Zx: $p = \left[\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{32}\right]$

$F_6 = [0, 0.1, 0.11, 0.111, 0.1111, 0.11111]$

$i = [1, 2, 3, 4, 5, 6]$ $\boxed{15:00}$

$\boxed{j=5}$

$F_j, \; j > i, \; \boxed{i=4}$ $F_j = 0.1111$ $\boxed{070248462}$

$F_i = 0.1110$

$l_j \geq l_i$, CODEWORD F_j DIFERS FROM CODEWORD FOR F_i AT LEAST ONECE IN THE FIRST l_i PLACES. NO CODEWORD IS PREFIX OF ANY OTHER CODEWORD.

$\boxed{\text{PROBLEM 29}}$ OPTIMAL CODES FOR DYADIC DISTRIBUTIONS

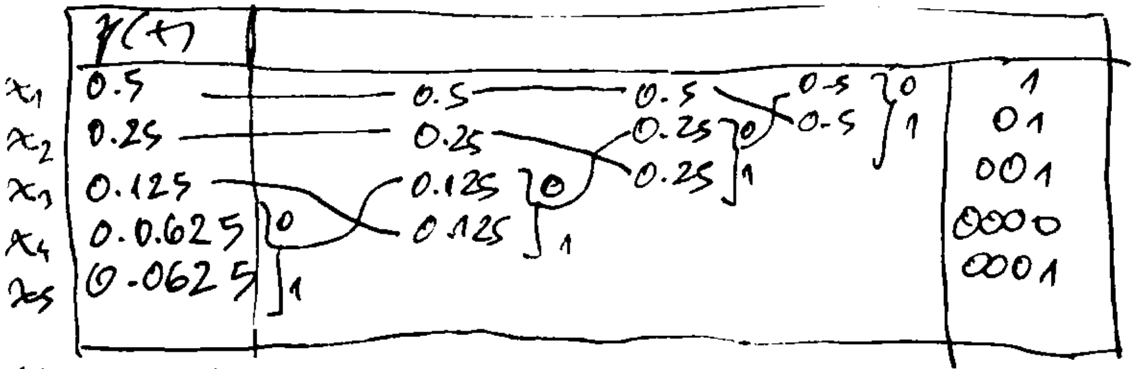
FOR HUFFMAN CODE TREE, DEFINE THE PROBABILITY OF A NODE AS SUM OF PROBABILITIES OF ALL THE LEAVES UNDER THAT NODE. LET THE RANDOM VARIABLE $\mathcal{X}$ BE DRAWN FROM A DYADIC DISTRIBUTION, I.E. $p(x) = 2^{-i}$, FOR SOME i, FOR ALL $x \in \mathcal{X}$. NOW CONSIDER BINARY HUFFMAN CODE FOR THIS DISTRIBUTION

(a) ARGUE THAT FOR ANY NODE IN THE TREE, THE PROBABILITY OF THE LEFT CHILD IS EQUAL TO THE PROBABILITY OF THE RIGHT CHILD.

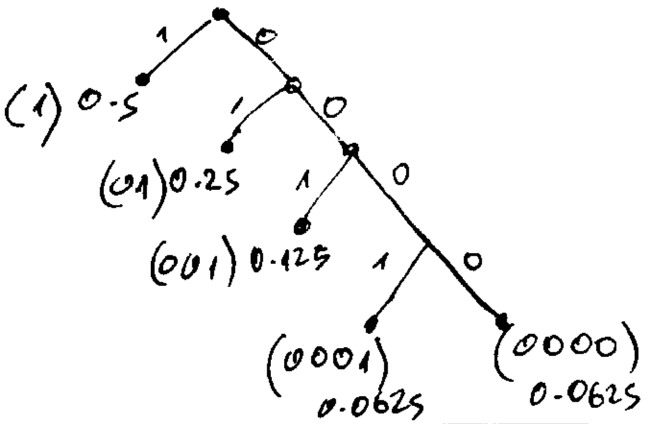
(b) LET $X_1, X_2, \ldots, X_n$ BE DRAWN $\sim p(x)$. USING THE HUFFMAN CODE FOR $p(x)$ WE MAP $X_1, X_2, \ldots X_n$, TO SEQUENCE OF BITS $Y_1, Y_2, \ldots Y_k(X_1, X_2, \ldots X_n)$. THE LENGHT OF THIS SEQUENCE WILL DEPEND ON THE OUTCOME $X_1, X_2, \ldots X_n$. USE PART (a) TO ARGUE THAT SEQUENCE $Y_1, Y_2, \ldots$ FORMS SEQUENCE OF FAIR COIN FLIPS, I.E., THAT $Pr\{Y_i = 0\} = Pr\{Y_i = 1\} = 1/2$, INDEPENDENT OF $Y_1, Y_2, \ldots, Y_{i-1}$.

THE ENTROPY RATE OF THE CODED SEQUENCE IS 1 bit PER SYMBOL.

(c) GIVE A HEURISTIC ARGUMENT WHY THE ENCODED SEQUENCE OF BITS FOR ANY CODE THAT ACHIEVES THE ENTROPY BOUND CANNOT BE COMPRESSIBLE AND THEREFORE SHOULD HAVE AN ENTROPY RATE OF 1 BIT PER SYMBOL.

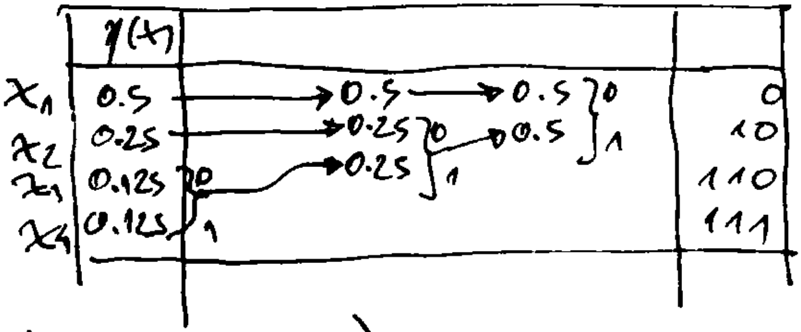


(a)



- OČIGLEDNO E DEKA VEROJATNOSTA NA LEFT CHILD E GOLAKVA NA VEROJATNOSTA NA RIGHT CHILD.

(B)



$x_3x_2x_1x_4 = 110|10|0|111 =$

$= z_1z_2z_3z_4z_5z_6z_7z_8z_9$

$H(x) = \frac{1}{2}ld2 + \frac{1}{4}ld4 + 2\cdot\frac{1}{8}ld8$

$= \frac{1}{2} + \frac{1}{2} + \frac{1}{4}3 = 1.75$

$H(x_1, x_2, \ldots x_4)$

e.G. $H(X^{4=4}) = H(X^4) = P(x_1x_1x_1x_1)\cdot ld\frac{1}{P(x_1x_1\cdots x_1)} +$

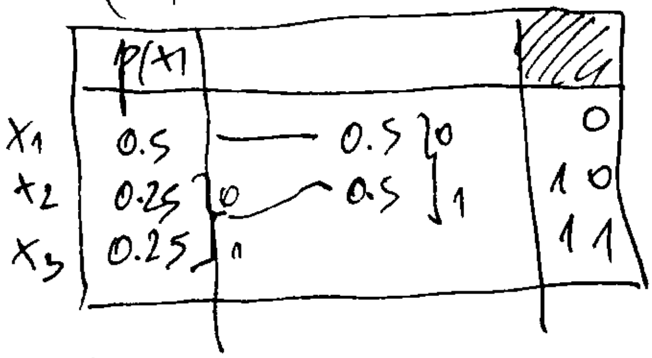
$+ P(x_1x_1x_1x_2)\cdot ld\frac{1}{P(x_1x_1x_1x_2)} + \ldots P(x_4x_4x_4x_4)\, ld\frac{1}{P(x_4x_4x_4x_4)} =$

1/2

$= \left|\left(\frac{1}{2}\right)^4\cdot ld\, 2^4 = \cancel{4}\cdot\frac{1}{\cancel{16}4} = \frac{1}{4}\right| = \frac{1}{4}ld4 + \frac{1}{2^3}\cdot\frac{1}{4}\cdot ld32 + \ldots$

$+ \left(\frac{1}{8}\right)^4 ld8^4$

$4^4 = 256$ KOMBINACII



$3^3 = 27$ KOMBINACII

$x_1x_1x_1$	1/8	$x_2x_1x_1$	1/16	$x_3x_1x_1$	1/16
$x_1x_1x_2$	1/16.	$x_2x_1x_2$	1/32.	$x_3x_1x_2$	1/32
$x_1x_1x_3$	1/16	$x_2x_1x_3$	1/32.	$x_3x_1x_3$	1/32
$x_1x_2x_1$	1/16	$x_2x_2x_1$	1/32	$x_3x_2x_1$	1/32
$x_1x_2x_2$	1/32.	$x_2x_2x_2$	1/64	$x_3x_2x_2$	1/64
$x_1x_2x_3$	1/32.	$x_2x_2x_3$	1/64	$x_3x_2x_3$	1/64
$x_1x_3x_1$	1/16.	$x_2x_3x_1$	1/32.	$x_2x_3x_1$	1/32
$x_1x_3x_2$	1/32.	$x_2x_3x_2$	1/64	$x_2x_3x_2$	1/4
$x_1x_3x_3$.	1/32.	$x_2x_3x_3$	1/64	$x_2x_3x_3$	1/4

$H(X) = 6\cdot\frac{1}{16}ld16 + 12\frac{1}{32}ld32 +$

$+ 8\frac{1}{64}ld64 + \frac{1}{8}ld8 =$

$= 6\cdot\frac{1}{16}\cdot 4 + 12\cdot\frac{1}{32}\cdot 5 + \frac{1}{8}\cdot 6 + \frac{3}{8} = \frac{6}{4} + \frac{3\cancel{6}}{\cancel{16}8}\cdot 5 + \frac{3}{4} + \frac{3}{8} = \frac{3}{2} + \frac{15}{8} + \frac{3}{4} + \frac{3}{8}$

$$H(X) = \frac{12+15+6+3}{8} = \frac{27+9}{8} = \frac{36}{8} = \frac{18}{4} = \frac{9}{2}$$

- ENTROPY RATE OF THE CODED SEQUENCE Y_c:

$$H(\mathcal{X}) = \lim_{n\to\infty} \frac{1}{n} H(x_1, x_2, \dots x_n) = \lim_{n\to\infty} H(x_n|x_{n-1} \dots x_1)$$

- DA DOKAZAM DEKA: $P_r\{Y_i=0\} = P_r\{Y_i=1\} = 1/2$ I NE ZAVISI OD $Y_1, Y_2, \dots Y_{i-1}$

i = BEGINING OF x_2 OR x_3

$P(Y|X)$

$p(x)$	$x \backslash Y$	$P_r(Y_i=0\|x=x_j)$	$P_r(Y_i=1\|x=x_j)$	$j=1,2,3$
1/2	$x_1=0$	1	0	
1/4	$x_2=10$	1/2 (*)	1/2 (**)	
1/4	$x_3=11$	$\frac{1}{2}$	1/2	

PRETHODNIOT SYMBOL.

$$P(X,Y) = P(X)P(Y|X)$$

$P(X,Y)$

$X \backslash Y$	$P_r(Y_i=0)$	$P_r(Y_i=1)$	$P(X)$
x_1	1/2	0	1/2
x_2	1/8	1/8	1/4
x_3	1/8	1/8	1/4
$P(Y)$	3/4	1/4	

(*) (**) $P_r(Y_i=0|X=X_2) = P_r(x_j=x_1|x_{j-1}=x_2) = P_r(x_j=x_1) = \frac{1}{2}$

$$P_r(Y_i=1|X=x_2) = P_r(x_j=x_2|x_{j-1}=x_2) + P_r(x_i=x_3|x_{j-1}=x_2) =$$
$$= P_r(x_j=x_2) + P_r(x_j=x_3) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

ISTOTO VAZI I ZA:

$P_r(Y_i=0|X=X_3) = 1/2$ i.e. $P_r(Y_i=1|x=x_3) = 1/2$

-OVA NE E DOBRO ZOSTO Y_i NE SEKOGAS E NA POZICIJA "1" OD x_2 I X_3. MOZE DA ADE I NA POZICIJA "2". ZATOA:

$p(x)$	$x_{j-1} \backslash x_j$	$x_j=0$	$x_j=10$	$x_j=11$
1/2	$x_{j-1}=0$	1/2	1/4	1/4
1/4	$x_{j-1}=10$	1/2	1/4	1/4
1/4	$x_{j-1}=11$	1/2	1/4	1/4

$$\boxed{P(x_j|x_{j-1}) = P(x_j)}$$

x_{j-1}	x_j	$Y_{j1}=0$	$Y_{j2}=0$	$Y_{j1}=1$	$Y_{j2}=1$
$x_1=0$	$x_1=0$ $x_2=10$ $x_3=11$	$\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$ 0 0	0 $\frac{1}{2}\cdot\frac{1}{8}=\frac{1}{16}$ 0	0 $\frac{1}{8}+\frac{1}{2}\frac{1}{4}=\frac{1}{4}$	$\frac{1}{2}\cdot\frac{1}{4}=\frac{1}{8}$
$x_2=10$	$x_1=0$ $x_2=10$ $x_3=11$	$\frac{1}{4}\cdot\frac{1}{2}=\frac{1}{8}$ 0 0	0 $1/4\cdot 1/4=\frac{1}{16}$	$\frac{1}{4}\cdot\frac{1}{4}\cdot 2=\frac{1}{8}$	$\frac{1}{4}\cdot\frac{1}{4}=\frac{1}{16}$
$x_3=11$	$x_1=0$ $x_2=10$ $x_3=11$	$\frac{1}{4}\cdot\frac{1}{2}=\frac{1}{8}$	$\frac{1}{4}\cdot\frac{1}{4}=\frac{1}{16}$	$\frac{1}{4}\cdot\frac{1}{4}\cdot 2=$ $1/8$	$\frac{1}{4}\cdot\frac{1}{4}=\frac{1}{16}$

$$P(Y_i=0|Y_{i-1}=0)=$$
$$P(x_{j-1}=0)\cdot P(x_j=0|x_{j-1}=0)$$
$$+P(x_{j-1}=10)\cdot P(x_j=0|x_{j-1}=10)$$
$$=\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{4}\cdot\frac{1}{2}=$$
$$=\frac{1}{4}+\frac{1}{8}=3/8$$

$P(x_{j-1}=0)$

$P(Y_i|Y_{i-1})$

$Y_{i-1} \backslash Y_i$	0	1
0	$\frac{3}{8}$	$\frac{3}{8}$
1	$3/8$	$\frac{3}{8}$

$$P(Y_i=1|Y_{j-1}=0)=\underset{P(x_{j-1}=0)}{P}(x_j=0|x_{j-1}=0)+$$
$$P(x_{j-1}=10)P(x_j=10|x_{j-1}=10)+P(x_j=11|x_{j-1}=0)+$$
$$P(x_{j-1}=10)P(x_j=11|x_{j-1}=10)=\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{16}$$

[illegible]

$$=\frac{1}{8}+\frac{1}{16}+\frac{1}{8}+\frac{1}{16}=2\left(\frac{1}{8}+\frac{1}{16}\right)=2\,\frac{2+1}{16}=\frac{3}{8}$$

$$P(Y_i=0|Y_{i-1}=1)=P(x_{j-1}=11)\cdot P(x_j=0|x_{j-1}=11)+P(x_2)=\frac{1}{4}\cdot\frac{1}{2}+\frac{1}{4}$$

$$\underline{P(Y_i=0|Y_{i-1}=1)}=\frac{1}{8}+\frac{1}{4}=\frac{3}{8}$$

$$P(Y_i=1|Y_{i-1}=1)=P(x_3)+P(x_{j-1}=11)\cdot P(x_j=10|x_{j-1}=11)+$$
$$P(x_j=11|x_{j-1}=11)=\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{4}=\frac{1}{4}+\frac{1}{16}+\frac{1}{16}=\frac{3}{8}$$

[illegible]

$P(Y_{i-1},Y_i)$

$Y_{i-1} \backslash Y_i$	0	1	$P(Y_{i-1})$
0			
1			
$P(Y_i)$			

$$\left.\begin{aligned}P(Y_i=0)&=P(Y_{i-1}=0)\cdot\frac{3}{8}+P(Y_{i-1}=1)\cdot\frac{3}{8}=\\ P(Y_i=1)&=P(Y_{i-1}=0)\cdot\frac{3}{8}+P(Y_{i-1}=1)\cdot\frac{3}{8}\end{aligned}\right\}\ (1)$$

$H(X^3)=\frac{9}{2}$　$H(X^4)=7$　$H(X^5)=9.3750$

$H(X^6)=?$　$\frac{n^2}{n-1}=\frac{16}{3}=?$

NUMBER SEQUENCE

[4.5; 7; 9.3750...]

- MI TEKNA DEKA KOGA X_j IMA DVE POZICII VEROJATNOSTA DA BIDE Y_i DA BIDE NA PRVATA E $\frac{1}{2}$ I VEROJATNOSTA NA VTORATA E 1/2

(w) — POINTER OF Y_i POSITION

$$P(Y_i=1|Y_{i-1}=0) = P(X_{j-1}=0)\,P(X_j=\underset{P(w=1)}{10}|X_{j-1}=0)\cdot P(w=1)$$

$$+ P(X_{j-1}=10)\,P(X_j=10|X_{j-1}=10)\cdot P(w=1) + V P(X_{j-1}=0)\cdot P(X_j=11|X_{j-1}=0)$$

$$+ P(X_{j-1}=10)\,P(X_j=11|X_{j-1}=10)\cdot P(w=1)$$

$P(w=1)$ VEROJATNOST DEKA POINTEROT VO DADENIOT MOMENT JE NA POZICIJA 1

1/8   1/16   1/16   1/32

$$P(Y_i=1|Y_{i-1}=0) = \frac{1}{2}\cdot\frac{1}{4}\cdot + \frac{1}{4}\cdot\frac{1}{4}\cdot + \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{2} = \frac{3}{16}$$

① $P(Y_i=0)+P(Y_i=1)=1$ $P(Y_{i-1}=0)+P(Y_{i-1}=1)=1$

$$1-P(Y_i=1) = \left(1-P(Y_{i-1}=1)\right)\cdot\frac{3}{8} + P(Y_{i-1}=1)\cdot\frac{3}{8}$$

$$1-P(Y_i=1) = \frac{3}{8} - \cancel{\frac{3}{8}P(Y_{i-1}=1)} + \cancel{P(Y_{i-1}=1)\cdot\frac{3}{8}}$$

$$P(Y_i=1) = 1-\frac{3}{8} = 5/8$$

$$\frac{1}{8}+\frac{1}{16}+\frac{1}{16}+\frac{1}{32} = \frac{4+4+1}{32}$$

- NIŠTO OVA GORE NE ČINI ZATOA ĆE IZVEDUVAM PAK:

$$P(Y_i=0|Y_{i-1}=0) = P(X_j=0|X_{j-1}=0) + P(X_j=0|X_{j-1}=10) =$$

$$= 1/2 + \frac{1}{2} = 1\ ?$$

- IZGLEDA OVA PRETHODNO ŠTO REŠAVAV SE ZDRUŽENI VEROJATNOSTI. DOVOLNO DOKAŽUV DEKA:

$P(Y_{i-1}, Y_i)$

Y_{i-1} \ Y_i	0	1	$P(Y_{i-1})$
0	1/4	1/4	1/2
1	1/4	1/4	1/2
$P(Y_i)$	1/2	1/2	

$H(X^6) = 11.625$ $[4.5; 7; 9.3750; 11.625 \ldots]$ NE MOŽE DA SE NAJDE NA INTERNET

- EDITION 2 SOLUTION

(a) FOR DYADIC DISTRIBUTION HUFFMAN CODE ACHIEVES ENTROPY BOUND. THE CODE TREE CONSTRUCTED WITH HUFFMAN ALGORITHM IS COMPLETE TREE WITH LEAVES AT DEPTH l_i WITH PROBABILITY $p_i = 2^{-l_i}$.

FOR SUCH A COMPLETE BINARY TREE WE CAN PROVIDE THE FOLLOWING PROPERTIES:

THE PROBABILITY OF ANY INTERNAL NODE AT DEPTH „K" IS 2^{-K}. PROOF BY INDUCTION:

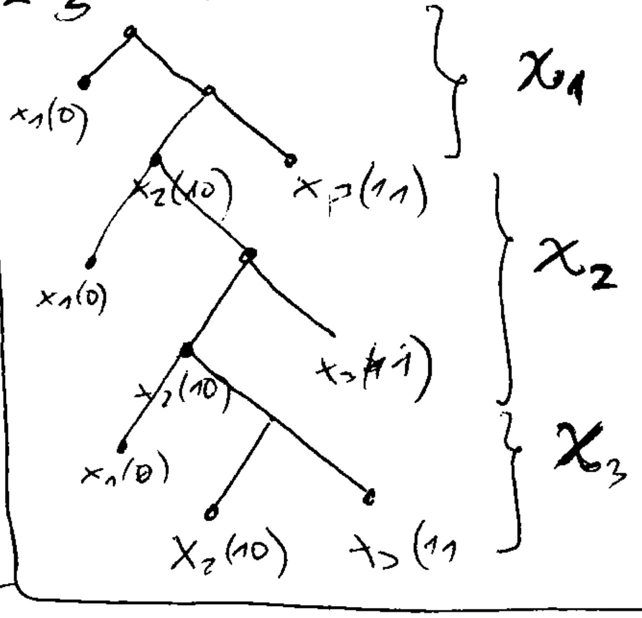
- CLEARLY IT IS TRUE FOR A TREE WITH 2 LEAVES
- ASSUME IT IS TRUE FOR ALL TREES WITH „n" LEAVES
- FOR ANY TREE WITH n+1 LEAVES, AT LEAS TWO OF THE LEAVES HAVE TO BE SIBLINGS ON THE TREE (OTHERWISE THE TREE WOULD NOT BE COMPLETE). LET THE LEVEL OF THESE SIBLINGS BE „j". THE PROBABILITY OF THEIR PARENT IS:

$$2^{j} + 2^{j} = 2^{j-1} \quad (\text{PARENT IS AT LEVEL } j-1)$$

WE CAN NOW REPLACE THE TWO SIBLINGS WITH THEIR PARENT. BUT NOW WE HAVE TREE WITH „n" LEAVES WHICH SATISFIES THE REQUIRED PROPERTY. THUS THE PROPERTY IS TRUE FOR ALL COMPLETE BINARY TREES.

	p(x)		C(x)
x_1	0.5	0.5	0
x_2	0.25	0 0.5	10
x_3	0.25	1	11

$X_1X_2X_3 = x_1x_3x_2$



THUS THE PROBABILITY OF FIRST BIT BEING 1 IS 1/2, AND AT ANY INTERNAL NODE, PROBABILITY OF NEXT BIT BEING 1 IS EQUAL TO THE PROBABILITY OF NEXT BIT BEING 0. HENCE TE BITS PRODUCED BY THE CODE ARE I.I.D BERNOULLI (1/2), AND THE ENTROPY RATE OF THE CODED SEQUENCE IS:

$$H(\mathcal{X}) = \lim_{n\to\infty} \frac{1}{n} H(X_1, X_2, \ldots X_n) = \lim_{n\to\infty} \frac{1}{n}\, nH(p) = 1$$

$\boxed{30}$ RELATIVE ENTROPY IS COST OF MISCODING: LET THE RANDOM VARIABLE X HAVE FIVE POSSIBLE OUTCOMES $\{1,2,3,4,5\}$. CONSIDER TWO DISTRIBUTIONS $p(x)$ AND $q(x)$ ON THIS RANDOM VARIABLE:

SYMBOL	$p(x)$	$q(x)$	$C_1(x)$	$C_2(x)$
1	1/2	1/2	0	0
2	1/4	1/8	10	100
3	1/8	1/8	110	101
4	1/16	1/8	1110	110
5	1/16	1/8	1111	111

EN100 Ver 3.02

(a) CALCULATE $H(p)$, $H(q)$, $D(p\|q)$ AND $D(q\|p)$

b.) THE LAST TWO COLUMNS ABOVE REPRESENT CODES FOR THE RANDOM VARIABLE. VERIFY THAT THE AVERAGE LENGHT OF C_1 UNDER p IS EQUAL TO ENTROPY $H(p)$. THUS C_1 IS OPTIMAL FOR p. VERIFY THAT C_2 IS OPTIMAL FOR q

c.) NOW ASSUME THAT WE USE CODE C_2 WHEN THE DISTRIBUTION IS p. WHAT IS THE AVERAGE LENGHT OF THE CODEWORDS. BY HOW MUCH DOES IT EXCEDE THE ENTROPY p?

d) WHAT IS THE LOSS IF WE USE CODE C_1 WHEN THE DISTRIBUTION IS q?

$$H_p(X) = \sum_{i=1}^{5} p(x_i) \operatorname{ld} \frac{1}{p(x_i)} = \frac{15}{8} = 1.875$$

$$H_q(X) = \sum_{i=1}^{5} q(x_i) \operatorname{ld} 1/q(x_i) = 2$$

$$D(p\|q) = \sum_{i=1}^{5} p(x_i) \operatorname{ld} \frac{p(x_i)}{q(x_i)} = \frac{1}{8} \qquad D(q\|p) = \sum q(x_i) \operatorname{ld} \frac{q(x_i)}{p(x_i)}$$

b) $$E[l_p(x)] = \sum_{x\in X} l_p(x)\, p(x) = \frac{15}{8} = 1.875 = H_p(x)$$

$$E[l_q(x)] = \sum_{x\in X} l_q(x)\, q(x) = 2 = H_q(x)$$

c) $$E_p[l_q(x)] = \sum_{i=1}^{5} l_q(i)\cdot p_i = \sum_{i=1}^{5} \operatorname{ld} \frac{1}{q(i)}\, p_i = \sum_{i=1}^{5} p_i \operatorname{ld} \frac{p_i}{q_i}\frac{1}{p_i}$$

$$= \sum_{i=1}^{5} p_i \operatorname{ld} \frac{1}{p_i} + \sum_{i=1}^{5} p_i \operatorname{ld} \frac{p_i}{q_i} = H_p(x) + D(p\|q) = \frac{15}{8} + \frac{1}{8} = 2$$

d.) $$E_q[l_p(x)] = \sum_{i=1}^{5} l_p(i)\, q_i = H_q(x) + D(q\|p) = 2 + \boxed{\frac{1}{8}} = \frac{17}{8}$$

THE LOSS IF WE USE CODE C_1 WHEN THE DISTRIBUTION IS q IS EQUAL TO: $D(q\|p) = 1/8$

PROBLEM 5.31 NONSINGULAR CODES. THE DISCUSSION IN THE TE FOCUSED ON INSTANTENOUS CODES, WITH EXTENSION TO UNIQUELY DECODABLE CODES. BOTH THESE ARE REQUIRED I CASES WHEN CODE IS TO BE USED REPEATEDLY TO ENCODE A SEQUENCE OF OUTCOMES OF RANDOM VARIABLE. BUT WHEN WE NEED TO ENCODE ONLY ONE OUTCOME AND WE KNOW WHEN WE REACHED

THE END OF A CODEWORD, WE DON'T NEED UNIQUE DECODABILITY - THE FACT THAT THE CODE IS NONSINGULAR WOULD SUFFICE. FOR EXAMPLE, IF THE RANDOM VARIABLE X TAKES ON THREE VALUES: a, b AND c, WE COULD ENCODE THEM BY 0, 1, AND 00. SUCH CODE IS NONSINGULAR BUT NOT UNIQUELY DECODABLE.

IN THE FOLLOWING ASSUME THAT WE HAVE RANDOM VARIABLE X WHICH TAKES ON $m =$ VALUES WITH PROBABILITIES $p_1, p_2, \cdots p_m$ AND THE PROBABILITIES ARE

$p_1 \geq p_2 \geq \cdots p_m$

(a) BY VIEWING THE NONSINGULAR BINARY CODE AS TERNARY CODE WITH THREE SYMBOLS 0, 1, AND STOP SHOW THAT EXPECTED LENGTH OF THE NONSINGULAR CODE $L_{1:1}$ FOR A RANDOM VARIABLE X SATISFIES THE FOLLOWING INEQUALITY:

$$L_{1:1} \geq \frac{H_2(X)}{ld 3} - 1,$$

WHERE $H_2(X)$ IS THE ENTROPY OF X IN BITS. THUS, THE AVERAGE LENGTH OF A NONSINGULAR CODE IS AT LEAST A CONSTANT FRACTION OF THE AVERAGE LENGTH OF AN INSTANTENOUS CODE.

(b) LET LINST BE THE EXPECTED LENGTH OF THE BEST INSTANTENOUS CODE AND $L^*_{1:1}$ BE THE EXPECTED LENGTH OF THE BEST NONSINGULAR CODE FOR X. ARGUE THAT:

$$L^*_{1:1} \leq L^*_{INST} \leq H(X) + 1$$

(c) GIVE A SIMPLE EXAMPLE WHERE THE AVERAGE LENGTH OF THE NON-SINGULAR CODE IS LESS THEN ENTROPY.

(d) THE SET OF CODEWORDS FOR A NONSINGULAR CODE IS: $\{0, 1, 00, 01, 10, 11, 000, \ldots\}$. SINCE $L_{1:1} = \sum_{i=1} p_i l_i$, SHOW THAT THIS IS MINIMIZED IF WE ALLOT THE SHORTEST CODEWORDS TO THE MOST PROBABLE SYMBOLS. THUS, $l_1 = l_2 = 1$, $l_3 = l_4 = l_5 = l_6 = 2$, ETC. SHOW THAT IN GENERAL $l_i = \left\lceil ld\left(\frac{i}{2}+1\right)\right\rceil$, AND THEREFORE $L^*_{1:1} =$

$$= \sum_{i=1}^{m} p_i \left\lceil ld\left(\frac{i}{2}+1\right)\right\rceil$$

(e) PART (d) SHOWS THAT IT IS EASY TO FIND OPTIMAL NONSINGULAR CODE FOR DISTRIBUTION. HOWEVER, IT IS A LITTLE MORE TRICKY TO DEAL WITH AVERAGE LENGTH OF THIS CODE. WE NOW BOUND THIS AVERAGE LENGTH. IT FOLLOWS FROM (d) THAT $L^*_{1:1} \geq \tilde{L} \triangleq \sum_{i=1}^{m} p_i ld\left(\frac{i}{2}+1\right)$. CONSIDER THE DIFFERENCE:

GO TO: P. 13

$p(x)$		c_i	c_n	c_3
$1/2$	$1/2\ ]0$	0	0	0
$1/4\ ]0$	$1/2\ ]1$	10	1	1
$1/4\ ]1$		11	00	2

$$H(X) = \frac{1}{2} ld 2 + \frac{1}{4} ld 4 + \frac{1}{4} ld 4$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$$

$$E(l_{c_i}) = \frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 2 + \frac{1}{4}\cdot 2 = \frac{3}{2} = 1.5 \text{bits}$$

$$E(l_{c_n}) = \frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 1 + \frac{1}{4}\cdot 2 = 1 + \frac{1}{4} = \frac{5}{4} \text{bits}$$

$$E(l_{c_i}) = 1.5 \qquad E(l_{c_n}) = 1.25$$

(a) $$\boxed{L_{1:1} \geq \frac{H_2(X)}{ld 3} - 1}$$

$$\boxed{H_3(X) = \sum_{i=1}^{m} p(x) \log_3 \frac{1}{p(x)}}$$

$$\boxed{H_3(X) = \frac{H_2(X)}{ld 3}}$$

$$y = \log_3 x \qquad \boxed{3^y = x}$$

$$ld\, 3^y = ld\, x \qquad \boxed{y = \frac{ld\, x}{ld\, 3}} \qquad L_{1:1} = E[l_3(X)]$$

$$H_3(X) \leq E[l_3(X)] \leq H_3(X) + 1$$

$$E[l_3(X)] = \frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 1 + \frac{1}{4}\cdot 1 = 1$$ TERNARY DIGITS

$$19038 \cdot (1-x) = 10440 \qquad 1 - x = \frac{10440}{19038}$$

$$x = 1 - \frac{10440}{19038} = 1 - 0.55 = \underline{\underline{45\%}}$$

(b)

$p(x)$	$c(x)$	$l(x)$
p_1	0	1
p_2	1	1
p_3	00	2
	01	2
	⋮	
p_m		l_m

SO KLASICNO BINARNO KODIRANJE

$p_1 \sim 0 \quad p_2 \sim 1 \quad p_3 \sim 10 \quad p_4 \sim 11$

MAKSIMALNOTO m E:

$$m = 2^{l_m} \rightarrow l_m = ld(m)$$

e.g $p_8 \sim 111 \qquad \underline{m = 2^{l_m} = 2^3 = 8}$

$p_1 - 0$
$p_2 - 1$
$p_3 - 00$
$p_4 - 01$
$p_5 - 10$
$p_6 - 11$
$p_7 - 000$
$p_8 - 001$
$p_9 - 010$
$p_{10} - 011$
$p_{11} - 100$
$p_{12} - 101$
$p_{13} - 110$
$p_{14} - 111$

$$m = 2^1 + 2^2 + 2^3 = 2 + 4 + 8 = 14$$

$$m = \sum_{i=1}^{m} 2^{l_i} = \sum_{l=l_1}^{l_m} 2^l = \sum_{l=1}^{l_m} 2^l = 2^{l_m} \sum_{l=1}^{l_m} \frac{2^l}{2^{l_m}}$$

$$m = \sum_{l=0}^{l_m} 2^l - 1 = \frac{1-2^{l_m+1}}{1-2} - 1 = 2^{l_m+1} - 1 - 1 = 2^{l_m+1} - 2$$

$$2^{l_m+1} = m+2 \qquad 2^{l_m} \cdot 2 = m+2 \qquad 2^{l_m} = \frac{m}{2} + 1$$

$$\boxed{l_m = ld\left(\frac{m}{2}+1\right)} \qquad \boxed{l_i = ld\left(\frac{i}{2}+1\right)}$$

OVA VAZI ZA POSLEDNIOT x_i VO GRUPATA NA x_j KOI IMAAT ISTA DOLZINA, ZA DRUGITE:

$$l_i = \left\lceil ld\left(\frac{i}{2}+1\right)\right\rceil$$

NA PRIMER:

$$x_7 \sim p_7 \to c(x_7) = 000$$

$$\left\lceil ld\left(\frac{7}{2}+1\right)\right\rceil = \left\lceil ld(4.5)\right\rceil = \left\lceil 2.16993\right\rceil = 3$$

$$\boxed{L^*_{1:1} = \sum_{i=1}^{m} p_i \cdot \left\lceil ld\left(\frac{i}{2}+1\right)\right\rceil} \Rightarrow$$

OPTIMAL CODE AVERAGE LENGHT

$$\left(D^{-l_i}\right)' = \left(e^{-l_i \ln(D)}\right)'$$

$$\min \sum_x l(x)p(x) \qquad \text{s.t.} \quad \sum_x D^{-l(x)} \leq 1$$

$$\frac{d}{dl_i}\left(\sum_i p_i l_i + \lambda \sum_i D^{-l_i}\right) = \sum_i \frac{d}{dl_i}\left(p_i l_i + \lambda D^{-l_i}\right) =$$

$$= \sum_i \left(p_i - \lambda \cdot \ln D \cdot e^{-l_i \ln(D)}\right) = \sum_i \left(p_i - \lambda \cdot \ln D \cdot D^{-l_i}\right) = 0$$

$$p_i - \lambda \cdot \ln D \cdot D^{-l_i} = 0 \qquad p_i = \lambda \ln D \cdot D^{-l_i} \qquad D^{-l_i} = \frac{p_i}{\lambda \cdot \ln D}$$

$$\sum D^{-l_i} = \sum \frac{p_i}{\lambda \cdot \ln D} = 1 \Rightarrow \boxed{\lambda = \frac{1}{\ln D}} \Rightarrow$$

$$p_i = D^{-l_i} \Rightarrow \boxed{l_i^* = ld\ 1/p_i}$$

(6) $L^*_{1:1} \leq L^*_{INST} \leq H(X) + 1$

$$L_{1:1} \geq \frac{H_2(X)}{ld 3} - 1 \qquad \boxed{H_2(X) \leq (L_{1:1}+1) \cdot ld 3}$$

$$H(X) \leq L^*_{INST} \leq H(X) + 1 \qquad H(X) - 1 \leq L^*_{INST} \leq H(X)$$

FOR BINARY CODE

$$\underline{H(X) \leq L_{1:1} + 1} \qquad L_{1:1} \geq H_2(X) - 1 = H(X) - 1$$

FROM pp. 9 $E[lc_i] = 1.5 = H(X)$

$$E[lc_n] = 1.25 \geq \frac{H_2(X)}{ld 2} - 1 = 1.5 - 1 = 0.5$$

(c)

$p(x)$	$c_n(x)$	$l_n(x)$
1/2	0	1
1/4	1	1
1/8	00	2
1/8	01	2

$$H(X) = \frac{1}{2}ld2 + \frac{1}{4}ld4 + \left(\frac{1}{8}\cdot 3\right)\cdot 2 =$$
$$= \frac{1}{2} + \frac{1}{2} + \frac{2}{4} = 1 + \frac{3}{4} = \frac{7}{4} \quad ⊛$$
$$E[l_n(X)] = \frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 1 + \left(\frac{1}{\cancel{8}4}\cdot 2\right)\cdot\cancel{2} =$$
$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{2} = 1.25$$

$E[l_n(X)] = 1.25 \leq 7/4 = 1.75 = H(X)$ NONSINGULAR CODE COULD DO BETTER THEN ENTROPY.

