma(2k+n-m+1)}{\Gamma(2k)}} = \frac{1}{(2k)_{n-m+1}} =$$

$$\boxed{f(m,n) = \begin{cases} (2k)_{n-m+1} & m \leq n+1 \\ \dfrac{1}{(2k+n-m+1)_{m-n-1}} & m > n+1 \end{cases}}$$

$$m-1-n+1=0$$

$$= (2k+n-m+1)_{m-}$$

MMV

OVA E SUŠTINSKATA ZAMENA KOJA GI OTSTRANI SINGULA-
RITETITE VO ZBIRITE ZA Pea

$$\frac{\Gamma(2k+n+1-m+i)}{\Gamma(2k+n+1)} = (2k+n+1)_{i-m} = \frac{1}{(2k+n+1+i-m)_{m-i}}$$

$$i \le m$$

$$= \frac{1}{(2k+n+i-m+1)_{m-i}} = \frac{\Gamma(2k+n+i-m+1)}{\Gamma(2k+n+1)}$$

MMV

$$\frac{\Gamma(m-n-2k)}{\Gamma(-n-2k)} = (-n-2k)_m \qquad (z)_n = \frac{\Gamma(z+n)}{z}$$

ff 70

$$(-n)_m = \frac{(-1)^n \cdot \Gamma(n+1)}{\Gamma(n-m+1)}$$

$$(-n-2k)_m = \frac{(-1)^n \cdot \Gamma(n+2k+1)}{\Gamma(n+2k-m+1)} = \frac{(-1)^n}{\text{pochammer}(n+2k+1,-m)}$$

$$= (-1)^n \cdot \text{pochammer}(n+2k-m+1,m)$$

ff.69

$$P(\alpha \le x) = \int_{-\infty}^{x} p(x)\,dx \qquad M(-s) = \int_{-\infty}^{\infty} p(x) e^{-xs}\,dx$$

$$p(x) = \frac{dP(x)}{dx} \Big/ \mathcal{L} \qquad \mathcal{L}[p(x)] = M(-s) \quad \hat{p}(x) = M(-s)$$

$$\mathcal{L}[p(x)] = \mathcal{L}\left[\frac{dP(x)}{dx}\right] = s \cdot P(s) \Rightarrow M(-s) = s\,P(s)$$

$$\frac{M(-s)}{s} = \hat{P}(x) \qquad \boxed{P(x) = \mathcal{L}^{-1}\left[\frac{M(-s)}{s}\right]}$$

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^n}(2n-1)!! = \frac{\sqrt{\pi}}{2^n} \cdot \frac{(2n)!}{2^n \cdot n!} = \frac{\sqrt{\pi}\,\Gamma(2n+1)}{2^{2n} \cdot \Gamma(n+1)}$$

$$\Gamma\left(n+2k+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{n+2k}} \frac{(2n+4k)!}{2^{n+2k} \cdot (n+2k)!} = \frac{\sqrt{\pi}\;\Gamma(2n+4k+1)}{2^{2n+4k}\;\Gamma(n+2k+1)}$$

$$\frac{\Gamma\left(\frac{m}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{m}{2}+1\right)} = \frac{\sqrt{\pi}\,\Gamma(m+1)}{\Gamma\left(\frac{m}{2}+1\right) 2^{[illegible]} \cdot \Gamma\left(\frac{m}{2}+1\right)} = \frac{\sqrt{\pi}\;\Gamma(m+1)}{\Gamma^2\left(\frac{m}{2}+1\right) 2^m}$$

$$\frac{(-z)_n}{(-1)^n} = \frac{\Gamma(n-z)}{\Gamma(-z)\,(-1)^n}$$

$$(1-n-2k,\, m-1) = \frac{\Gamma(m-1+1-n-2k)}{\Gamma(1-n-2k)}$$

$$\Rightarrow (1-n-2k,\, m-1) = \frac{\Gamma(m-n-2k)}{\Gamma(1-n-2k)} =$$

$$(1-n-2k,\, m-1) = \Big(-\underbrace{(2k+n-1)}_{z}\Big)_{m-1} \qquad \boxed{z \triangleq 2k+n-1}$$

$$(-z)_{m-1} = (-z)\cdot(-z+1)\cdot(-z+2)\cdots(-z+m-2) =$$

$$= (-1)^{m-1} z(z-1)(z-2)\cdots(z-m+2) =$$

$$= [-(n+2k-1)]\,[-(n+2k-1)+1]\,[-(n+2k-1)+2]\cdots[-(n+2k-1)+m-2]$$

$$= [-(n+2k)+1]\,[-(n+2k)+2]\,[-(n+2k)+3]\cdots[-(n+2k)+m-1]$$

$$= \frac{[-(n+2k)]\cdot[-(n+2k)+1]\cdots[-(n+2k)+m-1]}{-(n+2k)} = \frac{[-(n+2k)]_m}{-(n+2k)}$$

$$\text{pochhammer}(2k+n-m+1,\, m) = \frac{\Gamma(2k+n+1)}{\Gamma(2k+n-m+1)}$$

$$\boxed{(-n-2k)_m = \frac{(-1)^m\,\Gamma(n+2k+1)}{\Gamma(n+2k-m+1)}} \rightarrow \text{VIDI N.70}$$

$$\boxed{(-z)_{m-1} = \frac{(-z-1)_m}{-z-1}} \rightarrow \text{NOV IZRAZ ZA UJOJKEDA}$$

$$MGF = E[e^{sx}]$$

$$\lim_{s\to\infty} \frac{s^n\left(m^2(\bar{\gamma}s+b+1)^{n+2k+1} + c\bar{\gamma}\right)}{(\bar{\gamma}s+b+1)^{n+2k+1}} =$$

$$= \lim \frac{m^2\left(\bar{\gamma}s\cdot s^{\frac{n}{n+2k+1}} + b+1\right)^{n+2k+1} + c\bar{\gamma}\cdot s^n}{(\bar{\gamma}s+b+1)^{n+2k+1}}, \qquad n=0,\ldots,m-1,\quad k=0,\ldots,m-1$$

$$S^{n}\left[n^{-2}\Gamma(m)\,\Gamma(k+1)\,\Gamma(n+1)\,\Gamma(m-n)\left(\bar{\delta}1+b+1\right)^{n+2k+1}+\right.$$

$$\left.+(-1)^{k+m+n}\,\Gamma(m-k)\,\Gamma(n+2k+1)\,(2k+n-m+1)_{(m-n-1)}\,(b+1)^{n}\,b^{k}\,\bar{\delta}\right]$$

$$(2k+n+1-m)_{m-n-1}=\begin{cases}(-1)^{m-n-1}\dfrac{\Gamma(m-n-2k-\cancel{1}+\cancel{1})}{\Gamma(\cancel{n}-\cancel{n}-2k-\cancel{1}-\cancel{m}+\cancel{n}+1+\cancel{1})} & \text{IF } 2k+n+1<m\\[2ex] \dfrac{\Gamma(2k+\cancel{n}+\cancel{1}-\cancel{n}+\cancel{m}-\cancel{n}-\cancel{1})}{\Gamma(2k+n+1-m)} & \text{IF } 2k+n+1>m\end{cases}$$

$$\boxed{(-n)_m=\frac{(-1)^{n}\,\Gamma(n+1)}{\Gamma(n-m+1)}}$$

$$(2k+n+1-m)_{m-n-1}=\begin{cases}(-1)^{m-n-1}\dfrac{\Gamma(m-n-2k)}{\Gamma(-2k+1)}=(-1)^{m-n-1}\cdot 1 & \text{IF } m>2k+1\\[2ex] (*)\ \dfrac{\Gamma(2k)}{\Gamma(2k+n+1-m)} & \text{IF } \underline{2k+n+1>m}\end{cases}$$

$$\frac{\Gamma(m-n-2k)}{\Gamma(-2k+1)}=\frac{\Gamma(m-n-1-2k+1)}{\Gamma(-2k+1)}=\frac{\Gamma(m-n-2k)}{(1-2k)\,\Gamma(-2k)}=$$

$$=\frac{1}{1-2k}\,(-2k)_{m-n}=\frac{(-1)}{2k-1}\,(-2k)_{m-n}=-\frac{(-2k)_{m-n}}{2k-1}$$

ZA $k=0$ $\quad 1-2k>0$ NO TOGAS $(0)_{m-n}=0$

$$\Gamma(n+1)=(n+1)\cdot\Gamma(n)$$

$W(0,2,3)$

$$(*)=\frac{\Gamma(2k)}{\Gamma(2k+n+1-m)}=\frac{1}{\text{pochammer}(2k,\underbrace{n+1-m}_{\leq 0})}=$$

$n\leq m-1$ $\quad\boxed{m\geq n+1}$ $\quad n+1-m\leq 0$

$$=\left|\ \text{pochhammer}(x,n)=\frac{1}{\text{pochhamer}(x+n,-n)}\right|=\text{pochhamer}(2k+n+1-m,\ m-n-1)$$

$$=\frac{\Gamma(2k)}{\Gamma(2k+n+1-m)}$$

$$\Gamma(n+1)=n\cdot\Gamma(n)=n\cdot(n-1)\cdot(n-2)\cdots 1=\Gamma(n+1)$$

$$\frac{\Gamma(m-n-2k)}{\Gamma(-2k+1)}=\frac{\Gamma(m-n-2k)}{(-2k)\cdot\Gamma(-2k)}$$

$m > 2k+n+1$ $\quad \boxed{m-n-2k>1}$ KOMETI GO OVA!!

$$(1) = \frac{\Gamma(m-n-2k)}{\Gamma(-2k+1)} = \frac{\Gamma(-2k+1+m-n-1)}{\Gamma(-2k+1)} = \boxed{(-2k+1)_{m-n-1}}$$

$$\boxed{m-n-1 > 2k}$$

$$n_{max} = m-1$$

$$2k < m - n - 1$$

$$2k < \cancel{m} - \cancel{m} + \cancel{1} - \cancel{1}$$

$n=0$, $\quad m-1 > 2k$

$n=1$, $\quad m-2 > 2k$

$n=2$ $\quad m-3 > 2k$

$n=m-1$ $\quad 0 > 2k \Rightarrow \boxed{k=0}$

$$(1)_{m-n-1} = \frac{\Gamma(m-n-1+1)}{\Gamma(1)} = \Gamma(m-n)$$

$$\left(-(2k-1)\right)_{m-n-1} = \frac{(-1)^{m-n-1}\,\Gamma(2k-\cancel{1}+\cancel{1})}{\Gamma(2k-\cancel{1}-m+n+\cancel{1}+1)} =$$

$$= \frac{(-1)^{m-n-1}\,\Gamma(2k)}{\Gamma(2k-m+n+1)} \quad //$$

$$(-2k+1)_{m-n-1}$$

$\boxed{k=0}$ $\quad (1) = (1)_{m-n-1} = \Gamma(m-n)$

$k>0$ $\quad (1) = \dfrac{(-1)^{m-n-1}\,\Gamma(2k)}{\Gamma(2k+n+1-m)}$

$m > 2k+n+1$ $\quad \boxed{n < m-2k-1}$

$$(-2k+1)_{\cancel{m}-\cancel{m}+2k+\cancel{1}-\cancel{1}} = (-2k+1)_{2k}$$

$(-1)^{m-n-1}\,\Gamma(m-n)$ $\quad w\underset{k,\,n,\,m}{(1,0,4)}$ $\quad \Gamma(m-1) = \Gamma(4) = 3! = 6$

ЗАТРАИ ОГРАНИЧУВАЊЕТО $m > 2k+n+1$ ОВОЈ ПОЧЕТОК

ПОШТО САМО ЗА $\boxed{k=0}$

$$\prod_{i=1}^{m}(2i-1) = (2m-1)!! = \frac{(2m)!}{2^m m!} = \frac{\Gamma(2m+1)}{2^m\cdot\Gamma(m+1)}$$

$$P_{eas} = \frac{1}{2d^m\,\Gamma(m+1)}\cdot\frac{b^m+1}{\delta^m}\cdot\frac{\Gamma(2m+1)}{2^m\,\Gamma(m+1)} = \frac{\Gamma(2m+1)\cdot(b^m+1)}{2^{m+1}\,\Gamma^2(m+1)\,d^m\delta^m}$$

$$\frac{\bar{\gamma}^4 s^4 (1+\sigma^3)}{(\bar{\gamma}s+\sigma+1)^7}$$

$$\frac{\bar{\gamma}^{n+2} s^{n+2}(1+\sigma^n)}{(\bar{\gamma}s+\sigma+1)^{2n+2}} \sim \frac{\bar{\gamma}^{n+2} s^{n+2}(1+\sigma^n)}{\bar{\gamma}^{2n+2} s^{2n+2}}$$

$$s^n \cdot \frac{\bar{\gamma}^{n+2} s^{n+2}(1+\sigma^n)}{\bar{\gamma}^{2n+2} s^{2n+2}} = \frac{\cancel{\bar{\gamma}^{n+2}} \cancel{s^{2n+2}}(1+\sigma^n)}{\bar{\gamma}^{\cancel{2n+2}} \cancel{s^{2n+2}}} = \frac{1+\sigma^n}{\bar{\gamma}^n}$$

$$A = \frac{1}{\sqrt{\sigma}\,\|H\|_F^2} \qquad \gamma = \bar{\gamma}\frac{\frac{1}{\sigma}\|G\|^2}{\frac{1}{\sigma}\frac{\|G\|^2}{\|H\|^2} + 1}$$

$$\gamma = \bar{\gamma}\frac{\frac{1}{\sigma}\|G\|^2\|H\|^2}{\frac{1}{\sigma}\|G\|^2 + \|H\|^2} = \bar{\gamma}\frac{\|G\|^2\|H\|^2}{\|G\|^2 + \sigma\|H\|^2}$$

$$\bar{\gamma} = \frac{1}{\frac{1}{\|H\|^2} + \frac{\sigma}{\|G\|^2}} \qquad \frac{1}{\gamma} = \frac{1}{\|H\|^2} + \frac{\sigma}{\|G\|^2}$$

$\boxed{\text{RETURN TO T.M. COVER BOOK}}$

$\boxed{\text{PROBLEM 2.31}}$ CONDITIONAL ENTROPY. UNDER WHAT CONDITIONS $H(X|g(Y)) = H(X|Y)$?

- SUFFICIENT STATISTIC

$$X \to T(X) \to Y$$

$$P(Y|X, T(X)) = P(Y|T(X))$$

$$X \to Y \to Z$$

MARKOVITY

$$\cancel{P(X|Y,Z) = \frac{P(X|Y)\cdot P(X|Z)}{}}$$

$$\cancel{P(X|Y,Z) = \frac{P(X,Y,Z)}{P(Y,Z)}}$$

$$P(X|Y) = P(Z|Y)$$

$$P(X,Z|Y) = \frac{P(X,Y,Z)}{P(Y)} = \frac{P(X,Y)\cdot P(Z|X,Y)}{P(Y)}$$

$$P(X,Z|Y) = P(X|Y)\cdot P(Z|Y)$$

MARKOVITY FOR GIVEN "Y" Z IS INDEPENDENT OF X.

FROM PROBLEM 2.4

$$H(X, g(X)) = H(X) + H(g(X)|X) = H(g(X)) + H(X|g(X))$$

$$\geq H(g(X)) \qquad X = f[g(X)] \Rightarrow H(X|g(X)) = 0$$

$H(X|g(Y)) = H(X|Y)$ IF $g(Y)$ IS SUFFICIENT STATISTIC OF "Y"

$$X \to g(Y) \to Y \qquad \underline{P(X|Y, g(Y)) = P(X|g(Y))}$$

SUFICIENT STATISTIC:

$$I(\theta; X) = I(\theta; T(X)) \qquad \theta \to T(X) \to X$$

$$I(\theta; X) = \underline{H(\theta) - H(\theta|X)} = H(X) \cdot H(X|\theta)$$

$$I(\theta; T(X)) = H(T(X)) - H(T(X)|\theta) = \underline{H(\theta) - H(\theta|T(X))}$$

$*=**$ $\quad \cancel{H(\theta)} + H(\theta|X) = \cancel{H(\theta)} - H(\theta|T(X))$

$$\underbrace{H(\theta|X) = H(\theta|T(X))}$$

OVA E DOKAZ DEKA AKO IMAME MARKOV CHAIN

$$X \to g(Y) \to Y$$

T.E. AKO $g(Y)$ E SUFFICIEN STATISTIC FOR X

(Y IS INDIPENDENT OF X FOR GIVEN $g(Y)$)

$$I(X; Y) = I(X; g(Y)) \quad \} \text{ VALID FOR SUFFICIENT STATISTIC}$$

$$\underline{\cancel{H(X)} - H(X|Y) = H(X) - H(X|g(Y))} \Rightarrow$$

$$\boxed{H(X|Y) = H(X|g(Y))}$$

2.32 Fano We are given following joint distribution (X, Y):

$X \backslash Y$	a	b	c	$P(X)$
1	1/6	1/12	1/12	1/3
2	1/12	1/6	1/12	1/3
3	1/12	1/12	1/6	1/3
$P(Y)$	1/3	1/3	1/3	

LET $\hat{X}(Y)$ BE AN ESTIMATOR OF X (based on Y) AND LET $P_e = Pr(\hat{X}(Y) \neq X)$

(a) FIND MINIMUM PROBABILITY OF ERROR ESTIMATOR $\hat{X}(Y)$ AND THE ASSOCIATED P_e.

(b) EVALUATE FANO'S INEQUALITY FOR THIS PROBLEM AND COMPARE.