(a)

$p(x)$	$c(x)$	$l'_n(x)$
1/2	02	2
1/4	12	2
1/8	002	3
1/8	012	3

$$E[l'_n(X)] = 2\cdot\frac{1}{2} + 2\cdot\frac{1}{4} + 2\left(3\cdot\frac{1}{8}\right) =$$
$$= 1 + \frac{1}{2} + \frac{3}{4} = \frac{4+2+3}{4} = \frac{9}{4}$$
$$= 2.25$$

⊛

$$H(X) = \frac{7}{4} \qquad H_3(X) \leq E[l'_n(X)] \leq H_3(X)+1$$

OVA VAZI ~~ZATOA~~ STO SO ~~DODAVANJE~~ NA STOP SIMBOL NONSINGULARNIOT ~~KOD~~ STANUVA PREFIX KOD.

• ~~REALNATA~~ PROSECNA DOLZINA NA NESINGULARNIOT KOD E

$$L'_{1:1} = E[l'_n(X)] \qquad E[l'_n(X)] = \sum_{i=1}^{m} l'_n(x_i)\,p(x_i) =$$
$$= \sum_{i=1}^{m}(l_n(x_i)+1)\,p(x_i) = \sum_{i=1}^{m} l_n(x_i)\cdot p(x_i) + \sum_{i=1}^{m} p(x_i) = L_{1:1}+1$$
$$L_{1:1}+1 \geq H_3(X) \qquad \boxed{L_{1:1} \geq \frac{H_2(X)}{ld3} - 1}$$

(b) (EDITION 2 SOLUTION) SINCE INSTANTENOUS CODE IS ALSO NONSINGULAR CODE, THE BEST NONSINGULAR CODE IS AT LEAST AS GOOD AS BEST INSTANTENOUS CODE. SINCE THE BEST INSTANTENOUS CODE HAS AVERAGE LENGTH $\leq H(X)+1$, WE HAVE: $L^*_{11} \leq L^*_{INST} \leq H(X)+1$.

(c) EXAMPLE $|X| = 3$. IN THIS EXAMPLE SIMPLEST NON-SINGULAR CODE IS: 0, 1, 00. IF EACH OF THIS SYMBOLS IS EQUALY LIKELY: $H(X) = 3\cdot\frac{1}{3}ld3 = 1.58$ bits WHEREAS A. LENGHT OF THE CODE IS: $1\cdot\frac{1}{3} + 1\cdot\frac{1}{3} + 2\cdot\frac{1}{3} = \frac{4}{3} = 1.33$

$E[l(X)] = 1.33 \leq H(X) = 1.58$ THUS NON-SINGULAR CODE COULD DO BETTER THAN THE ENTROPY.

PROOF OF LEMA 5.8.1 (REVISITED)

IF $p_j > p_k$ THEN $l_j \leq l_k$

- SUPOSE NOT TRUE I.E IN C_m' CODE WE HAVE CODES j & k INTERCHANGED THEN:

$$\underbrace{L(C_m^*)}_{\text{OPTIMAL CODE}} - L(C_m') = \sum p_i l_i^* - \sum p_i l_i' =$$

$$= \underline{p_j l_j} + \underline{p_k l_k} - \underline{p_j l_k} - \underline{p_k l_j} = l_j(p_j - p_k) + l_k(p_k - p_j)$$

$$= l_j(p_j - p_k) - l_k(p_j - p_k) = (p_j - p_k)(l_j - l_k) \leq 0$$

$$(p_j - p_k) \geq 0 \Rightarrow l_j - l_k \leq 0 \Rightarrow \underline{\underline{l_j \leq l_k}}$$ PROVED!

BY THE PROOF OF THIS LEMA WE HAVE PROVEDED THAT BY ALLOTING OF SHORTEST CODEWORDS TO A MORE PROBABLE SYMBOLS WE MINIMIZE THE AVERAGE CODE LENGTH.

<u>HARDY - LITTLEWOOD - POLYA INEQUALITY</u>:

- IF $a < b$, $c < d$ then $a \cdot d + b \cdot c < ac + bd$

- GENERAL VERSION OF HARDY - LITTLEWOOD - POLYA: IF WE WERE GIVEN TWO SETS OF NUMBERS, $A = \{a_j\}$, $B = \{b_j\}$ EACH OF SIZE m, AND LET $a_{[i]}$ BE THE i-TH LARGEST ELEMENT OF A AND $b_{[i]}$ BE THE i-TH LARGEST ELEMENT OF SET B THEN:

$$\boxed{\underset{(*)}{\sum_{i=1}^{m} a_{[i]} b_{[m+1-i]}} \leq \underset{(\#)}{\sum_{i=1}^{m} a_i b_i} \leq \underset{(\$)}{\sum_{i=1}^{m} a_{[i]} b_{[i]}}}$$

E.G. $A = \{a_1\ a_2\}$ $\quad B = \{b_1\ b_2\}$ $\quad m = 2$

$a_1 > a_2$ $\quad b_1 > b_2$

$[A] = \{\underset{a_{[1]}}{a_1}\ \underset{a_{[2]}}{a_2}\}$ $\quad [B] = \{\underset{b_{[1]}}{b_2},\ \underset{b_{[2]}}{b_1}\}$

$a_1 \cdot b_1 + a_2 \cdot b_2 \leq a_1 b_2 + a_2 b_1$ $\quad$ (POTEKNUVA OD (#), ($)

$a_1 b_1 + a_2 b_2 \leq a_1 b_1 + a_2 b_2$ $\quad$ (POVLEKUVA OD (*), (#))

e.G. $A = \{a_1\ a_2\ a_3\}$ $\quad B = \{b_1\ b_2\ b_3\}$ $\quad m = 3$

$[A] = \{\underset{a_{[1]}}{a_1}\ \underset{a_{[2]}}{a_2}\ \underset{a_{[3]}}{a_3}\}$ $\quad [B] = \{\underset{b_{[1]}}{b_2}\ \underset{b_{[2]}}{b_3}\ \underset{b_{[3]}}{b_1}\}$

$$a_1 \cdot b_1 + a_2 b_3 + a_3 \cdot b_2 \leq a_1 b_1 + a_2 b_2 + a_3 b_3 \leq$$

$$\leq a_1 b_2 + a_2 b_3 + a_3 b_1$$

LET: $C_k = \sum_{j=1}^{k-1} 2^j$ THEN ALL SOURCE SYMBOLS: $C_k + 1,\ C_k + 2,\ \ldots,\ C_k + 2^k = C_{k+1}$

VIDI PR. 9 : $\gamma_1 - 0$; $\gamma_2 - 1$; $\gamma_3 = 00$; $\gamma_4 = 01$; $\gamma_5 = 10$; $\gamma_6 = 11$

ZNAČI ZA $K=2$ (DOLŽINA NA KODNIOT ZBOR)

$c_2 = \sum_{j=1}^{1} 2^j \qquad c_2 = 2^1 = 2 \qquad c_2+1, c_2+2, c_2+3, c_2+4 = 2+1, 2+2, 2+3, 2+4 = 3,4,5,6$ } [redacted] KODNITE ZBOROVI SO ONIE INDEX-I IMAAT DOLŽINA $\underline{K=2}$.

$$c_k + 2^k = 2^1 + 2^2 + \dots + 2^k = 2\sum_{i=0}^{k-1} 2^i = 2\cdot\frac{2^k-1}{2-1} = 2 2^k - 2$$

$\boxed{c_k = 2\cdot 2^k - 2 - 2^k = 2^k - 2}$ e.g. $k=2$ $\boxed{c_k = c_2 = 4-2=2}$

• OTTUKA SITE KODNI ZBOROVI SO INDEX „i" VO OPSEGOT:

$c_k+1 \ \dots \ c_k + 2^k$ T.E

$c_k+1 \le i \le c_k + 2^k \qquad 2^k - 2 + 1 \le i \le 2^k - 2 + 2^k$

$2^k - 1 \le i \le 2 \cdot 2^k - 2$; $\boxed{2^k - 1 \le i \le 2^{k+1} - 2}$

IMAAT ISTA DOLŽINA NA KODNITE ZBOROVI EDNAKVA NA „K"

$2^k + 1 \le i+2 \le 2^{k+1}$, $k \le ld(i+2) \le k+1$; $ld2$

$k-1 \le ld(i+2) - 1 \le k$, $k-1 \le ld\frac{i+2}{2} \le k$

AKO JA ISPUŠTIŠ „1" DOBIVAŠ $2^k < i+2 \le 2^{k+1}$

ZNAČI: $k-1 < ld\frac{i+2}{2} \le k \Rightarrow k = \lceil ld\frac{i+2}{2}\rceil$

$ld\frac{i+2}{2} \le k \le ld\frac{i+2}{2}+1 \qquad k-1 \le ld\frac{i+2}{2}$, $ld\frac{i+2}{2} \le k$

SEPAK MISLAM DEKA MOJOT DOKAZ NA PR. 10 E PODOBAR.

- THUS BEST NON-SINGULAR CODE ASSIGNS CODEWORD LENGHT $l_i^* = \lceil ld\left(\frac{i+2}{2}\right)\rceil$ TO SYMBOL „i". THEREFORE

$$L_{11}^* = \sum p_i \left\lceil ld\left(\frac{i+2}{2}\right)\right\rceil$$

(e) CONTINUES FROM PR. 8. CONSIDER THE DIFERENCE:

$$F(p) = H(X) - \tilde{L} = -\sum_{i=1}^{m} p_i \, ld\, p_i - \sum_{i=1}^{m} p_i \, ld\left(\frac{i}{2}+1\right)$$

PROVE BY THE METHOD OF LAGRANGE MULTIPLIERS THAT THE MAXIMUM OF $F(p)$ OCCURS WHEN $p_i = c/(i+2)$ WHERE $c = 1/(H_{m+2} - H_2)$ AND H_k IS SUM OF HARMONIC SERIES $H_k = \sum_{i=1}^{k} \frac{1}{i}$

$\tilde{N}_1$

$$L^*_{1:1} = \sum_{i=1}^{m} p_i \left\lceil ld\left(\frac{i}{2}+1\right)\right\rceil \qquad \left\lceil ld\left(\frac{i}{2}+1\right)\right\rceil \geq ld\left(\frac{i}{2}+1\right) \Rightarrow$$

$$\boxed{L^*_{1:1} \geq \tilde{L} \triangleq \sum_{i=1}^{m} p_i\, ld\left(\frac{i}{2}+1\right)}$$

$$F(p) = -\sum_{i=1}^{m} p_i\, ld\, p_i - \sum_{i=1}^{m} p_i\, ld\left(\frac{i}{2}+1\right)$$

s.t. $\sum p_i = 1$

$$F_1(p) = -\sum_{i=1}^{m} p_i\, ld\, p_i - \sum_{i=1}^{m} p_i\, ld\left(\frac{i}{2}+1\right) + \lambda\left(\sum_{i=1}^{m} p_i - 1\right)$$

$$\frac{dF_1(p)}{dp_i} = 0 \qquad -\sum_{i=1}^{m}\left(ld\, p_i + 1\right) - \sum_{i=1}^{m} ld\left(\frac{i}{2}+1\right) + \lambda m = 0$$

$$-\sum_{i=1}^{m}\left(ld\, p_i + 1 + ld\left(\frac{i}{2}+1\right) - \lambda\right) = 0$$

$$ld\, p_i + 1 + ld\left(\frac{i}{2}+1\right) - \lambda = 0 \qquad \lambda = ld\, p_i\left(\frac{i}{2}+1\right) + 1$$

$$\lambda = ld\, 2p_i\left(\frac{i}{2}+1\right) = ld\, i p_i + 2p_i = ld\,(i+2)\,p_i$$

$$p_i \cdot (i+2) = 2^{\lambda} \qquad \boxed{p_i = \frac{2^{\lambda}}{i+2}}$$

ODI OD TUKA NA ★

$$\sum_{i=1}^{m} \frac{2^{\lambda}}{(i+2)} = 1 \qquad 2^{\lambda}\underbrace{\sum_{i=1}^{m}\frac{1}{i+2}}_{S} = 1$$

$$S = \sum_{i=1}^{m}\frac{1}{i+2} \qquad S = \sum_{n=1}^{m}\frac{1}{n} = ?$$

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \dots \frac{1}{m}$$

$$S = \sum_{n=1}^{m}\frac{x^{n-1}}{} \qquad \int S\, dx = \sum_{n=1}^{m}\frac{x^n}{n}$$

$$S = \sum_{n=1}^{m} x^{n-1} = \sum_{n=0}^{m-1} x^n = \frac{x^m - 1}{x-1}; \qquad \int\frac{x^n-1}{x-1}dx = \sum_{n=1}^{n}\frac{x^n}{n}$$

$$\lim_{x\to 1}\frac{x^m-1}{x-1} = \lim_{x-1\to 0}\frac{(y+1)^m-1}{y+1-1} = \lim_{y\to 0}\frac{(y+1)^m-1}{y}$$

- $$\lim_{y\to 0}\frac{n(y+1)^{n-1}}{1} = n\cdot 1^{n-1} = n$$

$$\int\frac{x^n-1}{x-1}\,dx\Big|_{x=1} = \sum_{n=1}^{m}\frac{x^n}{n}\Big|_{x=1} \qquad \sum_{n=1}^{m}\frac{1}{n} = \int\underbrace{\lim_{x\to 1}\frac{x^n-1}{x-1}}_{m}\,dx$$

$$\sum_{n=1}^{m}\frac{1}{n} = \int m\,dx = m\cdot x\Big|_{x=1} = m \quad (?)$$

$$2^\lambda\sum_{i=1}^{m}\frac{1}{i+2} = 1\,; \quad 2^{-\lambda} = \sum_{i=1}^{m}\frac{1}{i+2} \qquad \lambda = -ld\sum_{i=1}^{m}\frac{1}{i+2}$$

$$\boxed{\Gamma(x) = \int_{0}^{\infty} t^{x-1}\cdot e^{-t}\,dt}$$

$$\Gamma(x) = (x-1)!$$

$$\Gamma(3) = (3-1)! = 2! = 2$$

$$\psi = \frac{d}{dx}\ln\Gamma(x) = \frac{\frac{d}{dx}\Gamma(x)}{\Gamma(x)}$$

- EULER'S CONSTANT :

$$\gamma = \lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{1}{i} - \ln(n)\right]$$

$$\psi(s+1) = -\gamma - \sum_{k=1}^{\infty}\frac{(-1)^k}{k}\binom{s}{k}\,; \qquad \boxed{p_i = \frac{2^\lambda}{i+2}}$$

★

$$\boxed{2^\lambda = C} \qquad C = \frac{1}{H_{m+2}-H_2} \qquad \boxed{H_k = \sum_{i=1}^{k}\frac{1}{i}}$$

$$\sum_{i=1}^{m}\frac{2^\lambda}{(i+2)} = 1\,; \quad 2^\lambda = \frac{1}{\sum_{i=1}^{m}\frac{1}{i+2}}\,; \quad S = \sum_{i=1}^{m}\frac{1}{i+2}\,;$$

j = i + 2
i = 1
j = 3
i = m
j = m + 2

$$H_{m+2} = \sum_{i=1}^{m+2}\frac{1}{i}\,; \quad H_2 = \sum_{i=1}^{2}\frac{1}{i}\,; \quad S = \sum_{j=3}^{m+2}\frac{1}{j}$$

$$S = \sum_{j=1}^{m+2}\frac{1}{i} - \sum_{j=1}^{2}\frac{1}{j} = \underline{H_{m+2} - H_2}$$

- ZNAČI, DOKAŽAV DEKA :

$$C = 2^\lambda = \frac{1}{\sum_{i=1}^{m}\frac{1}{i+2}} = \frac{1}{H_{m+2}-H_2}\,; \quad \text{T.E.} \quad \boxed{p_i = \frac{1}{(i+2)\left[H_{m+2}-H_2\right]}}\,;$$

$F(\gamma) = -\sum \gamma_i \, ld\, \gamma_i - \sum \gamma_i \, ld \left(\frac{i+2}{2}\right)$; $\gamma_i = \frac{1}{H_{n+2}-H_2} \cdot \frac{1}{i+2}$;

$F(\gamma) = -\sum_{i=1}^{n} \gamma_i \, ld\, \gamma_i \left(\frac{i+2}{2}\right) = -\sum_{i=1}^{n} \gamma_i \, ld \frac{1}{H_{n+2}-H_2} \cdot \frac{1}{\cancel{i+2}} \cdot \frac{\cancel{i+2}}{2} =$

$= -\sum_{i=1}^{n} \gamma_i \, ld \frac{1}{2(H_{n+2}-H_2)} = \sum_{i=1}^{n} \gamma_i \, ld\, 2(H_{n+2}-H_2) =$

$= ld\, 2(H_{n+2}-H_2)$

- DOKAZ DEKA SE RABOTI ZA MAXIMUM

$\frac{dF_1(\gamma_i)}{d\gamma_i} = -\sum_{i=1}^{n} \left[ld\, \gamma_i + 1 + ld \left(\frac{i}{2}+1\right) - \lambda \right)\Big]'$

$\frac{d^2 F_1(\gamma)}{d\gamma^2} = -\sum_{i=1}^{n} \frac{1}{\gamma_i} < 0 \Rightarrow$ MAXIMUM

- ALTERNATIVNO REŠENIE SO KORISTENJE NA RELATIVNA ENTROPIJA

$F(\gamma) = -\sum \gamma_i \, ld\, \gamma_i - \sum \gamma_i \, ld \frac{i+2}{2}$; $q_i = \frac{1}{H_{n+2}-H_2} \cdot \frac{1}{i+2}$;

$F(\gamma) = -\sum \gamma_i \, ld\, \gamma_i - \sum \gamma_i \, ld \frac{1}{2} \frac{1}{(H_{n+2}-H_2) q_i} =$

$= -\sum \gamma_i \, ld\, \gamma_i - \sum \gamma_i \, ld \frac{1}{q_i} - \sum \gamma_i \, ld \frac{1}{2(H_{n+2}-H_2)} =$

$= ld\, 2(H_{n+2}-H_2) - \sum \gamma_i \, ld \frac{\gamma_i}{q_i} = ld\, 2(H_{n+2}-H_2) - \underbrace{D(\gamma \| q)}_{\geq 0}$

IF $\gamma = q \Rightarrow \boxed{F(\gamma) = F_{max}(\gamma) = ld\, 2(H_{n+2}-H_2)}$

f) Complete the arguments for:

$$\boxed{H(X) - L^*_{1:1} \leq H(X) - \tilde{L} \leq ld\left(2(H_{n+2}-H_2)\right)}$$

NOW IT IS WELL KNOWN (SEE, E.G. KNUTH, „ART OF COMPUTER PROGRAMMING", VOL. 1) THAT $H_K \doteq \ln K$

$\boxed{H_K = \ln k + \gamma + \frac{1}{2k} - \frac{1}{12k^2} + \frac{1}{120k^4} - \varepsilon}$ $\boxed{0 < \varepsilon \leq \frac{1}{252n^6}}$

γ - EULER'S CONSTANT $\gamma = 0\ 5772$

EITHER USING THIS OR SIMPLE APPROXIMATION: $H_K \leq \ln k + 1$ WHICH CAN BE PROVED BY INTEGRATION OF $\frac{1}{x}$ IT CAN BE SHOWN THAT: $H(X) - L^*_{1:1} \leq ld\, ld\, n+2$

THUS WE HAVE: $H(X) - \mathrm{ld}\,\mathrm{ld}\, m - 2 \leq L^*_{1:1} \leq H(X) + 1$

A NON-SINGULAR CODE CANNOT DO MUCH BETTER THAN AN INSTANTENOUS CODE !!!

$$L^*_{1:1} = \sum_{i=1}^{m} p_i \left\lceil \mathrm{ld}\left(\frac{i+2}{2}\right)\right\rceil \geq \tilde{L} \triangleq \sum_{i=1}^{m} p_i \,\mathrm{ld}\left(\frac{i+2}{2}\right)$$

$$H(X) - L^*_{1:1} \leq H(X) - \tilde{L} \leq \mathrm{ld}\, 2(H_{m+2} - H_2)$$

$H_k \leq \ln k + 1$; $\mathrm{ld}\, 2\left(H_{m+2} - \frac{3}{2}\right) \leq \mathrm{ld}(2 \cdot H_{m+2}) \leq$

$\leq \left| \begin{array}{ll} H_{m+2} \leq \ln(m+2) + 1 \,, & y = \ln k;\ e^y = k \\ y\,\mathrm{ld}e = \mathrm{ld}k & y = \frac{\mathrm{ld}k}{\mathrm{ld}e} = \ln k \end{array} \right|$

$$\leq \mathrm{ld}\left(2\,\frac{\mathrm{ld}(m+2)}{\mathrm{ld}e} + 2\right) = \mathrm{ld}\left[\frac{\mathrm{ld}(m+2)^2 + 2\,\mathrm{ld}e}{\mathrm{ld}e}\right] =$$

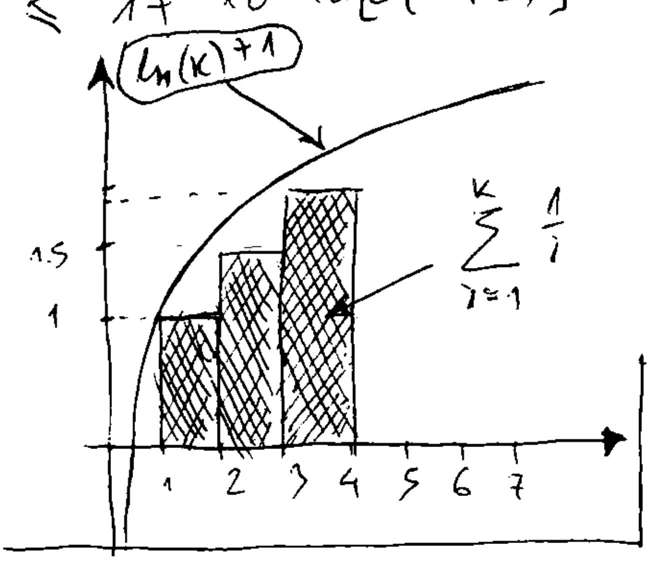
$$= \mathrm{ld}\,\frac{\mathrm{ld}[(m+2)^2 \cdot e^2]}{\mathrm{ld}\,e} \quad ; \quad \boxed{\mathrm{ld}\,\mathrm{ld}e = \mathrm{ld}\, 2^{\mathrm{ld}\,\mathrm{ld}e}}$$

$$\mathrm{ld}\,2(H_{m+2} - H_2) = \mathrm{ld}\,2 + \mathrm{ld}(H_{m+2} - H_2) \leq 1 + \mathrm{ld}(H_{m+2}) = ⊛$$

$$H_{m+2} = \ln(m+2) + 1 = \frac{\mathrm{ld}(m+2)}{\mathrm{ld}(e)} + 1$$

$$⊛ \leq 1 + \mathrm{ld}\,\frac{[\mathrm{ld}(m+2) + \mathrm{ld}e]}{\mathrm{ld}e} = 1 + \mathrm{ld} \cdot \mathrm{ld}[e(m+2)] - \mathrm{ld}\,1.41$$

$$\leq 1 + \mathrm{ld}\,\mathrm{ld}[e(m+2)] = 1 + \mathrm{ld}\left[\mathrm{ld}e + \mathrm{ld}(m+2)\right] \quad ?$$



$$1 + \frac{1}{2} + \frac{1}{3} = \frac{6+3+2}{6} = \frac{11}{6}$$

$$\int_1^k \frac{1}{x}\,dx = \ln(x)\Big|_1^k = \ln(k) + 0$$

$$\sum_{i=1}^{k} \frac{1}{i} \leq \ln(k) + 1$$

$$\mathrm{ld}\,2(H_{m+2} - H_2) = \mathrm{ld}\,2\left(H_{m+2} - \frac{1}{2} - 1\right) \leq$$

$\left| H_{m+2} \leq \mathrm{ld}(m+2) + 1 \right| \leq \mathrm{ld}\,2(\mathrm{ld}(m+2) + \ldots$

$$\leq \mathrm{ld}\,2(\mathrm{ld}(m+2)) \leq \mathrm{ld}\,2\,\mathrm{ld}\,m^2 \quad ①$$

/① IS TRUE FOR $m \geq 2$/ ; $① = \mathrm{ld}(4\,\mathrm{ld}m) = \mathrm{ld}4 + \mathrm{ld}\,\mathrm{ld}m =$

$= 2 + \mathrm{ld}\,\mathrm{ld}m$; $H(X) - L^*_{1:1} \leq \mathrm{ld}\,\mathrm{ld}\,m + 2$ $\boxed{H(X) - \mathrm{ld}\,\mathrm{ld}\,m - 2 \leq L^*_{1:1}}$

$L^*_{1:1} = \sum p_i\,\mathrm{ld}\,\frac{i+2}{2}$ $\quad p_i = \frac{1}{H_{m+2} - H_2} \cdot \frac{1}{i+2}$ ④

$p_i = \frac{1}{H_{m+2} - H_2} \cdot \frac{1}{i+2}$ $\qquad H_{m+2} \leq ld(m+2) + 1$

$p_i = \frac{1}{H_{m+2} - H_2} \cdot \frac{1}{i+2} \geq \frac{1}{ld(m+2) + \cancel{1} - \cancel{1} - \frac{1}{2}} \frac{1}{i+2}$

$p_i \geq \frac{1}{ld(m+2) - \frac{1}{2}} \cdot \frac{1}{i+2}$ $\qquad i+2 \geq \frac{1}{p_i\left(ld(m+2) - \frac{1}{2}\right)}$

$$L^*_{1:1} = \sum_{i=1}^{m} p_i \left\lceil ld \frac{i+2}{2} \right\rceil \leq \underbrace{\sum_{i=1}^{m} p_i \, ld \frac{i+2}{2}}_{\tilde{L}} + 1 = \tilde{L} + 1$$

$H(X) - \tilde{L} \leq ld\, ld\, m + 2$ $\qquad \tilde{L} \geq H(X) - ld\, ld\, m - 2$

$L^*_{1:1} \leq \tilde{L} + 1$ $\qquad L^*_{1:1} - 1 \leq \tilde{L}$

$L^*_{1:1} \geq \tilde{L}$

- VO SOLUTIONS OF 2nd edition TVRDAT DEKA ·

$$H(X) - ld\, ld\, m - 2 \leq L^*_{1:1} \underbrace{\leq H(X) + 1}$$

OVOJ DEL NE MOŽAM DA GO DOKAŽAM !!! 1

$H(X) - L^*_{1:1} \leq H(X) - \tilde{L} \leq H(X) - L^*_{1:1} + 1$

$L^*_{1:1} \leq \tilde{L} + 1 \leq L^*_{1:1} + 1$

- ŠTE EDNAŠ DOKAZ NA (f)

- OD (e) SLEDUVA DEKA

$H(X) - \tilde{L} \leq ld\left[2\left(H_{m+2} - H_2\right)\right] = ld\left[2\left(H_{m+2} - \frac{1}{2} - 1\right)\right] \leq$

$\leq \left| H_{m+2} \leq ld(m+2) + 1 \right| \leq ld\left[2\left(ld(m+2) + \cancel{1} - \frac{1}{2} - \cancel{1}\right)\right] \leq$

$ld\left(2\, ld(m+2)\right) \leq ld\, 2\, ld\, m^2 = ld\, 4 (ld\, m) = 2 + ld\, ld\, m$

$\boxed{H(X) - 2 - ld\, ld(m) \leq \tilde{L} \leq L^*_{1:1}}$

$L^*_{1:1} \leq H(X) + 1$

$+ 1$

OD (b) SLEDUVA DEKA: $L_{1:1} \leq H + 1$

$L^*_{1:1} = \sum p_i \left\lceil ld\left(\frac{i+2}{2}\right)\right\rceil \leq \sum_{i=1}^{m} p_i\, ld \frac{i+2}{2} + \tilde{L}$

$\tilde{L} \geq L^*_{1:1} - 1$

$L^*_{1:1} = \sum_{i=1}^{m} p_i \left\lceil ld \frac{i+2}{2} \right\rceil \geq$

$\sum_{i=1}^{m} p_i\, ld \frac{i+2}{2} = \tilde{L}$ $\qquad \boxed{L^*_{1:1} \geq \tilde{L}} \Rightarrow H(X) - 2 - ld\, ld\, m \leq L_{1:1} \leq H(X) + 1$

PROBLEM 5.33 Huffman vs. Shannon. A RANDOM VARIABLE X TAKES ON THREE VALUES WITH PROBABILITIES 0.6, 0.3 AND 0.1.

(a) WHAT ARE THE LENGTHS OF THE BINARY HUFFMAN CODEWORDS FOR X. WHAT ARE THE LENGTHS OF THE BINARY SHANNON CODEWARDS:

$$l(x) = \left\lceil ld \frac{1}{p(x)} \right\rceil$$ FOR X?

(B) WHAT IS THE SMALLEST INTEGER D SUCH THAT THE EXPECTED SHANNON CODEWORD LENGTH WITH D-ARY ALPHABET EQUALS THE EXPECTED HUFFMAN CODEWORD LENGHT WITH D-ARY ALPHABET?

---

①

$p(x)$		$C(x)$	$l_H(x)$	Shannon	$l_s(x)$
0.6 →	0.6 }0	0	1	$\lceil 0.74 \rceil$	1
0.3 }0 →	0.4 }1	10	2	$\lceil 1.737 \rceil$	2
0.1 }1		11	2	$\lceil 3.32 \rceil$	4

Za $D=3$

$l_s(x) = l_H(x)$
$= 1$

$\boxed{?}$

Ⓑ $$E[l_s(x)] = \sum p(x) \left\lceil \log_D \frac{1}{p(x)} \right\rceil \leq \sum p(x) \log_D \frac{1}{p(x)} + 1$$

$$H(X) \leq E[l_H(x)] \leq H(X) + 1$$

$\boxed{a}$

$$p(x) = \frac{1}{D^n} = D^{-l(x)} \qquad E[l(x)] =$$

$l(x) = 1 \qquad p_1 = 1/3 \qquad p_2 = 1/3 \qquad p_3 = 1/3$

$\boxed{a}$

$(0.1)_{10} \to (0.00011)_2$; $(0.3)_{10} \to (0.01001)_2$;
$(0.6)_{10} \to (0.10011)_2$; EXPECTED NUMBER OF FAIR BITS REQUIRED BY THE OPTIMAL ALGORITHM ARE:

$$H(X) \leq E[T] \leq H(X) + 2$$

(20a)

$H(X) = 0.6\, ld \frac{1}{0.6} + 0.3\, ld \frac{1}{0.3} + 0.1\, ld \frac{1}{0.1} = 1.3$

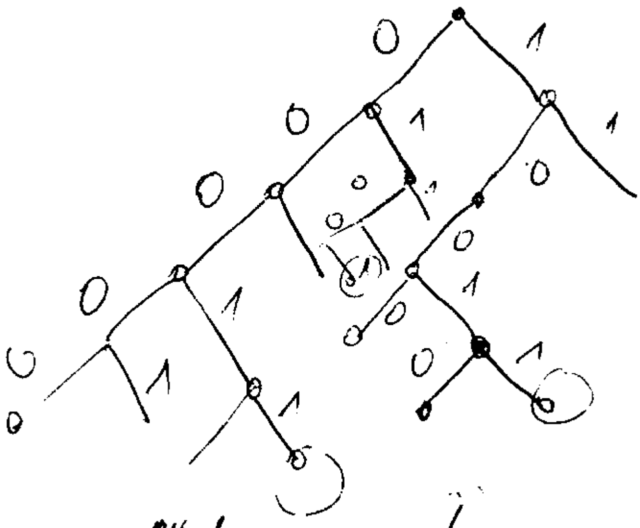
$H(X) \leq E[T] < H(X) + 2$ $\quad$ $\underline{E[T] < 4}$

$X = \{x_1, x_2, x_3\}$ $\quad$ $Y = \{p_1, p_2, p_3\}$

$C(x_1) = 11$ $\quad$ $C(x_2) = 1001$ $\quad$ $C(x_3) = 10011$

$\log_3(0.6)^{-1} = \underline{0.46}$ $\quad$ $\log_3(0.3)^{-1} = \underline{1.096}$ $\quad$ $\log_3(0.1)^{-1} = \underline{2.096}$

$l_S^3 = [1, 2, 3]$ $\quad$ $l_H^3 = [1, 1, 1]$



$\left\lceil \log_3\left(\frac{1}{p(x)}\right) \right\rceil = \log_3 \frac{1}{p(x)}$

$\log_D \frac{1}{p(x)} = l_1$

$D^{l_1} = \frac{1}{p(x)}$

$p(x) = D^{-l_1}$ $\quad$ $\underline{D^{-l_1} = 0.6 \quad D^{-l_2} = 0.3 \quad D^{-l_3} = 0.1}$

$\underline{l_1, l_2, l_3}$ - INTEGER ; D - INTEGER

• $\boxed{\text{SOLUTION 2}}$ FOR $D > 2$ THE HUFFMAN CODE FOR THREE SYMBOLS HAVE LENGTH 1. SHANNON CODE FOR ~~EACH~~ SYMBOL SHOULD BE ALSO !!