$X \to Y \to \hat{X}$ $\quad P_e = Pr\{\hat{X} \neq X\}$

FANO INEQUALITY: $\boxed{H(P_e) + P_e \, ld|\mathcal{X}| \geq H(X|\hat{X}) \geq H(X|Y)}$

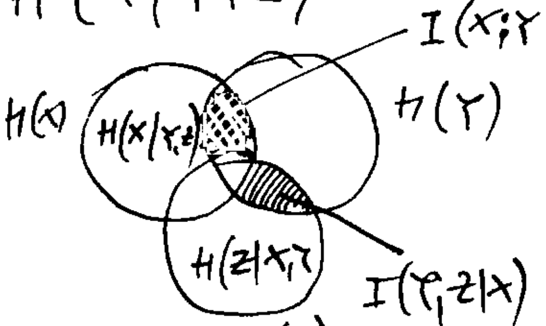
$1 + P_e \, ld|\mathcal{X}| \geq H(X|Y)$ $\quad P_e \geq \frac{H(X|Y)-1}{ld|\mathcal{X}|}$

Proof: $H(P_e) + P_e \, ld|\mathcal{X}| \geq H(X|\hat{X})$

$$\epsilon = \begin{cases} 1 & \text{if } \hat{X} \neq X \\ 0 & \text{if } \hat{X} = X \end{cases}$$

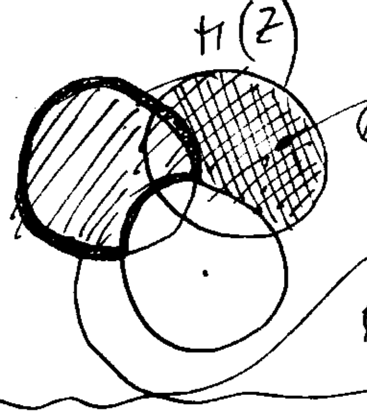
$$H(\epsilon, X|\hat{X}) = H(\epsilon|\hat{X}) + H(X|\hat{X}, \epsilon)$$

$$H(X,Y|Z) = H(X|Z) + H(Y|Z,X)$$



$\underline{I(X;Y|Z)} = H(X|Z) - H(X|Z,Y)$

$I(X,Y;Z) = I(X;Z) + I(Y;Z|X)$



$H(X,Y|Z)$

$- H(X|Z)$

$H(Y|X,Z)$

$$H(X,\epsilon|\hat{X}) = H(X|\hat{X}) + \underbrace{H(\epsilon|X,\hat{X})}$$

$\epsilon = f(X,\hat{X}) \Rightarrow H(\epsilon|X,\hat{X}) = 0$

$$= \underbrace{H(\epsilon|\hat{X})}_{\leq H(P_e)} + \underbrace{H(X|\hat{X},\epsilon)}_{\leq P_e \, ld|\mathcal{X}|} \leq H(P_e) + P_e \, ld|\mathcal{X}|$$

CONDITIONING REDUCES ENTROPY: $H(\epsilon|\hat{X}) \leq H(\epsilon) = H(P_e)$

$P_e = Pr(\hat{X} \neq X) = Pr(\epsilon = 1)$

$$H(X|\hat{X},\epsilon) = \underbrace{P_r(\epsilon=0)}_{=(1-P_e)} \cdot \underbrace{H(X|\hat{X},\epsilon=0)}_{=0} + \underbrace{P_r(\epsilon=1)}_{P_e} \underbrace{H(X|\hat{X},\epsilon=1)}_{\leq H(X) \leq ld|\mathcal{X}|}$$

$H(X,\epsilon|\hat{X}) \leq H(P_e) + P_e \, ld|\mathcal{X}|$ $\quad H(X|\hat{X}) \leq H(P_e) + P_e \, ld|\mathcal{X}|$

$$D(p\|q) = \sum_{x\in X} p(x)\,\mathrm{ld}\frac{p(x)}{q(x)} \qquad q(x) = \frac{1}{|X|} \quad x\in\{x_1, x_2 \dots x_N\}, \; N=|X|$$

$$D(p\|q) \geq 0 \qquad \sum_{x\in X} p(x)\,\mathrm{ld}\, p(x) - \sum_{x\in X} p(x)\,\mathrm{ld}\, q(x) \geq 0$$

$$-\sum_{x\in X} p(x)\,\mathrm{ld}\, p(x) \leq -\sum_{x\in X} p(x)\,\mathrm{ld}\frac{1}{|X|} = -\mathrm{ld}\frac{1}{|X|}\sum_{x\in X} p(x)$$

$$\boxed{H(X) \leq \mathrm{ld}|X|} \qquad H(X,Y) = \sum_{x\in X}\sum_{y\in Y} p(x,y)\,\mathrm{ld}\frac{1}{p(x,y)}$$

$$H(X|Y) = \sum_{y\in Y} p(y)\cdot H(X|Y=y)$$

DATA PROCESSING INEQUALITY: $I(X;\hat{X}) \leq I(X;Y)$

$$H(X) - H(X|\hat{X}) \leq H(X) - H(X|Y) \Rightarrow \underline{H(X|\hat{X}) \geq H(X|Y)}$$

$$H(P_e) + P_e\,\mathrm{ld}|X| \geq H(X|\hat{X}) \geq H(X|Y)$$

- COROLLARY: $X, Y \quad p = \Pr(X \neq Y)$

$$\boxed{H(p) + p\,\mathrm{ld}|X| \geq H(X|Y)}$$

- FOR ANY TWO X & Y if $g(Y)$ TAKES VALUES IN X
- COROLLARY $\quad P_e = \Pr(X \neq \hat{X}) \quad \hat{X}: Y \to X$

$$H(P_e) + P_e\,\mathrm{ld}(|X|-1) \geq H(X|Y)$$

$$H(X,\epsilon|\hat{X}) = H(X|\hat{X}) = \underbrace{H(\epsilon|\hat{X})}_{\leq H(P_e)} + \underbrace{H(X|\hat{X},\epsilon)}_{\leq P_e\,\mathrm{ld}(|X|-1)}$$

$$H(X|\hat{X},\epsilon) = \Pr(\epsilon=0)\underbrace{H(X|\hat{X},\epsilon=0)}_{0} + \underbrace{\Pr(\epsilon=1)H(X|\hat{X},\epsilon=1)}_{\leq P_e\,\mathrm{ld}(|X|-1)}$$

ZATOA ŠTO $\hat{X}$ ZEMA VREDNOSTI OD „X" VNUTRO MOŽNOSTI SE $|X|-1$, A NE $|X|$ ZOŠTO TOGAŠ NEMA DA IMA $\epsilon = 1$

$$H(X|\hat{X}) \leq H(P_e) + P_e\,\mathrm{ld}(|X|-1)$$

$$\boxed{H(P_e) + P_e\,\mathrm{ld}(|X|-1) \geq H(X|Y)}$$

Remark: $x \in \{1, 2, \dots m\}$ $\quad p_1 \geq p_2 \geq \dots \geq p_m$

$\hat{x} = 1 \Rightarrow$ BEST GUESS OF x

PROBABILITY OF ERROR: $P_e = 1 - p_1$

$$H(P_e) + P_e \,\mathrm{ld}(m-1) \geq H(X)$$

$$(p_1, p_2, \dots p_m)^* = \left(1-P_e, \frac{P_e}{m-1}, \dots \frac{P_e}{m-1}\right)$$

$$H(X) = -\sum_{x \in X} p(x) \,\mathrm{ld}\, p(x) = -(1-P_e)\,\mathrm{ld}(1-P_e) -$$

$$-\sum \frac{P_e}{m-1} \,\mathrm{ld} \frac{P_e}{m-1} = (P_e - 1)\,\mathrm{ld}(1-P_e) - \cancel{(m-1)} \frac{P_e}{\cancel{m-1}} \,\mathrm{ld} \frac{P_e}{m-1}$$

$$= (P_e - 1)\,\mathrm{ld}(1-P_e) - P_e \,\mathrm{ld} P_e + P_e \,\mathrm{ld}(m-1) =$$

$$\cancel{= P_e \,\mathrm{ld}(1-P_e) - \mathrm{ld}(1-P_e) - P_e \,\mathrm{ld} P_e + P_e \,\mathrm{ld}(m-1)}$$

$$= \underbrace{-(1-P_e)\,\mathrm{ld}(1-P_e) - P_e \,\mathrm{ld} P_e}_{H(P_e)} + P_e \,\mathrm{ld}(m-1)$$

$$\boxed{H(X) = H(P_e) + P_e \,\mathrm{ld}(m-1)}$$

- LET X & X' ARE IID

$$\boxed{\Pr(X = X') = \sum_x p^2(x)}$$

LEMMA 2.10.1: $\Pr(X = X') \geq 2^{-H(X)}$

$X \sim p(x)$ $\quad$ JENSEN'S INEQUALITY: $E[f(X)] \geq f[E(X)]$

$$E[2^{\mathrm{ld}\, p(X)}] = \underbrace{E[p(X)]}_{\sum_x p^2(x)} \geq 2^{E[\mathrm{ld}\, p(X)]} = 2^{-H(X)}$$

$$\Rightarrow \sum_x p^2(x) \geq 2^{-H(X)} \qquad \boxed{P(X = X') \geq 2^{-H(X)}} \text{ DOCAZANO}$$

$$2^{-H(X)} = 2^{+\sum p \,\mathrm{ld}\, p} \leq \sum p(x)\, 2^{+\mathrm{ld}\, p(x)} = \sum p(x) \cdot p(x) = \sum_x p^2(x)$$

Corollary: $x \sim q(x)$ $\quad x' \sim r(x)$ $\quad x, x' \in \mathcal{X}$

$$\Pr(x = x') \geq 2^{-H(q) - D(q\|r)} \quad ; \quad \Pr(x = x') \geq 2^{-H(r) - D(r\|q)}$$

$$2^{-H(q) - D(q\|r)} = 2^{+\sum q \, ld\, q - \sum q \, ld \frac{q}{r}} = 2^{\sum q(x)\, ld\, r(x)}$$

$$\leq \sum_x q(x)\, 2^{ld\, r(x)} = \sum q(x) \cdot r(x) = P(x = x')$$

$$P(x = x') \geq 2^{-H(q) - D(q\|r)} \qquad \text{PROOFED!!!}$$

$p(x,y)$

X\Y	a	b	c	P(X)
1	1/6	1/12	1/12	1/3
2	1/12	1/6	1/12	1/3
3	1/12	1/12	1/6	1/3
P(Y)	1/3	1/3	1/3	
	P(a)	P(b)	P(c)	

$$H(P_e) + P_e\, ld|X| \geq H(X|Y)$$

$$\boxed{X \to Y \to \hat{X}}$$

$$P_e = \Pr(\hat{X}(Y) \neq X)$$

$$E = \begin{cases} 1 & \hat{X} \neq X \\ 0 & \hat{X} = X \end{cases}$$

$P(1,a) = 1/6 \quad P(1,b) = 1/12 \quad P(1,c) = 1/12$

$$P_e(1) = \underbrace{P(b)}_{1/3} \cdot \underbrace{P(1|b)}_{\frac{P(1,b)}{P(b)}} + P(c) \cdot P(1|c) = P(1,b) + P(1,c) = 1/12 + 1/12 = 1/6$$

(*1)

$$1 - P_e = P(a)\,P(1|a) + P(b)\,P(2|b) + P(c)\,P(3|c) = \underbrace{P(1,a) + P(2,b) + P(3,c)}_{\text{VEROJATNOST NA TOČNA ESTIMACIJA}} = 3 \cdot \frac{1}{6} = \frac{1}{2}$$

!!!

$$P_e = 1 - \frac{1}{2} = \frac{1}{2}$$

$$H\left(\frac{1}{2}\right) + \frac{1}{2}\, ld\, 3 \geq H(X|Y)$$

$$H(X|Y) = \underbrace{P(Y=a)}_{1/3} \cdot H(X|Y=a) + P(Y=b) \cdot H(X|Y=b) + P(Y=c)\, H(X|Y=c)$$

$$= \frac{1}{3} H(X|Y=a) + \frac{1}{3} H(X|Y=b) + \frac{1}{3} H(X|Y=c)$$

$$H(X,Y) = -\sum_x \sum_y p(x,y)\, ld\, p(x,y)$$

$$H(X|Y) = \sum_y p(y) \cdot H(X|Y=y)$$

$$H(X,Y) = -\sum_{x\in X}\sum_{y\in Y} p(x,y)\, ld\, p(x,y)$$

$$H(X|Y) = -\sum_{x\in X}\sum_{y\in Y} p(x,y)\, ld\, p(x|y)$$

$$H(X|Y=y) = -\sum_{x} p(x|y)\, ld\, p(x|y)$$

$$H(X|Y) = \overline{H(X|Y=y)} = -\sum_{y\in Y} p(y) \sum_{x} p(x|y)\, ld (x|y)$$

$$= -\sum_{y}\sum_{x} p(x,y)\, ld\, p(x|y)$$

$p(x|y)$

X \ Y	a	b	c
1	1/2	1/4	1/4
2	1/4	1/2	1/4
3	1/4	1/4	1/2

$$-H(X|Y) = P(1,a)\, ld\, P(1|a) + P(1,b)\, ld P(1|b) + P(1,c)\, ld P(1|c) + P(2,a)\, ld P(2|a) + P(2,b)\, ld P(2|b) + P(2,c)\, ld\, P(2|c) + P(3,a)\, ld P(3|a) + P(3,b)\, ld P(3|b) + P(3,c)\, ld P(3|c) =$$

$$H(X|Y) = \left(\frac{1}{6}\cdot \cancel{ld 2}^{1} + \frac{1}{12}\cancel{ld 4}^{2} + \frac{1}{12}\cancel{ld 4}^{2}\right)\cdot 3 = \left(\frac{1}{6}+\frac{1}{6}+\frac{1}{6}\right)\cdot 3$$

$$= \frac{3}{6}\cdot 3 = \frac{3}{2}$$

$$1 + \frac{1}{2} ld 3 \geq \frac{3}{2} = 1.5$$

$$1 + 0.5\cdot 1.6 = 1 + 0.8 = 1.8 > 1.5$$

1 (ZA $Pe = \frac{1}{2}$)

$$\cancel{H(Pe)} + Pe\, ld|X| \geq H(X|Y) \qquad Pe \geq \frac{H(X|Y)-1}{ld|X|}$$

$$Pe \geq \frac{1.5-1}{ld(3)} = \frac{0.5}{ld 3} = \underline{\underline{0.315}} \qquad \underline{0.5 > 0.315}$$

• OD REŠENIETO VO HW2s2011 KORISTO E:

$$\hat{X}(Y) = \begin{cases} 1 & y=a \\ 2 & y=b \\ 3 & y=c \end{cases} \qquad Pe = Pr\{\hat{X}(Y) \neq X\} = 6\cdot\frac{1}{12} = \frac{1}{2}$$

OVA E ALTERNATIVEN NAČIN DA JA NAJDEŠ VEROJATNOSTA NA GREŠKA. LOGIČNO !!!

 OVA E VSUŠNOST ANALOGNO NA: N12.(x1) SAMO ŠTO OVDE DIREKTNO PRESMETUVAŠ Pe, A NE 1-Pe.

$\boxed{2.33}$ FANO'S INEQUALITY; $P_r(X=i)=p_i \quad i=1,2,\dots m$

$p_1 > p_2 > p_3 > \dots p_m \qquad \left.\hat{X}=1\right\}$ MINIMAL PROBABILITY PREDICTOR.

$P_e = 1-p_1$. MAXIMIZE $H(p)$ SUBJECT TO CONSTRAINT: $1-p_1=P_e$ TO FIND A BOUND OF P_e IN TERMS OF H. THIS IS FANO INEQUALITY IN ~~THE~~ ABSENCE OF CONDITIONING.

$$(p_1, p_2, \dots p_m) = \left(1-P_e, \frac{P_e}{m-1}, \frac{P_e}{m-1}, \dots \frac{P_e}{m-1}\right)$$

VEROJATNOSTA NA GREŠKA E: $(m-1)\cdot\frac{P_e}{m-1}=P_e$

$$H(X) = -\sum_{i=1}^{m} p_i \, ld\, p_i = -(1-p_e)\, ld\,(1-p_e) - \sum_{i=1}^{} \frac{P_e}{m-1}\, ld \frac{P_e}{m-1}$$

$$= -(1-p_e)\, ld(1-p_e) - \cancel{(m-1)}\frac{p_e}{\cancel{m-1}}\, ld\frac{P_e}{m-1} =$$

$$= \underbrace{-(1-p_e)\, ld(1-p_e) - p_e\, ld\, p_e}_{H(p_e)} + P_e\, ld(m-1)$$

$$H(X) = H(P_e) + P_e\, ld(m-1)$$

$$\boxed{P_e = \frac{H(X)-H(P_e)}{ld(m-1)}}$$

$$\boxed{P_{e\,max} = \frac{H(X)-1}{ld(m-1)}}$$

$H(P_e) + P_e\, ld(|X|-1) \geq H(X|Y)$

X & Y STATISTICALLY INDEPENDENT.

$1 + P_e\, ld((|X|-1) \geq H(X)$

$$P_e \geq \frac{H(X)-1}{ld(|X|-1)}$$

$$\frac{d P_e}{d H} = \frac{d}{dH}\left(\frac{H-1}{ld(m-1)}\right) = 0 \qquad \frac{1}{ld(m-1)} = 0$$

$$\frac{dH(x)}{dp_e} = \frac{1}{ln 2}\frac{d}{dp_e}\cdot\left[p_e\, ln(m-1) - p_e\, ln(p_e) - (1-p_e)\, ln(1-p_e)\right]$$

$$= \frac{1}{ln 2}\left[ln(m-1) - ln(p_e) - \cancel{p_e}\frac{1}{\cancel{p_e}} + ln(1-p_e) + \cancel{(1-p_e)}\frac{1}{\cancel{(1-p_e)}}\right]$$

$$= \frac{1}{ln 2}\left[ln(m-1) - ln(p_e) + ln(1-p_e)\right] = 0$$

$$ln\frac{1-p_e}{p_e} = -ln(m-1) \qquad ln\frac{p_e}{1-p_e} = ln(m-1)$$

$$\frac{p_e}{1-p_e} = m-1 \qquad p_e = (1-p_e)m - (1-p_e) = m - mP_e - 1 + \cancel{p_e}$$

$mP_e = m-1$ $\qquad \boxed{p_e = \frac{m-1}{m}}$

МАКСИМАЛНА ЕНТРОПИЈА СЕ ДОБИВА ЗА:

$$\boxed{p_e = \frac{n-1}{n}} \qquad H(X) = H(p_e) + p_e \, ld(n-1)$$

$$H(X) = H\left(\frac{n-1}{n}\right) + \frac{n-1}{n} \, ld \frac{n-1}{n} =$$

$$= -\left(1-\frac{n-1}{n}\right) ld\left(1-\frac{n-1}{n}\right) - \cancel{\frac{n-1}{n} \, ld \frac{n-1}{n}} + \cancel{\frac{n-1}{n} \, ld \frac{n-1}{n}}$$

$$= -\frac{1}{n} \, ld\left(\frac{n-n+1}{n}\right) = -\frac{1}{n} \, ld\frac{1}{n} = \frac{1}{n} \, ld\, n$$

$$p(x) = (p_1, p_2, \dots, p_n)$$

$$H(X) = \sum_{i=1}^{n} p_i \, ld\, p_i = -\underbrace{p_1 \, ld\, p_1}_{p_e ld p_e} - \sum_{i=2}^{n} p_i \, ld\, p_i =$$

$$= -p_1 \, ld\, p_1 - \underbrace{p_e \sum_{i=2}^{n} \frac{p_i}{p_e} \, ld \frac{p_i}{p_e}}_{=S} \qquad S = \sum_{i=2}^{n} \left(p_i \, ld\, p_i - p_i \, ld\, p_e\right)$$

$$S = \sum_{i=2}^{n} p_i \, ld\, p_i - ld\, p_e \underbrace{\sum_{i=2}^{n} p_i}_{=p_e} = \sum_{i=2}^{n} p_i \, ld\, p_i - p_e \, ld\, p_e =$$

$$= -p_1 \, ld\, p_1 - \underbrace{p_e \sum_{i=2}^{n} \frac{p_i}{p_e} \, ld \frac{p_i}{p_e} - p_e \, ld\, p_e}_{= \sum_{i=2}^{n} p_i \, ld\, p_i} =$$

$$p_1 = 1 - p_e$$

$$= \underbrace{-p_1 \, ld\, p_1 - p_e \, ld\, p_e}_{H(p_e)} - p_e \underbrace{\sum_{i=2}^{n} \frac{p_i}{p_e} \, ld \frac{p_i}{p_e}}_{H\left(\frac{p_1}{p_e}, \dots, \frac{p_n}{p_e}\right)}$$

$$= H(p_e) + p_e \underbrace{H\left(\frac{p_1}{p_e}, \frac{p_2}{p_e}, \dots, \frac{p_n}{p_e}\right)}_{\le \, ld(n-1)} \le H(p_e) + p_e \, ld(n-1)$$

$$\boxed{H(X) \le H(p_e) + p_e \, ld(n-1)}$$

→ Секоја случајна променлива чија вредност може да се претпостави со веројатност на грешка p_e мора да го исполнува ова неравенство.

· Долната граница на p_e е:

$$\boxed{p_e \ge \frac{H(X) - 1}{ld(n-1)}}$$

$\boxed{\text{PROBLEM 2.34}}$ ENTROPY OF INITIAL CONDITIONS. PROVE THAT $H(X_0|X_n)$ IS NONDECREASING WITH n FOR ANY MARKOV CHAIN.

$$X_0 \to X_1 \to X_2 \qquad P_R(X_2|X_1,X_0) = P_R(X_2|X_1)$$

$$P(X|Y) \qquad = P(Z|Y)$$

$$X \to Y \to Z$$

$$P(X,Z|Y) = \frac{P(X,Y,Z)}{P(Y)} = \frac{P(X,Y)\,P(Z|X,Y)}{P(Y)}$$

$$\boxed{P(X,Z|Y) = P(X|Y)\cdot P(Z|Y)}$$

$$\boxed{X \to Y \to Z}$$

MINATO SEGAŠNOST IDNINA

VIDI WIKIPEDIA ARTICLE

$$H(Y|X) = -\sum_{x\in\mathcal{X}}\sum_{y\in Y} p(y|x)\,\mathrm{ld}\,p(x,y)$$

$$H(Y|X) = \sum_{x} p(x)\sum_{y} p(Y|X=x)\,\mathrm{ld}\,p(X=x,Y)$$

$$H(Z|X) = \sum_{x}\sum_{z} p(z|x)\,\mathrm{ld}\,p(x,y)$$

$$H(X|Z) = \sum_{x}\sum_{z} p(\underline{x|z})\,\mathrm{ld}\,p(x,y)$$

$$\boxed{P(X|Y,Z) = P(X|Y)}$$

$$I(X;Y) \geq I(X;Z) \qquad \cancel{H(X)} - H(X|Y) \geq \cancel{H(X)} - H(X|Z)$$

$$H(X|Y) \leq H(X|Z) \qquad H(X|Z) \geq H(X|Y) = \frac{H(X,Z|Y)}{P(\cancel{Z}|Y)}$$

$$X_0 \to X_1 \to X_2 \qquad H(X_0|X_2) \geq H(X_0|X_1)$$

$$X_0 \to X_1 \to X_2 \to X_3 \qquad H(X_1|X_3) \geq H(X_1|X_2)$$

$$I(X;Y) \geq I(X;Z) \qquad H(\cancel{Y}) - H(Y|X) \geq H(Z) - H(Z|X)$$

$$H(X_1,X_2|X_3) = H(X_1|X_3) + H(X_2|X_3,X_1)$$

$$H(X,Y|Z) = H(X|Z) + H(Y|X,Z)$$

$$H(X,Z|Y) = H(X|Y) + H(Z|X,Y) = H(X|Y) + H(Z|Y)$$

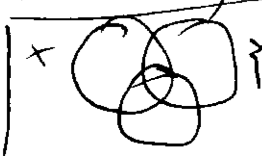
$H(P_e) + P_e \log(|X|) \geq P(X|Y) \geq P(X|Z)$ $\boxed{X \to Y \to Z}$

$X_0 \to X_1 \to X_2 \to X_3$ $\qquad H(X_1|X_3) \geq H(X_1|X_2)$

$X_0 \to X_1 \to X_2$ $\qquad \underline{H(X_0|X_2) \geq H(X_0|X_1)}$

$X_0 \to X_1 \to X_2 \to X_3 \to X_4 \to$

$H(X_2|X_4) \geq H(X_2|X_3)$



$I(X;Y|Z) = H(X|Z) - H(X|Z,Y)$

$I(X_0;X_3) \leq I(X_0;X_2) \leq I(X_0,X_1)$

$\cancel{H(X_0)} - H(X_0|X_3) \leq \cancel{H(X_0)} - H(X_0|X_2)$

$H(\cancel{X_0}) - H(X_0|X_2) \leq \cancel{H(X_0)} - H(X_0|X_1)$

$H(X_0|X_3) \geq \underline{H(X_0|X_2)}$ $\qquad \underline{H(X_0|X_2)} \geq H(X_0|X_1)$

$\boxed{H(X_0|X_3) \geq H(X_0|X_2) \geq H(X_0|X_1)}$

$I(X_0;X_3|X_1X_2) = H(X_0|X_1X_2) - H(X_0|X_1X_2X_3) = H(X_0|X_1) - H(X_0|X_1X_3) = I(X_0;X_3|X_1)$

• $X_0 \to X_1 \to X_2 \to X_3$

$\boxed{I(X_0;X_1,X_2,X_3)} = I(X_0;X_1) + \underbrace{I(X_0;X_2|X_1)}_{=\emptyset} + I(X_0;X_3|X_1,X_2)$

$= I(X_0;X_1) + I(X_0;X_3|X_1)$
$= I(X_0;X_1,X_3)$

X_0 & X_2 STATIST. NEZAVISNI GIVEN X_1 = $\boxed{\text{MMV}}$

(*)

$= I(X_0;X_2) + \underset{(1)}{I}(X_0;X_1|X_2) + I(X_0;X_3|X_1,X_2)$

$= I(X_0;X_3) + I(X_0;X_2|X_3) + I(X_0;X_1|X_2,X_3)$

$I(X_0;X_1) + \cancel{I_0(X_0;X_3|X_1,X_2)} = I(X_0;X_2) + I(X_0;X_1|X_2) + \cancel{I(X_0;X_3|X_1,X_2)}$

$I(X_0;X_1) = I(X_0;X_2) + I(X_0;X_1|X_2)$ $\qquad \underline{I(X_0;X_1) \geq I(X_0;X_2)}$

$I(X_0;X_1|X_2) = H(X_0|X_2) - H(X_0|X_1,X_2) = H(X_0|X_2) - H(X_0|X_1)$

(*) $= I(X_0;X_2) + H(X_0|X_2) - \cancel{H(X_0|X_1)} + \underbrace{\cancel{H(X_0|X_1,X_2)}}_{=H(X_0|X_1)} - H(X_0|\cancel{X_1},X_2,X_3)$

(1) $= I(X_0;X_3) + H(X_0|X_3) - \cancel{H(X_0|X_2,X_3)} + \cancel{H(X_0|X_2,X_3)} - H(X_0|\cancel{X_1,X_2,X_3})$

$\boxed{I(X_0;X_2) + H(X_0|X_2) = I(X_0;X_3) + H(X_0|X_3)}$

$$H(x_0|x_2) - H(x_0|x_3)$$

$$I(x_0;x_2) = H(x_0) - H(x_0|x_2)$$

$$I(x_0;x_2) - I(x_0;x_3) = \underline{\underline{H(x_0|x_3)}} - H(x_0|x_2)$$

$$I(x_0;x_3) = H(x_0) - H(x_0|x_3)$$

$$H(x_0|x_3) \leq H(x_0)$$

$$I(x_0;x_2) - I(x_0;x_3) \leq \underbrace{H(x_0) - H(x_0|x_2)}_{I(x_0;x_2)} \qquad I(x_0;x_3) \geq 0$$

$$I(x_0;x_2) \geq \emptyset$$

$$I(x_0;x_2) - I(x_0;x_3) \leq \underline{I(x_0;x_2)}$$

$$I(x_0,x_1;x_2,x_3) = H(x_0,x_1) - H(x_0,x_1|x_2,x_3) =$$

$$= H(x_0) + H(x_1|x_0) - H(x_1|x_2,x_3) - H(x_0|x_1,x_2,x_3)$$

$$H(X,Y|Z) = H(X|Z) + H(Y|X,Z)$$

$$H(X,Y,Z) = H(Z) + H(X,Y|Z)$$

$$I(x_0,x_1;x_2,x_3) = H(x_0) + H(x_1|x_0) - H(x_1|x_2) - H(x_0|x_1,x_3)$$

$$= I(x_0;x_1,x_3) -$$

$$I(x_1;x_0x_2) = \cancel{I(x_1;x_0)} = I(x_1;x_0) + I(x_1;x_2|x_0)$$

$$I(x_1;x_0|x_2) = H(x_1|x_2) - H(x_1|x_0,x_2)$$

$$I(x_1;x_2|x_0) = H(x_1|x_0) - H(x_1|x_0,x_2)$$

$$I(x_0,x_1,x_2;x_3) = I(x_0;x_3) + I(x_1;x_3|x_0) + I(x_2;x_3|x_0,x_1) =$$

$$I(X,Y;Z) = I(X;Z) + I(Y;Z|X)$$

$$= I(x_1;x_3) + I(x_2;x_3|x_1) + I(x_0;x_3|x_1,x_2) =$$

$$= I(x_2;x_3) + I(x_0;x_3|x_2) + I(x_1;x_3|x_0,x_2) \qquad = I(x_1;x_0x_2)$$

$$I(x_1;x_3|x_0,x_2) = \underbrace{H(x_1|x_0,x_2)}_{H} - H(x_1|x_0,x_2,x_3) \qquad \to \emptyset$$

$$H(x_1|x_0,x_3)$$

$$I(x_1;x_0,x_2,x_3) = I(x_1;x_0) + I(x_1;x_2|x_0) + I(x_1;x_3|x_0,x_2)$$

$$I(x_1;x_3|x_0,x_2) = H(x_1|x_0,x_2) - H(x_1|x_0x_2x_3) = H(x_1|x_0,x_2) - H(x_1|x_0x_2) = \emptyset$$

$$I(x_1;x_2x_0x_3) = I(x_1;x_2) + \underbrace{I(x_1;x_3|x_2)}_{0} + I(x_1;x_0|x_2x_3)$$

$$I(x_1;x_2x_0x_3) = I(x_1;x_2) + I(x_1;x_0|x_2x_3)$$

$$I(x_1;x_0|x_2x_3) = \underbrace{H(x_1|x_2x_3)}_{H(x_1|x_2)} - \underbrace{H(x_1|x_0x_2x_3)}_{H(x_1|x_0x_2)} = I(x_1;x_0|x_2)$$

$$I(x_1;x_2x_0x_3) = I(x_1;x_2) + I(x_1;x_0|x_2) = I(x_1;x_0x_2)$$

$$\boxed{I(x_0;x_1x_2x_3) = I(x_0;x_1x_3)} \qquad \boxed{I(x_1;x_2x_0x_3) = I(x_1;x_0x_2)}$$

P88

$$\boxed{I(x_0;x_1) \geq I(x_0;x_2)} \qquad 0 \leq I(x_0;x_1,x_3) = I(x_0;x_1) + I(x_0;x_3|x_1)$$

$$I(x_0;x_3|x_1) = H(x_0|x_1) - H(x_0|x_1,x_3) \overset{?}{\leq} H(x_0) - H(x_0|x_3) = I(x_0;x_3)$$

$$I(x_0;x_1x_3) = I(x_0;x_1) + I(x_0;x_3|x_1) \leq I(x_0;x_2) + I(x_0;x_3|x_1)$$

$$I(x_0;x_1,x_2,x_3,x_4) = I(x_0;x_1) + \underbrace{I(x_0;x_2|x_1)}_{0} + \underbrace{I(x_0;x_3|x_1,x_2)}_{=I(x_0;x_3|x_1)} + I(x_0;x_4|x_1,x_2,x_3)$$

$$I(x_0;x_4|x_1,x_2,x_3) = \underbrace{H(x_0|x_1,x_2x_3)}_{H(x_0|x_1,x_3)} - \underbrace{H(x_0|x_1,x_2,x_3x_4)}_{H(x_0|x_1,x_3,x_4)} = I(x_0;x_4|x_1x_3)$$

$$I(x_0;x_1,x_2,x_3,x_4) = I(x_0;x_1) + I(x_0;x_3|x_1) + I(x_0;x_4|x_1x_3)$$
$$= I(x_0;x_1x_3x_4)$$

• FORMAL DEFINITION OF MARKOV CHAIN: A MARKOV CHAIN IS A SEQUENCE OF RANDOM VARIABLES $X_1, X_2, X_3 \dots$ WITH MARKOV PROPERTY, NAMELY THAT, GIVEN THE PRESENT STATE, THE FUTURE AND PAST STATES ARE INDEPENDENT: (#)

$$\boxed{Pr(X_{n+1}=x \mid X_1=x_1, X_2=x_2, \dots X_n=x_n) = Pr(X_{n+1}=x \mid X_n=x_n)}$$

$$\boxed{I(x_0;x_1,x_2,x_3) = I(x_0;x_1,x_3) = I(x_0;x_1) + I(x_0;x_3|x_1)}$$

$$I(x_0;x_3|x_1) = H(x_0|x_1) - H(x_0|x_1,x_3) = H(x_3|x_1) - H(x_3|x_1x_0)$$

$$I(x_0;x_1x_2) = I(x_0;x_1) + I(x_0;x_2|x_1) = I(x_0;x_1)$$

$$I(x_0;x_2|x_1) = H(x_0|x_1) - H(x_0|x_1,x_2) = H(x_0|x_1) - H(x_0|x_1) = 0$$

$$I(x_0;x_1x_2) = I(x_0;x_2) + I(x_0;x_1|x_2) = I(x_0;x_1) \qquad \boxed{I(x_0;x_1) \geq I(x_0;x_2)}$$

$$I(x_0; x_1, x_3) = I(x_0; x_3) + I(x_0; x_1 | x_3) =$$
$$= I(x_0; x_1) + I(x_0; x_3 | x_1)$$

$$I(x_0, x_1; x_2) = I(x_0; x_2) + I(x_1; x_2 | x_0)$$
$$I(x_1; x_2 | x_0) = H(x_1 | x_0) - H(x_1 | x_0 x_2) = H(x_1|x_0) - H(x_1|x_0 x_2)$$

$$I(x_0; x_1 | x_3) = H(x_0 | x_3) - H(x_0 | x_1 x_3)$$
$$I(x_0; x_3 | x_1) = H(x_0 | x_1) - H(x_0 | x_1 x_3)$$

$$I(x_0, x_1, x_2; x_3) = I(x_2; x_3) + \underbrace{I(x_1; x_3 | x_2)} + I(x_0; x_3 | x_1 x_2)$$
$$I(x_0, x_1, x_2; x_3) = I(x_2; x_3) + I(x_0; x_3 | x_1, x_2)$$
$$I(x_0; x_3 | x_1 x_2) = H(x_0 | x_1 x_2) - H(x_0 | x_1, x_2, x_3) =$$
$$= H(x_0 | x_1) - H(x_0 | x_1, x_3) = \underline{I(x_0; x_3 | x_1)}$$
$$I(x_0, x_1, x_2; x_3) = I(x_2; x_3) + I(x_0; x_3 | x_1)$$

$$\cancel{I(x_2; x_3) \leq I(x_1; x_3) \leq I(x_1; x_3) + I(x_0; x_3 | x_1) = I(x_0, x_1; x_3)}$$

$$\cancel{I(x_0, x_1; x_3) \geq I(x_0, x_1, x_2; x_3)}$$

$$I(x_0; x_1, x_2, x_3) = I(x_0; x_1) + I(x_0; x_3 | x_1) \quad ④$$
$$I(x_0; x_3 | x_1) = H(x_0 | x_1) - H(x_0 | x_1, x_3)$$
$$I(x_0; x_1) \geq I(x_0; x_2)$$

$$I(x_0; x_1, x_2 x_3) \geq I(x_0; x_2) + I(x_0; x_3 | x_1)$$
$$I(x_0; x_1, x_2, x_3) = I(x_0; x_2) + I(x_0; x_1 | x_2) + \underbrace{I(x_0; x_3 | x_1, x_2)}_{= I(x_0; x_3 | x_1)} =$$
$$= I(x_0; x_2) + I(x_0; x_1 | x_2) + I(x_0; x_3 | x_1)$$
$$\cancel{I(x_0; x_2)} + I(x_0; x_1 | x_2) + \cancel{I(x_0; x_3 | x_1)} \geq \cancel{I(x_0; x_2)} + \cancel{I(x_0; x_3 | x_1)}$$

$$\boxed{I(x_0; x_1 | x_2) \geq 0}$$

$$I(x_0; x_1, x_2, x_3) = I(x_0; x_3) + I(x_0; x_2 | x_3) + I(x_0; x_1 | x_2 x_3)$$
$$I(x_0; x_2 | x_3) = H(x_0 | x_3) - H(x_0 | x_2, x_3)$$

$$H(x_0,x_1,x_2) = H(x_0) + H(x_1|x_0) + H(x_2|x_0x_1) =$$
$$= H(x_0) + H(x_1|x_0) + H(x_2|x_1)$$
$$H(x_0,x_1,x_2,x_3) = H(x_0) + H(x_1|x_0) + H(x_2|x_1) + \underbrace{H(x_3|x_0x_1x_2)}_{= H(x_3|x_2)}$$

ZARADI NJO. (4)

MARKOV CHAIN:

$$H(x_0,\dots x_n) = \sum_{i=0}^{n} H(x_n|x_{n-1})$$

$$H(x_3,x_2,x_1,x_0) = H(x_3) + H(x_2|x_3) + H(x_1|x_2x_3) + H(x_0|x_1x_2x_3)$$
$$I(x_0;x_1) \geq I(x_0;x_2) \Rightarrow H(x_0|x_1) \leq H(x_0|x_2)$$
$$I(x_0;x_3) + I(x_0;x_2|x_3) + I(x_0;x_1|x_2x_3) = \underline{\underline{I(x_0;x_1x_2x_3)}}$$
$$I(x_0;x_1x_2x_3) = I(x_0;x_1) + I(x_0;x_3|x_1)$$
$$I(x_0;x_1x_2x_3) = \underbrace{I(x_0;x_2)}_{\leq I(x_0;x_1)} + I(x_0;x_3|x_2) + I(x_0;x_1|x_2,x_3)$$
$$I(x_0;x_3) + H(x_0|x_3) - \cancel{H(x_0|x_2x_3)} + \cancel{H(x_0|x_2x_3)} - \cancel{H(x_0|x_1x_2x_3)}$$
$$\leq I(x_0;x_1) + H(x_0|x_2) - \cancel{H(x_0|x_2x_3)} + \cancel{H(x_0|x_2x_3)} - \cancel{H(x_0|x_1x_2x_3)}$$

$$I(x_0;x_3) + H(x_0|x_3) \leq I(x_0;x_1) + H(x_0|x_2)$$
$$H(x_0|x_3) - H(x_0|x_2) \leq I(x_0;x_1) - I(x_0;x_3)$$
$$H(x_0|x_3) - H(x_0|x_2) = \underbrace{I(x_0;x_2) - I(x_0;x_3)}_{\geq 0} \Rightarrow$$

$$\boxed{H(x_0|x_3) \geq H(x_0|x_2)} \qquad \boxed{H(x_0|x_2) \geq H(x_0|x_1)}$$

JAS VSUŠNOST SAKAV OVA CELO VREME DA GO DOKAŽAM
AMA TOA E LOGIČNO !!!

$$I(x_0;x_1x_2x_3) = I(x_0;x_2) + I(x_0;x_3|x_2) + I(x_0;x_1|x_2x_3)$$
$$I(x_0;x_1x_2x_3) = I(x_0;x_3) + I(x_0;x_1|x_3) + I(x_0;x_2|x_1x_3)$$
$$I(x_0;x_1x_2x_3) = I(x_0;x_3) + I(x_0;x_2|x_3) + I(x_0;x_1|x_2x_3)$$

$$I(x_0;x_2) + I(x_0;x_3|x_2) = I(x_0;x_3) + I(x_0;x_2|x_3)$$
$$\underline{I(x_0;x_2) - I(x_0;x_3) = I(x_0;x_2|x_3) - I(x_0;x_3|x_2)} =$$
$$= H(x_0|x_3) - \cancel{H(x_0|x_2x_3)} - H(x_0|x_2) + \cancel{H(x_0|x_2x_3)} = H(x_0|x_3) - H(x_0|x_2)$$

ZNAČI ODMA SE DOKAŽUVA !!! NEMA POTREBA VO ENTROPII REZONNO RAZBIVANJE KAKO GORE !!!