$l_1 = \log_D \frac{1}{0.6} \leq 1$ $\quad$ $l_2 = \log_D \frac{1}{0.3} \leq 1$ $\quad$ $l_3 = \log_D \frac{1}{0.1} \leq 1$

$\log_D \frac{1}{0.6} = \log_D \frac{1}{0.1} \cdot \frac{1}{6} = \log_D \frac{1}{0.1} + \log_D \frac{1}{6} =$

$\log_D 10 \quad \log_D 6$ $\quad$ $l_1 = l_3 - \log_D 6$

$l_3 = l_1 + \log_D 6$ $\quad$ $l_3 > l_1$ $\quad$ ZNAČI AKO $l_3 \leq 1$ TOGAŠ SIGURNO $l_1 \leq 1$

$\log_D 10 = 1 \Rightarrow D^1 = 10 \Rightarrow \boxed{D = 10}$

HENCE FOR $D \geq 10$ SHANNON CODE IS ALSO OPTIMAL

PROBLEM 5.34 HUFFMAN ALGORITHM FOR TREE CONSTRUCTION

CONSIDER THE FOLLOWING PROBLEM. m BINARY SIGNALS $S_1, S_2, \ldots, S_m$ ARE AVAILABLE AT TIMES $T_1 \leq T_2 \leq \ldots T_m$ AND WOULD LIKE TO FIND THEIR SUM $S_1 \oplus S_2 \oplus \cdot \oplus S_m$ USING TWO INPUT GATES, EACH GATE WITH ONE TIME UNIT DELAY, SO THAT THE FINAL RESULT IS AVAILABLE AS QUICKLY AS POSSIBLE. A SIMPLE GREEDY ALGORITHM IS TO COMBINE THE EARLIEST TWO RESULTS, FORMING THE PARTIAL RESULT AT TIME $\max(T_1, T_2) + 1$. WE NOW HAVE A NEW PROBLEM WITH $S_1 \oplus S_2, S_3, \ldots, \underset{\text{CAN}}{S_m}$ AVAILABLE AT TIMES $\max(T_1, T_2) + 1, T_3, \ldots, T_m$. WE NOW SORT THIS LIST OF T'S AND APPLY THE SAME MERGING STEP AGAIN, REPEATING THIS UNTIL WE HAVE THE FINAL RESULT.

(a) ARGUE THAT THE FOREGOING PROCEDURE IS OPTIMAL, IN THAT IT CONSTRUCTS A CIRCUIT FOR WHICH THE FINAL RESULT IS AVAILABLE AS QUICKLY AS POSSIBLE.

(b) SHOW THAT THIS PROCEDURE FINDS THE TREE THAT MINIMIZES:

$$C(T) = \max_i (T_i + l_i)$$

WHERE T_i IS THE TIME AT WHICH THE RESULT ALLOTED TO THE i-TH LEAF IS AVAILABLE, AND l_i IS THE LENGTH OF THE PATH FROM THE i-TH LEAF TO THE ROOT

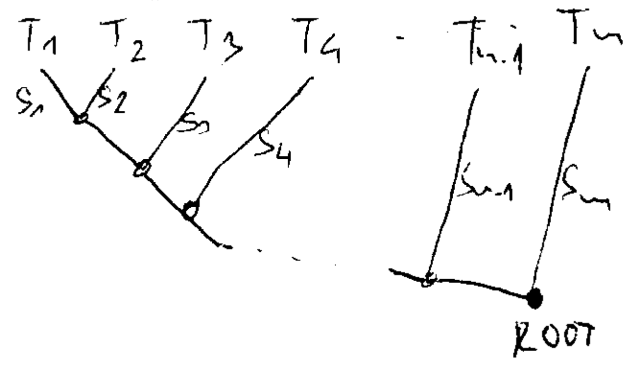
(c) SHOW THAT $C(T) \geq ld\left(\sum_i 2^{T_i}\right)$

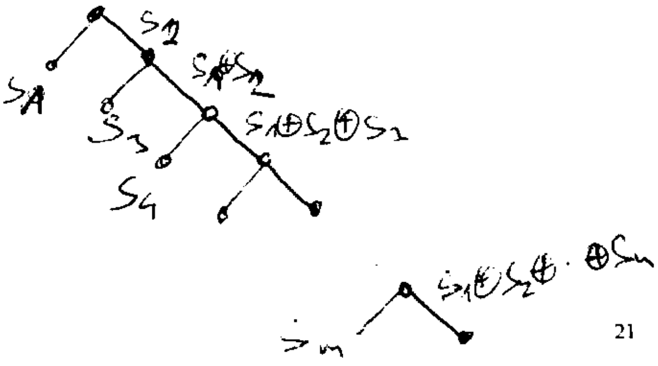
(d) SHOW THAT THERE EXISTS A TREE SUCH THAT

$$C(T) \leq ld\left(\sum_i 2^{T_i}\right) + 1$$

THUS $ld \sum_i 2^{T_i}$ IS ANALOG TO ENTROPY FOR THIS PROBLEM.

(a)





22a

$$\min E[l(x)] = \sum_x l(x)\, p(x) \qquad \text{s.t.} \quad \sum_x D^{-l(x)} \le 1$$

$$F(x) = \sum_x l(x)\, p(x) + \lambda \left[\sum_x D^{-l(x)} - 1 \right]$$

$$\frac{dF(x)}{dL} = 0 \qquad \sum_x p(x) + \lambda \sum_x \ln D\, D^{l(x)} = 0$$

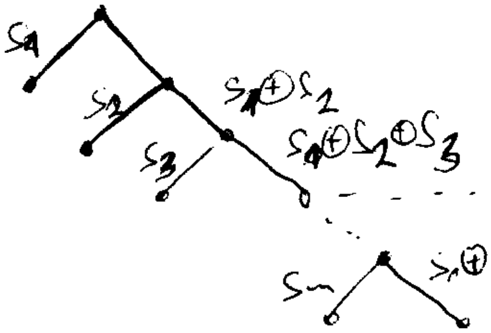
$$\left(D^{-x}\right)' = \left(e^{-x \ln D}\right)' = -\ln D \cdot e^{-x \ln D} = -\ln D \cdot D^{x}$$

$$\sum_x \left(p(x) - \lambda \ln D\, D^{-l(x)} \right) = 0 \qquad p(x) = \lambda \cdot \ln D \cdot D^{-l(x)}$$

$$D^{-l(x)} = \frac{p(x)}{\lambda \cdot \ln D} \qquad \sum \frac{p(x)}{\lambda \cdot \ln D} = \frac{1}{\lambda \cdot \ln D} \sum p(x) = \frac{1}{\lambda \ln D}$$

$$\frac{1}{\lambda \cdot \ln D} = 1 \quad \uparrow_{\text{constr.}} \qquad \boxed{\lambda = \frac{1}{\ln D}} \Rightarrow \boxed{p(x) = D^{-l(x)}}$$

a



$\max(T_1, T_2) + 1$

$\max(\max(T_1, T_2) + 1, T_3) + 1$

$\max(T, T_n) + 1 = T_E$

a

$$T_E = \max\left(\max\left(\cdots \max\left(\max(T_1, T_2) + 1, T_3\right) + 1\right)\right)$$

1) IF: $\max(T_1, T_2) + 1 > T_3$ AND SO ON :

$$T_E = T_2 + 1 + 1 + 1 \cdots + 1 = T_2 + (n-1)$$

$S_1 \oplus S_2$

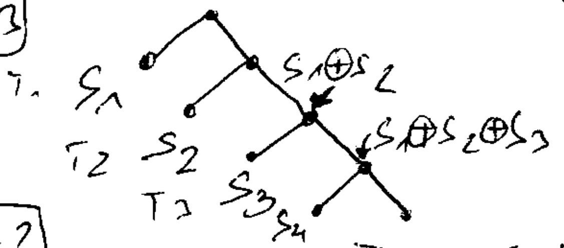
$S_1 \oplus S_2 \oplus S_3$

$S_1 \oplus S_2 \oplus S_3 \oplus S_4$

$S_1 \oplus S_2 \oplus S_3 \cdots \oplus S_n$

23a

$\boxed{m=3}$



$T_1 \quad T_2 \quad T_3 \quad T_4$

$S_1 \quad S_2 \quad S_3 \quad S_4$

$\boxed{T_t = ?}$

Ⓘ $T_1 = 1 \; ; \; T_2 = 1\,5 \; ; \; T_3 = 3 \; ; \; T_4 = 4 \; ;$

- $S = S_1 \oplus S_2$ AT: $\max(T_1 ; T_2) = T_2 = 1.5$
- $S = S_1 \oplus S_2 \oplus S_3$ AT: $\max(T_2+1, T_3)+1 = \max(2,5 ; 3) + 1 = 3+1 = 4$

ALTERNATIVNO POKOLKU: $T_2 = 2 \quad T_3 = 2.5$

AT $\max(2+1, 2.5)+1 = \max(3, 2.5)+1 = \underline{4}$

- $S = S_1 \oplus S_2 \oplus S_3 \oplus S_4$ AT: $\max(4, 4) + 1 = 5$

ALTERNATIVNO POKOLKU $T_5 = 3.5$

AT: $\max(4, 3.5) + 1 = 4+1 = 5$

SEKAKO E: 5

ALTERNATIVNO POKOLKU $T_5 = 6$

$\max(\underset{\uparrow}{4}, 6)+1 = 6+1 = 7$

OVA NE MOZE DA GI POMALKU OD T_3+1 TE

$\max(T_2+1, T_3)+1$

Ⓘ Ⓘ $T_1 = 0.1 \quad T_2 = 0.2 \quad T_3 = 0.3 \quad T_4 = 0.4$

- $S = S_1 \oplus S_2$ AT: $\max(T_1, T_2) = T_2 = 0.2$
- $S = S_1 \oplus S_2 \oplus S_3$ AT: $\max(T_2+1, T_3)+1 = \max(1.2; 0.3)+1 = \underline{2.2}$
- $S = S_1 \oplus S_2 \oplus S_3 \oplus S_4$ AT: $\max(T_2+1+1, T_4)+1 = \underline{3.2}$

$\boxed{a}$

$\underline{III}$ $T_1 = 5 \quad T_2 = 10 \quad T_3 = 15 \quad T_4 = 20$

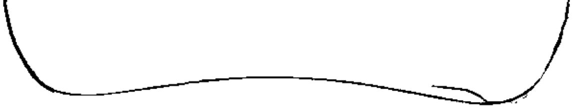
$S = S_1 \oplus S_2$ AT $\max(T_1 T_2) \overset{+1}{=} \max(5, 10) \overset{+1}{=} T_2 \overset{+1}{=} 10+1 = 11$

$S = S_1 \oplus S_2 \oplus S_3$ AT: $\max(T_2+1, T_3)+1 = \max(11, 15)+1 = 16$

$S = S_1 \oplus S_2 \oplus S_3 \oplus S_4$ AT: $\max(T_3+1, T_4)+1 = T_4+1 = 21$

- OD OVIE TRI PRIMERI MI IZGLEDA DEKA FINALNIOT REZULTAT ĆE BIDE DUDINE VO INTERVALOT:

$\boxed{T_2 + (m-1) \leq T \leq T_m + 1}$ AKO $T_m \geq T_{m-1}+1$



– RECALL HUFFMAN CODING AND WEIGHTED CODEWORDS.

ANY SET OF NUMBERS $p_i \geq 0$

HUFFMAN CODE MINIMIZES THE SUM OF WEIGHTED CODE LENGHTS. l_j

$$S_j = \sum_{i=1} S_i$$

} MODULO SUMA

l_j – BROJ NA SUMANDI

$$\boxed{\sum_{j=2}^{m} l_j \cdot T_j} \qquad \min \sum_{j=2}^{m} l_j T_j$$

ZA PR. 23a : $T_2+(n-1) \leq T \leq T_n+1$

$$\min \sum_{j=2}^{m} l_j \left[T_2+(j-1)\right)]$$

LOGIČNO E OVA !!! NA PRIMER

$$C(T) = \max_i (T_i + l_i) \quad \Big\} \quad T_i = T_n$$

$$t_n = \max(T_n+1, T_{n+1}) = T_n+1$$

[illegible] [illegible] $t_{n+1} = \max(t_n, T_{n+1})+1$

• AKO OD MOMENTOT $\boxed{i=n}$ PA NATAMU CELO VREME $T_n+(j-n)$ T.E. E POGOLEMO OD T_j

$$T_n+(j-n) \geq T_j \quad ZA \quad \underline{j = n+1, n+2, \dots m}$$

TOGAŠ: $C(T) = T_n + (m-n\cancel{)}+1)$

• AKO TOA NE E SLUČAJ T.E. SE JAVI NEKOE T_k ZA KOE VAŽI:

$$\max\left(T_n+(k-n\cancel{)}+1), T_k\right) = T_k$$

$$\boxed{C(T) = T_k + (m-k\cancel{)}+1)}$$

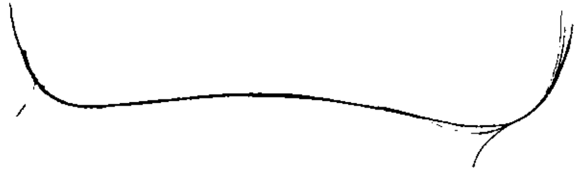
TOGAŠ E MINIMALNOTO ~~MAKSIMALNOTO~~ VREME ZA KOE ĆE SE POSTIGNE REZULTATOT

• NA KRAJ AKO:

$$\left(T_k + (m-k+1), T_m\right) = T_m$$

TOGAŠ REZULTATOT ĆE SE DOBIE VO: T_m+1 T.E.

$$C(T) = T_m + (m-m)+1 = \underline{T_m+1}$$



259

VO (b) STOJA DA VAZE DEKA REZULTATOT E :

① PRESMETAJ GI (NA PRIMER $n = 5$)

$T_1 + (n-1)+1 = T_1 + 5$; $T_2 + (n-2)+1 = T_2 + n - 1 = T_2 + 4$;

$T_3 + (5-3)+1 = T_3 + 3$; $T_4 + (5-4)+1 = \underline{T_4 + 2}$;

a

$T_5 + (5-5)+1 = T_5 + 0 + 1 = T_5 + 1$

SEGA OD: $\max(T_1+5 ; T_2+4 ; T_3+3 , T_4+2 , T_5+1)$

NAOGJAS MAKSIMUM. TOA VAZIVA

$$C(T) = \max_i (T_i + l_i)$$

(c) $C(T) \geq ld\left(\sum_i 2^{T_i}\right)$

$$2^{T_1} + 2^{T_2} + 2^{T_3} + 2^{T_4} + 2^{T_5} = ?$$

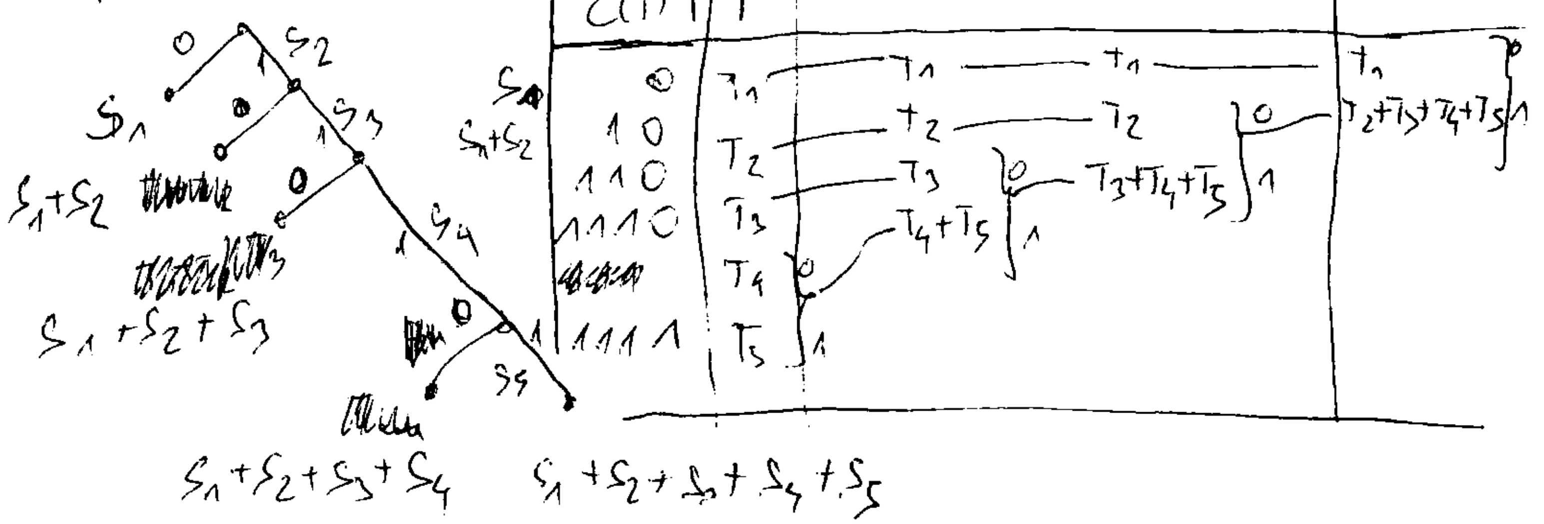
E.G. $T_1 = 1$, $T_2 = 3$, $T_3 = 4$, $T_4 = 8$, $T_5 = 9$

$C := T_i + (n-i) + 1$ $\quad i = 1, 2, \ldots 5$;

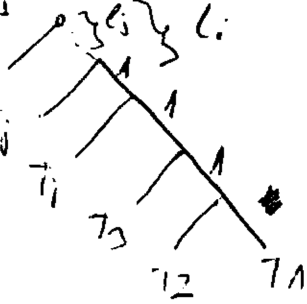
$C = [6, 7, 7, 10, 10]$; $\qquad 2^{T_i} = [2, 8, 16, 256, 512]$

$C(T) = \max[C] = 10$ $\qquad ld\left(\sum_{i=1}^{5} 2^{T_i}\right) = 9.637$

$T_1 \rightarrow$ ANAZOGNO NA P_i



$T_i < T_j$ then $l_i \geq l_j$



PROOF BY CONTRADICTION:

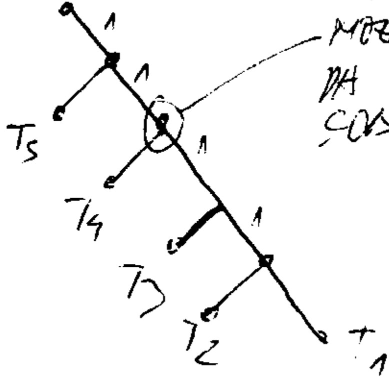
IF $T_i < T_j$ $l_i < l_j$

$$\max(T_i + l_i, T_j + l_j) \geq \max(T_i + l_j, T_j + l_i)$$

$c_i = T_i + (m-i) + 1$ $c_1 = T_1 + (5-1) + 1 = T_1 + 5$

$\boxed{c_2 = T_2 + 4}$ $c_3 = T_3 + 3$ $c_4 = T_4 + 2$ $c_5 = T_5 + 1$

$$\boxed{C(T) = \max_i (T_i + l_i)}$$



MOŽE DOCVATA DA IMICILA COSILANTE!!!

INDUKCIJA

- $T_1 \ T_2 \ T_3 \ T_4 \ T_5$

$c_2 = \max(T_1, T_2) + 1$ $\boxed{i = 2}$

$\boxed{i = j}$ $c_j = \max(c_{j-1}, T_j) + 1$

$i = j+1$ $c_{j+1} = \max(c_j, T_{j+1}) + 1 = \max(\max(c_{j-1}, T_j) + 1, T_{j+1}) + 1$

$|$ IF $c_{j-1} < T_j | = \max(T_j + 1, T_{j+1}) + 1 = |$ IF $T_j + 1 < T_{j+1} | = \underline{T_{j+1} + 1}$

- AKO VO GENERALEN SLUČAJ $C_{j-1} > T_j \ \forall j \Rightarrow$

$c_2 = T_1 + 1$

PLOT $i=j$

$c_j = c_{j-1} + 1 = \underbrace{T_1 + (j-1)}$

$i = j+1$ $c_{j+1} = c_j + 1 = T_1 + j - 1 + 1 = \underline{T_1 + j}$

DOKAZANO!!!

(c) $p = \frac{2^{T_i}}{\sum 2^{T_i}}$ $r_2 = \frac{2^{-l_i}}{\sum 2^{-l_i}}$ $c_1 = \sum 2^{T_i}$ $c_2 = \sum 2^{-l_i} \leq 1$

$$C(T) = \max_i (T_i + l_i) = \max_i (\text{ld}\, p \cdot c_1 \quad \text{ld}\, r \cdot c_2) =$$

$$= \max_i (\text{ld}\, c_1 \quad \text{ld}\, c_2 + \text{ld}\frac{p}{r}) = \text{ld}\, c_1 - \text{ld}\, c_2 + \max\left(\text{ld}\frac{p}{r}\right) \geq$$

$$\geq \text{ld}\, c_1 - \text{ld}\, c_2 + D(p \| r)$$

$$\max_i \left(ld \frac{p}{r} \right) \geq \sum_i p \, ld \frac{p}{r} = D(p \| r)$$

MAXIMUM IS ALWAYS GREATER THEN THE AVERAGE !!!

$$C(T) \geq ldC_1 - ldC_2 + D(p\|r) \qquad -ldC_2 > 0 \;;\; D(p\|r) > 0$$

$$\boxed{C(T) \geq ldC_1 = ld \sum_i 2^{T_i}}$$

DOKAŽANO!!!

(d) SHOW THAT THERE EXIST THE SUCH THAT

$$C(T) \leq ld \left(\sum_i 2^{T_i} \right) + 1$$

- FOR $p_i = r_i$ $\quad C_2 = 1$; $\quad C(T) \geq ldC_1 - \underbrace{ldC_2}_{0} + 0 \Rightarrow$

$$C(T) \geq ld \left(\sum 2^{T_i} \right) \Big\} \; \underline{\text{LOWER BOUND}}$$

$$l_i = \left\lceil ld \frac{1}{p_i} \right\rceil = \left\lceil ld \frac{\sum 2^{T_i}}{2^{T_i}} \right\rceil = \underline{\left\lceil ld \sum 2^{T_i} \right\rceil - T_i}$$

$$l_i + T_i = \left\lceil ld \sum 2^{T_i} \right\rceil \leq \underline{ld \sum_i 2^{T_i} + 1} \quad \forall i$$

$$\sum 2^{-l_i} = \sum 2^{-\left\lceil ld \frac{\sum 2^{T_i}}{2^{T_i}} \right\rceil} = \sum 2^{-\left\lceil ld \frac{1}{p_i} \right\rceil} \leq \sum 2^{-ld \frac{1}{p_i}}$$

$$\leq \sum p_i = 1$$

$$\rightarrow \underline{l_i + T_i \leq ld \left(\sum 2^{T_i} \right) + 1} \quad \forall i \Rightarrow \max_i (l_i + T_i) \leq ld \left(\sum 2^{T_i} \right) + 1$$

$$\boxed{ld \left(\sum 2^{T_i} \right) \leq \max_i (l_i + T_i) \leq ld \left(\sum 2^{T_i} \right) + 1}$$

- NAJMALOTO VREME ZA DOBIVANJE NA REZULTATOT E VO GRANICI $ld \left(\sum 2^{T_i} \right) \ldots ld \left(\sum 2^{T_i} \right) + 1$. ZNAČI ZA OVOJ PROBLEM $ld \left(\sum 2^{T_i} \right)$ E EQUIVALENT NA ENTROPIJA.

$\boxed{5.35}$ GENERATING RANDOM VARIABLES. ONE WANTS TO GENERATE A RANDOM VARIABLE X

$$X = \begin{cases} 1 & \text{WITH PROBABILITY } p \\ 0 & \text{WITH PROBABILITY } 1-p \end{cases}$$

28a

YOU ARE GIVEN FAIR COIN FLIPS $Z_1, Z_2, \dots$. LET N BE THE (RANDOM) NUMBER OF FLIPS NEEDED TO GENERATE X. FIND THE GOOD WAY TO GENERATE X. SHOW THAT $E[N] \le 2$.

$$p = \sum_{i=1}^{N} z_i \cdot 2^{-i} = z_1 2^{-1} + z_2 \cdot 2^{-2} + \dots + z_N 2^{-N}$$

AKO SE SE CONICI

$$! \enclose{circle}{\le} \sum_{i=1}^{N} 2^{-i} = \frac{1}{2} + \frac{1}{2^2} + \dots \frac{1}{2^N} = \frac{1}{2}\left(1 + \frac{1}{2} + \dots \frac{1}{2^{N-1}}\right)$$

$$S = 1 + \frac{1}{2} + \dots + \frac{1}{2^{N-1}}$$

$$\frac{1}{2} S = \frac{1}{2} + \frac{1}{4} + \dots \frac{1}{2^N}$$

$$\left(1 - \frac{1}{2}\right) S = 1 - \frac{1}{2^N}$$

$$\boxed{S = \frac{1 - 2^{-N}}{1 - 2}}$$

$$p \le \frac{1}{2} \frac{1 - \frac{1}{2^N}}{1 - \frac{1}{2}} = \left(1 - \frac{1}{2^N}\right)$$

$$E[N] \le \sum_{N=1}^{\infty} N \left(1 - \frac{1}{2^N}\right) = \sum_{N=1}^{\infty} N - \sum_{N=1}^{\infty} N \frac{1}{2^N} =$$

$$= \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad n = 3 \to \frac{3(3+1)}{2} = 6 \qquad =$$

$$= \frac{N(N+1)}{2} - \frac{\frac{1}{2}}{\left(1 - \frac{1}{2}\right)^2} = \frac{N(N+1)}{2} - \frac{\frac{1}{2}}{\frac{1}{4}} = \frac{N(N+1)}{2} - 2$$

- NAMESTO „p" DA ČEMAM „$1-p$"

$$1 - p \le 1 - \frac{1}{2^N} \qquad p \le \frac{1}{2^N} \qquad E[N] \le \sum_{N=1}^{\infty} N \frac{1}{2^N} = \frac{\frac{1}{2}}{\left(1 - \frac{1}{2}\right)^2}$$

$$\boxed{E[N] \le \frac{\frac{1}{2}}{\frac{1}{4}} = 2}$$

- EDITION 2 SOLUTION:

EXAMPLE $X = \begin{cases} 1 & p = 0.7 \\ 0 & 1 - p = 0.3 \end{cases}$

$p = 0.7 \qquad q = 1 - p = 0.3$

$U_1 = 0.\underline{1011}0011 \qquad U_2 = 0.\underline{0100}1$

$\underline{1011}\,\underline{1011}\,\underline{0100}\,1011\,\underline{0100100}$

$\frac{1}{2} + \frac{1}{8} + \frac{1}{16} = \frac{8+2+1}{16} = \frac{11}{16} = 0.68$

$0.0100 \to \frac{1}{2^2} = \frac{1}{4} = 0.25$

5.36 OPTIMAL WORD LENGTHS.

(a) CAN $l=(1,2,2)$ BE THE LENGTH OF BINARY HUFFMAN CODE. WHAT ABOUT $(2,2,3,3)$?

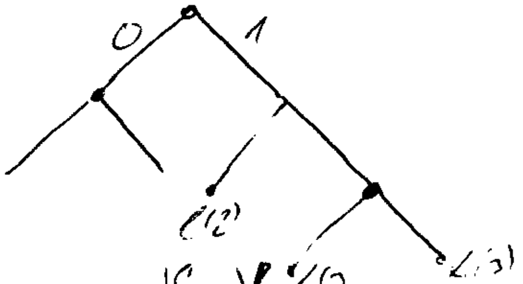
(b) WHAT WORD LENGTHS $l=(l_1,l_2,\ldots)$ CAN ARISE FROM BINARY HUFFMAN CODES

(a)

	p			C(x)	l
x_1	0.5		0.5 }0	0	1
x_2	0.25	}0	0.5 }1	10	2
x_3	0.25	}1		11	2

$l = (1,2,2)$ CAN BE THE LENGTH OF BINARY HUFFMAN CODE

$l = (2,2,3,3)$



$$\sum_{i=1}^{4} D^{-l(i)} \le 1$$

$$\sum_{i=1}^{4} 2^{-l_i} \le 1$$

✓ $\boxed{l=(2,2,3,3)}$

	p(x)			C	l(x)
x_1	0.5	0.5	0.6 }0	1	1
x_2	0.3	0.3 }0	0.5 }1	00	2
x_3	0.2 }0	0.3 }1		010	3
x_4	0.1 }1			011	3

$$\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8} =$$

$$\frac{1}{2}+\frac{1}{4}=\frac{3}{4}<1$$

$$\sum_{i=1}^{4} D^{-l_i} = 1$$

$$l=(1,2,3,3) \Rightarrow$$

FOR D-ADIC DISTRIBUTION

$$\sum_{i=1}^{4} p_i = 1 \qquad \boxed{p_i = D^{-l_i}}$$

$$\sum_{i=1}^{4} 2^{l_i} = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8} = 1$$

(b) ZA DADENATE DOLZINI $l=(l_1,l_2,\ldots,l_n)$ MORA DA POSTOI D-ADIC DISTRIBUCIJA IS $\sum 2^{l_i}=1$ ILI OCEKUVANATA DOLZINA NA KODOT (DOLZINA NA DRVO-TO) E EDNAKVA NA ENTROPIJATA

$$E[l(x)] = H(x)$$

$\boxed{30a}$

38 FIND HUFFMAN D-ARY CODE: $(p_1 p_2 p_3 p_4 p_5 | p_6) = \left(\frac{6}{25}, \frac{6}{25}, \frac{4}{25}, \frac{4}{25}, \frac{3}{25}, \frac{2}{25}\right)$

(a)

X	p					C(x)	l
x_1	6/25	6/25	8/25	11/25	14/25 }0	01	2
x_2	6/25	6/25	6/25	8/25 }0	11/25 }1	10	2
x_3	4/25	5/25	6/25 }0	6/25 }1		000	3
x_4	4/25	4/25 }0	5/25 }1			001	3
x_5	3/25 }0	4/25 }1				110	3
x_6	2/25 }1					111	3

$E[l(x)] = \sum l(x) \cdot p(x) = 2.52$ $\qquad H(X) = 2.493$

(b)

X	p		C(x)	l(x)
x_1	6/25	9/25 }0	1	1
x_2	6/25	6/25 }1	2	1
x_3	4/25	6/25 }2	3	1
x_4	4/25 }0	4/25 }3	00	2
x_5	3/25 }1		01	2
x_6	2/25 }2		02	2
x_7	0 }3		03	2

$E[l_4(x)] = \sum_x l(x) p(x) = 136$ $\qquad H_4(x) = 1.24644$

5.39 ENTROPY OF ENCODED BITS. LET $C: X \to \{0,1\}^*$ ~~BE~~ A NONSINGULAR BUT UNIQUELY DECODABLE CODE

LET X HAVE ENTROPY $H(X)$.

(a) COMPARE $H(C(X))$ TO $H(X)$

(b) COMPARE $H(C(X^n))$ TO $H(X^n)$

X	p	C(x)	p(c(x))
a	1/2	0	1/2
b	1/4	00	1/4
c	1/4	11	1/4

$H(X) = \frac{1}{2} \operatorname{ld} 2 + 2 \frac{1}{4} \operatorname{ld} 4 = \frac{1}{2} + 1 = \frac{3}{2}$

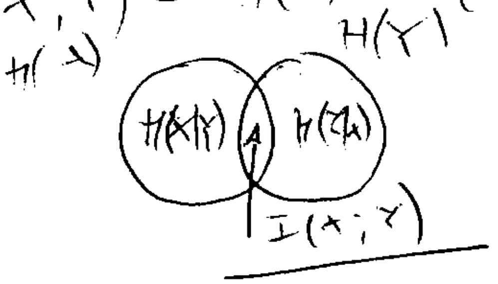
$H(C(X)) = 3/2 = H(X)$!?

EDITION 2 SOLUTIONS:

(a) SINCE THE CODE IS NON-SINGULAR, THE FUNCTION $X \to C(X)$ IS ONE-TO-ONE HENCE:

$$H(X) = H(C(X))$$

$$H(X,Y) = H(X) + H(Y|X) = H(Y) + H(X|Y)$$



$$I(X;Y) = H(X) - H(X|Y)$$
$$= H(Y) - H(Y|X)$$

$$\boxed{Y = f(X)}$$

$$H(X,Y) = H(X) + 0 = H(Y) + H(X|Y)$$

$$\boxed{H(X) \geq H(Y)}$$

(b) $H(X^n) \geq H(C(X^n))$ SINCE $X^n \to C(X^n)$ IS MANY TO ONE

PROBLEM 5.40 CODE RATE

LET X BE A RANDOM VARIABLE WITH ALPHABET $\{1,2,3\}$ AND DISTRIBUTION:

$$X = \begin{cases} 1 & \text{WITH } p = 1/2 \\ 2 & \text{WITH } p = 1/4 \\ 3 & \text{WITH } p = 1/4 \end{cases}$$

THE DATA COMPRESSION CODE FOR X ASSIGNS CODEWORDS:

$$C(x) = \begin{cases} 0 & x = 1 \\ 10 & x = 2 \\ 11 & x = 3 \end{cases}$$

LET $X_1, X_2, \ldots$ BE INDEPENDENT IDENTICALLY DISTRIBUTED ACCORDING TO THIS DISTRIBUTION AND LET $Z_1Z_2Z_3\ldots = C(X_1)C(X_2)\ldots$ BE THE STRING OF BINARY SYMBOLS RESULTING FROM CONCATENATING THE CORRESPONDING CODEWORDS. FOR EXAMPLE: 122 BECOMES 01010.

(a) FIND THE ENTROPY RATE $H(X)$ AND THE ENTROPY RATE $H(Z)$ IN BITS PER SYMBOL. NOTE THAT Z IS NOT COMPRESSIBLE FURTHER.

(b) NOW LET THE CODE BE: $C(X) = \{00, 10, 01\}$ AND FIND THE ENTROPY RATE $H(Z)$.

(c) LET THE CODE BE: $C(X) = \{00, 1, 01\}$ AND FIND THE ENTROPY RATE $H(Z)$.

$\boxed{324}$

(a) $H(\mathcal{X}) = \lim_{n\to\infty} \frac{1}{n} H(X_1, X_2 \dots X_n) = \lim_{n\to\infty} H(X_n | X_1^{n-1})$

$H(\mathcal{X}) = \cancel{\frac{1}{n}} \cdot \cancel{n} \cdot H(X) = \frac{1}{2} ld 2 + \frac{1}{4} ld 4 + \frac{1}{4} ld 4 = 3/2$

$H(\mathcal{Z}) = ?$ $\quad H(Z) = \lim_{n\to\infty} \frac{1}{n} \left[H(C(X_1)) \, H(C(X_2)) \cdot H(C(X_n)) \right]$

$= \frac{1}{n} \cdot n \cdot H(C(X_1)) = H[C(X_1)]$

$H(Z) = -\left(P(0) \cdot ld\, P(0) + P(1) \cdot ld\, P(1) \right)$ $\quad P(0) = ?$ $\quad P(1) = ?$

$P(0) = P(X=1) + P(X=2) \cdot \frac{1}{2} = \frac{1}{2} + \frac{1}{8} = \frac{5}{8}$

$P(1) = P(X=2) \cdot \frac{1}{2} + P(X=3) = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$

$H(Z) = \frac{5}{8} \cdot ld \frac{8}{5} + \frac{3}{8} \cdot ld \frac{8}{3} = \underline{0.95}$

$Y = f(X)$

$H(X,Y) = H(X) + \underbrace{H(Y|X)}_{0} = H(Y) + H(X|Y)$ $\quad \boxed{H(X) \geq H(Y)}$

(b) $C(X) = \{00, 10, 01\}$

$P(0) = \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$P(1) = \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{2} \frac{1}{4} = \frac{1}{8} + \frac{1}{8} = \frac{2}{8} = \frac{1}{4}$

$H(Z) = \frac{3}{4} ld \frac{4}{3} + \frac{1}{4} \cdot 2 = 0.81128$

$P(0) = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{2} + \frac{1}{8} = \frac{5}{8}$

(c) $C(X) = \{00, 1, 01\}$

$P(1) = \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$ $\quad \underline{H(Z) = 0.95}$

EDITION 2 SOLUTIONS:

(a) $H(Z) = H(Bern(1/2)) = 1$ ↗ {PROBABILITIES ARE DIADIC AND THE CODE IS OPTIMAL HENCE $H(Z) = 1$

(b) $H(Z) = \lim_{n\to\infty} \frac{H(Z_1, Z_2, \dots Z_n)}{n} = \lim_{n\to\infty} \frac{H(X_1, X_2 \dots X_{n/2})}{n}$

$H(Z) = \frac{n/2}{n} \cdot H(X) = \frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4}$

$\boxed{33a}$

SUPPOSE WE ENCODE FIRST „n" SYMBOLS $x_1, x_2, \ldots x_n$

$$z_1 z_2 \cdots z_m = C(x_1)\,C(x_2) \cdots C(x_n)$$

$m = L(C(x_1)) + L(C(x_2)) + \cdots + L(C(x_n))$ – TOTAL LENGTH OF THE ENCODED SEQUENCE IN BITS, „L" IS BINARY LENGHT FUNCTION.

$$\underline{\underline{n H(X) =}}\ \underbrace{H(x_1 x_2, \ldots x_n) = H\Big(z_1 z_2 \cdots z_{\sum_1^n L(C(x_i))}\Big)}$$

SINCE CONCATENATED WORD SEQUENCE IS INVERTABLE FUNCTION OF $(x_1, \ldots x_n)$

$$H\Big(z_1 z_2 \cdots z_{\sum_1^n L(C(x_i))}\Big) = E\Big[\sum_{i=1}^{n} L(C(x_i))\Big] \cdot H(Z) = \underline{n E[L(C(X))]\, H(Z)}$$

$$H(Z) = \frac{H(X)}{E[L(C(x_1))]}, \quad E[L(C(x_1))] = \frac{1}{2}\cdot 2 + \frac{1}{4} + 8\cdot\frac{1}{4} = \frac{4+1+2}{4} = 7/4$$

$$H(Z) = \frac{3/2}{7/4} = \frac{4\cdot 2}{14} = \frac{6}{7}$$

OVA E SAMO Z A NE KANONSKO !!