$H(x_0|x_3) - H(x_0|x_2) = I(x_0;x_2) - I(x_0;x_3) \geq 0$ (#1) $\boxed{H(x_0|x_3) \geq H(x_0|x_2)}$

$I(x_0;x_2) \leq I(x_0;x_3)$ $\qquad H(x_0|x_3) - H(x_0|x_2) \leq I(x_0;x_1) - I(x_0;x_3)$

$I(x_0;x_1x_2x_3x_4) = I(x_0;x_3) + I(x_0;x_4|x_3) + I(x_0;x_1|x_3x_4) + I(x_0;x_2|x_1x_3x_4)$

$I(x_0;x_1x_2x_3x_4) = I(x_0;x_4) + I(x_0;x_3|x_4) + I(x_0;x_2|x_3x_4) + I(x_0;x_1|x_2x_3x_4)$

$I(x_0;x_1x_2x_3x_4) = I(x_0;x_3) + I(x_0;x_4|x_3) + I(x_0;x_2|x_3x_4) + I(x_0;x_1|x_2x_3x_4)$

$I(x_0;x_4) - I(x_0;x_3) + I(x_0;x_3|x_4) - I(x_0;x_4|x_3) = 0$

$I(x_0;x_3) - I(x_0;x_4) = I(x_0;x_3|x_4) - I(x_0;x_4|x_3) =$

$= H(x_0|x_4) - \cancel{H(x_0|x_3x_4)} - H(x_0|x_3) + \cancel{H(x_0|x_3x_4)}$

$\underbrace{I(x_0;x_3) - I(x_0;x_4)}_{\geq 0} = H(x_0|x_4) - H(x_0|x_3)$

$\boxed{H(x_0|x_4) - H(x_0|x_3) \geq 0}$

- CELO VREME RASTE NEIZVESNOSTA = 0 (#2) $\boxed{H(x_0|x_4) \geq H(x_0|x_3)}$

$I(x_0;x_1x_2) = I(x_0;x_1) + I(x_0;x_2|x_1)$

$I(x_0;x_1,x_2) = I(x_0;x_2) + I(x_0;x_1|x_2)$

$I(x_0;x_1) + I(x_0;x_2|x_1) = I(x_0;x_2) + I(x_0;x_1|x_2)$

$I(x_0;x_1) - I(x_0;x_2) = I(x_0;x_1|x_2) - \underbrace{I(x_0;x_2|x_1)}_{0} \geq 0$

$\boxed{I(x_0;x_1) \geq I(x_0|x_2)}$

$I(x_0;x_1) - I(x_0;x_2) = H(x_0|x_2) - \cancel{H(x_0|x_1x_2)} - H(x_0|x_1) + \cancel{H(x_0|x_1x_2)}$

$\boxed{I(x_0;x_1) - I(x_0;x_2) = H(x_0|x_2) - H(x_0|x_1)}$ $\qquad \boxed{H(x_0|x_2) \geq H(x_0|x_1)}$ (#3)

$I(x_0;x_2) = I(x_0;x_1) + H(x_0|x_1) - H(x_0|x_2)$

• ~~ZNAČI~~ DOBIVAME:

$$\boxed{\begin{aligned} I(x_0;x_2) - I(x_0;x_3) &= I(x_0;x_2|x_3) - I(x_0;x_3|x_2) \\ I(x_0;x_3) - I(x_0;x_4) &= I(x_0;x_3|x_4) - I(x_0;x_4|x_3) \\ I(x_0;x_1) - I(x_0;x_2) &= I(x_0;x_1|x_2) - I(x_0;x_2|x_1) \end{aligned}}$$

(*) = 0 MARKOV CHAIN IF YOU KNOW x_1 x_0 & x_2 ARE INDEPENDENT!!...

b) TREBALO (*) DA VAŽI I ZA:

$I(x_0;x_3|x_2) = 0$ $\qquad I(x_0;x_4|x_3) = 0$ $\}$ SIGURNO JE NULA!!!

ZATOA ŠTO $I(x_1;x_3|x_2) = 0$ $\quad x_3$ E STATISTIČKI NEZAVIS

OD x_1 AKO SE ZNAE x_2 A NE STANUVA ZBOR ZA „x_0".

OVA E DOKAZ DEKA: $\boxed{I(x_0;x_2) - I(x_0;x_3) \geq 0}$ & $I(x_0;x_3) - I(x_0;x_4) \geq 0$

$\$1, \$2, \$3 \Rightarrow H(x_0|x_2) \geq H(x_0|x_1)$; $H(x_0|x_3) \geq H(x_0|x_2)$;
$H(x_0|x_4) \geq H(x_0|x_3)$; $\Rightarrow$

$$\boxed{H(x_0|x_4) \geq H(x_0|x_3) \geq H(x_0|x_2) \geq H(x_0|x_1)}$$

• VO OPŠT SLUČAJ:

$$I(x_0;x_{n-1}) - I(x_0;x_n) = I(x_0;x_{n-1}|x_n) - \underbrace{I(x_0;x_n|x_{n-1})}_{=0}$$

$$\Rightarrow \boxed{I(x_0;x_{n-1}) - I(x_0;x_n) \geq 0}$$ OD EDNA STRANA! T.E $\boxed{I(x_0;x_{n-1}) \geq I(x_0;x_n)}$

OD DRUGA STRANA:

$$I(x_0;x_{n-1}|x_n) - I(x_0;x_n|x_{n-1}) = H(x_0|x_n) - \cancel{H(x_0|x_{n-1},x_n)} - H(x_0|x_{n-1}) + \cancel{H(x_0|x_{n-1},x_n)}$$

$$H(x_0|x_n) - H(x_0|x_{n-1}) \geq 0 \qquad \boxed{H(x_0|x_n) \geq H(x_0|x_{n-1})} \quad \$4$$

$$\boxed{H(x_0|x_n) \geq H(x_0|x_{n-1}) \geq \dots \geq H(x_0|x_4) \geq H(x_0|x_3) \geq H(x_0|x_2) \geq H(x_0|x_1)}$$

DOKAŽANO!!! NEIZVESNOSTA RASTE KAKO SE ZGOLEMUVA „n"

OVA MI TEŠKA DA GO SPOREDAM ŠO IGRATA RASIPAN TELEFON. KAKO RASTE „n" TAKA SE POMALKU X_n ZNAE ZA X_0 !!!

OD IZRAZOT : $I(x_0;x_{n-1}) \geq I(x_0;x_n)$ MOŽE ODMA DA GO DOBIEŠ OVA:

$$\cancel{H(x_0)} - H(x_0|x_{n-1}) \geq \cancel{H(x_0)} - H(x_0|x_n)$$

$\boxed{H(x_0|x_{n-1}) \leq H(x_0|x_n)}$ T.E $\boxed{H(x_0|x_n) \geq H(x_0|x_{n-1})}$

$\boxed{\text{PROBLEM 2.35}}$ RELATIVE ENTROPY IS NOT SYMETRIC

LET RANDOM VARIABLE X HAVE TRI POSSIBLE OUTCOMES $\{a, b, c\}$. CONSIDER TWO DISTRIBUTIONS ON THIS RANDOM VARIABLE:

SYMBOL	$p(x)$	$q(x)$
a	1/2	1/3
b	1/4	1/3
c	1/4	1/3

CALCULATE $H(p)$, $H(q)$, $D(p\|q)$ AND $D(q\|p)$. VERIFY THAT IN THIS CASE, $D(p\|q) \neq D(q\|p)$

$$H(p) = \frac{1}{2} ld 2 + \frac{1}{4} ld 4 + \frac{1}{4} ld 4 = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2} = 1.5$$

$$H(q) = 3 \cdot \frac{1}{3} ld 3 = ld 3 = 1.58 \; ; \; D(p\|q) = \sum_x p \, ld \frac{p}{q} = \frac{1}{2} ld \frac{\frac{1}{2}}{\frac{1}{3}} + \frac{1}{4} ld \frac{\frac{1}{4}}{\frac{1}{3}} + \frac{1}{4} ld \frac{3}{4} = \frac{1}{2} ld \frac{3}{2} + \frac{1}{2} ld \frac{3}{4} = \frac{1}{2} ld \frac{9}{8} = 0.085$$

$$D(q\|p)=\sum_x q\,\mathrm{ld}\,\frac{q}{p}=\frac{1}{3}\mathrm{ld}\frac{2}{3}+\frac{1}{3}\mathrm{ld}\frac{4}{3}+\frac{1}{3}\mathrm{ld}\frac{4}{3}=\frac{1}{3}\mathrm{ld}\frac{32}{27}.$$

$$=\frac{1}{3}\mathrm{ld}\left(\frac{64}{27}\right)\cdot\frac{1}{2}=\frac{1}{3}\mathrm{ld}\left(\frac{4}{3}\right)^3+\frac{1}{3}\mathrm{ld}\frac{1}{2}=\underline{\mathrm{ld}\frac{4}{3}-1/3}=0.082$$

$$D(q\|p)=0.082<0.085=D(p\|q)$$

$\boxed{\text{PROBLEM 2.36}}$ $x\in\{0,1\}$ $p=\left\{\frac{1}{2},\frac{1}{2}\right\}$ $q=\left\{\frac{1}{3},\frac{2}{3}\right\}$

$$D(p\|q)=\frac{1}{2}\mathrm{ld}\frac{\frac{1}{2}}{\frac{1}{3}}+\frac{1}{2}\mathrm{ld}\frac{\frac{1}{2}}{\frac{2}{3}}=\frac{1}{2}\mathrm{ld}\frac{3}{2}+\frac{1}{2}\mathrm{ld}\frac{1}{2}=\underline{\frac{1}{2}\mathrm{ld}\frac{3}{4}}\ \#$$

$$D(q\|p)=\frac{1}{3}\mathrm{ld}\frac{\frac{1}{3}}{\frac{1}{2}}+\frac{2}{3}\mathrm{ld}\frac{\frac{2}{3}}{\frac{1}{3}}=\frac{1}{3}\mathrm{ld}\frac{2}{3}+\frac{2}{3}\mathrm{ld}2=\underline{\frac{1}{3}\mathrm{ld}\frac{8}{3}}$$

TREBA p I q DA SE SIMETRIČNI T.E

$$p=\left\{\frac{1}{3},\frac{2}{3}\right\}\quad q=\left\{\frac{2}{3},\frac{1}{3}\right\}$$

$$D(p\|q)=\frac{1}{3}\mathrm{ld}\frac{\frac{1}{3}}{\frac{2}{3}}+\frac{2}{3}\mathrm{ld}\frac{\frac{2}{3}}{\frac{1}{3}}=\underline{\frac{1}{3}\mathrm{ld}\frac{1}{2}+\frac{2}{3}\mathrm{ld}2}$$

$$D(q\|p)=\frac{2}{3}\mathrm{ld}\frac{\frac{2}{3}}{\frac{1}{3}}+\frac{1}{3}\mathrm{ld}\frac{\frac{1}{3}}{\frac{2}{3}}=\frac{2}{3}\mathrm{ld}2+\frac{1}{3}\mathrm{ld}\frac{1}{2}$$

ISTO ĆE VAŽI ZA: $p=\left\{\frac{1}{4},\frac{3}{4}\right\}$ $q=\left\{\frac{3}{4},\frac{1}{4}\right\}$

- VO GENERALEN SLUČAJ GO IMAM VO MultilogHMM capacity.mw.

$\boxed{\text{PROBLEM 2.37}}$ Relative Entropy. LET X,Y,Z BE THREE RANDOM VARIABLES WITH JOINT PROBABILITY MASS FUNCTION $p(x,y,z)$. THE RELATIVE ENTROPY BETWEEN THE JOINT DISTRIBUTION AND THE PRODUCT OF MARGINALS IS:

$$D(p(x,y,z)\|p(x)p(y)p(z))=E\left[\mathrm{ld}\frac{p(x,y,z)}{p(x)p(y)p(z)}\right]$$

EXPAND IN TERM OF ENTROPIES. WHEN IS THIS QUANTITY ZERO?

$$I(X,Y)=\sum_x\sum_y p(x,y)\,\mathrm{ld}\frac{p(x,y)}{p(x)p(y)}=\sum_{xy}p(x,y)\,\mathrm{ld}\frac{p(x,y)}{p(y)}+$$

$$-\sum_{xy}p(x,y)\,\mathrm{ld}\,p(x)=\sum_{xy}p(x,y)\,\mathrm{ld}\,p(x|y)-\sum_x\mathrm{ld}\,p(x)\sum_y p(x,y)$$

$$=-H(X|Y)-\sum_x p(x)\,\mathrm{ld}\,p(x)=H(X)-H(X|Y)$$

$$(*) = E\left[ld \frac{p(x,y,z)}{p(x)p(y)p(z)} \right] = \sum_{xyz} p(x,y,z)\, ld \frac{p(x,y,z)}{p(x)} + \sum_{xyz} p(xyz)\, ld \frac{1}{p(y)}$$

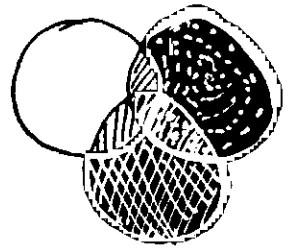
$$+ \sum_{xyz} p(xyz)\, ld \frac{1}{p(z)} = \sum_{xyz} p(x,y,z)\, ld\, p(yz|x) + H(Y) + H(Z)$$

$$= -H(Y,Z|X) + H(Y) + H(Z)$$

$$H(X,Y,Z) = H(X) + H(Y,Z|X)$$
$$H(Y,Z|X) = H(X,Y,Z) - H(X)$$

MORE OR GO ZAMENIS

VIDI HW2S !!!

X  Y Z

$$(*) = \sum_{xyz} p(xyz)\, ld \frac{p(xyz)}{p(x)}$$

$$\boxed{H(Y,Z|X) = H(Y|X) + H(Z|X,Y)}$$

$(*) =$  $+$ 

$$D\big(p(xyz)\,\|\,p(x)p(y)p(z)\big) = H(Y) + H(Z) - H(Y,Z|X)$$

$$= H(Y) + H(Z) - H(Y|X) - H(Z|X,Y) = I(X;Y) + H(Z) - H(Z|X,Y) =$$

$$= \underline{\underline{I(X;Y) + I(Z;X,Y)}} =$$

$$= I(X;Y) + I(X,Y;Z) = \underline{I(X;Y) + I(X;Z) + I(Y;Z|X)}$$

PROBLEM 2.28 THE VALUE OF QUESTION. LET $X \sim p(x)$ $x = 1, 2, \ldots, m$. WE ARE GIVEN A SET $S \subseteq \{1, 2, \ldots m\}$. WE ASK WHETHER $X \in S$ AND RECEIVE THE ANSWER:

$$Y = \begin{cases} 1 & \text{IF } X \in S \\ 0 & \text{IF } X \notin S \end{cases}$$

SUPPOSE THAT $Pr\{X \in S\} = \alpha$. FIND THE DECREASE IN UNCERTAINTY $H(X) - H(X|Y)$. APPARENTLY ANY SET „S" WITH A GIVEN „α" IS AS GOOD AS ANY OTHER.

$$H(X|Y) = \underbrace{Pr(Y=0)}_{1-\alpha} \cdot H(X|Y=0) + \underbrace{Pr(Y=1)}_{\alpha} H(X|Y=1)$$

$$H(X) = \sum_{x=1}^{m} p(x)\, ld\, p(x) = \sum_{x \in X} p(x)\, ld\, p(x)$$

$$H(X|Y) = (1-\alpha) H(X|Y=0) + \alpha H(X|Y=1) = -(1-\alpha) \sum_{x \notin S} p(x)\, ld\, p(x) -$$

$$- \alpha \sum_{x \in S} p(x)\, ld\, p(x)$$

$$H(X) - H(X|Y) = -\sum_{x \in X} p(x)\, ld\, p(x) + \sum_{x \notin S} p(x)\, ld\, p(x) - \alpha \sum_{x \notin S} p(x)\, ld\, p(x) +$$

$$+ \alpha \sum_{x \in S} p(x)\, ld\, p(x) = -\sum_{x \in S} p(x)\, ld\, p(x) + \alpha \sum_{x \in S} p(x)\, ld\, p(x) - \alpha \sum_{x \notin S} p(x)\, ld\, p$$

$$H(X)-H(X|Y)=(\alpha-1)\sum_{x\in S} p\,\mathrm{ld}\,p-\alpha\sum_{x\notin S} p\,\mathrm{ld}\,p \qquad (*)$$

• DA POKAŽAM OBRATNO:

$$H(X|Y)=\alpha\cdot H(X|Y=0)+(1-\alpha)H(X|Y=1)=$$

$$=\alpha\sum_{x\in S} p\,\mathrm{ld}\left(\frac{1}{p}\right)+\left((1-\alpha)\sum_{x\notin S} p\,\mathrm{ld}\left(\frac{1}{p}\right)=\alpha\sum_{x\in S} p\,\mathrm{ld}\frac{1}{p}+\sum_{x\notin S} p\,\mathrm{ld}\frac{1}{p}-\right.$$

$$-\alpha\sum_{x\notin S} p\,\mathrm{ld}\frac{1}{p}$$

$$H(X)-H(X|Y)=\sum_{x\in X} p\,\mathrm{ld}\frac{1}{p}-\alpha\sum_{x\in S} p\,\mathrm{ld}\frac{1}{p}-\sum_{x\notin S} p\,\mathrm{ld}\frac{1}{p}+\alpha\sum_{x\notin S} p\,\mathrm{ld}\frac{1}{p}$$

$$=\sum_{x\in S} p\,\mathrm{ld}\frac{1}{p}-\alpha\sum_{x\in S} p\,\mathrm{ld}\frac{1}{p}+\alpha\sum_{x\notin S} p\,\mathrm{ld}\frac{1}{p}=$$

$$=(1-\alpha)\sum_{x\in S} p\,\mathrm{ld}\left(\frac{1}{p}\right)+\alpha\sum_{x\notin S} p\,\mathrm{ld}\frac{1}{p}=\underbrace{(\alpha-1)\sum_{x\in S} p\,\mathrm{ld}\,p-\alpha\sum_{x\notin S} p\,\mathrm{ld}\,p}_{\text{ISTIOT REZULTAT !!!}}$$

• UŠTE EDNAŠ KAKO ŠTO TREBA:

$$H(X)-H(X|Y)=\sum_{x\in X} p\,\mathrm{ld}\frac{1}{p}-\left[(1-\alpha)\sum_{x\notin S} p\,\mathrm{ld}\frac{1}{p}+\alpha\sum_{x\in S} p\,\mathrm{ld}\frac{1}{p}\right]$$

$$=\sum_{x\in X} p\,\mathrm{ld}\frac{1}{p}-\sum_{x\notin S} p\,\mathrm{ld}\frac{1}{p}+\alpha\sum_{x\notin S} p\,\mathrm{ld}\frac{1}{p}-\alpha\sum_{x\in S} p\,\mathrm{ld}\frac{1}{p}=$$

$$=\underline{\sum_{x\in S} p\,\mathrm{ld}\frac{1}{p}}+\alpha\sum_{x\notin S} p\,\mathrm{ld}\frac{1}{p}-\underline{\alpha\sum_{x\in S} p\,\mathrm{ld}\frac{1}{p}}=\underbrace{(1-\alpha)\sum_{x\in S} p\,\mathrm{ld}\frac{1}{p}+\alpha\sum_{x\notin S} p\,\mathrm{ld}\frac{1}{p}}_{\text{ISTO KAKO }(*)}$$

$$\underline{\sum_{x\in S} p\,\mathrm{ld}\frac{1}{p}}+\sum_{x\notin S} p\,\mathrm{ld}\frac{1}{p}-(1-\alpha)\sum_{x\notin S} p\,\mathrm{ld}\frac{1}{p}-\underline{\alpha\sum_{x\in S} p\,\mathrm{ld}\frac{1}{p}}=$$

$$=(1-\alpha)\sum_{x\in S} p\,\mathrm{ld}\frac{1}{p}-(1-\alpha)\sum_{x\notin S} p\,\mathrm{ld}\frac{1}{p}+\sum_{x\notin S} p\,\mathrm{ld}\frac{1}{p}$$

$$(*)=(1-\alpha)\sum_{x\in S} p\,\mathrm{ld}\left(\frac{1}{p}\right)+\alpha\sum_{x\notin S} p\,\mathrm{ld}\frac{1}{p}=(1-\alpha)\sum_{x\in S} p\,\mathrm{ld}\frac{1}{p}+\alpha\left(\sum_{x\in X} p\,\mathrm{ld}\frac{1}{p}-\sum_{x\in S} p\right.$$

$$=\sum_{x\in S} p\,\mathrm{ld}\frac{1}{p}-\alpha\sum_{x\in S} p\,\mathrm{ld}\frac{1}{p}+\alpha\sum_{x\in X} p\,\mathrm{ld}\frac{1}{p}-\alpha\sum_{x\in S} p\,\mathrm{ld}\frac{1}{p}$$