$\boxed{a}$

• SE NAVRAĆAM NA SLEDNATA NA ČLANOT SO REZULTATITE OD „Lee". SOMNITELNI MI SE REZULTATITE ZA $\underline{\underline{N+1+N}}$

$$C_1 = \frac{p_1}{n_c^s R_c^s} = \frac{1}{N R_c^s} = \frac{1}{N \cdot \frac{K}{T}} \qquad \boxed{C_2 = \frac{p_2}{n_c^R R_c^R} = \frac{1}{1\cdot 1} = 1}$$

$$\boxed{C_1 = \frac{T}{N\cdot K}}\,,$$

$$P_e^{ur} = \frac{1}{2}\left(\frac{2}{\rho C_1 + 2C_1 + 2}\right)^N \sum_{i=0}^{N-1} \frac{(N-1+i)!}{i!\,(N-1)!} \left(\frac{2}{\rho + 2/C_1 + 2}\right)^i +$$

$$+ \frac{1}{2}\left(\frac{2}{\rho + 2/C_1 + 2}\right)^N \sum_{j=0}^{N-1} \frac{(N-1+j)!}{j!\,(N-1)!} \left(\frac{2}{\rho C_1 + 2 C_1 + 2}\right)^j =$$

$$= \frac{1}{2}\,\frac{2NK}{\rho T + 2T + 2NK} \sum_{i=0}^{N-1} \frac{(N-1+i)!}{i!\,(N-1)!} \left(\frac{2NK}{\rho NK + 2T + 2NK}\right)^i +$$

$\boxed{34a}$

$$+\frac{1}{2}\left(\frac{2C_1}{C_1\varrho+2+2C_1}\right)^N\sum_{j=0}^{N-1}\frac{(N-1+j)!}{j!\,(N-1)!}\left(\frac{2NK}{\varrho T+2T+2NK}\right)^j=$$

$$=\frac{1}{2}\left(\frac{2NK}{(\varrho+2)T+2NK}\right)^N\sum_{i=0}^{N-1}\frac{(N-1+i)!}{i!\,(N-1)!}\left(\frac{2T}{(\varrho+2)T+2NK}\right)^i+$$

$$+\frac{1}{2}\left(\frac{2T}{\varrho T+2NK+2T}\right)^N\sum_{j=0}^{N-1}\frac{(N-1+j)!}{j!\,(N-1)!}\left(\frac{2NK}{(\varrho+2)T+2NK}\right)^j$$

$$=\frac{1}{2}\sum_{i=0}^{N-1}\frac{(N-1+i)!}{i!\,(N-1)!}\frac{(2T)^i(2NK)^N}{[(\varrho+2)T+2NK]^{i+N}}+\frac{1}{2}\sum_{j=0}^{N-1}\frac{(N-1+i)!}{i!\,(N-1)!}\frac{(2NK)^i(2T)^N}{[(\varrho+2)T+2NK]^{i+N}}$$

$$=\frac{1}{2}\sum_{i=0}^{N-1}\frac{(N-1+i)!}{i!\,(N-1)!}\frac{1}{[(\varrho+2)T+2NK]^{i+N}}\left[(2NK)^N(2T)^i+(2NK)^i(2T)^N\right]$$

$$=\frac{1}{2}\sum_{i=0}^{N-1}\frac{(N-1+i)!}{i!\,(N-1)!}\frac{(2T\cdot 2NK)^i}{[(\varrho+2)T+2NK]^{i+N}}\left[(2NK)^{N-i}+(2T)^{N-i}\right]$$

$$P_e^{ur}\overset{\varrho\to\infty}{=}\frac{1}{2}\left(\frac{2}{\varrho C_1}\right)^N+\frac{1}{2}\left(\frac{2}{\varrho}\right)^N=\frac{1}{2}\left(\frac{2NK}{\varrho T}\right)^N+\frac{1}{2}\left(\frac{2}{\varrho}\right)^N$$

$$P_e^{ur}=\frac{1}{2}\sum_{i=0}^{N-1}\frac{(N-1+i)!}{i!\,(N-1)!}\frac{(4TNK)^i}{[(\varrho+2)T+2NK]^{i+N}}\left[(2NK)^{N-i}+(2T)^{N-i}\right]$$

$\boxed{\text{OK}}$

$N\times N\times N$

$$P_e^{ur}=\left(\frac{2}{\varrho C_1+4}\right)^{N^2}\sum_{i=0}^{N^2-1}\frac{(N^2-1+j)!}{i!\,(N^2-1)!}\left(\frac{2}{\varrho C+4}\right)^i=$$

$$=\left(\frac{2NK}{\varrho T+4NK}\right)^{N^2}\sum_{i=0}^{N^2-1}\frac{(N^2-1+j)!}{i!\,(N^2-1)!}\left(\frac{2NK}{\varrho T+4NK}\right)^i;\quad P_e^{ur}\overset{\varrho\to\infty}{=}\frac{2NK}{\varrho T}$$

PROBLEM 5.44 | HUFFMAN. FIND THE WORD LENGTHS OF THE OPTIMAL BINARY ENCODING OF

$$X = \left(\frac{1}{100}, \frac{1}{100}, \dots, \frac{1}{100}\right)$$

X				c(x)	l(x)
x_1	1/4	1/2	1/2 0	00	2
x_2	1/4	1/4 0	1/2 1	01	2
x_3	1/4 0	1/4 1		10	2
x_4	1/4 1			11	2

$$\boxed{E[l(X)] = \sum_{i=1}^{4} 2 \cdot \frac{1}{4} = 2}$$

$$\frac{1}{4} + \frac{2}{4} + \frac{\cancel{3}}{4} = \frac{3}{4} = 2.25$$

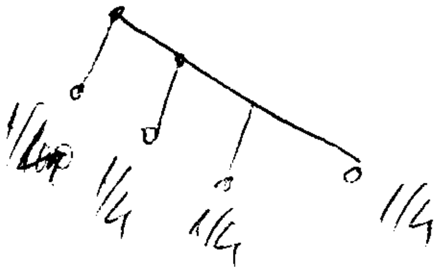
$3/4 = 0.75$

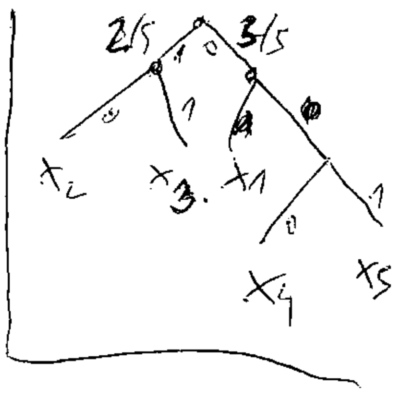
x_1	1/5	2/5	2/5	3/5 0	01	1
x_2	1/5	1/5	2/5 0	2/5 1	10	
x_3	1/5	1/5 0	1/5 1		11	
x_4	1/5 0	1/5 1			000	
x_5	1/5 1				001	

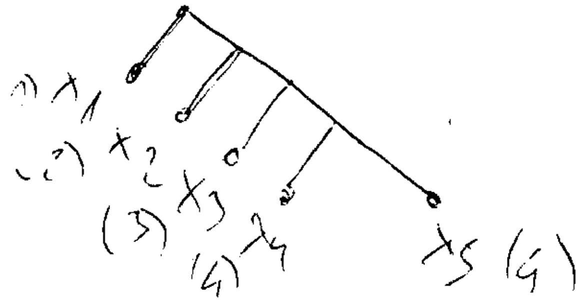
$$\left(\frac{1}{5} 2\right) 3 + \left(\frac{1}{5} \cdot 3\right) 2 = \frac{6}{5} + \frac{6}{5} = \frac{12}{5} = 2.4$$

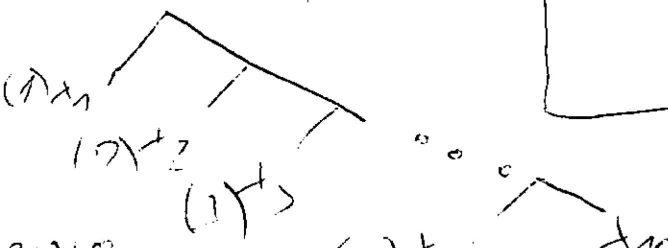
$H(X) = ld\, 5 = 2.32$

$$ld\left(\frac{1}{100}\right)^{-1} = 6.64$$









$$\frac{1}{5} + \frac{2}{5} + \frac{3}{5} + 2\,\frac{4}{5} = \frac{1+2+3+8}{5}\ \frac{14}{5} = 2.8$$

$+\,99/100$

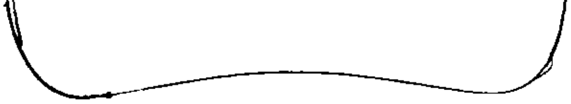
$+\,99/100$

$$\frac{1}{100}(1+2+\dots+99) + \frac{99}{100} = \frac{1}{100}\ \frac{N(N+1)}{2} = \frac{99(100)}{200} = 49.5 +$$

$$+\ 99/100 = 49.5 + 0.99 = 50.49$$

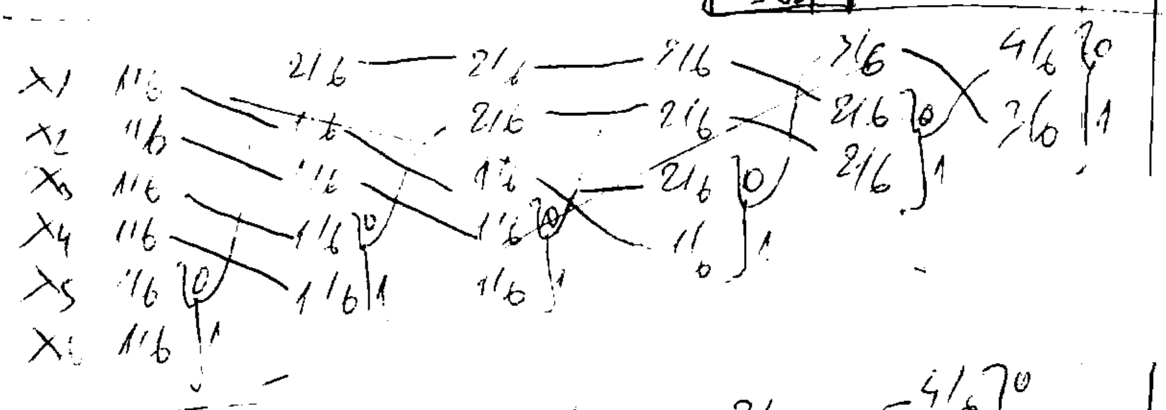
GENERALIZE

$$\frac{1}{N}\ \frac{(N-1)\cdot N}{2} + \frac{N-1}{N} = \frac{N^2 - N + 2N - 2}{2N} = \frac{N^2 + N - 2}{2N}$$



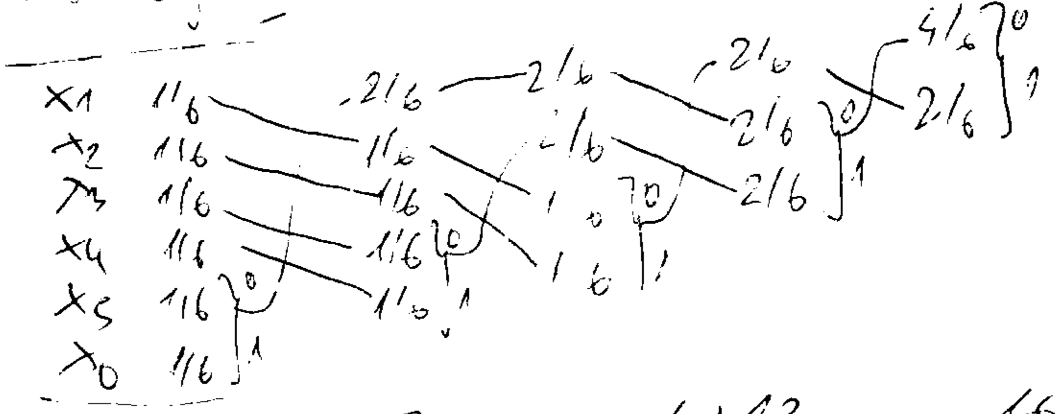
.3.6a

c(x)



000
001

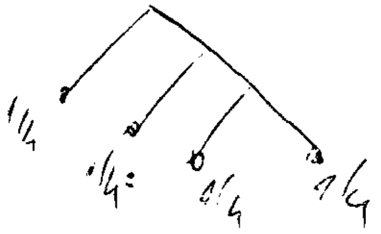
10 C
11 2
010 3
011 3
000 3
001 3

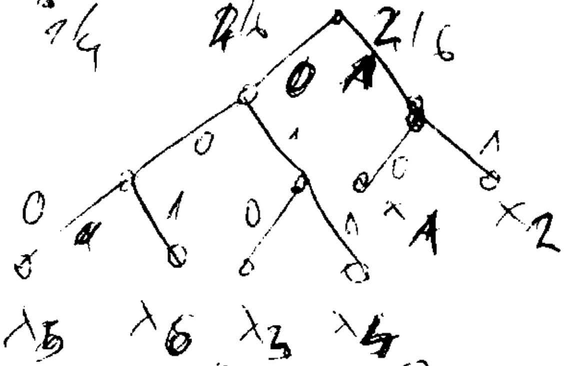


$$\Sigma \ \frac{2}{6} + 4 \cdot \frac{3}{6} = \frac{4+12}{6} = \frac{16}{6} = \underline{2.66}$$

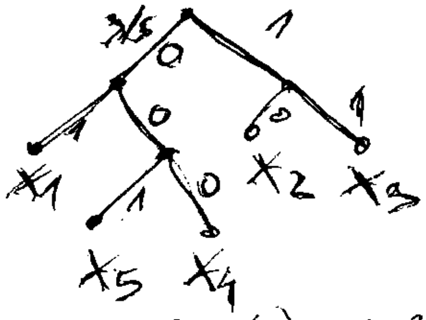
$$E[T] = 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{2} =$$

$$= \frac{1+2+6}{4} = \frac{9}{4} = \underline{2.25}$$

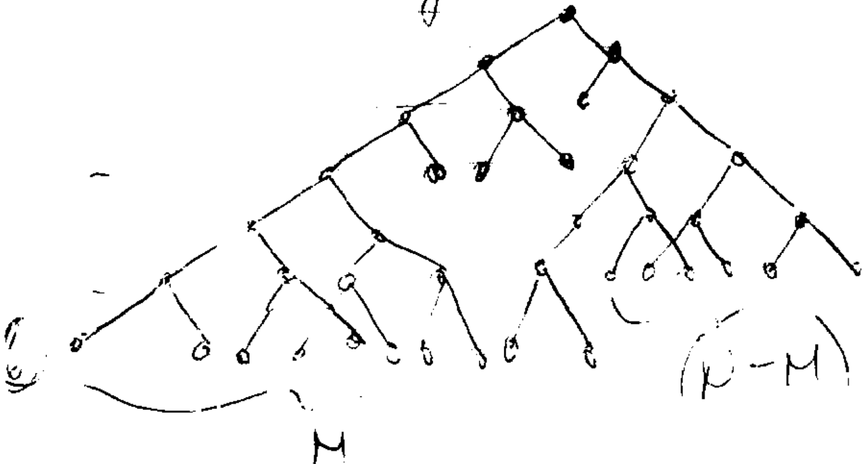




$\log_2(6) = 2.585$



$\log_2(5) = 2.32$



$\boxed{~~37~~a}$

$\boxed{N=100}$ $M = 2^n$ $2^6 = 64$ $2^5 = 32$

$$\frac{6}{100} \cdot 64 + \frac{5}{100} 36 = \frac{1}{100}(6 \cdot 64 + 5 \cdot 36)$$

$$\boxed{M = 2^{\lfloor ld\,100 \rfloor} + \sum_{i=1}^{n} 2^i \quad n \leq \lfloor ld\,100 \rfloor} \Rightarrow \text{AKO } H(X) = X.Y \quad Y > 5$$

$$E[l] = \min \left(\frac{7M + 6(N-M)}{N} \right)$$

$\lceil ld\,100 \rceil$ $\lfloor ld\,100 \rfloor$

$$\text{s.t.} \quad E[l] \geq H(X) = ld\,N$$

- So PLODA VO MAKE DUBIV $n-1$ i.e

$$M = 2^6 + 2 = 64 + 2 = 66$$

$$E[l] = \min \frac{7 \cdot 66 + 6 \cdot 34}{100} = 6.66000$$

- TEST ~~ZA~~ $\boxed{N = 6}$ $ld\,6 = 2.585$

$$M = 2^{\lfloor 2.585 \rfloor} + \sum_{i=1}^{n} 2^i \overset{n=1}{=} 2^2 + 2 = 4 + 2 = 6 \,\#$$

$$\overset{n=0}{=} 2^2 + ~~2~~ = 4$$

$$E[l] = \min \frac{\lceil ld\,6 \rceil \cdot M + \lfloor ld\,6 \rfloor (6-M)}{6} = \frac{3 \cdot 4 + 2 \cdot 2}{6} = \frac{16}{6} = 2.66$$

CONTINUE FROM 15.1.17

$\boxed{a}$

BROJOT NA PODMNOZESTVA VO DADENO MNOZESTVO SE [illegible]

DETERMINISTIC QUESTIONS

$$n = \sum_{i=0}^{n} \binom{n}{i} = 2^n \quad \boxed{n=3} \quad \{1\ 2\ 3\} \quad \{3\ \emptyset\} \quad \{\emptyset\ 1\} \quad \{2,3\}$$

$m = 2^n \rightarrow$ NE MOZE n_1 DA BIDE NEJAKN.

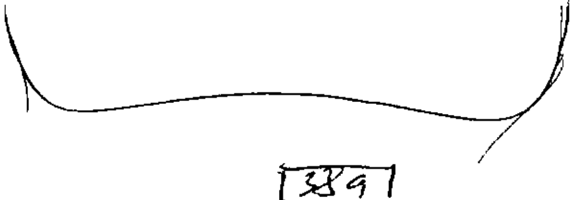
Z-TOA: $\{1,2,3,4\} = \{1\ 2\} + \{3\ 4\}$ S_1 S_2

i.e. $= \{1\}, \{2\}$

1° YES?

2° $X \in S_2$ YES

3° $X \in S_1$ YES END



$\boxed{389}$

$\{1,2,3,4,5,6,7,8\}$

$X \in S_1$? YES

$X \in S_{11}$? YES

$X \in \{1\}$? YES

$\{1,2,3,4\}$ S_1 $\{5,6,7,8\}$ S_2

$\{1\ 2\}$ $\{3\ 4\}$ S_{11} S_{12}

$\{1\}$ $\{2\}$ $\{3\}$ $\{4\}$

S_{111} S_{112} S_{121} S_{122}

KLASIČEN SLICE CODING !!! ⊛

$\boxed{4}$

- X IS UNIFORMLY DISTRIBUTED OVER $\{1, 2, \dots m\}$ i.e

$p = \{\frac{1}{m}, \frac{1}{m}, \dots \frac{1}{m}\}$ $\quad p(X=x_1) = \frac{1}{n} = p(X=x_2)$

- THE PROBLEM STATES ALL 2^n SUBSETS ARE EQUALLY LIKELY

OVA E NORMALNO ZATOA ŠTO IMAŠ UNIFORMNA RASPREDELBA !!!

e.g. (REFER TO ⊛) $\quad P(S_1) = \frac{1}{8}\cdot 4 = \frac{1}{2} \quad P(S_2) = \frac{1}{8}\cdot 4 = \frac{1}{2}$

$P(S_{11}) = \frac{1}{4}\cdot\frac{1}{2} = \frac{1}{8} \quad P(S_{12}) = \frac{1}{4}\cdot\frac{1}{2} = \frac{1}{8}$

- BROJOT NA DETERMINISTIČKI PRAŠANJA TREBA DA BIDE EDNAKOV NA $n = \cancel{2}$ ⊛ $m = 2^n = 8$ $\underline{n=3}$

- DA PRETPOSTAVIME DEKA IZVLEČENIOT $X=2$

$\{1,2,3,4\}$ $S_1(0)$ $\quad \{5,6,7,8\}$ $S_2(1)$

$\{1,2\}, \{3,4\}$ $S_{11}(0)$ $S_{12}(1)$ $\quad \{5,6\}, \{7,8\}$ $S_{21}(0)$ $S_{22}(1)$

$\{1\}\{2\}$ $\{3\}\{4\}$ $\quad \{5\}\{6\}$ $\{7\}\{8\}$

S_{111} S_{112} S_{121} S_{122} $\quad S_{211}$ S_{212} S_{221} S_{222}

$X \in S_2$ No

$X \in S_{12}$ No

$X \in S_{11}$ No

SO 3 PRAŠANJA OTKRIVAŠ DEKA E IZVLEČEN 2

- PRISTAPOT E IDENTIČEN SO ONOJ VO PROBLEM 5.9

ŠE DOKAZUVA SO INDUKCIJA

$n=0$ $\{1\}$ 0 PRAŠANJE

$n=1$ $\{1, 2\}$ 1 PRAŠANJE

$n=2$ $\{1,2,3,4\}$ 2 PRAŠANJA $(2^n=4)$

...

$n=k$ $\{1,2,3, \dots 2^k\}$ → k PRAŠANJA ⊛

- PRETPOSTAVKA DEKA ⊛ E TOČNO →

$n=k+1$ $\{1,2, \dots 2^k, 2^k+1, 2^k+2, \dots 2^{k+1}\}$

- SO EDNO DOPOLNITELNO PRAŠANJE GORNOTO MNOŽESTVO GO

3/4

SLEDUVA NA MNOZESTVO KOE IMA 2^k ELEMENTI:

$$\underbrace{\{1, 2, \dots 2^k\}}_{\text{ILI SE IZBIRA OVA}} \quad \underbrace{\{2^k+1, 2^k+2, \dots 2^{k+1}\}}_{\text{ILI SE IZBIRA OVA MNOZESTVO}}$$

A NATAMOSNO [illegible] DELA OD VAKVO MNOZESTVO SO „K" ELEMENTI, AKO EDEN ELEMENT SE IZBERE SO NAJMALKU „SLUCAEN IZBOR", TOJ MOZE DA SE IDENTIFIKUVA SO „K" PRASANJA. PROSECNATA DOLZINA NA PRASANJA SO KOJA SE IDENTIFIKUVA IZBRANIOT ELEMENT E:

$$H(X) \le E[Q] \le H(X)+1 \qquad H(X) = n \cdot \frac{1}{n}\, ld\, n = ld\, n$$

$$ld\, 2^n \le E[Q] \le ld\, 2^n + 1 \qquad \boxed{n \le E[Q] \le n+1}$$

(B) SUPPOSE THAT A=1. WHAT IS THE PROBABILITY THAT OBJECT 2 KEEPS THE SAME ANSWERS FOR „K" QUESTIONS AS DOES OBJECT „1"?

$$P(k) = 1 - \sum_{i=1}^{k+1} \left(\frac{1}{2}\right)^i = 1 - \left[\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^{k+1}}\right] = \frac{\frac{1}{2} - \left(\frac{1}{2}\right)^{k+1}}{1 - \frac{1}{2}} = \frac{2}{2}\left(1 - \left(\frac{1}{2}\right)^k\right.$$

e.g. $k = \dots \quad k \le n-1$

GO IZBIRA „N" = ZBOG ZA K=N SE DOBIVA RAZLIKA [illegible]

$$\boxed{P(k) = \frac{1}{2^k}}$$

$$P(k) = \frac{1}{2^2} = \frac{1}{4}$$

[illegible] ①

e.g. $k = 3 \quad P(k) = \frac{1}{2^3} = \frac{1}{8} \qquad \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$

$1 - \frac{1}{2} - \frac{1}{4} = \frac{4-2-1}{4} = \frac{1}{4} \qquad S_2 \ S_{12} \ S_{11}$

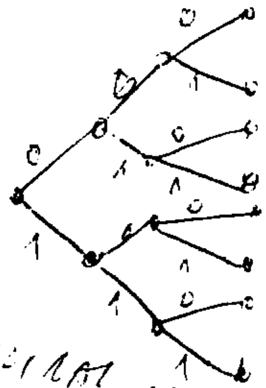
(C) EXPECTED NUMBER OF OBJECTS IN $\{2, 3, \dots n\}$ THAT HAVE THE SAME ANSWERS TO THE QUESTIONS AS DOES CORRECT OBJECT (1).

$\{2, 3, 4, 5, 6, 7, 8\}$

$\{2, 3, 4\} \ \{5, 6, 7, 8\}$

$\{2\} \ \{3, 4\} \ \{5, 6\} \ \{7, 8\}$

$\{1\}\{2\} \ \{3, 4\} \ \{5\}\{6\} \ \{7\} \ \{8\}$



BROJOT NA OBJEKTI E

$$\sum_{i=1}^{n-1} i \frac{1}{n-1} = 1 \qquad \neq$$

MISLAM SAMO OBJEKTOT „2"

PRASANJETO BI GO FORMULIRAL KAKO OPISUVANJE NA INDEKS. DO „1" MOZE DA SE STIGNE NA SLEDNIVE NACINI.

(D)
NE NE NE (0 0 0)
✓ NE NE DA (0 0 1)
NE DA NE (0 1 0)
✓ NE DA DA (0 1 1)
✓ DA NE DA (1 0 1)
✓ DA DA DA (1 1 1)
DA NE NE (1 0 0)
DA DA NE (1 1 0)

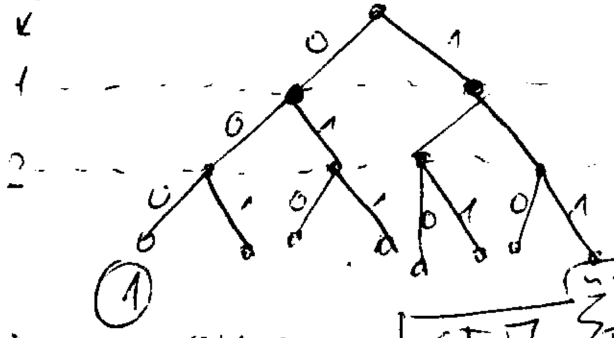
IMAJKI GO OVA VO PREDVID, AKO SE NAVRATAM NA (A), BI DOBIL DEKA $P(X=1) \cdot \frac{1}{2} \quad \frac{1}{1} \cdot \frac{1}{2} = \frac{1}{2}$ EDNA OD 4TE MOZNI.

$\boxed{40a}$

- ZNAČI PO „1" IMA 2^{n-1} MOŽNI PATI. POSLEDNOTO PRAŠANJE TREBA DA BIDE ODGOVOR DA = 1 A SITE DRUGI PRETHODNI ĆI IMAAT SITE MOŽNI KOMBINACII.
- ODGOVOR NA (b) MISLAM E: $n-1$ OBJEKTI

(d) MISLAM DEKA ~~ZA~~ n-TI SE OTKRIVA VISTINSKIOT OBJEKT ZOŠTO ~~ZA~~ TOA SE POVEĆI „n" PRAŠANJA

(e) KJE SE NAVRATAM NA (c). AKO SE NAPREŠ DRVOTO NA POTENCIJALNI POSLEDNI NA NIVOTO „k" OBJEKTI E:



$$E[N] = P(K=1)\cdot \overset{2^{n-1}-1}{3} + P(K=2)\cdot 1$$

$$= \frac{1}{2}\cdot 3 + \frac{1}{4}\cdot 1 = \frac{6+1}{4}$$

$$= \frac{7}{4} = 1.75 \;/\!/$$

VO GENERALEN SLUČAJ ĆE IMAŠ:

$$\boxed{E[N] = \sum_{i=1}^{n-1} P(K=i)\cdot\left(2^{n-i}-1\right)}$$

TEST: $E[N] = \sum_{i=1}^{n-1} P(K=i)\left(2^{n-i}-1\right)$ $\quad P(K) = \frac{1}{2^K}$

$\boxed{n=3}$ $E[N] = \sum_{i=1}^{2} P(K=i)\left(2^{n-i}-1\right) = \frac{1}{2}\cdot\left(2^{3-1}-1\right) + \frac{1}{4}\underbrace{\left(2^{3-2}-1\right)}_{1}$

$$= \frac{1}{2}\cdot 3 + \frac{1}{4} = 1.75$$

$\boxed{n=4}$ $E[N] = \frac{1}{2}\cdot\underbrace{\left(2^3-1\right)}_{7} + \frac{1}{4}\underbrace{\left(2^2-1\right)}_{3} + \frac{1}{8}\underbrace{\left(2-1\right)}_{1}$

- GENERALNA FORMULA ZA E[N]:

$$E[N] = \sum_{i=1}^{n-1} \frac{1}{2^i}\left(2^{n-i}-1\right) = \sum_{i=1}^{n-1}\left(2^{n-2i} - 2^{-i}\right) =$$

$$= 2^n \sum_{i=1}^{n-1} (4)^{-i} - \sum_{i=1}^{n-1} 2^{-i} = 2^n \frac{\frac{1}{4}-\left(\frac{1}{4}\right)^n}{\frac{3}{4}} - \frac{\frac{1}{2}-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}$$

41a

$$\boxed{\mathcal{E}[N] = 2^n\frac{1}{3}\left(1-\frac{1}{4^{n-1}}\right) - \left(1-\frac{1}{2^{n-1}}\right) = \frac{2^n}{3} - \frac{2^n}{3\cdot 2^{2n-2}} - 1 + 2^{-n+1} = \frac{2^n}{3} - \frac{1}{3\cdot 2^{n-2}} - 1 + 2^{1-n}}$$

EXPECTED NUMBER OF WRONG OBJECTS

(d) $\boxed{n+\sqrt{n}}$

$$\boxed{\mathcal{E}[N] = \frac{2^{n+\sqrt{n}}}{3} - \frac{1}{3\cdot 2^{n+\sqrt{n}-2}} - 1 + 2^{1-n-\sqrt{n}}}$$

→ EXPECTED NUMBER OF WRONG OBJECTS AGREEING WITH THE ANSWER !!