PRIMER: $x \in \{1,2,3\}$ $\quad S=\{1,2\}$ $\quad p=\{\frac{1}{2}\,\frac{1}{4}\,\frac{1}{4}\}$

$$(*) = (1-\alpha)\sum_{x\in S} p\,\mathrm{ld}\frac{1}{p} + \alpha\sum_{x\notin S} p\,\mathrm{ld}\frac{1}{p} + (1-\alpha)\sum_{x\notin S} p\,\mathrm{ld}\frac{1}{p} - (1-\alpha)\sum_{x\notin S} p\,\mathrm{ld}\frac{1}{p}$$
$$+\alpha\sum_{x\in S} p\,\mathrm{ld}\frac{1}{p} - \alpha\sum_{x\in S} p\,\mathrm{ld}\frac{1}{p}$$

$$= (1-\alpha)\sum_{x\in X} p\,\mathrm{ld}\frac{1}{p} + \alpha\sum_{x\in X} p\,\mathrm{ld}\frac{1}{p} - \sum_{x\notin S} p\,\mathrm{ld}\frac{1}{p} + \alpha\sum_{x\notin S} p\,\mathrm{ld}\frac{1}{p} - \alpha\sum_{x\in S} p\,\mathrm{ld}\frac{1}{p}$$

$$Y = \begin{cases} 1 & x\in S \\ 0 & x\notin S\end{cases} \qquad H(X) = \frac{1}{2}\mathrm{ld}2 + \frac{1}{4}\cdot 2 + \frac{1}{4}\cdot 2 = \frac{3}{2}$$

$$H(X|Y) = (1-\alpha)H(X|Y=0) + \alpha H(X|Y=1) = \dots$$

$$= (1-\alpha)\sum_{x\notin S} p\,\mathrm{ld}\frac{1}{p} + \alpha\sum_{x\in S} p\,\mathrm{ld}\frac{1}{p} =$$

$$= (1-\alpha)\left[\qquad \frac{1}{4}\mathrm{ld}4\right] + \alpha\left[\frac{1}{2}\mathrm{ld}2 + \frac{1}{4}\mathrm{ld}4\right] =$$

$$= (1-\alpha)\cdot\frac{1}{2} + \alpha\left[\frac{1}{2}+\frac{1}{2}\right] = (1-\alpha)\frac{1}{2} + \alpha = \frac{1}{2}+\frac{\alpha}{2}$$

$$= \frac{1}{2}(1+\alpha)$$

$$H(X) - H(X|Y) = \frac{3}{2} - \frac{1}{2} - \frac{\alpha}{2} = 1 - \frac{\alpha}{2}$$

- $S=\{2,3\}$

$$H(X|Y) = (1-\alpha)\left[\frac{1}{2}\mathrm{ld}2\right] + \alpha\left[\frac{1}{4}\mathrm{ld}4 + \frac{1}{4}\mathrm{ld}4\right] =$$

$$= \frac{1}{2}(1-\alpha) + \alpha = \frac{1}{2} - \frac{\alpha}{2} + \alpha = \frac{1}{2} + \frac{\alpha}{2}$$

NAVISTINA USLOVNATA ENTROPIJA NE ZAVISI OD IZBOROT NA PODMNOZESTVOTO „S"

$$H(X) - H(X|Y) = (1-\alpha)\sum_{x\in S} p\,\mathrm{ld}\frac{1}{p} + \alpha\sum_{x\notin S} p\,\mathrm{ld}\frac{1}{p} =$$

$x \in \{1,2,3,\dots m\}$ $\qquad S=\{1,2\}$

$$= (1-\alpha)\sum_{i=1}^{2} p_i\,\mathrm{ld}\frac{1}{p_i} + \alpha\sum_{i=3}^{m} p_i\,\mathrm{ld}\frac{1}{p_i} + \alpha\sum_{i=1}^{2} p_i\,\mathrm{ld}\frac{1}{p_i} - \alpha\sum_{i=1}^{2} p_i\,\mathrm{ld}\frac{1}{p_i}$$

$$= (1-2\alpha)\sum_{i=1}^{2} p_i \operatorname{ld}\frac{1}{p_i} + \alpha H(X)$$

$$H(X|Y) = (1-\alpha)\sum_{i=3}^{m} p_i \operatorname{ld}\frac{1}{p_i} + \alpha\sum_{i=1}^{2} p_i \operatorname{ld}\frac{1}{p_i}$$

$$H(X) - H(X|Y) = \sum_{i=1}^{m} p_i \operatorname{ld}\frac{1}{p_i} - (1-\alpha)\sum_{i=3}^{m} p_i \operatorname{ld}\frac{1}{p_i} - \alpha\sum_{i=1}^{2} p_i \operatorname{ld}\frac{1}{p_i}$$

$$= \sum_{i=1}^{2} p_i \operatorname{ld}\frac{1}{p_i} + \alpha\sum_{i=3}^{m} p_i \operatorname{ld}\frac{1}{p_i} - \alpha\sum_{i=1}^{2} p_i \operatorname{ld}\frac{1}{p_i} =$$

$$= (1-\alpha)\underbrace{\sum_{i=1}^{2} p_i \operatorname{ld}\frac{1}{p_i}}_{x\in S} + \alpha\underbrace{\sum_{i=3}^{m} p_i \operatorname{ld}\frac{1}{p_i}}_{x\notin S} =$$

$$= (1-\alpha)\sum_{x\in S} p \operatorname{ld}\frac{1}{p} + \alpha\sum_{x\notin S} p \operatorname{ld}\frac{1}{p}$$

$$(1-\alpha)\cdot\left[p_1 \operatorname{ld}\frac{1}{p_1} + p_2 \operatorname{ld}\frac{1}{p_2}\right] + \alpha\left[p_3 \operatorname{ld}\frac{1}{p_3} + p_4 \operatorname{ld}\frac{1}{p_4} + \dots + p_m \operatorname{ld}\frac{1}{p_m}\right]$$

$$\left[p_1 \operatorname{ld}\frac{1}{p_1} + p_2 \operatorname{ld}\frac{1}{p_2}\right] - \alpha\left[p_1 \operatorname{ld}\frac{1}{p_1} + p_2 \operatorname{ld}\frac{1}{p_2}\right] + \alpha\left[\dots\right]$$

$$(1-2\alpha)\left[p_1 \operatorname{ld}\frac{1}{p_1} + p_2 \operatorname{ld}\frac{1}{p_2}\right] + \alpha\underbrace{\sum_{i=1}^{m} p_i \operatorname{ld}\frac{1}{p_i}}_{H(X)} =$$

$$= \boxed{(1-2\alpha)H(S) + \alpha\cdot H(X)}$$

• SEGA MI TREBA DEKA $\Pr\{X\in S\}=\alpha$ MISLI NA EDNO VLEČENJE OD „X" = DA PROVERUVA DALI PRIPAĐA NA S: $\quad S\subseteq\{1,2,3,\dots,m\}$

$x\in\{1,2,\dots,m\}$ $\qquad p(x)=\{p_1,p_2,\dots,p_m\}$

$H(X)-H(X|Y)=\,?$ $\qquad Y=\begin{cases}1 & x\in S\\ 0 & x\notin S\end{cases}$

$$H(X|Y) = \underbrace{\Pr(Y=1)}_{\alpha}\cdot H(X|Y=1) + \underbrace{\Pr(Y=0)}_{1-\alpha}\cdot H(X|Y=0)$$

- $X \in \{1,2,3\}$ $\quad \underline{S=\{2\}}$ $\quad P_r(X\in S)=\alpha$
  $p = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\}$ $\quad Y = \begin{cases} 1 & X=2 \\ 0 & X\neq 2 \end{cases}$

$H(X|Y) = P_r(Y=1)\cdot H(X|Y=1) + P_r(Y=0)\cdot H(X|Y=0)$

$P_r(Y=1) = P_r(X\in S) \qquad P_r(Y=0) = P_r(X\notin S)$

$H(X|Y) = \alpha\cdot 0 + (1-\alpha)\sum_{X\notin S} p(x)\, ld\frac{1}{p(x)} =$

$= (1-\alpha)\left[\frac{1}{2}\cdot \cancel{ld}\, 2 + \frac{1}{4}\underbrace{ld 4}_{2}\right] = (1-\alpha)\left(\frac{1}{2}+\frac{1}{2}\right) = \underline{1-\alpha}$

- $X\in\{1,2,3\}$ $\quad S=\{2,3\}$ $\quad Y = \begin{cases} 1 & x\in\{2,3\} \\ 0 & x=1 \end{cases}$

$H(X|Y) = \alpha\cdot\sum_{x\in S} p\, ld\frac{1}{p} + (1-\alpha)\cdot 0 = \alpha\left[\frac{1}{4}\cdot 2 + \frac{1}{4}\cdot 2\right] = \frac{\alpha}{2}$

$H(X) = \frac{1}{2}\cdot 1 + 2\left(\frac{1}{4}\cdot 2\right) = \frac{1}{2} + 1 = \frac{3}{2}$

$H(Y|X) = \overline{H(Y|X=x)} = \sum_{x\in X} p(x)\sum_{y\in Y} p(y|x=x)\cdot ld\frac{1}{p(y|x=x)}$

$H(Y|X) = \sum_{x\in X}\sum_{y\in Y} p(x,y)\, ld\frac{1}{p(y|x)}$

$p(x,y)$

X \ Y	$x\notin S$ 0	$x\in S$ 1	$p(x)$
1	p_1	0	p_1
2	0	α	p_2
3	p_3	0	p_3
$p(Y)$	$1-\alpha$	α	

$S=\{2\}$ $\quad \boxed{p_2=\alpha}$

$\boxed{p_1+p_3 = 1-\alpha}$ $\quad p_3 = 1-\alpha-p_1$

$p(X|Y)$

X \ Y	0	1	
1	$\frac{p_1}{1-\alpha}$	0	
2	0	1	
3	$\frac{p_3}{1-\alpha}$	0	

$H(X|Y) = \sum_{x\in X}\sum_{y\in Y} p(x,y)\, ld\frac{1}{p(x|y)} =$

$= p_1\cdot ld\frac{1-\alpha}{p_1} + p_3\cdot ld\frac{1-\alpha}{p_3} + \alpha\cdot \cancel{ld}\, 1$

$= p_1\, ld\frac{1-\alpha}{p_1} + (1-\alpha-p_1)\, ld\frac{1-\alpha}{1-\alpha-p_1}$

$= p_1\, ld\frac{1}{p_1} + p_3\, ld\frac{1}{p_3} + \underline{p_1\, ld(1-\alpha) + p_3\, ld(1-\alpha)}$

$= \sum_{x\notin S} p\, ld\frac{1}{p} + (p_1+p_3)\, ld(1-\alpha) = \sum_{x\notin S} p\, ld\frac{1}{p} + (1-\alpha)\, ld(1-\alpha)$

$H(X) - H(X|Y) = \sum_{x\in X} p\, ld\frac{1}{p} - \sum_{x\notin S} p\, ld\frac{1}{p} + (1-\alpha)\, ld\frac{1}{(1-\alpha)}$

• $x \in \{1,2,3,4\}$ $\quad S \in \{2,3\}$ $\quad Y = \begin{cases} 1 & x \in \{2,3\} \\ 0 & x \notin \{2,3\} \end{cases}$

$p = \{p_1, p_2, p_3, p_4\}$ $\quad P_r(x \in S) = \alpha$

$p(x,y)$

$x \backslash y$	0	1	$p(x)$
1	p_1	0	p_1
2	0	p_2	p_2
3	0	p_3	p_3
4	p_4	0	p_4
$p(y)$	$1-\alpha$	α	

$$\boxed{\begin{aligned} p_1 + p_4 &= 1-\alpha \\ p_2 + p_3 &= \alpha \end{aligned}} \Rightarrow$$ ОВА Е ГЛАВНАТА ФИНТА !!!

$p(x|y)$

$x \backslash y$	0	1	
1	$\frac{p_1}{1-\alpha}$	0	
2	0	$\frac{p_2}{\alpha}$	
3	0	$\frac{p_3}{\alpha}$	
4	$\frac{p_4}{1-\alpha}$	0	

$$H(X|Y) = \sum_x \sum_y p(x,y)\, ld \frac{1}{p(x|y)} =$$

$$= p_1\, ld \frac{1-\alpha}{p_1} + p_4\, ld \frac{1-\alpha}{p_4} + p_2\, ld \frac{\alpha}{p_2} + p_3\, ld \frac{\alpha}{p_3}$$

$$= p_1\, ld \frac{1}{p_1} + p_4\, ld \frac{1}{p_4} + p_2\, ld \frac{1}{p_2} + p_3\, ld \frac{1}{p_3} +$$

$$+ p_1\, ld(1-\alpha) + p_4\, ld(1-\alpha) + p_2\, ld\,\alpha + p_3\, ld\,\alpha$$

$$= H(X) + \underbrace{(p_1+p_4)}_{=1-\alpha} ld(1-\alpha) + \underbrace{(p_2+p_3)}_{=\alpha} ld\,\alpha =$$

$$= H(X) - (1-\alpha)\, ld \frac{1}{(1-\alpha)} - \alpha\, ld \frac{1}{\alpha}$$

$$H(X) - H(X|Y) = \cancel{H(X)} - \cancel{H(X)} + (1-\alpha)\, ld \frac{1}{(1-\alpha)} + \alpha\, ld \frac{1}{\alpha} \Rightarrow$$

$$\boxed{H(X) - H(X|Y) = (1-\alpha)\, ld \frac{1}{(1-\alpha)} + \alpha\, ld \frac{1}{\alpha}}$$

- ОВА ГОРЕ МОЖЕ ДА СЕ ГЕНЕРАЛИЗИРА ЗА БИЛО КОЕ ПОДМНОЖЕСТВО „S“

$$\sum_{x \notin S} p(x) = 1-\alpha \qquad \sum_{x \in S} p(x) = \alpha$$

$$H(X|Y) = \sum_{x \in X} p(x)\, ld \frac{1}{p(x)} + [ld(1-\alpha)] \cdot \underbrace{\sum_{x \notin S} p(x)}_{1-\alpha} + [ld\,\alpha] \underbrace{\sum_{x \in S} p(x)}_{\alpha}$$

$$H(X|Y) = H(X) - (1-\alpha)\, ld \frac{1}{(1-\alpha)} - \alpha \cdot ld \frac{1}{\alpha}$$

$$\boxed{H(X) - H(X|Y) = (1-\alpha)\, ld \frac{1}{1-\alpha} + \alpha\, ld \frac{1}{\alpha}}$$ ЗНАЧИ СЕ ДОБИВА ИСТИОТ РЕЗУЛТАТ ЗА БИЛО КОЈ ИЗБОР НА ПОДМНОЖЕСТВОТО „S“.

MMV

• ОВА ЗНАЧИ ДЕКА:

$$I(X;Y) = H(X) - H(X|Y) = (1-\alpha)\, ld \frac{1}{1-\alpha} + \alpha\, ld \frac{1}{\alpha}$$

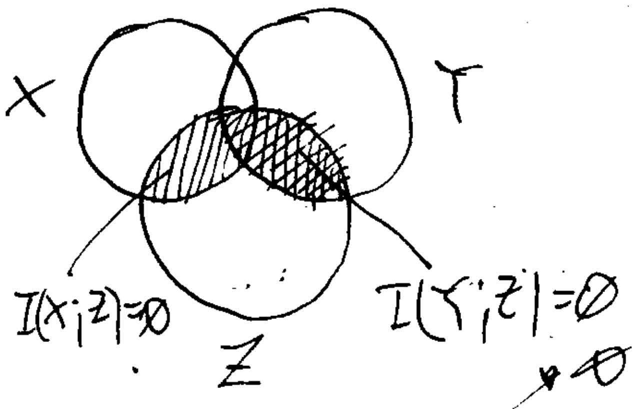
PROBLEM 2.39 ENTROPY AND PAIRWISE INDEPENDENCE. LET X, Y, Z BE THREE BINARY BERNOULLI$\left(\frac{1}{2}\right)$ RANDOM VARIABLES THAT ARE PAIRWISE INDEPENDENT I.E.

$$I(X;Y) = I(X;Z) = I(Y;Z) = 0$$

(a) UNDER THIS CONSTRAINT WHAT IS MINIMUM VALUE OF $H(X,Y,Z)$?

(b) GIVE AN EXAMPLE ACHIEVING THIS MINIMUM

$$f(k;\gamma) = \begin{cases} \gamma & k=1 \\ 1-\gamma & k=0 \end{cases} \qquad \boxed{f(k;\gamma) = \gamma^k (1-\gamma)^{1-k}}$$



$$H(X,Y,Z) = H(X) + H(Y|X) + \underline{H(Z|X,Y)}$$

$$I(X;Y) = H(X) - H(X|Y) = 0$$

$$\Rightarrow \boxed{H(X) = H(X|Y)}$$

$$\boxed{H(Y) = H(Y|X)}$$

$$I(X,Y;Z) = \cancel{I(X;Z)}^{\,0} + I(Y;Z|X) = \cancel{I(Y;Z)}^{\,0} + I(X;Z|Y)$$

$$= I(Y;Z|X) = I(X;Z|Y)$$

$$I(Y;Z|X) = H(Y|X) - H(Y|X,Z) = H(Y) - H(Y|XZ)$$

$$H(Z|X,Y) = -I(Y;Z|X) + H(Y)$$

$$I(Z;Y|X) = H(Z|X) - H(Z|X,Y) = H(Z) - H(Z|X,Y)$$

$$\boxed{H(Z|X,Y) = H(Z) - I(Z;Y|X)}$$

$I(X;Z) = 0 \Rightarrow$

$H(X) - H(X|Z) = 0$

$H(X|Z) = H(X)$

$$\boxed{\begin{aligned} H(X,Y,Z) &= H(X) + H(Y) + H(Z) - I(Z;Y|X) \\ &= H(X) + H(Y) + H(Z) - I(X,Y;Z) \end{aligned}} \quad (1)$$

$$I(X,Y;Z) = H(X,Y) - H(X,Y|Z) = H(X,Y) - H(X|Z) - H(Y|X,Z)$$

$$X \to Y \to Z \qquad H(X,Z|Y) = \frac{H(X,Y,Z)}{H(Z)} = \frac{H(X,Y)\,H(Z|X,Y)}{H(Y)}$$

$$= \underline{H(X|Y) \cdot H(Z|Y)}$$

$$I(X,Y;Z) = I(X;Z) + I(Y;Z|X) =$$

$$= I(Y;Z) + \underbrace{I(X;Z|Y)}_{0}$$

$$I(X;Y,Z) = I(X;Y) + \underbrace{I(X;Z|Y)}_{=0} = I(X;Z) + \underbrace{I(X;Y|Z)}_{\geq 0}$$

$$\boxed{I(X;Y) \geq I(X;Z)}$$

IF: $X \to Y \to Z \Rightarrow \underline{I(X,Y;Z) = I(Y;Z)}$

$$H(X,Y,Z) = H(X) + H(Y) + H(Z) - I(Y;Z) =$$

$$= H(X) + H(Y) + H(Z) - H(Y) + \underbrace{H(Y|Z)}_{=H(Y)} = H(X) + H(Y) + H(Z)$$

ОВА Е МАКСИМУМ

$$H(X,Y,Z) = H(X,Y) + H(Z|X,Y) = H(X) + H(Y|X) + H(Z|X,Y)$$

$$= H(X) + H(Y) + H(Z|X,Y)$$

МИНИМУМ ДОБИВАМЕ АКО $\underline{H(Z|X,Y) = 0}$

ТОА Е СЛУЧАЈ КОГА $\underline{Z = f(X,Y)}$

ВО ТОЈ СЛУЧАЈ: $\underline{H(X,Y,Z) = H(X) + H(Y)}$

$$\boxed{H(X) = H(Y) = -p \operatorname{ld} p - (1-p)\operatorname{ld}(1-p)} = H(Z)$$

(A) $\Rightarrow \quad H(X,Y,Z) = 3p \operatorname{ld}\frac{1}{p} + 3(1-p)\operatorname{ld}\frac{1}{1-p} - \underline{I(Y;Z|X)}$

$$I(X;Y) = \sum_X \sum_Y p(X,Y) \operatorname{ld} \frac{p(X,Y)}{p(X)p(Y)} = H(X) - H(X|Y)$$

$$I(Z;Y|X) = H(Z|X) - \overbrace{H(Z|X,Y)}^{0} = H(Z|X) = H(Z)$$

$$Z = f(X,Y)$$

ТАКА ДОАЃАМЕ ДО ИСТОТО: $\underline{H(X,Y,Z) = H(X) + H(Y)}$

$$\boxed{H(X,Y,Z) = 2p \operatorname{ld}\frac{1}{p} + 2(1-p)\cdot\operatorname{ld}\frac{1}{(1-p)}}$$

$p = \frac{1}{2} \Rightarrow H(X) = H(Y) = 1$

$\to H(X) + H(Y) = 2$

$$\boxed{H(X,Y,Z) \geq 2}$$

ПРИМЕР: $\underline{Z = (X+Y) \bmod 2}$

$H(Z|X,Y) = 0$ ЗОШТО АКО СЕ ПОЗНАВАТ X И Y СЕ ДОБИВА 1 и Z.