(e) SHOW THAT THE PROBABILITY OF ERROR (ONE OR MORE OBJECTS REMAINING) GOES TO ZERO AS $n \to \infty$.
USE MARKOV INEQUALITY.

$$\boxed{\Pr\{X \geq t\mu\} \leq \frac{1}{t}} \qquad \mu = \mathcal{E}[X]$$

GENERALIZATA FORMA ZA MARKOVOTO NERAVENSTVO & [illegible]
VO PROBLEM 3.1

$$\Pr\{X > \delta\} \leq \frac{\mathcal{E}[X]}{\delta} \qquad F(\xi \leq x) = \int p(x)\,dx$$

$$p(x) = \frac{dF(x)}{dx} \qquad \mathcal{E}[X] = \int_{-\infty}^{\infty} x\,p(x)\,dx = \int_{-\infty}^{\infty} x\,dF = \quad \text{IF: } p(x)=0,\ x<0$$

$$= \int_0^{\infty} x\,dF = \int_0^{\delta} x\,dF + \int_{\delta}^{\infty} x\,dF \geq \int_{\delta}^{\infty} x\,dF \geq \int_{\delta}^{\infty} \delta\,dF$$

$$= \delta\int_{\delta}^{\infty} dF = \delta\Pr(x>\delta) \Rightarrow \Pr(x>\delta) \leq \frac{\mathcal{E}[x]}{\delta}$$

. DA DA NAJDAM VEROJATNOSTA NA GREŠKA: MISLAM DEKA
ĶE SE DOBIE AKO SE POMNOŽI OČEKUVANIOT BROJ NA
GREŠKI PO VEROJATNOSTA „X" DA ~~ZAFATI~~ VLEZOT
OD $\{2, \ldots, n\}$ i.e.

$$\frac{1}{2^n} = P(\cancel{X}=2) \qquad p = \left\{\frac{1}{2^n}\ \frac{1}{2^n}\ \ldots\ \frac{1}{2^n}\right\}$$

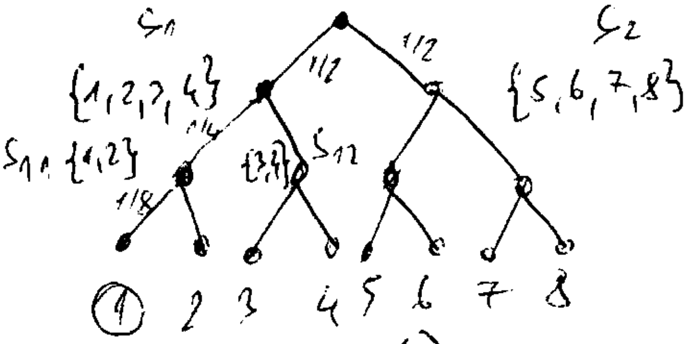
$$\Pr = \left(\frac{2^n}{3} - \frac{1}{3\,2^{n-2}} - 1 + 2^{1-n}\right)\frac{1}{2^n}$$

$$\Pr = \frac{1}{3} - \frac{1}{3\,2^{2n-2}} - \frac{1}{2^n} + \frac{1}{2^{2n-1}} \xrightarrow{n\to\infty} \frac{1}{3}$$

VEROJATNOSTA NA GREŠKA DA OSTANAT EDEN ili POVEKE ODGOVORI E
LITERM

$P(x) = \frac{1}{2^k}$

$$\sum_{i=1}^{n-1} \frac{1}{2^k} = \frac{\frac{1}{2} - \frac{1}{2^n}}{1 - \frac{1}{2}} = \frac{2}{2}\left(1 - \frac{1}{2^{n-1}}\right) = 1 - \frac{1}{2^{n-1}}$$



$P(S_{122})$

$$P(\varepsilon) = \underbrace{\frac{1}{2}}_{P(S_1)} \cdot \underbrace{\frac{1}{4}\frac{1}{8}}_{P(S_{12})} + \underbrace{\frac{1}{2}}_{P(S_1)} \cdot \underbrace{\frac{1}{4}}_{P(S_{11})} + ~~\frac{1}{2}\frac{1}{4}\frac{1}{8}~~$$

$$+ \underbrace{\frac{1}{2}}_{P(S_2)} + 1/2$$

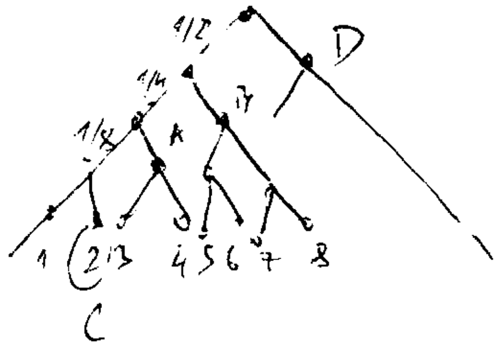
$$= \left(\frac{1}{2}\frac{1}{4}\frac{1}{8} + \frac{1}{2}\frac{1}{4}\right) = \frac{1}{2} + \frac{1}{4} + \frac{1}{4}\frac{1}{8}$$

$\frac{1}{2}\left(1 + \frac{1}{4} + \frac{1}{32}\right)$

$P(\varepsilon) =$ ~~$\frac{1}{2^{n-1}}\left(1 + \frac{1}{2^n}\right)$~~ $= \underline{0,~~28125~~ 6410}$

$n=2$ $P(\varepsilon) = 0,64063$

$\boxed{n\neq 2}$



$$P(\varepsilon) = \underbrace{\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{8}\cdot\frac{1}{16}}_{C} + \underbrace{\frac{1}{2}\frac{1}{4}\cdot\frac{1}{8}}_{A} + \underbrace{\frac{1}{2}\cdot\frac{1}{4}}_{B} + \underbrace{\frac{1}{2}}_{D}$$

3 MOŽNI GREŠKI

$$P(\varepsilon) = \frac{1}{2}\left(1 + \frac{1}{4} + \frac{1}{4}\frac{1}{8} + \frac{1}{4}\cdot\frac{1}{8}\frac{1}{16}\right)$$

$= 0\underline{64}163$

DO OVAA DECIMALA ĆE SE EKSPLOATIRA ZA $n \gg$

$$P(\varepsilon) = \frac{1}{2}\sum_{i=1}^{n} \frac{1}{(2\cdot 4\cdot 8\cdots)} + \frac{1}{2}$$

$4\cdot 8\cdot 16$

$(2k)!! = 2\cdot 4\cdot 6\cdots$

$$P(\varepsilon) = \frac{1}{2} + \frac{1}{2}\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{8} + \frac{1}{2}\frac{1}{4}\frac{1}{8}\cdot\frac{1}{16}$$

$$= \sum_{i=1}^{n} \prod_{j=1}^{i} \left(\frac{1}{2}\right)^j$$

$2^1, 2^3, 2^6, 2^{10}, 2^{15}, 2^{21}, 2^{28}$

$\frac{i(n+1)}{2}$

$[1, 3, 6, 10, 15, 21, 28, \ldots, i, \ldots] = \frac{1}{2}(i+1)^2 - \frac{1}{2}i - \frac{1}{2}$

$\boxed{439.}$

$$P(\varepsilon) = \sum_{i=1}^{n} \prod_{j=1}^{i} \left(\frac{1}{2}\right)^{j} = \sum_{i=1}^{n} 2^{-\frac{i(i+1)}{2}}$$

$n_1 = n + \sqrt{n}$ $\qquad n^2 + n = (n+\sqrt{n})^2 + n + \sqrt{n} = n^2 + 2n\sqrt{n} + 2n + \sqrt{n}$

$$= \underline{n(n+2) + \sqrt{n}(2n+1)}$$

$$\lim_{n\to\infty} P(\varepsilon) = \lim_{n\to\infty} \sum_{i=1}^{n} 2^{-\frac{n(n+1)}{2}}$$

$\boxed{9}$

$$P(\varepsilon) = \underbrace{P(k=1)\left[\underline{P(S_{11})}\cdot P(S_{112}) + \underline{P(S_{12})}\cdot P(S_{121}) + P(S_{12})\cdot P(S_{122})\right]}_{} + P(k=2)\cdot P_{...}$$

ОВА Е ЗА $n=3$ Т.Е $\{1, 2, 3, \ldots 8\}$ $m = 8$

$$P(\varepsilon) = \frac{1}{2}\left[\underbrace{\frac{1}{4}\cdot\frac{1}{8} + \frac{1}{4}\cdot\frac{1}{8} + \frac{1}{4}\cdot\frac{1}{8}}_{}\right] + \frac{1}{4}\cdot\frac{1}{8}$$

ПОТЕНЦИРАЊ ГРЕШКА 1 + ~~КЕДОПНОЗ~~

БРОЈОТ НА ПОТЕНЦИРАНИ ГРЕШКИ Е 3

$$P(\varepsilon) = P(k=1)\cdot 3\cdot\frac{1}{4}\cdot\frac{1}{8} + P(k=2)\cdot 1\cdot\frac{1}{8}$$

$$P(\varepsilon) = \sum_{i=1}^{3-1} P(k=i)\cdot\left(2^{n-i}-1\right)\left(\frac{1}{2}\right)^{\frac{i(i+1)}{2}+1} = P(k=1)\cdot\underbrace{\left(2^{3-1}-1\right)}_{}\left(\frac{1}{2}\right)^{2} + P(k=2)\cdot\underbrace{\left(2^{3-2}-1\right)}_{1}\frac{1}{2}$$

$$P(\varepsilon) = \frac{1}{2}\sum_{i=1}^{3-1} P(k=i)\left(2^{n-i}-1\right)\left(\frac{1}{2}\right)^{\frac{(n-i-1)(\ldots i - 3 + i)}{2}} = \frac{1}{2}\sum_{i=1}^{2} P(k=i)\left(2^{n-i}-1\right)\left(\frac{1}{2}\right)^{\frac{(n-i)\ldots}{2}}$$

$$= \frac{1}{2}\left[P(k=1)\cdot 3\cdot\left(\frac{1}{2}\right)^{\frac{(3-1)\cdot(2-2)}{2}} + P(k=2)\cdot 1\cdot\left(\frac{1}{2}\right)^{(1-2)\cdot 0}\right] \quad \#$$

$$\boxed{P(\varepsilon) = \frac{1}{4}\sum_{i=1}^{3-1} P(k=i)\left(2^{n-i}-1\right)\left(\frac{1}{2}\right)^{\frac{(n-i)(n-i+1)}{2}}} = \frac{1}{4}\left[P(k=1)\cdot 3\left(\frac{1}{2}\right)^{\frac{2\cdot 3}{2}} + P(k=1)\left(\frac{1}{2}\right)^{\frac{\ldots}{2}}\right.$$

$$\boxed{P(\varepsilon) = P(k=1)\cdot 3\cdot\frac{1}{4}\cdot\frac{1}{8} + P(k=2)\cdot\frac{1}{8}} \qquad \boxed{P(\varepsilon) = \frac{1}{4}\sum_{i=1}^{n-1} P(k=i)\cdot\left(2^{n-i}-1\right)\frac{1}{2}^{\frac{(n-i)\ldots}{2}}}$$

$\boxed{449}$

$n=4$

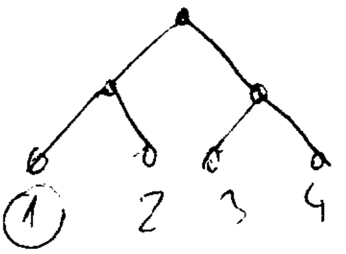
$$P(\mathcal{E}) = \frac{1}{2}\Big[\underbrace{\frac{1}{4}\cdot\frac{1}{8}\cdot\frac{1}{16} + \frac{1}{4}\cdot\frac{1}{8}\cdot\frac{1}{16} \cdots\cdots \frac{1}{4}\cdot\frac{1}{8}\cdot\frac{1}{16}}_{7\times}\Big] + \frac{1}{4}\Big[\frac{1}{8}\cdot\frac{1}{16} + \frac{1}{8}\cdot\frac{1}{16} + \frac{1}{8}\cdot\frac{1}{16}\Big]$$

$$+ \frac{1}{8}\Big[\frac{1}{16}\Big]$$

$$P(\mathcal{E}) = \frac{1}{4}\sum_{i=1}^{3} P(k=i)\cdot\left(2^{4-1}-1\right)\left(\frac{1}{2}\right)^{\frac{(4-i)(4-i+1)}{2}}$$

$$P(\mathcal{E}) = \frac{1}{4}\left[\frac{1}{2}\cdot 7\cdot\left(\frac{1}{2}\right)^{\frac{3\cdot4}{2}}\right] + \frac{1}{4}\left[\frac{1}{4}\cdot 3\cdot\left(\frac{1}{2}\right)^{\frac{2\cdot3}{2}}\right] + \frac{1}{8}\frac{1}{4}\left(\frac{1}{2}\right)^{\frac{1\cdot2}{2}}$$

$$= \frac{1}{4}\cdot\frac{1}{2}\cdot 7\cdot\frac{1}{64} + \frac{1}{4}\frac{1}{4}\cdot 3\cdot\frac{1}{8} + \frac{1}{8}\cdot\frac{1}{4}\cdot\frac{1}{2}$$



$\boxed{n=2}$ $P(\mathcal{E}) = P(k=1)\cdot\frac{1}{2^{k+1}} = \frac{1}{2}\cdot\frac{1}{4}$

$\boxed{n=3}$ $P(\mathcal{E}) = \frac{1}{2}\left[\frac{1}{4}\cdot\frac{1}{8}\cdot 3\right] + \frac{1}{4}\cdot\frac{1}{8}$

$\boxed{k=1}$ $\boxed{k=2}$

$\boxed{n=4}$ $P(\mathcal{E}) = \frac{1}{2}\left[\frac{1}{4}\cdot\frac{1}{8}\cdot\frac{1}{16}\cdot 7\right] + \frac{1}{4}\left[\frac{1}{8}\cdot\frac{1}{16}\cdot 3\right] + \frac{1}{8}\cdot\frac{1}{16}$

$k=1$ $k=2$ $k=3$

$$P(\mathcal{E}) = \sum_{i=1}^{n-1} P(k=i)\cdot\left(2^{n-i}-1\right)\cdot\prod_{j=i}^{n-1}\left(\frac{1}{2}\right)^{j+1}$$

$\boxed{i=1}$ $\prod_{j=i}^{4-1}\left(\frac{1}{2}\right)^{j+1} = \left(\frac{1}{2}\right)^{2}\left(\frac{1}{2}\right)^{3}\cdot\left(\frac{1}{2}\right)^{4} = \frac{1}{4}\cdot\frac{1}{8}\cdot\frac{1}{16}$

$\boxed{i=2}$ $\prod_{j=i}^{4-1}\left(\frac{1}{2}\right)^{j+1} = \left(\frac{1}{2}\right)^{3}\cdot\left(\frac{1}{2}\right)^{4} = \frac{1}{8}\cdot\frac{1}{16}$

$\boxed{i=1}$ $\prod_{j=1}^{n-i}\left(\frac{1}{2}\right)^{n-j+1} = \left(\frac{1}{2}\right)^{4}\cdot\left(\frac{1}{2}\right)^{3}\cdot\left(\frac{1}{2}\right)^{2} = \frac{1}{16}\cdot\frac{1}{8}\cdot\frac{1}{4}$

$\boxed{i=2}$ $\prod_{j=1}^{4-2}\left(\frac{1}{2}\right)^{4-j+1} = \left(\frac{1}{2}\right)^{4}\cdot\left(\frac{1}{2}\right)^{3} = \frac{1}{16}\cdot\frac{1}{8}$

MOŽE
I OVA
FORMA

$\boxed{45a}$

$$P(E) = \sum_{i=1}^{n-1} P(k=i) \left(2^{n-i} - 1\right) \prod_{j=1}^{n-i} \left(\frac{1}{2}\right)^{n-j+1}$$

$\frac{5(5+3)}{2} = \frac{54}{} = 20$

$\boxed{i=3}$ $\prod_{j=1}^{4-3} \left(\frac{1}{2}\right)^{4-j+1} = \left(\frac{1}{2}\right)^{4-1+1} = \left(\frac{1}{16}\right) = \left(\frac{1}{2}\right)^4$

$\boxed{\frac{4\cdot(4+3)}{2} = \frac{24}{} = 14}$

SEKVENCATA E: 4, 7, 9, ~~[illegible]~~

ZA: $\boxed{n=5}$ $\left(\frac{1}{2}\right)^5, \left(\frac{1}{2}\right)^5 \cdot \left(\frac{1}{2}\right)^4, \left(\frac{1}{2}\right)^5 \cdot \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^3, \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)$

SEKVENCATA E: $\underline{\underline{5, 9, 12, 14}}$

$\left(\frac{1}{2}\right)^{\overset{n}{\cancel{4}}} \left[\frac{1}{4} \cdot \frac{1}{8}\right], \left(\frac{1}{2}\right)^4 \left[\frac{1}{8}\right], \left(\frac{1}{2}\right)^4 [1]$

[redacted]

$5 [1, 4, 7, 9]$

$\boxed{n=4}$

$\left[\frac{1}{16}, \frac{1}{8} \cdot \frac{1}{16}, \frac{1}{4} \cdot \frac{1}{8} \cdot \frac{1}{16}\right] \cdot 16 = \left[1, \frac{1}{8}, \frac{1}{4} \cdot \frac{1}{8}\right]$ $\rightarrow$ 2^9

$\left[\frac{1}{16}, \frac{1}{8} \cdot \frac{1}{16}, \frac{1}{4} \cdot \frac{1}{8} \cdot \frac{1}{16}\right] \frac{512}{512} = \left[\overset{2^5}{32}, \overset{2^2}{4}, \overset{2^0}{1}\right] \frac{1}{512}$ $\rightarrow$ 2^{14}

$\boxed{n=5}$

$\left[\frac{1}{32}, \frac{1}{32} \cdot \frac{1}{16}, \frac{1}{32} \cdot \frac{1}{16} \cdot \frac{1}{8}, \frac{1}{32} \cdot \frac{1}{16} \cdot \frac{1}{8} \cdot \frac{1}{4}\right] \cdot \frac{16384}{16384} =$

$= \left[\overset{2^9}{512}, \overset{2^5}{32}, \overset{2^2}{4}, \overset{2^0}{1}\right] \frac{1}{2^{14}}$

$3n - n + 2$

$n = 1$ $\quad 3 \cdot 1 - 1 + 2 =$

2^5

$\boxed{S(n) = \frac{n(n+3)}{2}}$

$S(0) = 0 \quad S(1) = \frac{1 \cdot 4}{2} = 2$

$S(2) = \frac{2 \cdot 5}{2} = 5 \quad S(3) = \frac{3 \cdot 6}{2} = 9$ Δ

$\boxed{n=3}$ $\left[\frac{1}{4} \cdot \frac{1}{8}\right], \left[\frac{1}{8}\right]$ $\left[\frac{1}{48}, \frac{1}{8}\right] \cdot \frac{32}{32} = [1, 4] \cdot \frac{1}{2^5}$

$\boxed{n=2}$ $\left[\frac{1}{4}\right]_{n-1} = \left[\frac{1}{4}\right] \frac{4}{4} = [1] \cdot \frac{1}{2^2}$

$2^{(i-1)\frac{(i+2)}{2}}$ $[2, 5, 9, 14]$

$P(E) = \sum_{i=1} P(k=i) \left(2^{n-i} - 1\right)$ $\quad 2^{\frac{(n-1)(n+2)}{2}}$

|469|

$$\boxed{P(e) = \frac{1}{2^{\frac{(n-1)(n+2)}{2}}} \cdot \sum_{i=1}^{n-1} P(u=i)\left(2^{n-i}-1\right) \cdot 2^{\frac{(i-1)(i+2)}{2}}} \quad ☆$$

$\boxed{n=3}$ $$\frac{1}{2^{\frac{(n-1)(n+2)}{2}}} = 2^{-\frac{2\cdot 5}{2}} = 2^{-5} = \frac{1}{32}$$

$\boxed{i=1}$ $\frac{1}{2}\left(2^2-1\right)\cdot 2^0$ $\boxed{i=2}$ $\frac{1}{4}\cdot(1)\; 2^{\frac{1\cdot 4}{2}} = \frac{1}{4}\cdot 1\cdot 2^2$

$$\left[\frac{1}{2}\cdot 3,\ \frac{1}{4}\cdot 4\right]\cdot\frac{1}{32}$$

ok

$\boxed{n=4}$ $$2^{-\frac{(n-1)(n+2)}{2}} = 2^{-\frac{3\cdot 6}{2}} = 2^{-3\cdot 3} = 2^{-9} = \underline{1/512}$$

$\boxed{i=1}$ $\frac{1}{512}\cdot\frac{1}{2}\cdot 7\cdot 2^0$ ✓ ; $\boxed{i=2}$ $\frac{1}{512}\cdot\frac{1}{4}\cdot 3\cdot 2^2$ ✓ ;

$\boxed{i=3}$ $\frac{1}{512}\cdot\frac{1}{8}\cdot 1\cdot 2^5$ ✓

$\boxed{n=5}$ $$2^{-\frac{(n-1)(n+2)}{2}} = 2^{-\frac{4\cdot 7}{2}} = 2^{-14}$$

$\boxed{i=1}$ $\frac{1}{2^{14}}\cdot\frac{1}{2}\cdot 15\cdot 2^0$ ✓ $\boxed{i=2}$ $\frac{1}{2^{14}}\cdot\frac{1}{4}\cdot 7\cdot 2^2$ ✓ ,

$\boxed{i=3}$ $\frac{1}{2^{14}}\cdot\frac{1}{8}\cdot 3\cdot 2^5$ ✓ $\boxed{i=4}$ $\frac{1}{2^{14}}\cdot\frac{1}{16}\cdot 1\cdot 2^9$ ✓

Se POVRZUVA RELA ~~☆~~ E TOČNA !!!

$$\boxed{P(e) = 2^{-\frac{(n-1)(n+2)}{2}} \sum_{i=1}^{n-1} \frac{1}{2^i}\left(2^{n-i}-1\right) 2^{\frac{(i-1)(i+2)}{2}}}$$

MAKE $\boxed{\lim_{n\to\infty} P(n) \to e}$

$$(i-1)(i+2) = i^2+2i-i-2 = i^2+i-2$$

$$(n-1)(n+2) = n^2+n-2$$

$$P(e) = 2^{-(n^2+n-2)}\sum 2^{n-i}\, 2^{i^2-2} - 2^{-(n^2+n-2)}\sum_{i=1}^{n-1} 2^{i^2-2}$$

$$= 2^{-n^2+2}\sum_{i=1}^{n-1} 2^{-i}\, 2^{i^2-2} - \sum_{i=1}^{n-1}\frac{2^{i^2-2}}{2^{i^2+n-2}}$$

479

$$P(e) = \sum_{i=1}^{n-1} \frac{2^{i^2-i-2}}{\boxed{2^{n^2-2}}} - \sum_{i=1}^{n-1} \frac{2^{i^2-2}}{\boxed{2^{n^2+n-2}}} \xrightarrow[n\to\infty]{} \varepsilon$$

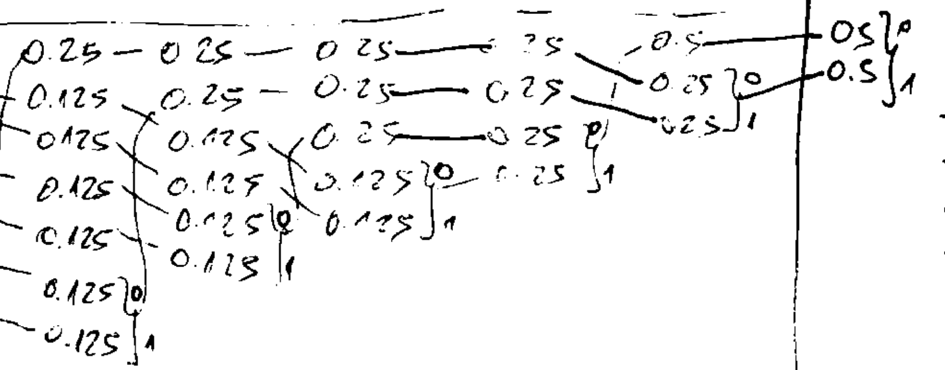
ZA $n \to \infty$ IMENITELITE POBRZU ODAT KON ∞ (POKAŽANO I VO MATLB) !!!

ZATOA $\lim_{n\to\infty} P(e) \to \varepsilon$

EDITION 2 SOLUTIONS

(a) OBVIOUSLY HUFFMAN CODES OF X ARE ALL OF LENGTH „n". HENCE WITH „n" DETERMINISTIC QUESTIONS WE CAN IDENTIFY AN OBJECT OUT OF 2^n CANDIDATES

X	p	C(X)
1	0.125	010
2	0.125	011
3	0.125	000
4	0.125	001
5	0.125	110
6	0.125	111
7	0.125	100
8	0.125	101



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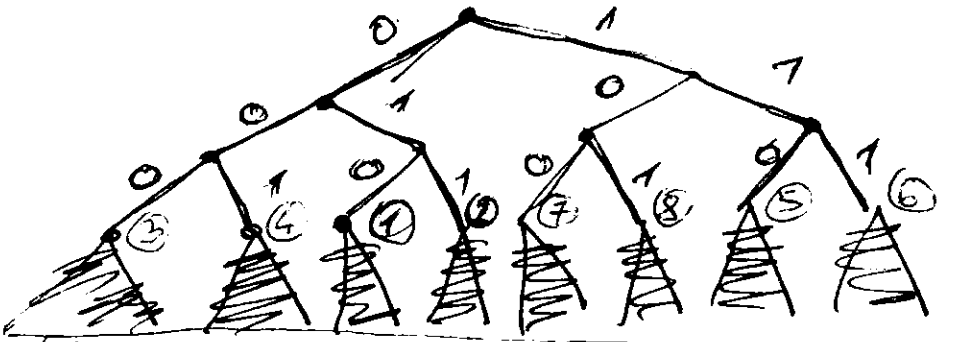


a

(b) TOTAL NUMBER OF SUBSETS 2^{n-1} } ČLENOVI ≥ 2 NA PODMNOŽESTVA

AKO GI SO ODREDENI ČLENOVI NULTIOT AOD NA PODMNOŽESTVATA E:

$\{1,2,3\}$; $n=3$, $2^{3-1} = 2^2 = 4$

$\{1,2\}$; $\{1,3\}$; $\{2,3\}$; $\{1,2,3\}$

THE PROBABILITY THAT „2" YIELDS SAME ANSWERS AS „1" IS

$$\left(\frac{2^{n-1}}{2^n}\right)^K = \left(\frac{2^2}{2^3}\right)^K = \frac{1}{2^K}$$

$\{1,2,3,4\}$ $\{5,6,7,8\}$ $k=1$

$\{1,2\}$ $\{3,4\}$ $\{5,6\}$ $\{7,8\}$ $k=2$

$\{1\}$ $\{2\}$ $\{3\}$ $\{4\}$ $\{5\}$ $\{6\}$ $\{7\}$ $\{8\}$

ZPg

SOGLASNO FOTOGRAFIJATA NA NASa.47 ZATO E DEKA AKO SE NAOĐAŠ NA NIVO $k=1$ VEROJATNOSTA PRAŠANJETO DALI „2" E NA S_1 E $1/2$; ISTO TAKA I PRAŠANJETO DALI „2" PRIPAĐA NA S_2 E $1/2$ NA NIVO $k=2$ DALI „2" E VO S_1 ILI S_2 PA VEĆE ZNAEŠ DALI „2" E VO ... ZATOA AKO $2 \in S_1$ TOGAŠ SE RAZGLEDUVAAT SAMO S_{11}, S_{12} VEROJATNOSTA $2 \in S_{11}$ E $\frac{1}{4}$. VO NIVO $K=3$ IZBOROT SE SVEDUVA (AKO $2 \in S_{11}$) SAMO MEĐU S_{111} I S_{112} KOI SE SO VEROJATNOST $1/8$. ZNAČI VEROJATNOSTA DA „2" DADE IST ODGOVOR KAKO 1 NA NIVO 1 E $1/2$ I NA NIVO 2 E $1/4$, NA NIVO 3 E $1/8$ T.E VO GENERALEN SLUČAJ $\frac{1}{2^K}$

= NAJNOT NASTAP NA NASa.39 ISTO DALI KODOT

$$P(K) = 1 - \sum_{i=1}^{k} \frac{1}{2^i} \quad k \le n-1 \quad = \frac{1}{2^k}$$

E.G. $n=3$ $\boxed{k=2}$

$$\boxed{P(k) = 1 - \overset{a}{\left(\frac{1}{2}\right)} - \overset{b}{\left(\frac{1}{4}\right)}}$$

FORMULATA E SO ODZEMEN PRISTAP. ČLENOT (a) E VKUPNOST VEROJATNOST DA NA NIVO 1 DONESE RAZLIČEN ODGOVOR A ČLENOT (b) NA NIVO 2 DA DOBIE RAZLIČEN ODGOVOR. POVEĆE E EDEN NA E RAZLIČEN ZA DA IMA RAZLIKA I ~~ZATO~~ SOBIRA A NE MNOŽI.

- GENERALNO MOŽE DA ODI I DO $K=n$

$$K=3 \qquad P(K) = 1 - \underbrace{\frac{1}{2} - \frac{1}{4} - \frac{1}{8}}_{\text{RAZLIČNI ODGOVORI}} = \frac{1}{8} = \frac{1}{2^3}$$

- FORMULATA $\left(\frac{2^{n-1}}{2^n}\right)^K$

2^{n-1} – BROJ NA KOMBINACIITE KOI MOŽE DA VRATAT PO IST ODGOVOR

2^n – VKUPEN BROJ NA KOMBINACII

- IF THE PROBLEM IS TREATED AS CHANNEL CODING PROBLEM THROUGH THE NOISELESS CHANNEL

OBJECT	CODEWORD
1	0000
2	0001

LENGTH OF THE STRING IS K

$\boxed{499}$

PRIMER: OBJECT

1 011

2 XYZ

KOLKAVA E VEROJATNOSTA 2 DA BIDE = 011

VKUPNIOT BROJ NA MOZNOSTI E:

000 011 110

001 100 111

010 101

$2^3 = 2^k = 8$ MOZNOSTI

SO OGLED NA TOA STO SITE SE PODEDNAKVO VEROJATNI VEROJATNOSTA 2 DA BIDE 011 E $\underline{\frac{1}{8} = \frac{1}{2^3}}$

(c) LET

$$\boxed{\ell_j = \begin{cases} 1 & \text{OBJECT } j \text{ YIELDS THE SAME ANSWERS FOR } k \text{ QUESTIONS AS OBJECT 1} \\ 0 & \text{OTHERWISE} \end{cases} \quad j = 2, 3, \ldots, n}$$

N - NUMBER OF OBJECTS WITH SAME ANSWERS AS 1 =

$$E[N] = E\left(\sum_{j=2}^{n} \ell_j\right) \underset{\text{STAT. NEZAVNI}}{=} \sum_{j=2}^{n} E[\ell_j] = \sum_{j=2}^{n} 2^{-k} = (n-1)2^{-k}$$

$$\boxed{E[N] = (2^{n} - 1)\, 2^{-k}}$$

$$E[N](n) = E[N](3) = (2^3-1)\frac{1}{8} = \frac{7}{8}$$

$k=2$

$$E[N] = (2^3-1)\frac{1}{4} = \frac{7}{4} = 1.75$$

NOVATA FORMULA

$$EN(n) = \frac{2^n}{3} - \frac{1}{32} \cdot 2^{-1} + 2^{1-n}$$

$$\underline{EN(n=3) = 1.75}$$

→ OVOJ IZRAZ MOZE DA SE OBJASNI SO IZRAZOT NA OBJEKTOT 1 KOJ IMA $(2^n - 1)$ KANDIDATI ISHODI SO VEROJATNOST 1/8

(d) $k = n + \sqrt{n}$

$$E[N] = (2^{n}-1)\, 2^{-n-\sqrt{n}} = (1 - 2^{-n})\, 2^{-\sqrt{n}} = (2^{-\sqrt{n}} - 2^{-n-\sqrt{n}})$$

(e) MARKOV INEQUALITY $P(X > a) \leq \frac{E[X]}{a}$

N - NUMBER OF WRONG OBJECTS REMAINING

$$P(N \geq 1) \leq \frac{E[N]}{1} = (2^n - 1)2^{-k} = (2^n - 1)2^{-n-\sqrt{n}} \underset{n\to\infty}{=} (2^{-\sqrt{n}} - 2^{-n-\sqrt{n}}) \leq 2^{-\sqrt{n}} \to 0$$

509

SE PRAŠUVAM ŠTO VO REALNOSTA MOJOT IZRAZ ZA VEROJATNOST ZA GREŠKA ZNAČI? / ZA IZRAZOT ZA SREDNA VREDNOST ME INTERESIRA. KJE ZA VOVEDAM REPREZENTACIJATA ZA NOISELESS CHANNEL.

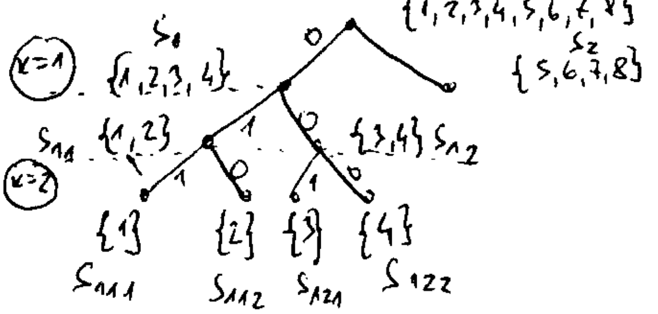
- PRVO ZA IZRAZOT ZA SREDNA VREDNOST:

OBJEKT	K →
1	0 1 1 1 0 1 1
2	0 1 0 0 1 0 1

VO MOMENTOT K=1, K=2 OBJEKTOT ① I OBJEKTOT ② IMAAT EDNAKVI ODGOVORI. KAKO SE ZGOLEMUVA „K" TAKA POČNUVA DA SE RAZLIKUVAAT. KAKO RASTE BROJOT NA OBJEKTI TAKA RASTE BROJOT NA OČEKUVANI OBJEKTI DA DADAT IST ODGOVOR KAKO I ①.

$$P(e) = P(k=1)\left[P(S_{11})P(S_{112}) + P(S_{12})\cdot P(S_{121}) + P(S_{12})P(S_{122})\right] + P(k=2)\left(P(S_{112})\right)$$

9



	K / OBJEKT	1	2	3
(X_1)	1	0	1	1
(X_2)	2	0	1	0
(X_3)	3	0	0	1
(X_4)	4	0	0	0

TABELATA DESNO SAKA DA KAŽE: VO MOMENTOT „K" VO PRIEMNIKOT PRISTIGNUVA „0". BEZ DA GI ZNAEŠ NAREDNITE DVA BITA PRIEMNIKOT NE MOŽE DA ZNAE DALI SE RABOTI ZA KODNIOT ZBOR $X_1, \underbrace{X_2, X_3}$ ILI X_4

VEROJATNOSTA DA SE SLUČI GREŠKA E NAMESTO ZA TOJ DA ODLUČI DEKA E PRIMEN ZBOROT X_2, X_3 ODNOSNO X_4. VO MOMENTOT $\boxed{K=2}$ PRIMEN E BIT „1". PRIEMNIKOT MOŽE DA MISLI DEKA SE RABOTI ZA ZBOROT X_1 ILI X_2. AKO SE ODLUČI ZA X_2 IMAME GREŠKA. VO MOMENTOT $\boxed{K=3}$ PRIMEN E BITOT „1". PRIEMNIKOT E SIGUREN DEKA E ISPRATEN ZBOROT (SIMBOLOT) X_1 (NEMA GREŠKA)

- DEFINITIVNO MOJOT IZRAZ E GREŠEN. MOŽEBI ŠTO SUMIRAM PO PREZIM „K" E NEŠTO ŠTO SE RAZLIKUVA OD NIVNIOT IZRAZ „NO I IZRAZITE ZA KONCEPTOT „K" MI SE POTOČNI. OSOBENO NE MI E JASNO IZVEDUVANJETO ZA EN. VO SUMATA ZEMAAT „N" OBJEKTI SEKAK PRETHODNIOT BIT VEĆE ELIMINIRAL POLA OD OBJEKTITE, A TOA NE GO ZEMAAT VO PREDVID.

5.19

$$ld\frac{1}{p_i} \leq l_i \leq ld\frac{1}{p_i}+1 \qquad ld\frac{1}{p_i} \leq l_i < ld\frac{2}{p_i} \quad / 2^{x}$$

$$\frac{1}{p_i} \leq 2^{l_i} < \frac{2}{p_i} \qquad (\;)^{-1} \qquad p_i \geq 2^{-l_i} \qquad \frac{p_i}{2} < 2^{-l_i}$$

$$p_i < 2^{-l_i+1} \qquad \underline{2^{-l_i} \leq p_i \leq 2^{-l_i+1}}$$

a

$$g(x) = \int_0^x f(y)\,dy \qquad \frac{dg(x)}{dx} = f(x)$$

FTC

$$P_b = E\left[P_{out}\left(\frac{x^2}{c}\right)\right] \longrightarrow$$ ∞ CLANAKOT NA ZHAO

$$P_{out} = Pr(\gamma \leq \gamma_0) = \int_0^{\gamma_0} p(\gamma)\,d\gamma$$

$$\boxed{\frac{dP_{out}}{d\gamma_0} = p(\gamma_0)}$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} \cdots$$

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!}$$

$$e^{-(b+1)x} \doteq 1 - (b+1)x = 1 - bx - x$$

a

CHAPTER 6 GAMBLING AND DATA COMPRESSION

THE SUM OF GROWTH RATE AND THE ENTROPY RATE IS A CONSTANT

6.1 THE HORSE RACE

ASSUME THAT m HORSES RUN IN A RACE. LET THE i-TH HORSE WIN WITH PROBABILITY p_i. IF THE HORSE i WINS, THE PAYOFF IS O_i FOR 1 (I.E. AN INVESTMENT OF 1 DOLLAR ON HORSE i RESULTS IN O_i DOLLARS IF HORSE i WINS AND 0 DOLLARS IF HORSE i LOSES).

- TWO WAYS OF DESCRIBING THE ODDS:

1° a-FOR-1 ODDS a = DOLLARS FOR WIN 0 FOR LOST

2° b-TO-1 ODDS b = DOLLARS FOR WIN 1 FOR LOST

E.G. FAIR ODDS ON COIN FLIPS: 2 FOR 1 <u>1 TO 1</u> KNOWN AS EVEN ODDS

52a

WE ASSUME THAT GAMBLER DISTRIBUTES ALL OF HIS WEALTH ACROSS THE HORSES.

b_i - FRACTION OF THE GAMBLER WEALTH INVESTED IN HORSE „i" $b_i \geq 0$ $\sum b_i = 1$. IF HORSE „i" WINS THE RACE, THE GAMBLER WILL RECEIVE O_i TIMES THE AMOUNT OF WEALTH BET ON HORSE i. ALL THE OTHER BETS ARE LOST

AT THE END OF THE GAME THE GAMBLER HAS WEALTH $b_i \cdot O_i$ WITH PROBABILITY p_i

---

THE WEALTH AT THE END OF THE RACE IS RANDOM VARIABLE, AND THE GAMBLER WISHES TO "MAXIMIZE" THE VALUE OF THIS RANDOM VARIABLE.

- LET S_n BE THE GAMBLER'S WEALTH AFTER „n" RACES. THEN:

$$S_n = \prod_{i=1}^{n} S(X_i)$$

$S(x) = b(x) \cdot o(x)$ - FACTOR BY WHICH THE GAMBLER'S WEALTH IS MULTIPLIED WHEN HORSE X WINS

DEFINITION: THE WEALTH RELATIVE $S(X) = b(X) \cdot o(X)$ IS THE FACTOR BY WHICH THE GAMBLER'S WEALTH GROWS IF HORSE X WINS IN RACE

DEFINITION: THE DOUBLING RATE OF A HORSE RACE IS

$$W(b, p) = E[\operatorname{ld} S(X)] = \sum_{k=1}^{m} p_k \operatorname{ld} b_k o_k$$

$\boxed{\text{THEOREM 6.1.1}}$ LET THE RACE OUTCOMES $X_1, X_2, \ldots$ BE i.i.D $\sim p(x)$. THEN THE WEALTH OF THE GAMBLER USING STRATEGY b GROWS EXPONENTIALLY AT RATE $W(b, p)$

$$\boxed{S_n = 2^{nW(b,p)}}$$

9

PROOF. FUNCTIONS OF RANDOM VARIABLES ARE ALSO INDEPENDENT, AND HENCE $\operatorname{ld} S(X_1), \operatorname{ld} S(X_2), \ldots,$ ARE I.I.D. THEN, BY THE WEEK LAW OF LARGE NUMBERS,

$$\frac{1}{n} \operatorname{ld} S_n = \frac{1}{n} \sum \operatorname{ld} S(X_i) = \frac{1}{n} \operatorname{ld} \prod_{i=1}^{n} S(X_i) = E[\operatorname{ld} S(X)]$$

$$\rightarrow \quad S_n = 2^{n \cdot E[\operatorname{ld} S(X)]} = 2^{n \cdot W(b,p)}$$

53a

DEFINITION: THE OPTIMUM DOUBLING RATE $W^*(p)$ IS THE MAXIMUM DOUBLING RATE OVER ALL CHOICES OF THE PORTFOLIO **b**:

$$W^*(p) = \max_{b} W(b,p) = \max_{b:\, b_i > 0,\, \sum b_i = 1} \sum_{i=1}^{m} p_i \, \mathrm{ld}\left(b_i o_i\right)$$

$$J(b) = \sum p_i \ln(b_i o_i) + \lambda\left(\sum b_i - 1\right)$$

$$\frac{\partial J(b)}{\partial b_i} = 0 \qquad \sum p_i \frac{1}{b_i \cancel{o_i}} \cancel{o_i} + \sum \lambda = 0$$

$$\sum \left(\frac{p_i}{b_i} + \lambda \right) = 0 \qquad \frac{p_i}{b_i} = -\lambda \qquad \boxed{b_i = -\frac{p_i}{\lambda}}$$

$$\sum b_i = 1 \qquad -\sum \frac{p_i}{\lambda} = 1 \Rightarrow \boxed{\lambda = -1} \quad \boxed{b_i = p_i}$$

$\boxed{b = p}$ → PROPORTIONAL GAMBLING OR KELLY GAMBLING

THEOREM 6.1.2 (PROPORTIONAL GAMBLING IS LOG-OPTIMAL
THE OPTIMUM DOUBLING RATE IS GIVEN BY:

$$W^*(p) = \sum p_i \, \mathrm{ld}\, o_i - H(p)$$

AND IS ACHIEVED BY THE PROPORTIONAL GAMBLING SCHEME $b_i^* = p_i$ (i.e. GAMBLER BETS ON EACH HORSE IN PROPORTION TO ITS PROBABILITY OF WINNING.)

PROOF: WE REWRITE $W(b,p)$:

$$W(b,p) = \sum p_i \, \mathrm{ld}\, b_i o_i = \sum p_i \, \mathrm{ld}\left(\frac{b_i}{p_i} \cdot p_i o_i\right) =$$

$$= \sum p_i \, \mathrm{ld}\, o_i - \sum p_i \, \mathrm{ld}\left(\frac{1}{p_i}\right) - \sum p_i \, \mathrm{ld} \frac{p_i}{b_i} =$$

$$= \sum p_i \, \mathrm{ld}\, o_i - H(p) - D(p \| b_i) \leq \sum p_i \, \mathrm{ld}\, o_i - H(p)$$

WITH EQUALITY IF $p = b$

EXAMPLE 6.1.1 CONSIDER THE CASE WITH TWO HORSES WHERE HORSE 1 WINS WITH PROBABILITY p_1 AND HORSE 2 WINS WITH PROBABILITY p_2. ASSUME EVEN ODDS (2 FOR 1 ON BOTH HORSES)

THEN THE OPTIMAL BETTING IS PROPORTIONAL BETTING i.e. ($b_1 = p_1$, $b_2 = p_2$). THE OPTIMAL DOUBLING RATE IS $W^*(p) = \sum p_i \, ld\, o_i - H(p)$, AND THE RESULTING WEALTH GROWS TO INFINITY AT THIS RATE:

$$S_n \doteq 2^{n(1-H_p)}$$

$$W^*(p) = \sum p_i \, ld(o_i) = \frac{1}{2} \cdot ld\, 2 + \frac{1}{2} \cdot ld\, 2 - H(p) = 1 - H(p)$$

→ THUS WE HAVE SHOWN THAT PROPORTIONAL BETTING IS GROWTH RATE OPTIMAL FOR A SEQUENCE OF I.I.D. HORSE RACES IF THE GAMBLER CAN REINVEST HIS WEALTH AND THERE IS NO ALTERNATIVE OF KEEPING SOME WEALTH IN CASH.

- SPECIAL CASE: FAIR ODDS i.e. THERE IS NO TRACK TAKE AND $\sum \frac{1}{o_i} = 1$, $\boxed{r_i = \frac{1}{o_i}}$ ⇒ INTERPRET AS PROBABILITY MASS FUNCTION OVER THE HORSES

1.6. BOOKIE'S ESTIMATE OF THE WIN PROBABILITIES.

DOUBLING RATE IS:

$$W(b,p) = \sum p_i \, ld\, b_i o_i = \sum p_i \, ld \frac{b_i}{p_i} \cdot \frac{p_i}{r_i} = D(p\|r_i) - D(p\|b)$$

$W(b,p) = D(p\|r) - D(p\|b)$ DOUBLING RATE IS THE DIFFERENCE BETWEEN THE DISTANCE OF THE BOOKIE'S ESTIMATE FROM THE TRUE DISTRIBUTION AND THE DISTANCE ~~BETWEEN~~ OF THE GAMBLER'S ESTIMATE FROM THE TRUE DISTRIBUTION ~~HENCE THE~~ GAMBLER CAN MAKE MONEY ONLY IF HIS ESTIMATE (AS EXPRESSED BY b) IS BETTER THAN THE BOOKIE'S.

- WHEN THE ODDS ARE: m-TO-1 $b = p$

$$W^*(p) = \sum p_i \, ld\, o_i - H(p) = \sum p_i \, ld \frac{p_i}{r_i} - H(p) =$$

$$= \sum p_i \, ld \frac{p_i}{1/m} - H(p) = D\left(p_i \Big\| \frac{1}{m}\right) - H(p) =$$

$$= \sum p_i \, ld(p_i \cdot m) - H(p) = \underbrace{\sum p_i \, ld\, p_i}_{-H(p)} + \sum p_i \, ld\, m - H(p) =$$

$$= ld\, m - 2H(p)$$ → GIESTO L'INESOUVATE !!!

=

$\boxed{659}$

$$W^*(p) = \sum p_i \, ld\, b_i o_i = \left| \underset{\text{OPTIMALITY}}{b=p} \right| = \sum p_i \, ld\, p_i \underset{\downarrow}{o_i} =$$

$$= \sum p_i \, ld\, o_i - H(p) = \sum p_i \, ld \frac{1}{r_i} - H(p) = \sum p_i \, ld \frac{p_i}{r_i}$$

$$= \sum p_i \, ld \frac{p_i}{1/m} = \underline{D(p \| \tfrac{1}{m})} = \sum p_i \, ld\, m - H(p)$$

$$W^*(p) = ld\, m \sum p_i - H(p) = \underline{\underline{ld\, m - H(p)}}$$

$\boxed{\text{THEOREM 6.1.3}}$ (CONSERVATION THEOREM) FOR UNIFORM FAIR ODDS

$$W^*(p) + H(p) = ld\, m$$

THUS THE SUM OF THE DOUBLING RATE AND THE ENTROPY RATE IS A CONSTANT.

EVERY BIT OF ENTROPY DECREASE DOUBLES THE GAMBLER'S WEALTH. LOW ENTROPY RACES ARE MOST PROFITABLE.

$$S_n = 2^{n[ld\, m - H(p)]} \qquad 2^{n\, ld\, m} \cdot 2^{-nH(p)}$$

E.G. $H_1(p) = 2$ $H_2(p) = 1 = \frac{1}{2} H_1(p)$

$$S_n^{(2)} = 2^{n\, ld\, m} \cdot 2^{-H_2(p)} = 2 \cdot 2^{n\, ld\, m}$$

$$S_n^{(1)} = 2^{n\, ld\, m} \cdot 2^{-2} = 2$$

$$\frac{S_n^{(2)}}{S_n^{(1)}} = \frac{\cancel{2^{n\, ld\, m}} \cdot 2^{-1}}{\cancel{2^{n\, ld\, m}} \cdot 2^{-2-1}} = \textcircled{2} \qquad \underbrace{S_n^{(2)} = 2 S_n^{(1)}}$$

EVERY BIT DECREASE OF ENTROPY DOUBLES THE GAMBLER'S WEALTH.

$\boxed{9}$

- LET $b(0)$ BE THE PROPORTION OF WEALTH HELD OUT AS CASH AND $b(1), b(2), \ldots b(m)$ BE THE PROPORTIONS BET ON VARIOUS HORSES. AT THE END OF THE RACE, THE RATIO OF FINAL WEALTH TO INITIAL WEALTH (THE WEALTH RELATIVE) IS:

$$S(X) = b(0) + b(X)\, o(X)$$

NOW THE OPTIMUM STRATEGY MAY DEPEND ON THE ODDS AND WILL NOT NECESSARY HAVE SIMPLE FORM OF PROPORTIONAL GAMBLING. THERE ARE 3 SUBCASES:

1) FAIR ODDS WITH RESPECT TO SOME DISTRIBUTION.

$$\sum \frac{1}{o_i} = 1 \qquad b_i = \frac{1}{o_i} \qquad S(X) = 1$$

IRRESPECTIVE OF WHICH HORSE WINS. WHATEVER MONEY THE GAMBLER KEEP ASIDE AS CASH CAN EQUALLY BE DISTRIBUTED OVER THE HORSES AND THE ASSUMPTION THAT THE GAMBLER MUST INVEST ALL HIS

$\boxed{569}$

DOESN'T CHANGE THE ANALYSIS. PROPORTIONAL BETTING IS OPTIMAL.

② SUPERFAIR ODDS $\sum \frac{1}{O_i} < 1$. IN THIS CASE, THE ODDS ARE EVEN BETTER THAN FAIR ODDS, SO ONE WOULD ALWAYS WANT TO ~~[illegible]~~ PUT ALL WEALTH INTO THE RACE. OPTIMAL STRATEGY IS PROPORTIONAL BETTING. CHOOSE „b" SO AS TO FORM „DUTCH BOOK":

$$b_i = c \cdot \frac{1}{O_i} \qquad c = \frac{1}{\sum \frac{1}{O_i}} \qquad O_i b_i = c$$

IRRESPECTIVE OF WHICH HORSE WINS

$$S(X) = b_i O_i = c = \frac{1}{\sum \frac{1}{O_i}} > 1$$

WITH PROBABILITY „1" (NO RISK)

$m = 1..3$

$$c = \frac{1}{\frac{1}{O_1} + \frac{1}{O_2} + \frac{1}{O_3}} = \frac{O_1 O_2 O_3}{O_1 + O_2 + O_3}$$

$$b_1 = \frac{O_1 O_2 O_3}{O_1 + O_2 + O_3} \frac{1}{O_1} = \frac{O_2 O_3}{O_1 + O_2 + O_3} \qquad b_1 O_1 = c$$

$$\boxed{b_2 = \frac{O_1 O_2 O_3}{O_1 + O_2 + O_3} \cdot \frac{1}{O_2} = \frac{O_1 O_3}{O_1 + O_2 + O_3}}$$

DUTCH BOOK, ALTHOUGH RISK-FREE, DOESN'T OPTIMIZE THE DOUBLING RATE !!!

③ SUBFAIR ODDS $\sum \frac{1}{O_i} > 1$ } REALISTIC

IN THIS CASE IT IS OPTIMAL TO BET ONLY SOME OF THE MONEY AND LEAVE THE REST ASIDE AS CASH. PROPORTIONAL GAMBLING IS NO LONGER LOG-OPTIMAL. OPTIMAL STRATEGY CAN BE FOUND USING KUHN-TUCKER CONDITIONS (PROBLEM 6.6.2), IT HAS SIMPLE WATER-FILLING INTERPRETATION.

6.2 GAMBLING AND SIDE INFORMATION

$\boxed{9}$

- SAKAM DA GO DOKAZAM IZRAZOT: (6.1)

$$S_n = \prod_{i=1}^{n} S(X_i) \qquad \underline{S(X_i) = b_i \cdot O_i}$$

$n = 1$

$$S_1 = b_i^{①} O_i^{1}$$

KONJ KOJ POBEDIL → TUKA

$n = 2$

$$S_2 = (S_1 b_i^2)\, O_i^2$$

b_i^2 } FRACTION OF GAMBLER'S WEALTH INVESTED IN HORSE „i"

[57a]

$$S_3 = \underbrace{(S_2 \cdot p_i^3)}_{b_1^3} \cdot o_1^3 = \underbrace{S_1}_{p_i^1 \cdot o_1^1} \cdot p_1^2 \cdot o_1^2 \cdot o_1^3 \cdot p_i^3 = p_i^1 o_1^1 \, p_i^2 o_1^2 \cdot p_i^3 o_1^3$$

$$= \prod_{j=1}^{3} p_i^j o_i^j$$

- VO GENERALEN SLUČAJ

$$\boxed{S_4 = \prod_{j=1}^{4} p_i^j o_i^j}$$

DOVEDE NA IZRAZOT (6.1)

$$\boxed{b_i = p_i}$$

[a]

FOR EXAMPLE THE GAMBLER MAY HAVE SOME INFORMATION ABOUT THE PERFORMANCE OF THE HORSES IN PREVIOUS RACES. THIS RESULTS IN THE INCREASE IN THE DOUBLING RATE DUE TO THAT INFORMATION.

LET HORSE $X \in \{1, 2, \dots, m\}$ WIN THE RACE WITH PROBABILITY $p(x)$ AND PAY ODDS OF $o(x)$ FOR 1. LET (X, Y) HAVE JOINT PROBABILITY MASS FUNCTION $p(x,y)$.

LET $b(x|y) \geq 0$, $\sum_x b(x|y) = 1$ BE AN ARBITRARY CONDITIONAL BETTING STRATEGY DEPENDING ON THE SIDE INFORMATION Y.

$$p(y) = \sum_{x \in \mathcal{X}} p(x,y) \qquad p(x) = \int p(x,y)\, dy$$

$$p(x,y) = p(y)\, p(x|y)$$

$b(x|y)$ PROPORTION OF WEALTH BET ON HORSE X WHEN Y IS OBSERVED

$b(x) \geq 0$, $\sum b(x) = 1$ DENOTE UNCONDITIONAL BETTING SCHEME

$$W^*(X) = \max_{b(x)} \sum_x p(x) \log b(x)\, o(x)$$

$$W^*(X|Y) = \max_{b(x|y)} \sum_{x,y} p(x,y) \log b(x|y)\, o(x)$$

$$\Delta W = W^*(X|Y) - W^*(X)$$

[illegible] HORSE RACE WEALTH

WE OBSERVE THAT FOR [illegible] $W^*(X|Y)$ [illegible] INFORMATION [illegible]

SHOW [illegible]

58a

AND LIKE $2^{nW^*(X)}$ WITHOUT SIDE INFORMATION.

THEOREM 6.2.1 THE INCREASE ΔW IN DOUBLING RATE DUE TO THE SIDE INFORMATION Y FOR HORSE X IS:

$$\boxed{\Delta W = I(X;Y)}$$

PROOF WITH SIDE INFORMATION MAXIMUM VALUE OF [illegible] INFORMATION Y IS ACHIEVED $W^*(X|Y)$ [illegible] PROPORTIONAL GAMBLING I.E. [illegible] IS

$$b^*(X|Y) = p(X|Y)$$

$$W^*(X|Y) = \max_{b(X|Y)} E[\log S] = \sum_{x,y} p(x,y) \log o(x) b(x|y) =$$

$$= \sum p(x,y) \log o(x) p(x|y) \quad \sum p(x,y) \log o(x) + \sum p(x,y) \log p(x|y)$$

[illegible] $p(x,y) \log \frac{1}{p(x,y)}$ [illegible] $= \quad =$

[illegible] $(X,Y) = H(X) +$ [illegible]

[illegible]

[illegible] $\log \frac{1}{p(X|Y)} = H(Y) +$ [illegible]

[illegible]

$= \sum p(y) \log$ [illegible] $= +$ [illegible]

$W^*(X)$ [illegible] $= H(X|Y)$ [illegible] $X) = I$ [illegible] $(X;Y)$

$\Delta W = W^*(X|Y) - W^*(X)$ [illegible]

THUS THE INCREASE [illegible] OF DOUBLING RATE DUE TO THE PRESENCE OF [illegible] Y [illegible] $I(X;Y)$

6.3 DEPENDENT HORSE RACES AND ENTROPY RATE

THE MOST COMON SIDE INFORMATION FOR HORSE RACES IS THE PAST PERFORMANCE OF THE HORSES. SUPPOSE THAT THE SEQUENCE OF HORSE RACE OUTCOMES FORMS A STOCHASTIC PROCESS. LET THE STRATEGY FOR EACH RACE DEPEND ON THE RESULTS OF PREVIOUS RACES. IN THIS CASE OPTIMAL DOUBLING RATE IS FOR UNIFORM FAIR ODDS

$$W^*(X_k|X_{k-1},X_{k-2},\dots,X_1) = \max_{b(X_k|X_1^{k-1})} E[\text{ld}\, o(X_k)\cdot b(X_k|X_1^{k-1})]$$

$$= \cancel{\max_b} E\left[\text{ld}\left(\frac{1}{1/m}\cdot p(X_k|X_1^{k-1})\right)\right] = \sum_{x_1,x_2\dots x_k} p(x_1^k)\cdot\text{ld}(m) +$$

$$+ \sum_{x_1,x_2,\dots x_k} p(x_1^k)\,\text{ld}\,p(x_k|x_1^{k-1}) = \underline{\text{ld}\,m - H(X_k|X_1^{k-1})}$$

MAXIMUM IS ACHIEVED FOR: $\underline{b^*(x_k|x_1^{k-1}) = p(x_k|x_1^{k-1})}$

AT THE END OF "n" RACES THE GAMBLER WEALTH IS:

$$\boxed{S_n = \prod_{i=1}^{n} S(X_i)}$$

$$\frac{1}{n}E[\text{ld}(S_n)] = \frac{1}{n}E\left[\text{ld}\left(\prod_{i=1}^{n} S(X_i)\right)\right] = \frac{1}{n}E\left[\sum_{i=1}^{n}\text{ld}\,S(X_i)\right] =$$

$$= \frac{1}{n}\sum_{i=1}^{n} E[\text{ld}\,S(X_i)] = \frac{1}{n}\sum_{i=1}^{n}\left(\text{ld}\,m - H(X_i|X_1^{i-1})\right) =$$

$$= \text{ld}\,m - \frac{1}{n}\underbrace{\sum_{i=1}^{n} H(X_i|X_1^{i-1})}_{H(X_1,X_2\dots X_n)} = \text{ld}\,m - \frac{H(X_1,X_2\dots X_n)}{n}$$

$$H(X_1,X_2) = H(X_1) + H(X_2|X_1) \qquad H(X_1X_2X_3) = H(X_1) + H(X_2X_3|X_1) =$$

$$= H(X_1) + H(X_2|X_1) + H(X_3|X_1X_2) = \sum_{i=1}^{3} H(X_i|X_1^{i-1})$$

$\frac{1}{n}H(X_1X_2\dots X_n)$ → AVERAGE ENTROPY PER RACE

FOR A STATIONARY PROCESS WITH ENTROPY RATE $H(\mathcal{X})$:

$$\boxed{\lim_{n\to\infty} \frac{1}{n} E[\log S_n] + H(\mathcal{X}) = \log m} \quad (*)$$

THE EXPECTATION IN (*) CAN BE REMOVED IF THE PROCESS IS ERGODIC

$$S_n \doteq 2^{nW} \text{ WITH PROBABILITY } 1$$

WHERE: $W = \log m - H(\mathcal{X})$ AND

$$H(\mathcal{X}) = \lim_{n\to\infty} \frac{1}{n} H(X_1, X_2, \ldots X_n)$$

EXAMPLE 6.3.1 (RED AND BLACK) IN THIS EXAMPLE, CARDS REPLACE HORSES AND THE OUTCOMES BECOME MORE PREDICTABLE AS TIME GOES ON. CONSIDER THE CASE OF BETTING ON THE COLOR OF THE NEXT CARD IN A DECK OF 26 RED AND 26 BLACK CARDS. BETS ARE PLACED ON WHETHER THE NEXT CARD WILL BE RED OR BLACK, AS WE GO THROUGH THE DECK. WE ALSO ASSUME THAT THE GAME PAYS 2-FOR-1.

4 - KARTI (2 CRVENI 2 CRNI)

$$\binom{4}{2} = \frac{4!}{2!2!} = \frac{24}{4} = 6$$

BBRR
BRBR
BRRB
RBRB
RRBB
RBBR

• WE CONSIDER TWO ALTERNATIVE BETTING SCHEMES:

1.) IF WE BET SEQUENTIALLY, WE CAN CALCULATE THE CONDITIONAL PROBABILITY OF THE NEXT CARD AND BET PROPORTIONALLY. THUS WE SHOULD BET: (RED, BLACK)

$$\underbrace{\left(\frac{1}{2}, \frac{1}{2}\right)}_{\text{FIRST CARD}}; \underbrace{\left(\frac{26}{51}, \frac{25}{51}\right)}_{\text{SECOND CARD IF FIRST IS BLACK}}; \text{ SO ON.}$$

2.) ALTERNATIVELY WE CAN BET ON THE ENTIRE SEQUENCE OF 52 CARDS AT ONCE. THERE ARE $\binom{52}{26}$ POSSIBLE SEQUENCES OF 26 RED AND 26 BLACK CARDS, ALL OF THEM EQUALLY LIKELY. THUS PROPORTIONAL BETTING IMPLIES THAT WE PUT $1/\binom{52}{26}$ OF MONEY ON EACH SEQUENCE.

61a

WE WILL ARGUE THAT THIS PROCEDURES ARE EQUIVALENT. FOR EXAMPLE, HALF THE SEQUENCES OF 52 CARDS START WITH RED, AND SO THE PROPORTION OF MONEY BET ON SEQUENCES THAT START WITH RED IN SCHEME 2 IS ALSO ONE ~~HALF~~, AGREEING WITH PROPORTION USED IN THE FIRST SCHEME. IN GENERAL, WE CAN VERIFY THAT BETTING $1/\binom{52}{26}$ OF THE MONEY ON EACH OF THE POSSIBLE OUTCOMES WILL AT EACH STAGE GIVE BETS THAT ARE PROPORTIONAL TO THE PROBABILITY OF RED AND BLACK AT THE STAGE. SINCE WE BET $1/\binom{52}{26}$ OF THE WEALTH ON EACH POSSIBLE OUTPUT SEQUENCE, AND A BET ON A SEQUENCE INCREASES WEALTH BY FACTOR 2^{52} ON THE SEQUENCE OBSERVED AND 0 ON ALL OTHERS, THE RESULTING WEALTH IS

$$S^*_{52} = \frac{2^{52}}{\binom{52}{26}} = 9.08$$ (*)

ZOŠTO 2^{52} PATI [illegible] ZA [illegible]
SEKVENCATA ŠTO VREDE [illegible] ZA KARTA
E SLEPA A ZA POGODENA KARTA [illegible]
E 2-FOR-1 [illegible] 1 MND (ODDS) E [illegible] 2 MND, A NA PRVO-POGODENATA KARTA [illegible] 2×2=4 MND ([illegible] BOGATSTVO [illegible]), NA SO PRVIOT POGODOK [illegible] $2\times2\times2 = 2^3 = 8$ MND; A NA TRETOTO POGODUVANJE [illegible] SO 52-TO POGODUVANJE [illegible] IMAŠ 2^{52}. SO OGLED NA TOA ŠTO [illegible] SI VLOŽIL $\left[1/\binom{52}{26}\right]\%$ OD BOGATSTVOTO (NORMALIZIRANOTO BOGATSTVO IZNESUVA 1) BOGATSTVOTO [illegible] SE [illegible] SEKVENCI [illegible] SO (*)

6.4 THE ENTROPY OF ENGLISH

[illegible] EXAMPLE OF AN [illegible]
ENGLISH TEXT [illegible]
[illegible]

6.5 DATA COMPRESSION AND GAMBLING

GOOD GAMBLER IS ALSO GOOD DATA COMPRESSOR. ANY SEQUENCE ON WHICH GAMBLER MAKES LARGE AMOUNT OF MONEY IS ALSO A SEQUENCE THAT CAN BE COMPRESSED BY LARGE ~~AMOUNT~~ FACTOR.

WE ASSUME THAT GAMBLER HAS MECHANICALLY IDENTICAL TWIN, WHO WILL BE USED FOR DATA DECOMPRESSION. THE DECODER WILL USE THE IDENTICAL TWIN TO GAMBLE ON ALL SEQUENCES, AND LOOK FOR THE SEQUENCE FOR WHICH THE SAME CUMULATIVE AMOUNT OF MONEY IS MADE. THIS SEQUENCE WILL BE CHOSEN AS THE DECODED SEQUENCE.

LET $x_1, x_2, \dots, x_n$ BE A SEQUENCE OF RANDOM VARIABLES THAT WE WISH TO COMPRESS. WE ASSUME THAT RANDOM VARIABLES ARE BINARY. GAMBLING ON THIS SEQUENCE WILL BE DEFINED BY SEQUENCE OF BETS.

$$b(x_{k+1}|x_1,x_2,\dots x_k) \geq 0 \qquad \sum_{x_{k+1}} b(x_{k+1}|x_1,x_2\dots x_k) = 1$$

WHERE $b(x_{k+1}|x_1,x_2,\dots x_k)$ IS PROPORTION OF MONEY BET AT TIME k ON THE EVENT THAT $X_{k+1}=x_{k+1}$ GIVEN THE OBSERVED PAST $x_1,x_2,\dots x_k$. BETS ARE PAID AT UNIFORM ODDS (2-FOR-1). THE WEALTH AT THE END OF THE SEQUENCE IS:

$$S_n = 2^n \prod_{k=1}^{n} b(x_k|x_1\dots x_{k-1}) = 2^n b(x_1,x_2,\dots x_n)$$

$$b(x_1,x_2,\dots x_n) = \prod_{k=1}^{n} b(x_k|x_{k-1},x_{k-2},\dots x_1)$$

SO SEQUENTIAL GAMBLING CAN ALSO BE CONSIDERED AS AN ASSIGNMENT OF PROBABILITIES (OR BETS) $b(x_1,x_2,\dots,x_n) \geq 0$, $\sum_{x_1^n} b(x_1,\dots x_n) = 1$ ON THE 2^n POSSIBLE SEQUENCES.

THIS GAMBLING ELICITS BOTH AN ESTIMATE OF THE TRUE PROBABILITY OF TEXT SEQUENCE:

$$\hat{p}(x_1,x_2,\dots) = \frac{S_n}{2^n} \quad \text{AS WELL AS ENTROPY:} \quad \hat{H} = \frac{1}{n}\, ld\, \hat{p}(x_1^n)$$

63a

OF THE TEXT FROM WHICH THE SEQUENCE IS DRAWN, WE WANT TO SHOW THAT HIGH VALUES OF WEALTH S_n LEAD TO HIGH DATA COMPRESSION.

• CONSIDER THE FOLLOWING DATA COMPRESSION ALGORITHM THAT MAPS TEXT: $X = x_1, x_2, \dots x_n \in \{0,1\}^n$ INTO CODE SEQUENCES $c_1 c_2 \dots c_k$, $c_i \in \{0,1\}$.
LET THE 2^n TEXT SEQUENCES BE ARRANGED IN LEXICOGRAPHICAL ORDER: FOR EXAMPLE, 0100101<0101101
THE ENCODER OBSERVES THE SEQUENCE: $X^n = (x_1, x_2, \dots x_n)$. HE THEN CALCULATES WHAT HIS WEALTH $S_n(x'(n))$ WOULD HAVE BEEN ON ALL SEQUENCES $x'(n) \le x(n)$ AND CALCULATES

$$F(X(n)) = \sum_{x'(n) \le x(n)} 2^{-n} S_n(x'(n)).$$

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CLEARLY $F(x(n)) \in [0,1]$

LET $k = \lceil n - \mathrm{ld}\, S_n(X(n)) \rceil$. NOW EXPRESS $F(x(n))$ AS BINARY DECIMAL TO k-PLACE ACCURACY:
$F(x(n)) = .c_1 c_2 \dots c_k$. THE SEQUENCE $c(k) = (c_1, c_2, \dots c_k)$ IS TRANSMITTED TO THE DECODER.

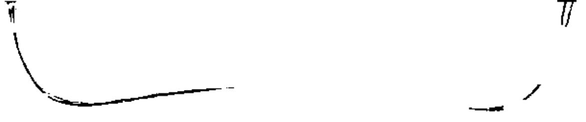
9

• THE DECODER TWIN CAN CALCULATE THE PRECISE VALUE $S(x'(n))$ ASSOCIATED WITH EACH 2^n SEQUENCES $x'(n)$. HE THUS KNOWS THE CUMULATIVE SUM OF $2^{-n} S(x'(n))$ UP THROUGH ANY SEQUENCE $x(n)$. HE TEDIOUSLY CALCULATES THIS SUM UNTIL IT FIRST EXCEEDS $.c(k)$. THE FIRST SEQUENCE $x(n)$ SUCH THAT THE CUMULATIVE SUM FALLS IN INTERVAL

$$[.c_1 \dots c_k, .c_1 \dots c_k + \left(\tfrac{1}{2}\right)^k]$$

IS DEFINED UNIQUELY, AND THE SIZE OF $S(x(n))/2^n$ GUARANTEES THAT THIS SEQUENCE WILL BE PRECISELY THE ENCODED $x(n)$. THUS, THE TWIN UNIQUELY RECOVERS $x(n)$.
THE NUMBER OF BITS REQUIRED IS: $k = \lceil n - \mathrm{ld}\, S(x(n)) \rceil$
FOR PROPORTIONAL GAMBLING:

$$\boxed{S(X(n)) = 2^n p(X(n))}$$



64a

$$S(X(n)) = 2^{n W(p,b)} \qquad W = \sum_x p(x) \operatorname{ld} b(x) \cdot \underline{o(x)}$$

PROPORTIONAL GAMBLING: $b(x) = p(x)$

$$W = \sum p(x) \operatorname{ld} m - \underbrace{\sum_x p(x) \operatorname{ld} p(x)}_{H(X)} = \underline{\overset{1}{\operatorname{ld} m} - H(X)} = 1 - H(X)$$

$$S(X(n)) = 2^n \cdot 2^{-nH(X)} = 2^n \cdot 2^{-n\sum p(x) \operatorname{ld} \frac{1}{p(x)}}$$

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$$E_K = \sum \lceil n - \operatorname{ld} S(X(n)) \rceil \cdot p(X) \le \sum (n - \operatorname{ld} S(X/n)) + 1) p(X)$$

$$= n - \sum_x \operatorname{ld}[S(X/n)]\, p(X) + 1 = \Big| S(X/n) = 2^n p(X) \Big| =$$

$$= \cancel{n} - \cancel{\sum n \cdot p(X)} - \underbrace{\sum p(x) \operatorname{ld} p(x)}_{-H(X)} + 1 = \underline{H(X) + 1}$$

$$E[K] = E[n - \operatorname{ld} S(X)] \le H(X) + 1$$

a

$$\begin{array}{cc} \frac{26}{52} & \frac{26}{52} \\ B & R \end{array} \Big] \begin{pmatrix} 1 \\ B \end{pmatrix} \left[\frac{25}{51} \mid \frac{26}{51} \right] \begin{pmatrix} 1 \\ B \end{pmatrix} \left[\frac{24}{50} \mid \frac{26}{50} \right] \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \left[0, 1 \right]$$

B

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6.6 GAMBLING ESTIMATE OF THE ENTROPY OF ENGLISH

ASSUME THAT ENGLISH CONSISTS OF 27 CHARACTERS
(26 LETTERS AND SPACE SYMBOL)

1. SHANNON GUESSING GAME

2. GAMBLING ESTIMATE WE LET HUMAN SUBJECT TO
BET ON THE NEXT LETTER IN A SAMPLE OF ENGLISH
TEXT. AS IN THE HORSE RACE, THE OPTIMAL BET IS
PROPORTIONAL TO THE CONDITIONAL PROBABILITY OF THE
NEXT LETTER. THE PAYOFF IS 27-FOR-1 ON CORRECT
LETTER. SINCE SEQUENTIAL BETTING IS EQUIVALENT
TO BETTING ON THE ENTIRE SEQUENCE, WE CAN WRITE
THE PAYOFF AFTER n LETTERS AS:

$\boxed{9}$

$$S_n = 27^n \cdot b(x_1, x_2, \ldots x_n)$$

- THUS AFTER n ROUNDS EXPECTED LOG WEALTH IS:

$$E\left[\frac{1}{n} ld S_n\right] = E\left[\frac{1}{n} ld 27^n\right] + E\left[ld b(x_1, x_2, \ldots x_n)\right]$$

$$= ld 27 + \frac{1}{n} E\left[ld b(x_1 x_2 \ldots x_n)\right] = ld 27 + \frac{1}{n} \sum_{x^n} p(x^n) ld b(x^n) = ld 27 + \frac{1}{n} \sum_{x_n} p(x^n) ld p(x^n) - \frac{1}{n} \sum_{x^n} p(x^n) ld \frac{p(x^n)}{b(x^n)} = ld 27 - \frac{H(X^n)}{n} - \frac{D(p \| b)}{n}$$

$$\leq ld 27 - \frac{H(X^n)}{n} \leq \underline{ld 27 - H(X)}$$

$$\boxed{H(X) \leq ld 27 - E\left[\frac{1}{n} ld S_n\right]}$$

SUMMARY

- DOUBLING RATE: $W(b,p) = E\left[ld[S(X)]\right] = \sum_{k=1}^{m} p_k \, ld[b_k o_k]$
- OPTIMAL DOUBLING RATE: $W^*(p) = \max_b W(b,p)$
- PROPORTIONAL GAMBLING IS OPTIMAL!
  $W^*(p) = \max_b W(b,p) = \sum p_i \, ld \, o_i - H(p)$
  $b^* = p$
- GROWTH RATE: WEALTH GROWS AS: $S_n \doteq 2^{nW^*(p)}$
- CONSERVATION LAW: $\boxed{H(p) + W^*(p) = ld \, m}$
- SIDE INFORMATION: IN HORSE RACE X, THE INCREASE ΔW IN DOUBLING RATE DUE TO SIDE INFORMATION Y IS:

$$\Delta W = I(X,Y)$$

PROBLEM 6.1 HORSE RACE. THREE HORSES RUN A RACE. A GAMBLER OFFERS 3-FOR-1 ODDS ON EACH HORSE. THESE ARE FAIR ODDS UNDER ASSUMPTION THAT ALL HORSES ARE EQUALLY LIKELY TO WIN THE RACE. THE TRUE WIN PROBABILITIES ARE KNOWN TO BE:

$$p = (p_1, p_2, p_3) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right).$$

LET $b = (b_1, b_2, b_3)$, $b_i \geq 0$, $\sum b_i = 1$, BE THE AMOUNT INVESTED IN EACH HORSE. THE EXPECTED$_2$ LOG WEALTH IS:

$$W(b) = \sum_{i=1} p_i \, ld \, 3 \cdot b_i$$

(a) MAXIMIZE OVER b TO FIND b^* AND W^*. THUS THE WEALTH ACHIEVED IN REPEATED HORSE RACES SHOULD GROW TO INFINITY LIKE 2^{nW^*} WITH PROBABILITY 1

(b) SHOW THAT IF INSTEAD WE PUT ALL OF OUR MONEY ON HORSE 1, THE MOST LIKELY WINNER, WE WILL EVENTUALLY GO BROKE WITH PROBABILITY 1.