2.40 DISCRETE ENTROPIES. LET X AND Y BE TWO INDEPENDENT INTEGER VALUED RANDOM VARIABLES. LET X BE UNIFORMLY DISTRIBUTED OVER $\{1,2,\dots 8\}$ AND LET $Pr\{Y=k\} = 2^{-k}$ $k=1,2,3\dots$

$Pr\{X\}$ $\boxed{p(X) = \left\{\frac{1}{8}, \frac{1}{8}, \dots \frac{1}{8}\right\}}$

(a) $H(X)=?$ (b) $H(Y)=?$ (c) $H(X+Y, X-Y)$

(a) $H(X) = \sum \frac{1}{8} \operatorname{ld} 8 = 8 \cdot \frac{1}{8} \operatorname{ld} 8 = 3$

(b) $\sum_{k=1}^{\infty} \frac{1}{2^k} \operatorname{ld} 2^k = \sum_{k=1}^{\infty} \frac{k}{2^k} = \sum_{k=1}^{\infty} k \cdot x^k$

$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$

$$\begin{array}{l} 1 + 2^1 + 2^2 + \dots 2^k + \dots = S \\ - \quad 2 + 2^2 + \dots 2^{k+1} = 2S \\ \hline S - 2S = 1 \qquad \boxed{S = \frac{1}{1-2}} \end{array}$$

$\int \frac{1}{1-x}\,dx = \#$

$\left(\frac{1}{1-x}\right)' = -1(1-x)^{-1-1} \cdot (-1) = \frac{1}{(1-x)^2}$

$\sum_{k=0}^{\infty} k x^{k-1} = \frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} k x^{k-1}$ $\quad x \cdot \sum_{k=1}^{\infty} k x^{k-1} = \frac{x}{(1-x)^2}$

$\boxed{\sum_{k=1}^{\infty} k x^k = \frac{x}{(1-x)^2}}$ $\quad H(Y) = \sum_{k=1}^{\infty} \frac{k}{2^k} = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = \frac{\frac{1}{2}}{\frac{1}{4}}$

$\boxed{H(Y) = 2}$

X \ Y	1	2	3	4	...	∞	p(X)
1							1/8
2							1/8
3							1/8
4							1/8
5							1/8
6							1/8
7							1/8
8							1/8
p(Y)	1/2	1/4	1/8	1/16			

$\boxed{H(X+Y, X-Y) = H(X+Y) + H(X-Y \mid X+Y)}$

$H(Z|X) = \sum_{x \in X} p(x) \cdot H(Z \mid X=x)$

$= \sum_{x \in X} p(x) \cdot H(X+Y \mid X=x) =$

$= \sum_{x \in X} p(x) \cdot H(Y) = \underline{H(Y)}$

$H(X+Y) = H(Z)$ $\quad I(X;Z) = H(X) - H(X|Z) =$

$= H(Z) - H(Z|X)$

$$\boxed{I(X;Z) = H(Z) - H(Y)}$$

$H(Z) = I(X;Z) + H(Y)$

$H(Z) \geq H(Y)$

IF X & Y ARE IDENTICALLY DISTRIBUTED: $H(Y) = H(X)$

$$\boxed{H(Z) \geq H(X)}$$

$$\boxed{Z = X+Y \qquad p(Z=z|X=x) = p(Y=z-x|X=x)}$$

$$H(Z|X) = \sum_{x\in\mathcal{X}} p(x) H(Z|X=x) = -\sum_{x\in\mathcal{X}} p(x) \sum_{z\in\mathcal{Z}} p(z|x) \operatorname{ld} p(z|x)$$

$$= -\sum_{x\in\mathcal{X}} p(x) \sum_{z\in\mathcal{Z}} p(Y=z-x|X=x) \operatorname{ld} p(Y=z-x|X=x) =$$

$$= -\sum_{x\in\mathcal{X}} p(x) \underbrace{\sum_{y\in\mathcal{Y}} p(y|X=x) \operatorname{ld} p(Y=y|X=x)}_{-H(Y|X=x)} =$$

$$= \sum_{x\in\mathcal{X}} p(x) \cdot H(Y|X=x) = H(Y|X)$$

$$\boxed{H(Z|X) = H(Y|X)}$$

$$I(X;Z) = H(X) - H(X|Z) = H(Z) - H(Z|X) = H(Z) - H(Y|X) =$$

$=$ | Y & X ARE INDEPENDENT | $= H(Z) - H(Y)$

$$\boxed{H(Z) \geq H(Y)} \rightarrow \boxed{H(Z) \geq H(X)}$$

$$H(Z|Y) = \sum_{y\in\mathcal{Y}} p(y) H(Z|Y=y) = -\sum_{y\in\mathcal{Y}} p(y) \sum_{z\in\mathcal{Z}} p(Z=z|Y=y) \operatorname{ld} p(Z=z|Y=y)$$

$$| p(Z=z|Y=y) = p(X=z-y|Y=y) | =$$

$$= -\sum_{y\in\mathcal{Y}} p(y) \underbrace{\sum_{z\in\mathcal{Z}} p(X=z-y|Y=y) \operatorname{ld} p(X=z-y|Y=y)}_{-H(X|Y=y)} =$$

$$= \sum_{y\in\mathcal{Y}} p(y) H(X|Y=y) = H(X|Y)$$

$$\boxed{H(Z|Y) = H(X|Y)}$$

$$I(Y;Z) = H(Z) - H(Z|Y) = H(Z) - H(X|Y) = H(Z) - H(X)$$

(*) $\boxed{H(Z|X) = H(Y|X)}$ ... $\boxed{H(Z|Y) = H(X|Y)}$

$I(X;Z) = H(X) - H(X|Z) = H(Z) - H(Z|X) = H(Z) - H(Y|X)$

Y & X INDEPENDENT

$I(X;Z) = H(Z) - H(Y)$

$H(Z) \geq H(Y)$

$I(Y;Z) = H(Z) - H(Z|Y) = H(Z) - H(X|Y) = H(Z) - H(X)$

$H(Z) \geq H(X)$

$H(Z) = H(Y) + I(X;Z) = H(Y) + H(X) + H(X|Z)$

$H(X,Y) = H(X) + H(Y|X) \leq \underline{H(X) + H(Y)}$

$H(Z) = H(Y) + H(X) + H(X|Z) \geq H(X,Y) + H(X|Z)$

$\boxed{H(X,Y) \leq H(X) + H(Y)}$ (1)

$H(X,Y) = H(X) + H(Z|X)$ ✓

$\underline{H(Z|X) = H(X,Y) - H(X)}$

$H(Z|X) \leq H(X,Y)$

$H(X,Y) = H(X,Z)$

$H(X,Z) = H(Z) + H(X|Z) \Rightarrow \boxed{H(X,Y) \geq H(Z)}$ (2)

(1) + (2) $\Rightarrow \boxed{H(Z) \leq H(X,Y) \leq H(X) + H(Y)}$

(*)

$H(X,Z) = H(Z) + H(X|Z) = H(X) + H(Z|X) =$

$= \underbrace{H(X) + H(Y|X)}_{= H(X,Y)} = \underline{H(X) + H(Y)}$

SADA DA VAŽE DOKA $\boxed{H(X,Y) = H(X,Z)}$ TOJAS:

$H(Z) \leq H(X,Z) = H(X,Y) \leq H(X) + H(Y)$

X, Y $\quad X \in \{1,2,3,\ldots 8\}$; $Y \in \{1,2,3,\ldots\}$

$p(X) = \left\{\frac{1}{8}, \frac{1}{8} \cdots \frac{1}{8}\right\}$ $\quad p(Y) = \left\{\frac{1}{2^k}\right\}$ $\quad k = 1, 2, \ldots$

$H(X+Y|X-Y) = H(Z|W) = -H(W) + H(W,Z)$

$Z = X+Y$

$W = X-Y$

$H(X+Y|X) = H(Y|X)$

$H(X-Y|X) = H(Y|X)$?

$$H(W|X) = \sum_{x\in X} p(x)\, H(W|X=x) = -\sum_{x\in X} p(x) \sum_{w\in W} p(W=w|X=x) \operatorname{ld} p(w\ldots$$

$W = Z - Y$

$$= -\sum_{x\in X} p(x) \underbrace{\sum_{w\in W} p(Y=x-w|X=x) \operatorname{ld} p(Y=x-w|X=x)}_{-H(Y|X=x)}$$

$$= \sum_{x\in X} p(x)\, H(Y|X=x) = \underline{H(Y|X)}$$

$$H(X+Y|X-Y) = \sum_{w\in W} p(w)\, H(X+Y|w)$$

$X \in \{1,2,\ldots 8\}$ $\qquad p(X) = \left[\frac{1}{8}, \frac{1}{8}, \ldots \frac{1}{8}\right]$

$Y \in \{1,2,\ldots 8\}$ $\qquad p(Y) = \left[\frac{1}{2}, \frac{1}{4}, \ldots \frac{1}{256}\right]$

$\underbrace{Z \in \{2,3,4,\ldots 16\}}_{|Z| = 15}$ $\qquad W \in \{-7,-6,\ldots 6,7\}$

$|W| = 15$

$\boxed{\frac{1}{8}\left(\frac{1}{2}+\frac{1}{2^2}\right) = \quad \frac{1}{8}\cdot\frac{1}{2}\left(1+\frac{1}{2}\right) = \quad \frac{1}{8}\cdot\frac{1}{2}\left(1-\frac{1}{2^2}\right)}$

$$p(Z=2) = p(X=1)\cdot p(Y=1) = \frac{1}{8}\cdot\frac{1}{2} = \frac{1}{16}$$

$$p(Z=3) = p(X=1)\cdot p(Y=2) + p(X=2)\cdot p(Y=1) = \frac{1}{8}\cdot\frac{1}{4} + \frac{1}{8}\cdot\frac{1}{2} = \frac{1}{32} + \frac{1}{16} = \frac{3}{32}$$

$$p(Z=4) = p(X=1)\cdot p(Y=3) + p(X=2)\cdot p(Y=2) + p(X=3)\cdot p(Y=1)$$

$$= \frac{1}{8}\cdot\frac{1}{2^3} + \frac{1}{8}\cdot\frac{1}{2^2} + \frac{1}{8}\cdot\frac{1}{2}$$ $\qquad \boxed{n = k-2 = 2}$

$$p(Z=\underset{=k}{4}) = \frac{1}{8}\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}\right) = \frac{1}{8}\cdot\frac{1}{2}\cdot\frac{1-q^{n+1}}{1-q} = \frac{1}{16}\cdot\frac{1-\left(\frac{1}{2}\right)^3}{1-\frac{1}{2}}$$

$$= \frac{1}{16}\cdot\frac{1-\frac{1}{8}}{1-\frac{1}{2}} = \frac{1}{16}\,\frac{\frac{7}{8}}{\frac{1}{2}} = \frac{1}{16}\,\frac{7}{4} = \frac{7}{64} = \frac{1}{8}\cdot\frac{7}{8}$$

$$p(Z=3) = \frac{1}{8}\frac{1}{2}\,\frac{1-\left(\frac{1}{2}\right)^2}{1-\frac{1}{2}} = \frac{1}{8}\,\frac{\frac{3}{4}}{\frac{1}{2}} = \frac{1}{8}\cdot\frac{3}{2} =$$

$s = 1 + q + q^2 + \ldots q^4$ $\qquad s(1-q) = 1 - q^{4+1}$

$q s = q + q^2 + q^3 + \ldots q^{4+1}$ $\qquad s = \frac{1-q^{4+1}}{1-q}$

$$\frac{1}{8}\left(\frac{1}{8}+\frac{1}{4}+\frac{1}{2}\right) = \frac{1}{8}\,\frac{1+2+4}{8} = \frac{7}{64}$$

$$p(Z=k) = \frac{1}{16}\,\frac{1-\left(\frac{1}{2}\right)^{k-2+1}}{1-\frac{1}{2}} = \frac{1}{16}\,\frac{1-\left(\frac{1}{2}\right)^{k-1}}{1-\frac{1}{2}} = \frac{1}{16}\,\frac{1-\left(\frac{1}{2}\right)^{k-1}}{\frac{1}{2}}$$

$$\boxed{p(Z=k) = \frac{1}{8}\left(1-\left(\frac{1}{2}\right)^{k-1}\right)} \qquad p(Z=2) = \frac{1}{8}\left(1-\frac{1}{2}\right) = \frac{1}{16}$$

OVA VAŽI SAMO ~~ZA~~ $K \leq 8$

NM2.114

$$p(Z=3) = \frac{1}{8}\left(1-\frac{1}{4}\right) = \frac{1}{8}\cdot\frac{3}{4} = \frac{3}{32}$$

$$p(Z=4) = \frac{1}{8}\left(1-\frac{1}{8}\right) = \frac{1}{8}\cdot\frac{7}{8} = \frac{7}{64}$$

$$p(Z=5) = p(X=1)\cdot p(Y=4) + p(X=2)\cdot p(Y=3) + p(X=3)\,p(Y=2) + p(X=4)\cdot p(Y=1) = \frac{1}{8}\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}\right) =$$

$$= \frac{1}{8}\left(\frac{8+4+2+1}{16}\right) = \frac{1}{8}\,\frac{15}{16} = \frac{15}{128}$$

$$p(Z=k) = \frac{1}{8}\left(1-\frac{1}{2^4}\right) = \frac{1}{8}\left(1-\frac{1}{16}\right) = \frac{15}{128}$$

IF $Y \in \{1,2,3,\ldots\infty\} \Rightarrow Z \in \{2,3,4,\ldots\infty)$

$$p(Z) = \left\{\frac{1}{8}\left(1-\frac{1}{2^{k-1}}\right)\right\}_{k=2,3,4,\ldots\infty}$$

$$H(Z) = -\sum_{k=2}^{\infty}\frac{1}{8}\left(1-\frac{1}{2^{k-1}}\right)\operatorname{ld}\frac{1}{8}\left(1-\frac{1}{2^{k-1}}\right)$$

$$H(Z) = -\frac{1}{8}\sum_{k=2}^{\infty}\left(1-\frac{1}{2^{k-1}}\right)\left[-\operatorname{ld}8+\operatorname{ld}\left(1-\frac{1}{2^{k-1}}\right)\right] =$$

$$= -\frac{1}{8}\sum_{k=2}^{\infty}\left[-\operatorname{ld}8+\operatorname{ld}\left(1-\frac{1}{2^{k-1}}\right)\right] + \frac{1}{8}\sum_{k=2}^{\infty}\frac{\operatorname{ld}8+\operatorname{ld}\left(1-\frac{1}{2^{k-1}}\right)}{2^{k-1}} =$$

$$H(X+Y,\,X-Y) = H(X+Y) + H(X-Y\,|\,X+Y) = H(X+Y) + H(X-Y)$$

$H(X-Y) = ?$ $\quad X \in \{1,2,\ldots 8\} \quad Y \in \{1,2,\ldots 8\}$

$$W = X-Y = -(Y-X) = -\{-7,-6,\ldots 0,1,2,3,\ldots\infty\} =$$

$$= \{-7,-6,\ldots,-3,-2,-1,0,1,2,3,\ldots 6,7\}$$

$$P(W=0) = P(X=1)\cdot P(Y=1) + P(X=2)\cdot P(Y=2) + \dots P(X=8)\cdot P(Y=8)$$

$$= \frac{1}{8}\left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots \frac{1}{2^8}\right) = \frac{1}{16}\left(1 + \frac{1}{2} + \dots + \frac{1}{2^7}\right)$$

$$= \frac{1}{16}\,\frac{1-2^{7+1}}{1-2} = \frac{1}{16}\,\frac{1-2^8}{1-2} = \frac{1}{16}\,\frac{1-\frac{1}{256}}{\frac{1}{2}} = \frac{1}{8}\left(1-\frac{1}{2^8}\right)$$

$$P(W=1) = P(X=2)\cdot P(Y=1) + \cancel{P(X=2)\,P(Y=2)} + P(X=3)\,P(Y=2)$$
$$+ P(X=4)\,P(Y=3) + P(X=5)\cdot P(Y=4) + P(X=6)\,P(Y=5) +$$
$$+ P(X=7)\,P(Y=6) + P(X=8)\cdot P(Y=7) = \frac{1}{8}\left(\frac{1}{2} + \frac{1}{2^2} + \dots \frac{1}{2^7}\right)$$

$$= \frac{1}{16}\left(1 + \frac{1}{2} + \dots + \frac{1}{2^6}\right) = \frac{1}{16}\,\frac{1-2^{7+1}}{1-2} = \frac{1}{8}\left(1-\frac{1}{2^7}\right)$$

$$P(W=2) = P(X=3)\cdot P(Y=1) + P(X=4)\,P(Y=2) + \dots$$
$$+ P(X=8)\cdot P(Y=6) = \frac{1}{8}\left(\frac{1}{2} + \frac{1}{2^2} + \dots \frac{1}{2^6}\right)$$

$$= \frac{1}{16}\left(1 + \frac{1}{2} + \dots + \frac{1}{2^5}\right) = \frac{1}{8}\left(1-\frac{1}{2^6}\right) = \frac{1}{8}\cdot\frac{63}{64}$$

$$P(W=3) = \frac{1}{8}\left(1-\frac{1}{2^5}\right) \qquad P(W=7) = \frac{1}{8}\left(1-\frac{1}{2^{8-k}}\right)$$

$$P(W=7) = \frac{1}{8}\left(1-\frac{1}{2}\right) = \frac{1}{16}$$

$$P(W=-7) = P(X=1)\cdot P(Y=8) = \frac{1}{8}\cdot\frac{1}{2^8} = \frac{1}{8}\cdot\frac{1}{256}$$

$$P(W=-6) = P(X=1)\cdot P(Y=7) + P(X=2)\cdot P(Y=8) = \frac{1}{8}\left(\frac{1}{2^7} + \frac{1}{2^8}\right)$$

$$P(W=-5) = P(X=1)\cdot P(Y=6) + P(X=2)\cdot P(Y=7) + P(X=3)\cdot P(Y=8) =$$
$$= \frac{1}{8}\left(\frac{1}{2^6} + \frac{1}{2^7} + \frac{1}{2^8}\right) = \frac{1}{8}\cdot\frac{1}{2^6}\left(1 + \frac{1}{2^1} + \frac{1}{2^2}\right) = \frac{1}{512}\,\frac{4+2+1}{4} = \frac{7}{2048}$$

$$P(W=-1) = \frac{1}{8}\left(\frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^8}\right) = \frac{1}{8}\cdot\frac{1}{2^2}\left(1 + \frac{1}{2} + \dots + \frac{1}{2^6}\right)$$

$$\Rightarrow P(W) = \frac{1}{8}\cdot 2^{+k-1}\left(1 + \frac{1}{2} + \dots + \frac{1}{2^{8+k-1}}\right) =$$

$$P(W=-1) = \frac{1}{8}\,\frac{64+32+16+8+4}{256} = \frac{30+36+1}{2048}$$

$$= \frac{2^{k-1}}{8}\,\frac{1-\left(\frac{1}{2}\right)^{8+k-1+1}}{\frac{1}{2}} = \frac{2^k}{8}\left(1-\frac{1}{2^{8+k}}\right) \quad \boxed{= \frac{127}{2048}}$$

$$P(W=-7) = \frac{2^{-7}}{8}\cdot\left(1-\frac{1}{2}\right) = \frac{2^{-7}}{8}\cdot\frac{1}{2} = \frac{1}{8\cdot 2^8}$$

$$P(W=K)=\begin{cases}\frac{1}{8}\left(1-\frac{1}{2^{8-K}}\right) & K\geq\emptyset\\ \frac{2^K}{8}\left(1-\frac{1}{2^{8+K}}\right) & K<\emptyset\end{cases}$$

?