(a) $b^* = p = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$ $\qquad S_n = 2^{nW(p,b)}$

$$W(b) = \frac{1}{2} ld \frac{3}{2} + \frac{1}{4} \cdot ld \frac{3}{4} + \frac{1}{4} ld \frac{3}{4} =$$

$$= \frac{1}{2} ld \frac{3}{2} + \frac{1}{2} ld \frac{3}{4} = \frac{1}{2} ld \frac{9}{8} = \underline{\underline{0.085}}$$

$$W(b,p) = E\left[\frac{1}{n} ld S_n\right] = / \Rightarrow S_n = \prod_{i=1}^{n} b_i \cdot O_i / =$$

$$= E\left[ld \prod_{i=1}^{n} b_i O_i\right] = \sum_{i=1}^{n} E\left[ld \, b_i O_i\right] \qquad ?$$

$$W(b,p) = \sum_{i=1}^{n} p_i \, ld \, b_i O_i$$

(a)

$$\frac{1}{n} ld S_n = \frac{1}{n} ld \prod_{i=1}^{n} S_i = \frac{1}{n} \sum_{i=1}^{n} ld S_i \xrightarrow{\text{IN PROBABILITY}} E[ld S(X)]$$

$$= \underline{W(b,p)}$$

$$S(X) = b(X) \cdot o(X)$$

$$ld S_n = nW(b,p) \qquad \boxed{S_n \doteq 2^{nW(b,p)}}$$

EDITION 2 SOLUTIONS

$$S_n = \prod_{i=1}^{n} 3 \cdot b(X_j) = \prod_{j=1}^{n} 2^{ld 3b(X_j)} = 2^{\sum ld 3b(X_j)} =$$

$$= 2^{n(\frac{1}{n} \sum ld 3b(X_j))} \xrightarrow{\text{LAW OF LARGE}} 2^{n E[ld 3b(X_i)]} = 2^{n W(b,p)}$$

(b) IF WE PUT ALL MONEY IN FIRST HORSE THE PROBABILITY THAT WE DONT GO BROKE IS IN n RACES IS:

$$P = \underbrace{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdots \frac{1}{2}}_{n} = \left(\frac{1}{2}\right)^n \quad n \to \infty \to P \to 0$$

ALTERNATIVELY:

$$b = (1,0,0) \qquad W(b) = \sum_{i=1}^{3} p_i \, ld \, o_i b_i =$$

$$= \frac{1}{2} (ld 3) + \frac{1}{4} \cdot \underbrace{ld 0 3}_{ld 0 \to -\infty} + \frac{1}{4} \cdot \underbrace{ld 0.3}_{-\infty} \to -\infty$$

$$S_n = 2^{+nW} = 2^{-\infty} \to 0 \,]$$ WEALTH IS $\overset{n}{\to} 0$

$\boxed{6.2}$ HORSE RACE WITH SUBFAIR ODDS. IF THE ODDS ARE BAD (DUE TO A TRACK TAKE) THE GAMBLER MAY WISH TO KEEP MONEY IN POCKET. LET $b(0)$ BE THE AMOUNT OF MONEY IN HIS POCKET AND LET $b(1)$, $b(2)$, ..., $b(m)$ BE THE AMOUNT BET ON HORSES 1, 2, ..., m WITH ODDS $o(1), o(2), \ldots, o(m)$, AND WIN PROBABILITIES $p(1), p(2), \ldots, p(m)$.

Ex 9

THUS THE RESULTING WEALTH IS: $S(x) = b(0) + b(x)o(x)$ WITH PROBABILITY $\gamma(x)$, $x = 1, 2, \ldots m$

(a) FIND b^* MAXIMIZING $E[\ln S]$ IF $\sum \frac{1}{o_i} < 1$

(b) DISCUSS b^* IF $\sum \frac{1}{o_i} > 1$ (THERE ISN'T ANY EASY CLOSED-FORM SOLUTION IN THIS CASE, BUT A "WATER-FILLING" SOLUTION RESULTS FROM THE APPLICATION OF KUHN-TUCKER CONDITIONS.)

(a) $\underline{b_i \cdot O_i = c}$ DUTCH BOOK $\quad S_m = \prod_{i=1}^{m} \underbrace{b_i O_i}_{S(x)} = \prod_i b$

$$c = \frac{1}{\sum \frac{1}{O_i}} > 1 \qquad \boxed{S(x) = c = \frac{1}{\sum \frac{1}{O_i}}} > 1$$

$$W(b,\gamma) = \sum_{i=1}^{m} \gamma_i \ln O_i \cdot b_i \qquad c: \sum_i b_i = 1$$

$$\frac{d}{db_i}\left[\sum_{i=1}^{m} \gamma_i \ln(O_i b_i) + \lambda\left(\sum_{i=1}^{m} b_i - 1\right)\right] = 0$$

$$\sum_{i=1}^{m} \gamma_i \frac{1}{\cancel{O_i} b_i} \cancel{O_i} + \lambda \sum_{i=1}^{m} 1 = 0 \qquad \sum_{i=1}^{m}\left(\frac{\gamma_i}{b_i} + \lambda\right) = 0$$

$$\frac{\gamma_i}{b_i} = -\lambda \qquad b_i = -\frac{\gamma_i}{\lambda} \qquad -\sum \frac{\gamma_i}{\lambda} = 1$$

$$-\frac{1}{\lambda}\sum_i \gamma_i = 1 \qquad -\frac{1}{\lambda} = 1 \qquad \boxed{\lambda = -1} \Rightarrow \boxed{\gamma_i = b_i}$$

$$S(x) = b(0) + b(x) \cdot o(x) \qquad \sum \frac{1}{O_i} < 1 \qquad O_i = \frac{c}{b_i}$$

$$E[\ln S(x)] = W(b,\gamma) \qquad b_i \cdot O_i = c \qquad \sum \frac{b_i}{c} < 1$$

$$\boxed{\sum b_i < c}$$

69a

$\max_{b} \left[\sum_{i=1}^{n} \gamma_i \, ld(O_i \cdot b_i) \right]$ st. $\sum_{i=1}^{n} b_i = 1 - b_0$

st. $\sum_{i=1}^{n} \frac{1}{O_i} < 1$

$$f = \sum_{i=1}^{n} \gamma_i \, ld(O_i b_i) + \lambda\left(\sum_{i=0}^{n} b_i - 1\right) + \mu\left(\sum_{i=1}^{n} \frac{1}{O_i} - 1\right)$$

$\frac{df}{db_i} = 0$ $i=0..n$ $f = \sum_{i=1}^{n} \gamma_i \, ld(O_i b_i) + \lambda b_0 + \lambda\left(\sum_{i=1}^{n} b_i - 1\right)$

$$\frac{df}{db_i} = \sum_{i=1}^{n} \gamma_i \frac{1}{O_i b_i} \cdot O_i + \lambda + \lambda \sum_{i=1}^{n} 1 = 0$$

$\sum_{i=1}^{n} \gamma_i \frac{1}{b_i} + \sum_{i=1}^{n} \lambda = -\lambda$ $\sum_{i=1}^{n} \left(\frac{\gamma_i}{b_i} + \lambda\right) = -\lambda$

$\sum_{i=1}^{n} \left(\frac{\gamma_i}{b_i} + \lambda\right) = -\lambda$ $\sum_{i=1}^{n} \left(\frac{\gamma_i}{b_i} + \lambda + \frac{\lambda}{n}\right) = 0$

$\frac{\gamma_i}{b_i} + \frac{(n+1)\lambda}{n} = 0$ $\frac{\gamma_i}{b_i} = -\frac{(n+1)\lambda}{n}$ $b_i = \frac{-n\gamma_i}{\lambda(n+1)}$

$\sum_{i=0}^{n} b_i = 1$ $-\sum_{i=0}^{n} \frac{n\gamma_i}{(n+1)\lambda} = 1$ $-\frac{n}{(n+1)\lambda} \sum_{i=1}^{n} \gamma_i = 1$

$\boxed{\lambda = -\frac{n}{n+1}}$ $\frac{\gamma_i}{b_i} \; 1 = 0$ $\boxed{\gamma_i = b_i \quad i=1..n}$

$b^* = [b_0, \gamma_1, \gamma_2, \ldots \gamma_n]$

$$f = \sum_{i=1}^{n} \gamma_i \, ld(b_0 + b_i O_i) + \lambda\left(\sum_{i=1}^{n} b_i - 1 + b_0\right) + \mu\left(\sum_{i=1}^{n} \frac{1}{O_i} - 1\right)$$

$\sum_{i=1}^{n} \gamma_i \frac{1}{b_0 + b_i O_i} \cdot O_i + \lambda \sum_{i=1}^{n} 1$ $\frac{\gamma_i O_i}{b_0 + b_i O_i} = -\lambda$

SOLUTIONS EDITION 2

$$\max_{b_i \geq 0} W(b,p) = E[\mathrm{ld}\, S(X)] = \sum_{i=1}^{m} p_i \,\mathrm{ld}(b_0 + b_i o_i) \qquad \sum_{i=0}^{m} b_i = 1$$

APPROACH 1 : RELATIVE ENTROPY

$$W(b,p) = \sum p_i \,\mathrm{ld}(b_0 + b_i o_i) = \sum p_i \,\mathrm{ld}\left(\frac{\frac{b_0}{o_i} + b_i}{\frac{1}{o_i}}\right) =$$

$$= \sum p_i \,\mathrm{ld}\left(\frac{\frac{b_0}{o_i} + b_i}{p_i}\right)\left(\frac{p_i}{\frac{1}{o_i}}\right) = \sum p_i \,\mathrm{ld}\, p_i o_i + \sum p_i \,\mathrm{ld} \frac{\frac{\frac{b_0}{o_i} + b_i}{\underbrace{\left(\sum \frac{b_0}{o_i} + b_i\right)}_{K}} K}{p_i}$$

$$= \sum p_i \,\mathrm{ld}\, p_i o_i + \sum p_i \,\mathrm{ld}\, K + \sum p_i \,\mathrm{ld} \frac{\left[\left(\frac{b_0}{o_i} + b_i\right) / K\right]}{p_i} =$$

$$= \sum p_i \,\mathrm{ld}\, p_i o_i + \overset{1}{\sum p_i} \,\mathrm{ld}\, K + \underbrace{\sum p_i \,\mathrm{ld} \frac{r_i}{p_i}}_{-D(p\|r)}$$

$$W(b,p) = \sum p_i \,\mathrm{ld}\, p_i o_i + \mathrm{ld}\, K - D(p\|r)$$

$$\boxed{r_i = \frac{\frac{b_0}{o_i} + b_i}{\sum \frac{b_0}{o_i} + b_i} = \frac{\frac{b_0}{o_i} + b_i}{K} \quad \textcircled{\$}}$$

NORMALIZED PORTFOLIO !!!

- IN ORDER TO MAXIMIZE $W(b,p)$ WE MUST MAXIMIZE $\mathrm{ld} K$ AND AT SAME TIME MINIMIZE $D(p\|r)$.

• LET US CONSIDER THE TWO CASES:

(a) $\sum \frac{1}{o_i} \leq 1$

$$K = \sum\left(\frac{b_0}{o_i} + b_i\right) = b_0 \sum \frac{1}{o_i} + \underbrace{\sum b_i}_{1-b_0}$$

$$\boxed{K = b_0\left(\sum \frac{1}{o_i} - 1\right) + 1}$$

$$\sum \frac{1}{o_i} < 1 \Rightarrow K_{max} = 1 \text{ FOR } \boxed{b_0 = 0}$$

- IN THIS CASE $b_0 = 0 \xrightarrow{} K=1 \Rightarrow \overset{\textcircled{\$}}{p_i = b_i} \Rightarrow r_i = p_i \Rightarrow$

$$D(p\|r) = 0 \Rightarrow W^*(p) = \sum p_i \,\mathrm{ld}\, p_i o_i$$

HENCE IN FAIR OR SUBFAIR GAMES, THE GAMBLER SHOULD INVEST ALL HIS MONEY IN THE RACE USING PROPORTIONAL GAMBLING, AND NOT LEAVE ANYTHING ASIDE AS CASH.

(b) $\sum \frac{1}{o_i} > 1$ i.e. SUBFAIR ODDS.

$$K = b_0\left(\sum \frac{1}{o_i} - 1\right) + 1 \qquad \sum \frac{1}{o_i} > 1 \Rightarrow$$

K_{max} FOR $b_0 = 1$

$$r = \frac{\frac{b_0}{o_i} + b_i}{\sum \frac{b_0}{o_i} + b_i} = \frac{\frac{b_0}{o_i} + b_i}{K}$$

$D(p\|r)$ CANNOT SIMULTANEOUSLY BE MINIMIZED.

$$W(b, p) = \sum p_i \operatorname{ld} p_i o_i + \operatorname{ld} K - D(p\|r)$$

$\boxed{p_i o_i \leq 1}$ $\quad \sum p_i \operatorname{ld} p_i o_i < 0$

- FOR $b_0 = 0$ BEST WE CAN DO IS PROPORTIONAL BETTING $r = p = b \Rightarrow D(p\|r) = 0$ $\quad \operatorname{ld} K = \operatorname{ld} 1 = 0$

$\Rightarrow W(b, p) < 0$ WORSE THEN $b_0 = 1$ WHERE WE HAVE 0 log RETURN

THIS INDICATES THAT ONE SHOULD LEAVE ALL ONE'S MONEY AS CASH.

- MORE RIGOROUS APPROACH USING CALCULUS:
- LET THE AMOUNT BET ON EACH HORSE IS $(b_1, b_2, \ldots b_m)$

$\sum_{i=1}^{m} b_i = 1$, i.e. THERE IS NO MONEY LEFT ASIDE. ARRANGE HORSES IN ORDER OF DECREASING $b_i o_i$ SO THAT THE m-TH HORSE IS WITH MINIMUM PRODUCT.

CONSIDER NEW PORTFOLIO:

$$\boxed{b_i' = b_i - \frac{b_m o_m}{o_i}} \qquad \begin{aligned} b_i o_i &\geq b_m o_m \\ b_i &\geq \frac{b_m o_m}{o_i} \quad b_i - \frac{b_m o_m}{o_i} \geq 0 \end{aligned}$$

$b_i' \geq 0 \quad \forall i$. WE KEEP THE REMAINING MONEY:

$$1 - \sum_{i=1}^{m} b_i' = 1 - \sum\left(b_i - \frac{b_m o_m}{o_i}\right) = 1 - 1 + \sum_{i=1}^{m} \frac{b_m o_m}{o_i} = \sum_{i=1} \frac{b_m}{o_i}$$

$$\boxed{\sum_{i=1} \frac{b_n O_n}{O_i}} \Big\} \quad \text{AMOUNT OF CASH WE KEEP}$$

- THE RETURN IF HORSE i WINS :

$$b_i' O_i = \left(b_i - \frac{b_n O_n}{O_i} \right) \cdot O_i + \sum_{i=1}^{n} \frac{b_n O_n}{O_i} =$$

$$= b_i O_i - b_n O_n + b_n O_n \sum_{i=1}^{n} \frac{1}{O_i} = b_i O_i + b_n O_n \left(\sum_{i=1}^{n} \frac{1}{O_i} - 1 \right)$$

$$> b_i O_i \quad \text{SINCE } \sum_{i=1}^{n} \frac{1}{O_i} > 1$$

$\boxed{b_i' O_i > b_i O_i} \Longrightarrow$ IRRESPECTIVE OF WHICH HORSE WINS, THE NEW PORTFOLIO DOES BETTER THEN OLD PORTFOLIO

APPROACH 2: CALCULUS

$$f = \sum_{i=1}^{n} \gamma_i \ln(O_i b_i + b_0) + \lambda \left(\sum_{i=0}^{n} b_i - 1 \right)$$

$$\frac{\partial f}{\partial b_i} \quad \sum_{i=1}^{n} \gamma_i \frac{O_i}{O_i b_i + b_0} + \lambda \sum_{\bigcirc{i=0}}^{n} 1 = 0 \qquad \sum_{i=1}^{n} \left(\gamma_i \frac{O_i}{O_i b_i + b_0} + \lambda \right) = -\lambda$$

$\boxed{i = 1 \ldots n}$

$\gamma_0 = 0$ 1A MORE DI 50 INVERIS TROVATA SUITO

$$\boxed{1^\circ} \; \boxed{\frac{\gamma_i O_i}{O_i b_i + b_0} = -\lambda}$$

$$\sum_{i=0}^{n} \left(\gamma_i \frac{O_i}{O_i b_i + b_0} + \lambda \right) = 0 \qquad \gamma_i = - \frac{\lambda (O_i b_i + b_0)}{O_i}$$

$$\gamma_i O_i = - \lambda O_i b_i - b_0 \lambda$$

$\boxed{2^\circ}$

$$\boxed{\sum_{i=0}^{n} b_i = 1} \Longleftarrow \text{SE POSIVA } \infty : \frac{\partial f}{\partial \lambda} = 0$$

$$\circledast \; \boxed{b_i = - \frac{\lambda b_0 + \gamma_i O_i}{\lambda O_i} \quad i > 0}$$

$$b_0 - \sum_{i=1}^{n} \frac{\lambda b_0 + \gamma_i O_i}{\lambda O_i} = 1 \qquad b_0 - \sum_{i=1}^{n} \frac{\lambda b_0}{\lambda O_i} - \sum_{i=1}^{n} \frac{\gamma_i}{\lambda} = 1$$

$$b_0 - b_0 \sum_{i=1}^{n} \frac{1}{O_i} - \frac{1}{\lambda} = 1 \qquad \boxed{b_0 - b_0 \sum_{1}^{n} \frac{1}{O_i} = \frac{1+\lambda}{\lambda}}$$

$$\boxed{\frac{\partial f}{\partial b_0}} = \sum_{i=1}^{n} \gamma_i \frac{1}{O_i b_i + b_0} + \lambda = 0 \Rightarrow$$

$\boxed{\gamma = 0}$

$$\boxed{3^\circ} \; \boxed{\sum_{i=1}^{n} \frac{\gamma_i}{O_i b_i + b_0} = -\lambda}$$

(*) $$\boxed{\frac{\gamma_i o_i}{o_i b_i + b_0} = -\lambda} \qquad \boxed{\frac{\gamma_i}{o_i b_i + b_0} = -\frac{\lambda}{o_i}} \Rightarrow (3°)$$

$$\sum_{i=1}^{n} \frac{\lambda}{o_i} = \lambda \quad \Rightarrow \quad \boxed{\lambda \sum_{i=1}^{n} \frac{1}{o_i} = \lambda} \ (*) \quad \boxed{\sum_{i=1}^{n} \frac{1}{o_i} = 1}$$

CLEARLY IN CASE WHEN $\sum_{i=1}^{n} \frac{1}{o_i} \neq 0$ THE ONLY SOLUTION TO THE EQUATION IS WHEN $\boxed{\lambda = 0}$.

- IN ADITION TO CONSTRAINT $\sum_{i=0}^{n} b_i = 1$ WE HAVE INEQUALITY CONSTRAINT. $\boxed{b_i \geq 0}$.

$$J(b) = \sum_{i=1}^{n} \gamma_i \, ld(b_0 + b_i o_i) + \lambda\left(\sum_{i=0}^{n} b_i - 1\right) + \sum_{i=0}^{n} \gamma_i b_i$$

ALLOWING INEQUALITY CONSTRAINTS, THE KARUSH-KUHN-TUCKER (KKT) APPROACH TO NON-LINEAR PROGRAMMING GENERALIZE THE METHOD OF LAGRANGE MULTIPLIERS, WHICH ALLOWS ONLY EQUALITY CONSTRAINT.

- $$\frac{dJ(b)}{db_i} = \sum_{i=1} \gamma_i \frac{1}{b_0 + b_i o_i} + \sum_{i=1}^{n} \lambda + \sum_{i=1} \gamma_i = 0$$

OVA NE E DOBRA FORMULACIJA VIDI LAGRANGE MULTIPLIERS VO STEVKATO

$$\boxed{\frac{\gamma_i o_i}{b_0 + b_i o_i} + \lambda + \gamma_i = 0}$$

$$\frac{\gamma_i o_i}{b_0 + b_i \gamma_i} = -\lambda - \gamma_i \qquad \boxed{\frac{\gamma_i}{b_0 + b_i o_i} = \frac{-\lambda - \gamma_i}{o_i}}$$

- $$\frac{dJ(b)}{db_0} = \sum_{i=1}^{n} \frac{\gamma_i}{b_0 + b_i \gamma_i} + \lambda + \gamma_0 = 0 \qquad \sum_{i=1}^{n} \frac{\gamma_i}{b_0 + b_i \gamma_i} = -\lambda - \gamma_0$$

- $$\frac{dJ(b)}{d\lambda} = \sum_{i=0}^{n} b_i - 1 = 0 \qquad \sum_{i=0}^{n} b_i = 1$$

$$\sum_{i=1}^{n} \frac{\lambda - \gamma_i}{o_i} = -\lambda - \gamma_0$$

$$\boxed{\lambda + \gamma_0 = \lambda \sum_{i=1}^{n} \frac{1}{o_i} + \sum_{i=1}^{n} \frac{\gamma_i}{o_i}}$$

WHICH INDICATES THAT IF $\sum \frac{1}{o_i} \neq 1$, AT LEAST ONE OF THE γ'S IS NONZERO, WHICH INDICATES THAT THE SOLUTION IS ON THE BOUNDARY OF THE REGION (THE CONSTRAINT HAS BECOME ACTIVE)

IN THE CASE OF SOLUTIONS ON THE BOUNDARY, WE HAVE TO USE KUHN-TUCKER CONDITIONS TO FIND THE MAXIMUM. [CHECK GALLAGER PG. 87]. THE CONDITIONS DESCRIBE THE BEHAVIOR OF THE DERIVATIVE AT THE MAXIMUM OF CONCAVE FUNCTION OVER CONVEX REGION. FOR ANY COORDINATE WHICH IS IN THE INTERIOR OF THE REGION, THE DERIVATIVE SHOULD BE 0. FOR ANY COORDINATE ON THE BOUNDARY, THE DERIVATIVE SHOULD BE NEGATIVE IN DIRECTION TOWARDS THE INTERIOR OF THE REGION. MORE FORMALLY FOR CONCAVE FUNCTION: $F(x_1, x_2, \ldots, x_n)$ OVER THE REGION $x_i \geq 0$,

$$\boxed{\frac{\partial F}{\partial x_i} \begin{array}{ll} \leq 0 & \text{IF } x_i = 0 \\ = 0 & \text{IF } x_i > 0 \end{array}}$$

$$F = \sum_{i=1}^{m} p_i \operatorname{ld}(b_0 + b_i o_i) + \lambda \sum_{i=0}^{m} b_i$$

$$\frac{dF}{db_i} = ! \quad i = 1 \ldots m$$

ONE GETS [illegible] GRADIENT. VIA STEWARD-LAGRANGE MULTIPLIERS

$\boxed{1°}$
$$\frac{p_i o_i}{b_0 + b_i o_i} + \lambda \begin{array}{ll} \leq 0 & \text{IF } b_i = 0 \\ = 0 & \text{IF } b_i > 0 \end{array}$$

$\boxed{2°}$
$$\sum_{i=1}^{m} \frac{p_i}{b_0 + b_i o_i} + \lambda \begin{array}{ll} \leq 0 & \text{IF } b_0 = 0 \\ = 0 & \text{IF } b_0 > 0 \end{array}$$

IF WE CAN FIND A SOLUTION TO KUHN-TUCKER CONDITIONS THE SOLUTION IS MAXIMUM OF THE FUNCTION IN THE REGION.

- LET US CONSIDER TWO CASES:

(a) $\sum \frac{1}{o_i} \leq 1$ WE EXPECT: $b_0 = 0$
$b_i = p_i$

1° $\frac{p_i o_i}{\cancel{o_i p_i}} + \lambda \leq 0$ IF $\lambda = -1$ $\boxed{1 \leq 1}$ } CONDITION 1° IS FULFILLED

2° $\sum_{i=1}^{m} \frac{1}{o_i} + \lambda \leq 0$ $\sum_{i=1}^{m} \frac{1}{o_i} \leq 1$ ✓

KUHN-TUCKER CONDITIONS ARE SATISFIED ⇒ THIS IS OPTIMAL PORTFOLIO FOR SUPERFAIR AND FAIR ODDS.

(b) WE TRY THE EXPECTED SOLUTION $b_0 = 1$ $b_i = 0$

1° $p_i o_i + \lambda \leq 0$ $p_i o_i - 1 \leq 0$ $p_i o_i \leq 1$

2° $\sum_{i=1}^{m} p_i + \lambda = 0 \Rightarrow \lambda = -1$

HENCE KHUN-TUCKER CONDITIONS ARE SATISFIED IF ALL $\boxed{\gamma_i O_i \le 1}$. UNDER THIS CONDITION, THE OPTIMUM SOLUTION IS NOT TO INVEST ANYTHING IN THE RACE BUT TO KEEP EVERYTHING IN CASH.

- IN CASE WHEN SOME $\gamma_i O_i > 1$, THE KHUN-TUCKER CONDITIONS ARE NO LONGER SATISFIED BY $b_0 = 1$. THE OPTIMUM SOLUTION MAY INVOLVE INVESTING IN SOME HORSES WITH $\gamma_i O_i \le 1$. THERE IS NO EXPLICIT FORM FOR THE SOLUTION IN THIS CASE.
- PROCEDURE FOR FINDING OPTIMUM DISTRIBUTION OF CAPITAL:

$$\boxed{\gamma_1 O_1 \ge \gamma_2 O_2 \ge \dots \ge \gamma_n O_n} \quad ✪$$

DEFINE:
$$C_k = \begin{cases} \dfrac{1 - \sum_{i=1}^{k} \gamma_i}{1 - \sum_{i=1}^{k} \frac{1}{O_i}} & \text{- IF } k \ge 1 \\ 1 & \text{- IF } k = 0 \end{cases}$$

DEFINE:
$$t = \min \{ n \mid \gamma_{n+1} O_{n+1} \le C_n \}$$

MINIMALNO „n" ZA KOJE VAŽI DA $\gamma_{n+1} O_{n+1} \le C_n$

CLEARLY $t \ge 1$ SINCE $\boxed{\gamma_1 O_1 > 1 = C_0}$

CLAIM: THE OPTIMAL STRATEGY FOR THE HORSE RACE WHEN THE ODDS ARE SUBFAIR AND SOME OF THE $\gamma_i O_i$ ARE GREATER THEN 1 IS:

SET: $b_0 = C_t$,

AND FOR $i = 1, 2, \dots, t$, SET
$$\boxed{b_i = \gamma_i - \frac{C_t}{O_i}}$$
AND FOR $i = t+1, \dots, n$ SET $b_i = 0$

- ABOVE CHOICE OF „b" SATISFIES KHUN-TUCKER CONDITIONS FOR „$\lambda = -1$".

$$2^\circ \Rightarrow \sum_{i=1}^{n} \frac{\gamma_i}{b_0 + b_i O_i} = \sum_{i=1}^{t} \frac{\cancel{\gamma_i}}{\cancel{C_t} + \cancel{\gamma_i} O_i - \cancel{C_t}} + \sum_{i=t+1}^{n} \frac{\gamma_i}{b_0}$$

$$= \sum_{i=1}^{t} \frac{1}{O_i} + \sum_{i=t+1}^{n} \frac{\gamma_i}{b_0} = \sum_{i=1}^{t} \frac{1}{O_i} + \frac{1}{C_t} \sum_{i=t+1}^{n} \gamma_i =$$

$$= \sum_{i=1}^{t} \frac{1}{O_i} + \frac{1}{C_t} \left(1 - \sum_{i=1}^{t} \gamma_i \right) = 1$$

6a: $1 \le i \le t$ KHUN-TUCKER CONDITIONS REDUC:

$$\frac{\gamma_i\theta_i}{b_0 + b_i\theta_i} + \lambda \quad \begin{matrix} \le 0 \text{ IF } b_i = 0 \\ = 0 \text{ IF } b_i > 0 \end{matrix}$$

$$\frac{\gamma_i\theta_i}{C_t + \theta_i\left(\gamma_i - \frac{C_t}{\theta_i}\right)} = \frac{\cancel{\gamma_i\theta_i}}{\cancel{\gamma_i\theta_i}} = 1$$

} CONDITION FULFILLED!! (FOR $b_i > 0$)

Za: $t+1 \le i \le m$ KHUN-TUCKER COND. REDUCE:

$$\frac{\gamma_i\theta_i}{b_0 + b_i\theta_i} = \frac{\gamma_i\theta_i}{b_0} = \underbrace{\frac{\gamma_i\theta_i}{C_t} \le 1}_{①}$$

(FOR $b_i = 0$)

BY DEFINITION OF ,t". HENCE THE KHUN-TUCKER CONDITIONS ARE SATISFIED AND THIS IS THE OPTIMAL SOLUTION

$$t = \min\{q \mid \gamma_{q+1}\theta_{q+1} \le C_q\}$$

③

$i = t+1$

$$p_{q+1}\theta_{q+1} \le C_q \qquad \boxed{\frac{\gamma_{q+1}\theta_{q+1}}{C_q} \le 1}$$

⊛ $$\frac{\gamma_{t+2}\theta_{t+2}}{C_{t+1}} \le 1$$

$$\frac{\gamma_{t+1}\theta_{t+1}}{C_t} \le 1 \qquad \overbrace{\gamma_{t+1}\theta_{t+1} > \gamma_{t+2}\theta_{t+2}}^{\text{P.75.★}}$$

$$\boxed{\frac{\gamma_{t+2}\theta_{t+2}}{C_t} \le \frac{\gamma_{t+1}\theta_{t+1}}{C_t} \le 1}$$

PROOF OF ①

STEWARD SEC. 14.8 LAGRANGE MULTIPLIERS.

- EXAMPLE IN 2d. FIND EXTREME VALUE OF $f(x,y)$ S.T. $g(x,y)=k$. IN OTHER WORDS, WE SEEK EXTREME VALUES OF $f(x,y)$ WHEN POINT (x,y) IS RESTRICTED TO LIE ON THE LEVEL CURVE $g(x,y)=k$. SO THE GRADIENT VECTORS OF $f(x,y)=c$ AND $g(x,y)=k$ ARE PARALLEL:

$$\boxed{\nabla f(x_0,y_0) = \lambda \nabla g(x_0,y_0)}$$

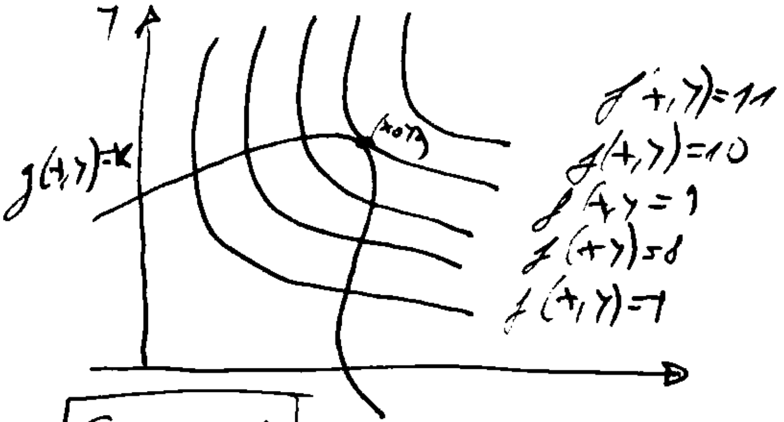
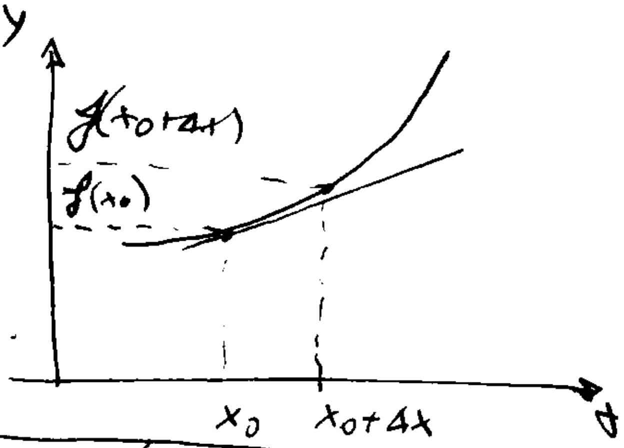


Figure 1



$$\boxed{f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}}$$

$$D_u f(x_0,y_0) = \lim_{h \to 0} \frac{f(x_0+ha,\ y_0+hb) - f(x_0 y_0)}{h}$$

$\vec{u} = \langle a, b \rangle$

DIRECTIONAL DERIVATIVE OF f IN DIRECTION u

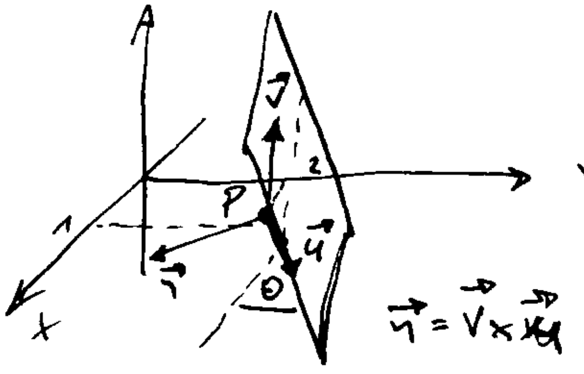
$u = \langle 1,0 \rangle = i \quad D_i f = f_x \qquad u = \langle 0,1 \rangle = j \quad D_j f = f_y$

$\vec{n} \cdot \langle x-x_0,\ y-y_0,\ z-z_0 \rangle = 0 \qquad \langle x_0, y_0, z_0 \rangle = \langle 1,2,0 \rangle$

$\vec{M} = \langle x_1, y_1, 0 \rangle \qquad x_1 = \cos\theta = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$y_1 = \sin\theta = \frac{1}{2}$

$\sqrt{x_1^2 + y_2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = \sqrt{1} = 1 \qquad \vec{M} = \langle \frac{\sqrt{3}}{2}, \frac{1}{2}, 0 \rangle$



$\vec{n} = \vec{V} \times \vec{M}$

$$\vec{V} = \begin{vmatrix} i & j & k \\ 1 & 2 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \end{vmatrix} = 0i + 0j + \left(\frac{1}{2} - \sqrt{3}\right)k$$

$$\vec{n} = \begin{vmatrix} i & j & k \\ 0 & 0 & \frac{1}{2}-\sqrt{3} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \end{vmatrix} = -\left(\frac{1}{2}-\sqrt{3}\right)\cdot\frac{1}{2}\, i + \left(\frac{1}{2}-\sqrt{3}\right)\frac{\sqrt{3}}{2}\, j$$

$$\vec{n} \cdot \langle x-x_0,\ y-y_0,\ z-z_0 \rangle = \langle \left(\sqrt{3} - \frac{1}{2}\right)\frac{1}{2}, \left(\frac{1}{2}-\sqrt{3}\right)\frac{\sqrt{3}}{2}, 0 \rangle \cdot \langle x-1, y-3, z-0 \rangle$$

$$\left(\sqrt{3} - \frac{1}{2}\right)\frac{1}{2}x + \left(\sqrt{3} - \frac{1}{2}\right)\frac{1}{2} + \left(\frac{1}{2} - \sqrt{3}\right)\frac{\sqrt{3}}{2}y - \cancel{3}\left(\frac{1}{2}-\sqrt{3}\right)\frac{\sqrt{3}}{\cancel{2}} = 0$$

$$+\frac{2\sqrt{3}-1}{4}x + \frac{(1-2\sqrt{3})\sqrt{3}}{4}y + \cancel{\frac{\sqrt{3}}{2}} + \frac{1}{4} - \cancel{\frac{\sqrt{3}}{2}} + 3 = 0$$

$$\frac{2\sqrt{3}-1}{4}x + \frac{(1-2\sqrt{3})}{4}\sqrt{3}y = \sqrt{3} - \frac{1}{4} - 3 = \sqrt{3} - \frac{13}{4}$$

$$\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)x + \left(\frac{\sqrt{3}}{4} - \frac{3}{2}\right)y = \sqrt{3} - \frac{13}{4}$$

$$< \frac{\sqrt{3}}{2}, \frac{1}{2}, 0 > \cdot < x-1, y-2, z-0 > = 0 \qquad \frac{\sqrt{3}}{2}x - \frac{\sqrt{3}}{2} + \frac{1}{2}y - 1 + 0 = 0$$

$$\frac{\sqrt{3}}{2}x + \frac{1}{2}y = \frac{\sqrt{3}}{2} + 1 \qquad \boxed{\sqrt{3}x + y = \sqrt{3} + 2}$$

IZGLEDA VAKA SUM GO DOBIVALE VO Stewartse W[illegible]

$\boxed{\text{Exp. 4}}$ $\quad f(x,y) = x^2y^3 - 4y \qquad \boxed{P(2,-1)}$

$$v = 2i + 5j \qquad |v| = \sqrt{4+25} = \sqrt{29}$$

$$D_v f(2,-1) \qquad \nabla f = f_x i + f_y j = 2xy^3 i + (3x^2y^2-4)j$$

$$D_v f(2,-1) = \nabla f \cdot \frac{v}{|v|} = 2xy^3 \cdot \frac{2}{\sqrt{29}} + (3x^2y^2-4)\frac{5}{\sqrt{29}} =$$

$$= \left| \begin{matrix} y=-1 \\ x=2 \end{matrix} \right| = -2\cdot 2\cdot 1 \frac{2}{\sqrt{29}} + \underbrace{(3\cdot 4\cdot 1 - 4)}_{12-4=8}\frac{5}{\sqrt{29}} = \frac{-8+40}{\sqrt{29}} = \frac{32}{\sqrt{29}}$$

$\boxed{\text{Exp. 6}}$ $\quad f(x,y) = x\cdot e^y$ (a.) RATE OF CHANGE $\overrightarrow{PQ}$

$P(2,0)$ $Q(\frac{1}{2}, 2)$

(b) IN WHAT DIRECTION $f(x)$ HAS MAXIMUM RATE OF CHANGE.

$$(a.) \quad v = \overrightarrow{PQ} = < \frac{1}{2}, 2 > - < 2, 0 > = < -\frac{3}{2}, 2 >$$

$$\nabla f = f_x i + f_y j = e^y i + x\cdot e^y j$$

$$\nabla f(2,0) = i + 2j \qquad D_v(2,0) = \langle 1, 2 \rangle < -\frac{3}{2}, 2 >$$

$$= -\frac{3}{2} + 4 = \frac{-3+8}{2} = \frac{5}{2} \; ?$$

$$\vec{u} = \frac{v}{|v|} = \frac{< -\frac{3}{2}, 2 >}{\sqrt{\frac{9}{4}+4}} = \frac{< -\frac{3}{2}, 2 >}{\sqrt{\frac{25}{4}}} = \frac{2}{5} < -\frac{3}{2}, 2 >$$

$$= < -\frac{3}{5}, \frac{4}{5} >$$

$D_u(2,0) = \langle 1,2 \rangle \langle -\frac{3}{5}, \frac{4}{5} \rangle = -\frac{3}{5} + \frac{8}{5} = \underline{\frac{5}{5} = 1}$

$D_u(2,0) = |\langle 1,2 \rangle| = \sqrt{1+4} = \sqrt{5}$

STEWART SE. 14.7 MAXIMUM AND MINIMUM VALUES

USAGE OF PARTIAL DERIVATIVES TO LOCATE MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES.