VIDI 4.111

NE E DOPIO TREBA DA SE ZEMAT $\{\in[1,2,\dots\infty]$

$$\frac{1}{8}\left(1-\frac{1}{2^{8-K}}\right)=\frac{1}{8}\left(1-\frac{2^K}{2^8}\right)=\frac{1}{8}\,\frac{2^8-2^K}{2^8}=\frac{2^8-2^K}{2048}$$

$$\frac{2^K}{8}\left(1-\frac{1}{2^{8+K}}\right)=\frac{2^K}{8}\,\frac{2^{8+K}-1}{2^{8+K}}=\frac{2^{8+K}-1}{2048}$$

$K\in[-7,-6,\dots-1]$

$m\in[1,\dots7]$ $\quad P(W)=\dfrac{2^m-1}{8\cdot2^8}$

$$\boxed{P(W=m-8)=\frac{2^m-1}{8\cdot2^8}}$$

$$H(X-Y)=-\sum_{m=1}^{7}\frac{2^m-1}{8\cdot2^8}\,ld\,\frac{2^m-1}{8\cdot2^8}$$

$$-\sum_{K=0}^{7}\frac{2^8-2^K}{2048}\,ld\,\frac{2^8-2^K}{2048}=$$

$$=-\frac{1}{2048}\sum_{m=1}^{7}(2^m-1)\,ld\left(\frac{2^m-1}{2048}\right)-\frac{1}{2048}\sum_{m=\emptyset}^{7}(2^8-2^m)\,ld\,\frac{2^8-2^m}{2048}$$

$$=-\frac{1}{2048}\left[\sum_{m=1}^{7}2^m\,ld\,\frac{2^m-1}{2048}-\sum_{m=1}^{7}ld\,\frac{2^m-1}{2048}+\sum_{m=0}^{7}2^8\,ld\,\frac{2^8-2^m}{2048}\right.$$
$$\left.-\sum_{m=0}^{7}2^m\,ld\,\frac{2^8-2^m}{2048}\right]$$

$$\boxed{H(X+Y)=-\frac{1}{8}\sum_{K=2}^{16}\left(1-\frac{1}{2^{K-1}}\right)ld\,\frac{1}{8}\left(1-\frac{1}{2^{K-1}}\right)}$$

$$H(X+Y)=-\frac{1}{8}\sum_{K=1}^{15}\left(1-\frac{1}{2^K}\right)ld\,\frac{1}{8}\left(1-\frac{1}{2^K}\right)=-\frac{1}{8}\sum_{K=1}^{15}\frac{2^K-1}{2^K}\,ld\left(\frac{2^K-1}{8\cdot2^K}\right)$$

$$H(X+Y)+H(X-Y) = -\frac{1}{8}\sum_{k=1}^{15}\left(1-\frac{1}{2^k}\right) ld\, \frac{1}{8}\left(1-\frac{1}{2^k}\right) -$$

$$-\sum_{k=1}^{7} \frac{2^k-1}{8\cdot 2^8}\, ld\, \frac{2^k-1}{8\cdot 2^8} - \sum_{k=0}^{7}\frac{2^8-2^k}{2048}\, ld\, \frac{2^8-2^k}{2048}$$

$$= \left| \frac{2^8-2^0}{2048}\cdot ld\, \frac{2^8-1}{2048} = \frac{255}{2048}\cdot ld\, \frac{255}{2048} \right| =$$

$$-\frac{1}{8}\sum_{k=1}^{15}\left(\frac{2^k-1}{2^k}\right) ld\, \frac{2^k-1}{8\cdot 2^k}$$

• MORA T DA IMA "∞" ČLENOVI ZA DA VEROJATNOSTITA $P(T) = \left\{\frac{1}{2^k}\ k=1,2,\ldots\infty\right\}$ BIDE EDNAKVA NA "1".

TOGAŠ:
$$H(Z) = -\sum_{k=2}^{\infty}\frac{1}{8}\left(1-\frac{1}{2^{k-1}}\right) ld\, \frac{1}{8}\left(1-\frac{1}{2^{k-1}}\right)$$

$$\boxed{H(Z) = -\frac{1}{8}\sum_{k=1}^{\infty}\left(1-\frac{1}{2^k}\right) ld\, \frac{1}{8}\left(1-\frac{1}{2^k}\right)}$$

$$\boxed{H(W) = -\sum_{n=0}^{7}\frac{2^n-1}{8\cdot 2^8}\, ld\, \frac{2^n-1}{8\cdot 2^8} - \sum_{n=-1}^{-\infty}\frac{1}{8}\frac{2^{8+k}-1}{2^8}\, ld\, \frac{2^{8+k}-1}{8\cdot 2^8}}$$

$$P(W=-7) = P(X=1)\cdot P(Y=8) + P(X=2)\cdot P(Y=9) + \cdots + P(X=8)\cdot P(Y=15)$$

$$= \frac{1}{8}\left(\frac{1}{2^8}+\frac{1}{2^9}+\cdots \frac{1}{2^{15}}\right) = \frac{1}{8}\cdot\frac{1}{2^8}\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^7}\right)$$

$$= \frac{1}{8}\cdot\frac{1}{2^8}\ \frac{1-\left(\frac{1}{2}\right)^8}{\frac{1}{2}} = \frac{1}{2^{10}}\left(1-\frac{1}{2^8}\right) = 2^{k-3}\left(1-\frac{1}{2^8}\right)$$

$$P(W=-1) = P(X=1)\cdot \underline{P(Y=2)} + P(X=2)\cdot P(Y=3) + \cdots P(X=8)\cdot P(Y=9)$$

$$= \frac{1}{8}\left(\frac{1}{2^2}+\frac{1}{2^3}+\cdots\frac{1}{2^9}\right) = \frac{1}{8}\cdot\frac{1}{2^2}\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^7}\right)$$

$$= \frac{1}{2^4}\cdot\left(1-\frac{1}{2^8}\right) = 2^{k-3}\left(1-\frac{1}{2^8}\right) = 2^{k-3}\,\frac{2^8-1}{2^8} = 2^{k-11}(2^8-1)$$

$$P(W=-2) = P(X=1)\cdot P(Y=3) + P(X=2)\cdot P(Y=4)\cdots P(X=8)\cdot P(Y=10)$$

$$= \frac{1}{8}\left(\frac{1}{2^3}+\frac{1}{2^4}+\cdots+\frac{1}{2^{10}}\right) = \frac{1}{8}\cdot\frac{1}{2^3}\left(1+\frac{1}{2}+\cdots\frac{1}{2^7}\right) = \frac{1}{2^5}\left(1-\frac{1}{2^8}\right)$$

$$= 2^{-13}(2^8-1)$$

$$P(W=k) = \begin{cases} \frac{1}{8}\left(1-\frac{1}{2^{8-k}}\right), & k \geq 0 \\ 2^{k-11}\left(2^{8}-1\right), & k < 0 \end{cases} = \begin{cases} 2^{k-11}\left(2^{8-k}-1\right) & k \geq 0 \\ 2^{k-11}\left(2^{8}-1\right) & k<0 \end{cases}$$

$$\frac{1}{8}\left(1-\frac{2^k}{2^8}\right) = \frac{1}{8}\,\frac{2^8-2^k}{2^8} = \frac{2^8-2^k}{2^{11}} = \underline{2^{k-11}\left(2^{8-k}-1\right)}$$

$$H(Z) = -\sum_{k=1}^{\infty} \frac{2^k-1}{8\cdot 2^k}\,\operatorname{ld}\left(\frac{1}{8}\,\frac{2^k-1}{2^k}\right)$$

$(2^4\cdot 3^3)^2 \quad 2^8\cdot 3^6$

$$H(W) = -\sum_{k=0}^{7} 2^{k-11}\operatorname{ld}\left[2^{k-11}\left(2^{8-k}-1\right)\right] - \sum_{k=-\infty}^{-1} 2^{k-11}(\underbrace{2^8-1}_{255})\operatorname{ld}2^{k-11}(\underbrace{2^8-1}_{255})$$

$$H(W) = -\sum_{k=0}^{7} 2^{k-11}\operatorname{ld}\left[2^{k-11}\left(2^{8-k}-1\right)\right] - \sum_{k=-\infty}^{-1}\left(255\cdot 2^{k-11}\right)\operatorname{ld}\left(252\cdot 2^{k-11}\right)$$

$$\sum_{k=-\infty}^{-1} 255\cdot 2^{k-11} = \left|\begin{matrix} n=-k \\ k=-\infty \\ n=\infty \\ k=-1 \\ n=1 \end{matrix}\right| = \sum_{n=1}^{\infty} 255\cdot 2^{-n-11} = \sum_{n=1}^{\infty}\frac{255}{2^{n+11}}$$

$$H(W) = -\sum_{k=0}^{7} 2^{k-11}\operatorname{ld}\left[2^{k-11}\left(2^{8-k}-1\right)\right] - \underbrace{\sum_{k=1}^{\infty}\frac{255}{2^{k+11}}\operatorname{ld}\frac{255}{2^{k+11}}}_{S_2}$$

11

$$S_2 = \frac{1}{2^{11}}\sum_{k=1}^{\infty}\frac{255}{2^k}\left[\operatorname{ld}255 - \operatorname{ld}2^{k+11}\right] = \frac{1}{2^{11}}\sum_{k=1}^{\infty}\frac{255}{2^k}\operatorname{ld}255 - \frac{1}{2^{11}}\sum_{k=1}^{\infty}\frac{255}{2^k}(k+11) = \frac{1}{2^{11}}255\cdot\operatorname{ld}255\sum_{k=1}^{\infty}\frac{1}{2^k} - \frac{255}{2^{11}}\sum_{k=1}^{\infty}\frac{k}{2^k} - \frac{255}{2^{11}}\cdot 11\sum_{k=1}^{\infty}\frac{1}{2^k}$$

2

$$\sum_{k=1}^{\infty}\frac{1}{2^k} = \sum_{k=0}^{\infty}\frac{1}{2^k} - 1 = \frac{1}{1-\frac{1}{2}} - 1 = 1$$

$$\sum_{k=1}^{\infty} n x^n = \frac{x}{(1-x)^2} \qquad \sum_{k=1}^{\infty}\frac{k}{2^k} = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = \frac{\frac{1}{2}}{\frac{1}{4}} = \frac{4}{2} = 2$$

$$S_2 = \frac{255\operatorname{ld}255}{2^{11}} - \frac{255}{2^{11}}\cdot 2 - \frac{255\cdot 11}{2^{11}}\cdot 1 = \frac{255}{2^{11}}\left(\operatorname{ld}255 - 2 - 11\right) = \frac{255}{2^{11}}\left(\operatorname{ld}255 - 13\right)$$

$$\sum_{k=0}^{7} 2^{k-11}(2^{8-k}-1)\, ld\, 2^{k-11}(2^{8-k}-1) = -\frac{14093}{2048}\ \frac{1}{2^{11}}\big(1193\, ld\,3 + 495\, ld\,5 + 255\, ld\,17 + 254\, ld\,127 + 476\, ld\,7 + 248\, ld\,31\big)$$

$$= \frac{1}{2^{11}}\left[ld\, 3^{1193}\cdot 5^{495}\cdot 17^{255}\cdot 127^{254}\cdot 7^{476}\cdot 31^{248}\right] - \frac{14093}{2048}$$

$$= ld\left[3^{\frac{1193}{2^{11}}}\cdot 5^{\frac{495}{2^{11}}}\cdot 17^{\frac{255}{2048}}\cdot 127^{\frac{254}{2048}}\cdot 7^{\frac{476}{2048}}\cdot 31^{\frac{248}{2048}}\right] - \frac{14}{2}$$

$$\boxed{H(W) = H(X-Y) = 3.394}$$

$$H(Z) = -\sum_{k=1}^{\infty} \frac{1}{8}\left(1-\frac{1}{2^k}\right) ld\left(\frac{1}{8}\left(1-\frac{1}{2^k}\right)\right) = ?$$

$$H(Z) = -\frac{1}{8}\Bigg[\underbrace{\sum_{k=1}^{\infty} ld\left[\frac{1}{8}\left(1-\frac{1}{2^k}\right)\right]}_{S1} - \underbrace{\sum_{k=1}^{\infty}\frac{1}{2^k}\, ld\left(1-\frac{1}{2^k}\right)}_{S2}\Bigg]$$

$$S_1 = \sum_{k=1}^{\infty}\left[ld\,\frac{1}{8} + ld\left(1-\frac{1}{2^k}\right)\right] = \sum_{k=1}^{\infty}\left[-3 + ld\left(1-\frac{1}{2^k}\right)\right.$$

$$\sum_{k=1}^{\infty} ln\left(1-\frac{1}{2^k}\right) = ? \qquad \sum_{k=1}^{\infty} ln\, x^k$$

$X \in [1,2,3,\ldots 8]$ $\quad Y \in [1,2,3,\ldots 8\ldots]$ $\quad Z = X+Y$

$Z \in [2,3,4,5,\ldots \infty]$ $\quad \frac{1}{8}\,\frac{2+1}{4} = \frac{3}{32}$

$$P(Z=2] = P(X=1)\cdot P(Y=1) = \frac{1}{8}\cdot\frac{1}{2}$$

$$P(Z=3] = P(X=1)\cdot P(Y=2) + P(X=2)\cdot P(Y=1) = \frac{1}{8}\left(\frac{1}{2}+\frac{1}{2^2}\right) =$$

$$= \frac{1}{8}\cdot\frac{1}{2}\left(1+\frac{1}{2}\right) = \frac{1}{8}\cdot\frac{1}{2}\cdot\frac{1-\left(\frac{1}{2}\right)^2}{1-\frac{1}{2}} = \frac{1}{8}\ \frac{1-\frac{1}{4}}{1} = \frac{1}{8}\cdot\frac{3}{4} = \frac{3}{32}$$

$$P(Z=4) = P(X=1)\cdot P(Y=3) + P(X=2)\cdot P(Y=2) + P(X=3)\cdot P(Y=1) =$$

$$= \frac{1}{8}\left[\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}\right] = \frac{1}{8}\cdot\frac{1}{2}\cdot\left[1+\frac{1}{2}+\frac{1}{2^2}\right] = \frac{1}{8}\cdot\left(1-\frac{1}{2^3}\right) = \frac{7}{64}$$

$$\frac{1}{16}\cdot\frac{4+2+1}{4} = \frac{1}{16}\cdot\frac{7}{4} = \frac{7}{64}$$

$$P(Z=8) = P(X=1)\cdot \underline{P(Y=7)} + P(X=2)\cdot P(Y=6) + \cdots + P(X=\cancel{8})\cdot P(Y=1)$$

$$= \frac{1}{8}\left(\frac{1}{2} + \frac{1}{2^2} + \cdots \frac{1}{2^7}\right) = \frac{1}{8}\cdot\frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{2^2} \cdots \frac{1}{2^6}\right) =$$

$$= \frac{1}{8}\cdot\frac{1}{\cancel{2}}\,\frac{1-\frac{1}{2^7}}{\cancel{\frac{1}{2}}} = \frac{1}{8}\left(1 - \frac{1}{2^7}\right)$$

$1 \leq K \leq 8$

$$P(Z=9) = P(X=1)\cdot P(Y=8) + P(X=2)\cdot P(Y=9) + \cdots P(X=7)\cdot P(Y=2) + P(X=8)P(Y=1)$$

$$= \frac{1}{8}\left(\frac{1}{2} + \frac{1}{2^2} + \cdots \frac{1}{2^8}\right) = \frac{1}{8}\cdot\frac{1}{2}\left(1 + \frac{1}{2} + \cdots \frac{1}{2^7}\right)$$

$$P(Z=9) = \frac{1}{8}\cdot\frac{1}{\cancel{2}}\,\frac{1-\frac{1}{2^8}}{\cancel{\frac{1}{2}}} = \frac{1}{8}\cdot\left(1 - \frac{1}{2^8}\right)$$

$$P(Z=10) = P(X=1)P(Y=9) + P(X=2)P(Y=8) + \cdots + P(X=8)P(Y=2)$$

$$= \frac{1}{8}\left(\frac{1}{2^2} + \frac{1}{2^3} + \cdots \frac{1}{2^9}\right) = \frac{1}{8}\,\frac{1}{2^2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^7}\right) = \frac{1}{8\cdot 2}\left(1 - \frac{1}{2^8}\right)$$

$$P(Z=11) = P(X=1)\cdot P(Y=10) + P(X=2)\cdot P(Y=9) + \cdots P(X=8)\cdot P(Y=3)$$

$$= \frac{1}{8}\left(\frac{1}{2^3} + \frac{1}{2^4} + \cdots \frac{1}{2^{10}}\right) = \frac{1}{8}\cdot\frac{1}{2^3}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^7}\right) = \frac{1}{8}\,\frac{1}{2^2}\left(1 - \frac{1}{2^8}\right)$$

$$\boxed{P(Z=k) = \frac{1}{8\cdot 2^{k-9}}\left(1 - \frac{1}{2^8}\right)}$$

$$P(Z=k) = \begin{cases} \frac{1}{8}\left(1 - \left(\frac{1}{2}\right)^{k-1}\right) & k \leq 8 \\ \frac{1}{8\, 2^{k-9}}\left(1 - \frac{1}{2^8}\right) & k > 8 \end{cases}$$

$$\boxed{P(Z=k) = \begin{cases} \frac{1}{8}\,\frac{2^{k-1}-1}{2^{k-1}} & k \leq 8 \\ \frac{1}{8\, 2^{k-1}}\left(2^8 - 1\right) & k > 8 \end{cases} = \begin{cases} \frac{2^{k-1}-1}{2^{k+2}} & k \leq 8 \\ \frac{255}{2^{k+2}} & k > 8 \end{cases}}$$

$k = 2, 3, \ldots \infty$

$$\boxed{P(W=k) = \begin{cases} 2^{k-11}\left(2^{8-k} - 1\right) & k \in \{0..7\} \\ 2^{k-11}\cdot 255 & k \in \{-\infty, \ldots -1\} \end{cases}}$$

$k \in \{0..7\} \rightarrow P_1(W=k)$

$k \in \{-\infty, \ldots -1\} \rightarrow P_2(W=k)$

$$H(W) + H(Z) = -\sum_{k=-\infty}^{-1} 2^{k-11}\cdot 255\, ld\left(2^{k-11}\cdot 255\right) - \sum_{k=0}^{7} 2^{k-11}\left(2^{8-k} - 1\right) ld\, 2^{k-11}\cdot 255$$

$$- \sum_{k=2}^{8} \frac{2^{k-1}-1}{2^{k+2}}\, ld\, \frac{2^{k-1}-1}{2^{k+2}} - \sum_{k>9}^{\infty} \frac{255}{2^{k+2}}\, ld\, \frac{255}{2^{k+2}}$$

$$-\sum_{k=2}^{8} \frac{2^{k-1}-1}{2^{k-1}} ld \frac{2^{k-1}-1}{2^{k-1}} = \left| \begin{matrix} n=k-2 \\ \underline{\underline{k=n+2}} \end{matrix} \right| = -\sum_{n=0}^{6} \frac{2^{n+1}-1}{2^{n+1}} ld \frac{2^{n+1}-1}{2^{n+1}}$$

$$-\sum_{k=0}^{7} 2^{k-11} \frac{2^8-2^k}{2^k} ld \frac{2^8-2^k}{2^{11}} = -\sum_{k=0}^{7} \frac{2^8-2^k}{2^{11}} ld \frac{2^8-2^k}{2^{11}}$$

$$-\sum_{k=-\infty}^{-1} 2^{k-11} \cdot 255\ ld\ 2^{k-11} \cdot 255 = \left| k=-n \right| = -\sum_{n=1}^{\infty} \frac{255}{2^{n+11}} ld \frac{255}{2^{n+11}}$$

$$-\sum_{n=1}^{7} \frac{255}{2^{n+11}} ld \frac{255}{2^{n+11}} \qquad \underline{\underline{n=0}} \qquad \boxed{\frac{255}{2^{11}} ld \frac{255}{2^{11}}}$$

$$P(Z=n+2) = \begin{cases} \frac{1}{8} \frac{2^{n+1}-1}{2^{n+1}} & n \le 6 \\ \frac{1}{8 \cdot 2^{n+1}} (2^8-1) & n > 6 \end{cases} = \begin{cases} \frac{2^{n+1}-1}{2^{n+4}} & n \le 6 \\ \frac{255}{2^{n+4}} & n > 6 \end{cases}$$

$$\boxed{n = 0, \ldots \infty}$$

$$H(Z) = -\sum_{n=0}^{6} \frac{2^{n+1}-1}{2^{n+4}} ld \frac{2^{n+1}-1}{2^{n+4}} - \sum_{n=7}^{\infty} \frac{255}{2^{n+4}} ld \frac{255}{2^{n+4}} \quad (*)$$

MOŽE DA GO PUŠTIŠ $n = 0 \div 7$ I OVDE $n = 8 \ldots \infty$

$$H_2(W) = -\sum_{k=-\infty}^{-1} 2^{k-11} \cdot 255 \cdot ld\ 2^{k-11} \cdot 255 = -\sum_{n=1}^{\infty} \frac{255}{2^{n+11}} ld \frac{255}{2^{n+11}} =$$

$$= \left| \begin{matrix} n+4 = m+11 \\ n = m+7 \\ m = n-7 \\ m=1 \quad n=8 \\ m=\infty \quad n=\infty \end{matrix} \right| = -\sum_{n=8}^{\infty} \frac{255}{2^{n+4}} ld \frac{255}{2^{n+4}}$$

$\cdot ld P_1$ $\quad ld P_1$

$$H_1(W) = -\sum_{k=0}^{7} 2^{k-11} (2^{8-k}-1) ld P_1 = -\sum_{k=0}^{7} \left(2^{k-11+8-k} - 2^{k-11}\right) ld P_1 = -\sum_{k=0}^{7} \left(2^{-3} - 2^{k-11}\right) ld P_1$$

$$= -\sum_{k=0}^{7} \left(\frac{1}{2^3} - 2^{k-11}\right) ld P_1 = -\sum_{k=0}^{7} \frac{1-2^{k-8}}{2^3} \cdot ld \left(\frac{1-2^{k-8}}{2^3}\right)$$

$$H(W) = -\sum_{k=0}^{7} \frac{1-2^{k-8}}{2^3} ld \left(\frac{1-2^{k-8}}{2^3}\right) - \sum_{k=8}^{\infty} \frac{255}{2^{k+4}} ld \frac{255}{2^{k+4}} \quad (\Delta)$$

$$\frac{1-2^{k-8}}{2^3} = \frac{1-\frac{2^k}{2^8}}{2^3} = \frac{2^8-2^k}{2^{11}}$$

PP115 (*) $\rightarrow$ $$H(Z) = -\sum_{k=0}^{7} \frac{2^{k+1}-1}{2^{k+4}} \operatorname{ld} \frac{2^{k+1}-1}{2^{k+4}} - \sum_{k=8}^{\infty} \frac{255}{2^{k+4}} \operatorname{ld} \frac{255}{2^{k+4}}$$

PP115 ① $\rightarrow$
$$H(W) = \underbrace{-\sum_{k=0}^{7} \frac{1-2^{k-8}}{2^3} \operatorname{ld} \frac{1-2^{k-8}}{2^3}}_{\$} - \sum_{k=8}^{\infty} \frac{255}{2^{k+4}} \operatorname{ld} \frac{255}{2^{k+4}}$$

$ \rightarrow$

$$k-8 = -(m+1) \qquad m+1 = 8-k \nearrow \boxed{m = 7-k} \qquad \begin{matrix} k=0 & m=7 \\ k=7 & m=0 \end{matrix} \qquad \frac{1-2^{k-8}}{2^3} = \frac{1-2^{-(m+1)}}{2^3} =$$

$$= \frac{2^{m+1}-1}{2^3 \cdot 2^{m+1}} = \frac{2^{m+1}-1}{2^{m+4}} \quad ✓$$

$$\$ = \sum_{k=0}^{7} \frac{1-2^{k-8}}{2^3} \operatorname{ld} \frac{1-2^{k-8}}{2^3} = \sum_{m=0}^{7} \frac{2^{m+1}-1}{2^{m+4}} \operatorname{ld} \frac{2^{m+1}-1}{2^{m+4}}$$

$$\boxed{H(W) = -\sum_{k=0}^{7} \frac{2^{k+1}-1}{2^{k+4}} \operatorname{ld} \frac{2^{k+1}-1}{2^{k+4}} - \sum_{k=8}^{\infty} \frac{255}{2^{k+4}} \operatorname{ld} \frac{255}{2^{k+4}}}$$

$\boxed{H(W) = H(Z)}$ LOGIČNO!!!

$$H(Z)+H(W) = -2\sum_{k=0}^{7} \frac{2^{k+1}-1}{2^{k+4}} \operatorname{ld} \frac{2^{k+1}-1}{2^{k+4}} - 2\underbrace{\sum_{k=8}^{\infty} \frac{255}{2^{k+4}} \operatorname{ld} \frac{255}{2^{k+4}}}_{S_1}$$

$$S_1 = \sum_{k=8}^{\infty} \frac{255}{2^{k+4}} \operatorname{ld} \frac{255}{2^{k+4}} = \left| \begin{matrix} n = k-7 \\ k = n+7 \end{matrix} \right| = \sum_{n=1}^{\infty} \frac{255}{2^{n+11}} \operatorname{ld} \frac{255}{2^{n+11}} =$$

$$= \left| \begin{matrix} N12.112 \\ (\text{DA}) \end{matrix} \right| = \frac{255}{2^{11}} (\operatorname{ld} 255 - 13)$$

$$\boxed{\begin{aligned} H(Z)+H(W) &= -2\sum_{k=0}^{7} \frac{2^{k+1}-1}{2^{k+4}} \operatorname{ld} \frac{2^{k+1}-1}{2^{k+4}} - \frac{255}{2^{10}} (\operatorname{ld} 255 - 13) \\ &= -\sum_{k=0}^{7} \frac{2^{k+1}-1}{2^{k+3}} \operatorname{ld} \frac{2^{k+1}-1}{2^{k+4}} - \frac{255}{2^{10}} (\operatorname{ld} 255 - 13) \end{aligned}}$$

$$H(Z)+H(W) = -\frac{1}{8}\,ld\,\frac{1}{16} - \frac{3}{16}\,ld\,\frac{3}{32} - \frac{7}{32}\,ld\,\frac{7}{64} - \frac{15}{64}\,ld\,\frac{15}{128}$$

$$-\frac{31}{128}\,ld\,\frac{31}{256} - \frac{63}{256}\,ld\,\frac{63}{512} - \frac{127}{512}\,ld\,\frac{127}{1024} - \frac{255}{1024}\,ld\,\frac{255}{2048}$$

$$-\frac{255}{2^{10}}\left(ld\,255 - 13\right) = +\frac{1}{8}\cdot 4 - \frac{3}{16}\,ld\,3 + \frac{3}{16}\cdot 5 - \frac{7}{32}\,ld\,7 +$$

$$+\frac{7}{32}\cdot 6 - \frac{15}{64}\,ld\,15 + \frac{15}{64}\cdot 7 - \frac{31}{128}\,ld\,31 + \frac{31}{128}\cdot 8 - \frac{63}{256}\,ld\,63$$

$$+\frac{63}{256}\,9 - \frac{127}{512}\,ld\,127 + \frac{127}{512}\,10 - \frac{255}{1024}\,ld\,255 + \frac{255}{1024}\cdot 11$$

$$-\frac{255}{2^{10}}\left(ld\,255 - 13\right) = 17 - \frac{3}{16}\,ld\,3 - \frac{7}{32}\,ld\,7 - \frac{15}{64}\,ld\,15 -$$

$$\frac{31}{128}\,ld\,31 - \frac{63}{256}\,ld\,63 - \frac{127}{512}\,ld\,127 - \frac{255}{1024}\,ld\,255 - \frac{255}{1024}\,ld\ldots$$

$$= 17 - \frac{96\,ld\,3 + 112\,ld\,7 + 120\,ld\,15 + 124\,ld\,31 + 126\,ld\,63 + 127\,ld\ldots}{512}$$

$$\overline{126\,ld\,7 + 126\,ld\,3} \quad + 255\,ld\ldots$$

$$= 17 - \frac{96\cdot ld\,3 + 112\,ld\,7 + 120\,ld\,3 + 120\,ld\,5 + 124\,ld\,31 + 126\cdot ld\,21 + 126\,ld\,3 + 127\,ld\,12\ldots}{512}$$

$$255\cdot ld\,15 + 255\,ld\,17$$

$$255\,ld\,5 + 255\,ld\,3$$

$$= 17 - \frac{723\,ld\,3 + 238\,ld\,7 + 375\,ld\,5 + 124\,ld\,31 + 127\,ld\,127 + 255\,ld\,17}{512}$$

$$\boxed{H(Z)+H(W) = 17 - \frac{723\,ld\,3 + 238\,ld\,7 + 375\,ld\,5 + 124\,ld\,31 + 127\,ld\,127 + 255\,ld\,17}{512}}$$

$$\boxed{H(X+Y) + H(X-Y) = 2H(X+Y) = \underline{6.787} = H(X+Y, X-Y)}$$

PROBLEM 2.41 RANDOM QUESTIONS. ONE WISHES TO IDENTIFY A RANDOM OBJECT $X \sim p(x)$. A QUESTION $Q \sim r(q)$ IS ASKED AT RANDOM ACCORDING TO $r(q)$. THIS RESULTS IN THE DETERMINISTIC ANSWER $A = A(x,q) \in \{a_1, a_2, \ldots\}$. SUPPOSE THAT X AND Q ARE INDEPENDENT. THEN $I(X;Q,A)$ IS THE UNCERTAINTY IN X REMOVED BY THE QUESTION-ANSWER (Q,A).

(a) SHOW THAT $I(X;Q,A) = H(A|Q)$. INTERPRET.

(b) NOW SUPPOSE THAT TWO I.I.D QUESTIONS $Q_1, Q_2 \sim r(q)$ ARE ASKED, ELICITING ANSWERS A_1 AND A_2. SHOW THAT TWO QUESTIONS ARE LESS VALUABLE THEN TWICE A SINGLE

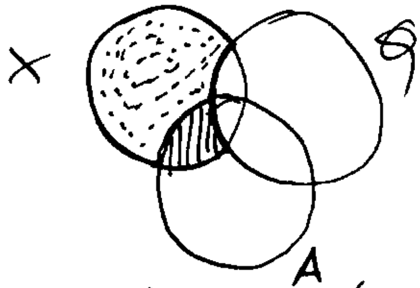
QUESTION IN SENSE THAT:

$$I(X;Q_1,A_1,Q_2,A_2) \le 2I(X;Q_1,A_1)$$

(a.) $X \sim \gamma(x)$ $\quad Q \sim r(q)$ $\quad A = A(x,q) \in \{a_1, a_2, \ldots\}$

X & Q ARE INDEPENDENT

$$I(X;Q,A) = \underbrace{I(X;Q)}_{\emptyset} + I(X;A|Q) = H(X) - \underbrace{H(X|Q)}_{H(X)} + I(X;A|Q)$$



$$\boxed{I(X;Q,A) = I(X;A|Q)}$$

$$I(X;A|Q) = \underbrace{H(X|Q)}_{} - \underbrace{H(X|Q,A)}_{}$$

$I(X;Y) = H(X) - H(X|Y)$

$H(X|Q) = H(X)$ (X & Q INDEPENDENT)

$I(X;A|Q) = H(X) - H(X|Q,A)$

$I(X;A|Q) = H(A|Q) - H(A|Q,X)$

AKO GO ZNAEŠ Q I X TOGAŠ GO ZNAEŠ I A:

$\Rightarrow H(A|Q,X) = \emptyset$

$$\Rightarrow \boxed{\boxed{I(X;A|Q) = H(A|Q)} \boxed{= I(X;Q,A)}}$$

DOKAZANO !!!

(b.) $Q_1, Q_2 \sim r(Q)$ $\quad Q_1$ & Q_2 ARE I.I.D.

$$\boxed{I(X;Q_1,A_1,Q_2,A_2) \le 2I(X;Q_1,A_1)}$$

$$I(X;\underbrace{Q_1,A_1}_{Y_1},\underbrace{Q_2,A_2}_{Y_2}) = \underbrace{I(X;Q_1,A_1)}_{=H(A_1|Q_1)} + I(X;Q_2,A_2|Q_1,A_1) =$$

$$= I(X;Q_1,A_1) + H(X|Q_1,A_1) - H(X|Q_1,A_1,Q_2,A_2)$$

~~$I(X;Q_2,A_2|Q_1,A_1) = H(Q_2,A_2|Q_1,A_1) - H(Q_2,A_2|X,A_1,Q_1)$~~

$$I(X;Q_2,A_2|Q_1,A_1) = H(X|Q_1,A_1) - \underbrace{H(X|Q_1,A_1,Q_2,A_2)}_{\ge\emptyset}$$

$$I(X;Q_2,A_2|Q_1,A_1) = H(Q_2,A_2|Q_1,A_1) - H(Q_2,A_2|Q_1,A_1,X)$$

$$= H(A_1|Q_1) + H(X|Q_1,A_1) - H(X|Q_1,A_1,Q_2,A_2) \le$$

$$\le H(A_1|Q_1) + H(X|Q_1,A_1) = H(Q_1,A_1,X) - H(Q_1)$$

$$I(X;Q_1A_1Q_2A_2) = \underbrace{I(X;Q_1)}_{0} + \underbrace{I(X;A_1|Q_1)}_{=H(A_1|Q)} + \underset{(*)}{I(X;Q_2|A_1Q_1)} + I(X;A_2|A_1Q_1Q_2)$$

$$= I(X;A_1|Q_1) + I(X;Q_2|A_1) + I(X;A_2|A_1Q_1Q_2)$$

$$I(X;Q_2|A_1) = H(X|A_1) - H(X|A_1Q_2) = H(Q_2|A_1) - H(Q_2|A_1X)$$

$$I(X;Q_1A_1A_2Q_2) = \underbrace{I(X;Q_1)}_{0} + I(X;A_1|Q_1) + I(X;A_2|Q_1A_1) + I(X;Q_2|Q_1A_1A_2)$$

$$= I(X;A_1|Q_1) + I(X;A_2|Q_1A_1) + I(X;Q_2|Q_1A_1A_2)$$

$$I(X;Q_2A_2|Q_1A_1) = \underbrace{I(X;Q_2|Q_1A_1)}_{I(X;Q_2|A_1)} + I(X;A_2|Q_1A_1Q_2)$$

$$I(X;Q_2|A_1) = H(X|A_1) - H(X|A_1Q_2) = H(Q_2|A_1) - H(Q_2|A_1X)$$

$$(*)\quad I(X;Q_2|A_1Q_1) = H(X|A_1Q_1) - H(X|A_1Q_1Q_2) =$$
$$= H(Q_2|A_1Q_1) - H(Q_2|A_1Q_1X) = H(Q_2) - H(Q_2) = 0$$

$$I(X;Q_1A_1Q_2A_2) = I(X;A_1|Q_1) + I(X;A_2|A_1Q_1Q_2)$$

$$I(X;A_2|A_1Q_1Q_2) = \underbrace{H(A_2|A_1Q_1Q_2)}_{A_2 \text{ NE ZAVISI OD } Q_1A_1} - H(A_2|A_1Q_1Q_2X) =$$
$$= H(A_2|Q_2) - H(A_2|Q_2X) = I(X;A_2|Q_2)$$

$$I(X;Q_1A_1Q_2A_2) = I(X;A_1|Q_1) + I(X;A_2|Q_2) =$$
$$= I(X;A_1Q_1) + I(X;A_2Q_2) = H(A_1|Q_1) + H(A_2|Q_2)$$

SO OGLED NA TOA ŠTO $I(X;AQ)$ E NEIZVESNOSTA VO X NAMALENA ZA PRAŠANJE, ODGOVOROT – AQ, LOGIČNO E POSLE VTOROTO PRAŠANJE, ODGOVOR POVEĆE DA E NAMALENA NEIZVESNOSTA T.E.: $I(X;A_1Q_1) \geq I(X;A_2Q_2)$

$$I(X;Q_1A_1Q_2A_2) = I(X;A_1Q_1) + I(X;A_2Q_2) \leq I(X;A_1Q_1) + I(X;A_1Q_1) = 2I(X;A_1Q_1)$$

ZNAČI $I(X;Q_1A_1Q_2A_2) \leq 2I(X;A_1Q_1)$ DOKAŽANO !!

2.42 INEQUALITIES. WHICH OF THE FOLLOWING INEQUALITIES ARE GENERALLY $\geq, =, \leq$? LABEL EACH WITH $\geq, =,$ OR $\leq$.

(a) $H(5X)$ vs. $H(X)$?

$$H(x) = \sum_{x \in X} p(x)\, ld\, p(x) \qquad H(Y) = \sum_{x \in X} p(x)\, ld\, p(x) = H(X)$$

$$Y = 5X \qquad \cancel{p(y) = \frac{p(x)}{\frac{dy}{dx}}} \quad \cancel{\boxed{x = \frac{y}{5}}} \quad \cancel{\boxed{p(y) = \frac{p\left(\frac{y}{5}\right)}{5}}}$$

$$X \in [x_1 \ x_2 \ \dots \ x_n]$$

$$p(x) = \{p(x_1), p(x_2), \dots p(x_n)\}$$

$$Y \in [5x_1, 5x_2, \dots 5x_n]$$

$$p(y) = \{p(x_1), p(x_2) \dots p(x_n)\}$$

$$\boxed{H(5X) = H(X)}$$

$$\boxed{H(g(x)|X) = 0}$$

(b) $I(g(x); Y)$ vs. $I(X, Y)$

$$I(Z;Y) = H(Z) - H(Z|Y) = H(g(x)) - H(g(x)|Y)$$

$$H(X, g(x)) = H(x) + \underbrace{H(g(x)|x)}_{0} = H(x) = H(g(x)) + \underbrace{H(x|g(x))}_{\geq 0} \geq H(g(x))$$

$$\Rightarrow \boxed{H(x) \geq H(g(x))}$$

$$I(g(x); Y) = H(g(x)) - H(g(x)|Y) \leq H(x) - H(g(x)|Y)$$

$$H(x, g(x)|Y) = H(x|Y) + \underbrace{H(g(x)|Y, X)}_{=0} =$$

ако го знаеш x го знаеш $g(x)$

$$= H(x, g(x)|Y) = H(g(x)|Y) + \underbrace{H(x|Y, g(x))}_{\geq 0} \geq H(g(x)|Y)$$

$$H(x|Y) \geq H(g(x)|Y)$$

$$I(g(x); Y) = H(Y) - H(Y|g(x))$$

$$H(x, Y|g(x)) = H(x|g(x)) + \overbrace{H(Y|x, g(x))}^{H(Y|X)} = H(Y|g(x)) + H(x|Y, g(x))$$

$$H(g(x), Y|x) = \underbrace{H(g(x)|x)}_{0} + H(Y|x, g(x)) =$$

$$= H(Y|x) + \underbrace{H(g(x)|Y, x)}_{0} \qquad \boxed{H(Y|x) = H(Y|x, g(x))}$$

$$H(g(x)) - H(g(x)|Y) \geq H(g(x)) - H(x|Y) \leq H(x) - H(x|Y)$$