THEOREM 14.7.2 IF $f(x,y)$ HAS LOCAL MAXIMUM OR MINIMUM AT (a,b) AND FIRST ORDER DERIVATIVES EXIST THERE, THEN $f_x(a,b)=0$ AND $f_y(a,b)=0$

EXMP 1: $f(x,y) = x^2+y^2-2x-6y+14$

$f_x(x,y) = 2x - 2$ $\quad f_y(x,y) = 2y-6$

$2x-2=0 \Rightarrow \boxed{x=1}$ $\quad 2y-6=0 \quad \boxed{y=3}$

CRITICAL POINT IS $(1,3)$.

$f(x,y) = (x-1)^2 - 1 + (y-3)^2 - 9 + 14 = 4 + (x-1)^2 + (y-3)^2$

$(x-1)^2 \geq 0 \quad \forall x \quad (y-3)^2 \geq 0 \quad \forall y \Rightarrow$

$\boxed{f(x,y) \geq 4 \quad \forall (x,y)}$ $\quad (1,3)$ IS ABSOLUTE ~~GLOBAL~~ ~~LOCAL~~ ~~MINIMUM.~~ MINIMUM

$\boxed{f(1,3) = 4}$

EXAMPLE 2 FIND THE EXTREME VALUES OF: $f(x,y) = y^2 - x^2$

$f_x(x,y) = -2x \quad f_y(x,y) = 2y \Big\} \quad \begin{matrix} x=0 \\ y=0 \end{matrix} \Big\}$ CRITICAL POINT.

$f(0,0) = 0 \qquad \boxed{x=0} \quad f(0,y) = y^2 \geq 0$

$y=0 \quad f(x,0) = -x^2 \leq 0 \quad \forall x$ — NO EXTREMA

- $(0,0)$ IS SADDLE POINT OF $f(x,y)$.
- **SECOND DERIVATIVE TEST** SUPPOSE THAT SECOND PARTIAL DERIVATIVES OF f ARE CONTINUOUS ON THE DISK WITH CENTER (a,b) AND SUPPOSE $f_x(a,b)=0 \quad f_y(a,b)=0$. [I.E. (a,b) IS CRITICAL POINT OF f] . LET:

$D = D(a,b) = f_{xx}(a b) f_{yy}(a b) - [f_{xy}(a b)]^2$

(a) IF $D>0$ AND $f_{xx}(a,b)>0$ THEN $f(a,b)$ IS LOCAL MIN.

(b) IF $D>0$ AND $f_{xx}(a,b)<0$ THEN $f(a,b)$ IS LOCAL MAX.

(c) IF $D<0$ THEN $f(a,b)$ IS NOT LOCAL MAX OR MINIMUM.

IN CASE (C) THE POINT (a,b) IS CALLED SADDLE POINT OF f.

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} \cdot f_{yy} - (f_{xy})^2$$

EXAMPLE 3 FIND LOCAL MAXIMUM, MINIMUM AND SADDLE POINTS OF : $f(x,y) = x^4 + y^4 - 4xy + 1$

SOLUTION : $D = f_{xx} f_{yy} - f_{xy}^2 = !$

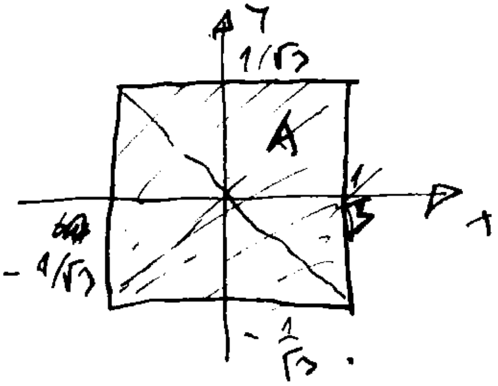
$f_x = 4x^3 - 4y \qquad f_{xx} = 12x^2$

$f_y = 4y^3 - 4x \qquad f_{yy} = 12y^2$

$f_{xy} = -4 \qquad \boxed{D = 144x^2y^2 - 16}$

$144x^2y^2 - 16 = 0 \qquad 144x^2y^2 = 16$

$x^2y^2 = \frac{1}{9} \qquad xy = \frac{1}{3}$



$\left(\frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}\right)^2 = \frac{1}{9}$

$|x| > \frac{1}{\sqrt{3}}$ & $|y| > \frac{1}{\sqrt{3}}$

$\Rightarrow D = d - 16 > 0$

$d > 16 \nearrow$

$d = 144x^2y^2 > \frac{144}{9} = 16$

$\left.|x| < \frac{1}{\sqrt{3}} \;\&\; |y| < \frac{1}{\sqrt{3}}\right\} \Rightarrow$

$4x^3 - 4y = 0 \qquad 4y^3 - 4x = 0$

$x^3 - y = 0 \qquad y^3 - x = 0 \qquad \underline{x = y^3}$

$y^9 - y = 0 \qquad y(y^8 - 1) = 0$

$y(y^4-1)(y^4+1) = y(y^2-1)(y^2+1)(y^4+1) = y(y-1)(y+1)(y^2+1)(y^4$

$\boxed{y_1=0,\ y_2=1,\ y_3=-1}$ CRITICAL POINTS

$x=y^3 \rightarrow \boxed{x_1=0 \quad x_2=1 \quad x_3=-1}$

$\boxed{(0,0),\ (1,1),\ (-1,-1)}$

$D(x,y) = 144x^2y^2-16 \qquad D(0,0)=-16 \Rightarrow$ (0,0) IS ~~[illegible]~~ SADDLE P.

$D(1,1) = 144-16 = 128>0 \qquad D(1,1)>0$ (1,1) IS LOCAL MIN

$D(-1,-1) = 144-16 = 128>0 \qquad D(-1,-1)>0$ (-1,-1) IS LOCAL MIN

$\rightarrow D(1,1)=128>0 \qquad f_{xx}(1,1) = 12\cdot1^2=12>0$ LOCAL MINIMUM

$D(-1,-1)=128>0 \qquad f_{xx}(-1,-1)=12\cdot1 = 12>0$ LOCAL MINIMUM

- RETURN TO LAGRANGE MULTIPLIERS:

NORMAL LINES OF $f(x,y)=c$ AND $g(x,y)=k$ ON FIGURE 1 (1177) ARE IDENTICAL HENCE THEIR GRADIENT VECTORS ARE PARALEL i.e:

$$\boxed{\nabla f(x_0,y_0) = \lambda \nabla g(x_0,y_0)}$$

- SUPPOSE THAT f HAS EXTREME POINT AT $P(x_0,y_0,z_0)$ ON THE SURFACE S AND LET C BE A CURVE WITH VECTOR EQUATION:

$r(t)=\langle x(t), y(t), z(t)\rangle$

THAT LIES ON S AND PASSES TROUGH P. IF t_0 CORESPONDS TO POINT P i.e. $r(t_0)=\langle x_0,y_0,z_0\rangle$ THE COMPOSITE FUNCTION:

$h(t)=f(x(t),y(t),z(t))$

REPRESENTS VALUES THAT f TAKES ON CURVE C. SINCE f HAS EXTREME VALUE AT $P(x_0,y_0,z_0)$ IT FOLLOWS THAT h HAS EXTREME AT t_0 SO $h'(t_0)=0$. WITH CHAIN RULE WE HAVE:

$$0 = h'(t_0) = f_x(x_0,y_0,z_0)\,x'(t_0) + f_y(x_0y_0z_0)\,y'(t_0) +$$
$$+ f_z(x_0,y_0,z_0)\,z'(t_0) = \nabla f(x_0,y_0,z_0)\cdot r'(t_0)$$

HENCE GRADIENT VECTOR $\nabla f(x_0,y_0,z_0)$ IS ORTOGONAL TO THE TANGENT VECTOR $r'(t_0)$.

GRADIENT VECTOR OF g, $\nabla g(x_0,y_0,z_0)$ IS ALSO ORTHOGONAL TO $r'(t_0)$, HENCE $\nabla f(x_0,y_0,z_0)$ AND $\nabla g(x_0,y_0,z_0)$ ARE PARALEL I.E. THERE EXIST "λ" SUCH AS:

$$\nabla f(x_0,y_0,z_0) = \lambda \nabla g(x_0,y_0,z_0) \qquad \oplus$$

THE NUMBER "λ" IN EQUATION $\oplus$ IS CALLED LAGRANGE MULTIPLIER.

• METHOD OF LAGRANGE MULTIPLIERS

TO FIND MAXIMUM AND MINIMUM VALUES OF $f(x,y,z)$ SUBJECT TO $g(x,y,z)=k$ [ASSUMING THAT THIS EXTREME VALUES EXIST AND $\nabla g \neq 0$ ON THE SURFACE $g(x,y,z)=k$]:

(a) FIND VALUES OF x,y,z AND λ SUCH THAT

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$

AND $g(x,y,z)=k$.

(b) EVALUATE "f" AT ALL THE POINTS (x,y,z) THAT RESULT FROM STEP (a). THE LARGEST OF THESE VALUES IS THE MAXIMUM VALUE OF "f"; THE SMALLEST IS MINIMUM VALUE OF "f".

- IF WE WRITE $\nabla f = \lambda \nabla g$ IN TERMS OF COMPONENTS:

$$f_x = \lambda g_x \quad f_y = \lambda g_y \quad f_z = \lambda g_z \quad g(x,y,z)=k$$

FOUR EQUATIONS WITH FOUR UNKNOWN: x, y, z, λ. IT IS NOT NECESSARY TO FIND EXPLICIT VALUES OF λ.

- FOR FUNCTIONS OF TWO VARIABLES:

$$\nabla f(x,y) = \lambda \nabla g(x,y) \qquad g(x,y)=k$$

$$f_x = \lambda g_x \quad f_y = \lambda g_y$$

EXAMPLE 1 A RECTANGULAR BOX WITHOUT A LID IS TO BE MADE FROM $12m^2$ OF CARDBOARD. FIND THE MAXIMUM VOLUME OF SUCH BOX:

SOLUTION: $V = xyz \qquad xy + 2xz + 2yz = 12$

$$g(x,y,z) = xy + 2xz + 2yz = 12$$

$\nabla V = \lambda \cdot \nabla g \qquad V_x = yz \qquad V_y = xz \qquad V_z = xy$

$g_x = y + 2z \qquad g_y = x + 2z \qquad g_z = 2x + 2y$

$yz = \lambda y + 2\lambda z$ (1°) $\quad xz = \lambda x + 2\lambda z$ (2°) $\quad xy = 2\lambda x + 2\lambda y$ (3°)

$xy + 2xz + 2yz = 12$ (4°)

(3°) $\rightarrow \quad x(y - 2\lambda) = 2\lambda y \qquad x = \dfrac{2\lambda y}{y - 2\lambda}$

(1°) $\rightarrow \quad y(z - \lambda) = 2\lambda z \qquad y = \dfrac{2\lambda z}{z - \lambda}$

(2°) $\rightarrow \quad x(z - \lambda) = 2\lambda z \qquad x = \dfrac{2\lambda z}{z - \lambda}$

(4°) $\Rightarrow \quad \dfrac{2\lambda y^2}{y - 2\lambda} + \dfrac{4\lambda z^2}{z - \lambda} + \dfrac{4\lambda z^2}{z - \lambda} = 12$

$$\frac{2\lambda \dfrac{4\lambda z^2}{(z-\lambda)^{\cancel{2}}}}{\dfrac{2\lambda z}{\cancel{z-\lambda}} - \dfrac{2\lambda(z-\lambda)}{\cancel{(z-\lambda)}}} + \frac{8\lambda z^2}{z - \lambda} = 12$$

$$\frac{\overset{4}{\cancel{8\lambda^2}} z^2}{(z-\lambda)\left[\cancel{2\lambda z} - \cancel{2\lambda z} + \cancel{2\lambda^2}\right]} + \frac{6\lambda z^2}{z - \lambda} = 12$$

$\dfrac{4z^2}{z - \lambda} + \dfrac{8\lambda z^2}{z - \lambda} = 12 \qquad 4z^2 + 8\lambda z^2 = 12z - 12\lambda$

$$\boxed{z_{1,2} = \frac{3 \pm \sqrt{7 - 12\lambda - 18\lambda^2}}{2 + 3\lambda}}$$

$(\underset{1}{\cancel{4}} + \underset{2}{8}\lambda)z^2 - \underset{3}{\cancel{12}}z + \underset{3}{\cancel{12}}\lambda = 0$

$\underset{2}{\cancel{8}}\lambda z^2 + \overset{3}{\cancel{12}}\lambda = \overset{3}{\cancel{12}}z - \cancel{4}z^2 \qquad 2\lambda z^2 + 3\lambda = 3z - z^2$

$$\boxed{(1 + 2\lambda)z^2 - 3z + 3\lambda = 0}$$

$\lambda(2z^2 + 3) = (3z - z^2) \qquad \boxed{\lambda = \dfrac{z(3 - z)}{2z^2 + 3}}$

$x \cdot$ (1°) $\Rightarrow \quad xyz = \lambda x(y + 2z)$

$y \cdot$ (2°) $\rightarrow \quad xyz = \lambda y(x + 2z)$

$z \cdot$ (3°) $\rightarrow \quad xyz = 2z \cdot \lambda(x + y)$

$\Rightarrow$ % $\quad \cancel{yx} + 2yz = \cancel{xy} + 2xz$

$\cancel{2}y\cancel{z} = \cancel{2}x\cancel{z} \Rightarrow \boxed{x = y}$

$\rightarrow$ % $\quad 2\cancel{z}x + 2\cancel{zy} = xy + \cancel{2yz}$

$y = 2z \qquad x = y = 2z \Rightarrow$ (4°) $\quad x^2 + 2x \cdot \dfrac{x}{2} + 2x \cdot \dfrac{x}{2} = 12$

$3x^2 = 12 \qquad x^2 = 4 \qquad \boxed{x = 2 \quad y = 2 \quad z = 1}$

EXAMPLE 2 FIND EXTREME VALUES OF: $f(x,y) = x^2 + 2y^2$ ON CIRCLE $\boxed{x^2 + y^2 = 1}$

90
14
76

2.8 DERIVATES .... 158/162 $r = 4$

11.10 TAYLOR & MAC 760 / 786 $r = 26$

VECTOR CALCULUS 1054 / 1080 $r = 26$

10. Differential Equations ... 622/612 $r = 10$

$f_x = 2x \quad f_y = 4y \quad 2x = 2\lambda x \quad x = 0$

$g_x = 2x \quad g_y = 2y \quad 4y = 2\lambda y \quad \lambda = 2$

$x = 0 \Rightarrow y = \pm 1 \quad (0,1).(0,-1)$

$y = 0 \Rightarrow x = \pm 1 \quad (1,0)(-1,0)$

$$\underbrace{f(0,1) = 2 \quad f(0,-1) = 2}_{MAX} \quad \underbrace{f(1,0) = 1 \quad f(-1,0) = 1}_{.MINIMUM}$$

EXAMPLE 3 FIND EXTREME VALUES OF $f(x,y) = x^2 + 2y^2$ ON THE DISK $x^2 + y^2 \leq 1$

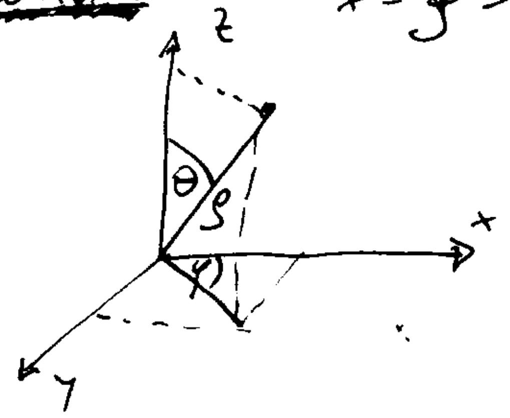
$x = 0, y = 0 \quad \underbrace{f(0,0) = 0}_{} < f(1,0) = f(-1,0) = 1$

$$\left.f(0,1) = f(0,-1) = 2\right\} \overset{MINIMUM}{MAXIMUM}$$

EXAMPLE 4 FIND THE POINTS ON THE SPHERE $x^2+y^2+z^2=4$ THAT ARE CLOSEST TO AND FARTHEST FROM POINT $(3,1,-1)$

SOLUTION: $x = \rho \cdot \sin\theta \cdot \cos\varphi \quad y = \rho \cdot \sin\theta \cdot \sin\varphi$

$z = \cos\theta$



$f(x,y,z) = x^2 + y^2 + z^2$

$f_x = 2x \quad f_y = 2y \quad f_z = 2z$

$d^2 = (x-3)^2 + (y-1)^2 + (z+1)^2$

$f(x,y,z) = (x-3)^2 + (y-1)^2 + (z+1)^2$

$g(x,y,z) = x^2 + y^2 + z^2 \quad \nabla f(x,y,z) = \nabla g(x,y,z)$

$f_x(\cdot) = 2(x-3); f_y(\cdot\cdot) = 2(y-1); f_z(\cdot\cdot) = 2(z+1);$

$g_x(\ldots) = 2x \quad g_y(\ldots) = 2y ; \quad g_z(\ldots) = 2z$

[illegible]

$2(x-3) = 2x \cdot \lambda \quad 2(y-1) = 2y \cdot \lambda \quad 2(z+1) = 2z \cdot \lambda$

$2x - 6 = 2x \cdot \lambda \quad 2x - 2x\lambda = 6 \quad 2x(1-\lambda) = 6 \quad x = \frac{3}{1-\lambda}$

$2y - 2 = 2y\lambda \quad 2y - 2y\lambda = 2 \quad 2y(1-\lambda) = 2 \quad y = \frac{1}{1-\lambda}$

$2z + 2 = 2z\lambda \quad 2z - 2z\lambda = -2 \quad 2z(1-\lambda) = -2 \quad z = -\frac{1}{1-\lambda}$

$x^2 + y^2 + z^2 = 4 \quad \frac{9}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} = 4$

$11 = 4(1-\lambda)^2 \quad (1-\lambda) = \pm\sqrt{\frac{11}{4}} \quad \lambda = 1 \pm \sqrt{\frac{11}{4}} = \begin{cases} -0.66 \\ +2.66 \end{cases}$

$x = \frac{3}{\pm\sqrt{\frac{11}{4}}} = \pm 3\sqrt{\frac{4}{11}} = \pm 6/\sqrt{11} \quad y = \pm\sqrt{\frac{4}{11}} = \pm 2/\sqrt{11} \quad z = \mp 2/\sqrt{11}$

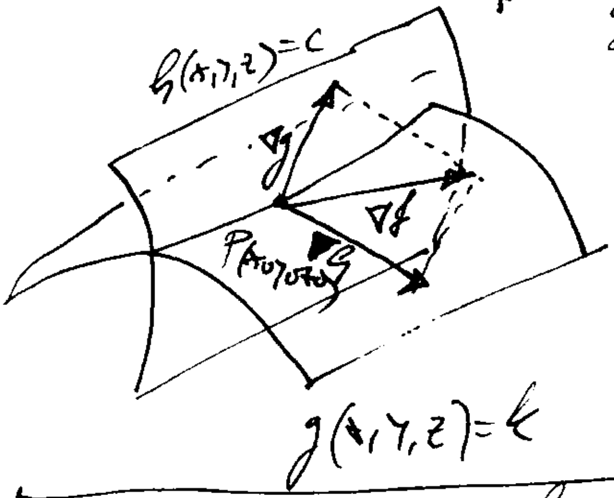
$\underbrace{\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right)}_{\text{MINIMUM}} \quad \underbrace{\left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)}_{\text{MAXIMUM}}$ } FROM Stewart5e WS9.4W GRAFIČKI

NO SE GLEDA / ANALITIČKI OD $f(x,y,z)$.

TWO CONSTRAINTS

3215945
944
943

ROMAN



$$\boxed{\Delta f(x_0, y_0, z_0) = \lambda \nabla g(x_0 y_0 z_0) + \mu \nabla h(x_0 y_0 z_0)}$$

$$\boxed{\begin{array}{lll} f_x = \lambda g_x + \mu h_x & f_y = \lambda g_y + \mu h_y & f_z = \lambda g_z + \mu g_z \\ g(x,y,z) = k & h(x,y,z) = c & \end{array}}$$

EXAMPLE 5 FIND THE MAXIMUM VALUE OF THE FUNCTION $f(x,y,z) = x + 2y + 3z$ ON THE CURVE OF INTERSECTION OF THE PLANE $x - y + z = 1$ AND THE CYLINDER $x^2 + y^2 = 1$

$f(x,y,z) = x + 2y + 3z$ $g_{(xyz)} = x - y + z$ $h(x,y) = x^2 + y^2 = 1$

$f_x = 1$ $f_y = 2$ $\underline{f_z = 3}$ $g_x = 1$ $g_y = -1$ $g_z = 1$

$h_x = 2x$ $h_y = 2y$ $h_z = 0$

$1 = \lambda + 2\mu x$ (1°) $2 = -\lambda + 2\mu y$ (2°) $\boxed{3 = \lambda}$ (3°)

$x - y + z = 1$ (4°)

$x^2 + y^2 = 1$ (5°)

(1°) → $x = \frac{1-\lambda}{2\mu}$ (2°) $y = \frac{2+\lambda}{2\mu}$

$x = |(3°)| = -\frac{2}{2\mu} = -\frac{1}{\mu}$ $\boxed{y = \frac{5}{2\mu}}$

(5°) $\frac{1}{\mu^2} + \frac{25}{4\mu^2} = 1$ $4\mu^2 = 4 + 25$ $\mu = \pm\sqrt{\frac{29}{4}} = \pm\frac{\sqrt{29}}{2}$

$x_{0_{1,2}} = \mp\frac{2}{\sqrt{29}}$ $y_{0_{1,2}} = \pm\frac{5}{2}\frac{2}{\sqrt{29}} = \pm\frac{5}{\sqrt{29}}$

$z_{0_{1,2}} = 1 - x_0 + y_0 = 1 \pm \frac{2}{\sqrt{29}} \pm \frac{5}{\sqrt{29}} = \underline{\underline{1 \pm \frac{7}{\sqrt{29}}}}$

$\boxed{f(x_{0_1}, y_{0_1}, z_{0_1}) = -\frac{2}{\sqrt{29}} + \frac{10}{\sqrt{29}} + \frac{21}{\sqrt{29}} + 3 =}$ $=$

$= 3 + \frac{31-2}{\sqrt{29}} = 3 + \frac{29}{\sqrt{29}} = \underline{\underline{3 + \sqrt{29}}}$ MAXIMUM

$f(x_{0_2}, y_{0_2}, z_{0_2}) = +\frac{2}{\sqrt{29}} - \frac{10}{\sqrt{29}} + 3 - \frac{21}{\sqrt{29}} = 3 + \frac{2-31}{\sqrt{29}} = 3 - \sqrt{29}$

MINIMUM

PROBLEM 6.3 **CARDS.** AN ORDINARY DECK OF CARDS CONTAINING 26 RED CARDS AND 26 BLACK CARDS IS SHUFFELED AND DELT OUT ONE CARD PER TIME WITHOUT REPLACEMENT. LET X_i BE THE COLOR OF THE i-TH CARD.

(a) DETERMINE $H(X_1)$

(b) DETERMINE $H(X_2)$

(c) DOES $H(X_K | X_1 X_2 \ldots X_{K-1})$ INCREASE OR DECREASE

(d) DETERMINE $H(X_1, X_2 \ldots X_{52})$.

(a) $p_1 = \frac{26}{52}$ $X_1 = \{\underset{R}{0}, \underset{B}{1}\}$ $P_1 = \{\frac{1}{2}, \frac{1}{2}\}$

$$H(x_1) = p_1(x_1=0)\cdot ld \frac{1}{p(x_1=0)} + p_1(x_1=1)\cdot ld \frac{1}{p(x_1=1)}$$

$$= \left(\frac{1}{2}\, ld \frac{1}{1/2}\right)\cdot 2 = ld\, 2 = 1$$

(b) $P(x_2=0) = ?$ $\quad P(x_2=1) = ?$

$$P(x_2=0) = P(x_1=0)\cdot \underbrace{P(x_2=0 \mid x_1=0)} + P(x_1=1)\cdot P(x_2=0/x_1=1)$$

$$= \frac{1}{2}\cdot\frac{25}{51} + \frac{1}{2}\cdot\frac{26}{51} = \frac{1}{2}\left(\frac{25+26}{51}\right) = \frac{1}{2}$$

$$P(x_2=1) = P(x_1=0)\cdot P(x_2=1 \mid x_1=0) + P(x_1=1)\cdot P(x_2=1/x_1=1)$$

$$= \frac{1}{2}\cdot\frac{26}{51} + \frac{1}{2}\,\frac{25}{51} = \frac{1}{2}\,\frac{51}{51} = \frac{1}{2}$$

$$H(x_2) = \frac{1}{2}\, ld\, 2 + \frac{1}{2}\, ld\, 2 = 1$$

(c) $P(x_3=0) = P(x_1=0\ x_2=0)\cdot P(x_3=0/00) +$

$$+ P(01)\cdot P(0|01) + P(10)\cdot P(0|10) + P(11)\, P(0|11) =$$

$$= \frac{1}{4}\,\frac{24}{50} + \frac{1}{4}\cdot\frac{25}{50} + \frac{1}{4}\cdot\frac{25}{50} + \frac{1}{4}\cdot\frac{26}{50} =$$

$$= \frac{1}{4}\,\frac{50+50}{50} = \frac{1}{4}\cdot\frac{100}{50} = \frac{1}{2}$$

$$H(Y|X) = \sum_{xy} p(xy)\, ld \frac{1}{P(Y|X)}$$

$$\boxed{H(X,Y) = H(X) + H(Y|X)}$$

$$H(X_3|x_1 x_2) = -\Big[P(00)\, ld\, P(0|00) + P(00)\, ld\, P(1|00) +$$

$$P(01)\cdot ld(0|01) + P(01)\, ld(1|01) + P(10)\, ld\, P(0|10) +$$

$$P(10)\cdot ld(1|10) + P(11)\, ld\, P(0|11) + P(11)\, ld(1|11)\Big] =$$

$$= -\Big[\frac{1}{4}\, ld \frac{24}{50} + \frac{1}{4}\, ld\left(\frac{26}{50}\right) + \frac{1}{4}\, ld \frac{25}{50} + \frac{1}{4}\, ld \frac{25}{50}$$

$$+ \frac{1}{4}\cdot ld \frac{25}{50} + \frac{1}{4}\, ld \frac{25}{50} + \frac{1}{4}\, ld \frac{26}{50} + \frac{1}{4}\, ld \frac{24}{50}\Big] =$$

$$= -\frac{1}{2}\left[ld \frac{24}{50} + 2\, ld \frac{25}{50} + ld \frac{26}{50}\right] = -\frac{1}{2}\, ld \frac{24\cdot 25^2\cdot 26}{(50)^4}$$

$$H(x_3|x_1x_2) = -\frac{1}{2}\,ld\,\frac{24\cdot 25^2\cdot 26}{51\cdot 10^4} = -\frac{1}{2}\,ld\,\frac{24\cdot 25\cdot 26}{625\cdot 10^4}$$

TREBA $P(x_1=0\,x_2=0)$

$$\boxed{\begin{aligned}H(x_2|x_1) &= -\Big[P(x_1=0)\,ld\,P(0|0) + P(0)\,ld\,P(1|0) + \\ &+ P(1)\cdot ld\,P(0|1) + P(1)\cdot ld\,P(1|1)\Big] = -\Big[\tfrac{1}{2}\,ld\,\tfrac{25}{51} + \\ &+ \tfrac{1}{2}\,ld\,\tfrac{26}{51} + \tfrac{1}{2}\cdot ld\,\tfrac{26}{51} + \tfrac{1}{2}\,ld\,\tfrac{25}{51}\Big] = \\ &= 2\cdot\tfrac{1}{2}\Big[ld\,\tfrac{25}{51} + ld\,\tfrac{26}{51}\Big] = ld\,\frac{25\cdot 26}{51^2}\end{aligned}}$$

SOGLASNO STR. 70 TREBA DELE 1/2 POMOŽO !!!

$$H(x_2|x_1) = 2{,}00055 \qquad H(x_3|x_1x_2) = 2{,}00116$$

TREBA $P(0000)$

$$\bullet\ H(x_4|x_1x_2x_3) = -\Big[P(000)\,ld\,P(0|000) + \underline{P(001)}\,ld\,P(0|001) + \dots$$

$$+ P(111)\,ld\,P(0|111) + P(000)\,ld\,P(1|000) + P(001)\,ld\,P(1|001) + \dots$$

ldC...]

$$+ P(111)\,ld\,P(1|111)\Big] = -\Big\{\frac{2}{8}\Big[\overset{0|000}{ld\,\frac{26}{49}} + \overset{0|001}{ld\,\frac{24}{49}} + \overset{0|010}{ld\,\frac{24}{49}} +$$

$$+ \overset{0|011}{ld\,\frac{25}{49}} + \overset{0|100}{ld\,\frac{24}{49}} + \overset{0|101}{ld\,\frac{25}{49}} + \overset{0|110}{ld\,\frac{25}{49}} + \overset{0|111}{ld\,\frac{26}{49}}\Big\}$$

$$\binom{3}{0} = 1 \qquad \binom{3}{2} = \frac{3!}{1!\cdot 2!} = \frac{6}{2} = 3 \qquad \binom{3}{1} = \frac{3!}{2!\,1!} = 3 \qquad \binom{3}{3} = 1$$

$$\boxed{1+3+3+1 = 8}$$

$$= -\frac{2}{8}\Big[\binom{3}{3}\,ld\,\frac{23}{49} + \binom{3}{2}\,ld\,\frac{24}{49} + \binom{3}{1}\,ld\,\frac{25}{49} + \binom{3}{0}\,ld\,\frac{26}{49}\Big]$$

$$H(x_4|x_1x_2x_3) = -\frac{2}{8}\sum_{i=0}^{k-1}\binom{k-1}{i}\,ld\,\frac{26-i}{52-(k-1)}$$

$$\boxed{H(x_k|x_1^{k-1}) = -\frac{2}{2^{k-1}}\sum_{i=0}^{k-1}\binom{k-1}{i}\,ld\,\frac{26-i}{52-k+1}}$$

$$\circledast\ \boxed{H(x_k|x_1^{k-1}) = -\frac{1}{2^{k-2}}\sum_{i=0}^{k-1}\binom{k-1}{i}\,ld\left(\frac{26-i}{52-k+1}\right)}\ \text{OK}$$

$$H(x_2|x_1) = -\frac{1}{4}\sum_{i=0}^{1}\binom{1}{i}\,ld\,\frac{26-i}{51} = -\frac{1}{4}\binom{1}{0}\,ld\,\frac{26}{51} - \frac{1}{4}\binom{1}{1}\,ld\,\frac{25}{51}$$

$$H(x_3 | x_1^2) = -\frac{1}{2}\binom{2}{0}^{1} ld\,\frac{26}{50} - \frac{1}{2}\binom{2}{1}^{2} ld\,\frac{25}{50} - \frac{1}{2}\binom{2}{2}^{1} ld\,\frac{24}{50}$$

$$= -\frac{1}{2}\left[ld\,\frac{26\cdot 25^2 \cdot 24}{50^3}\right] = 2.00181$$

$$\begin{array}{r} 52\cdot \\ 25 \\ \hline \boxed{27} \end{array}$$

$$H(x_4|x_1^3) = 2.00251 \qquad H(x_{10}|x_1^9) = 2.00707$$

$\Rightarrow H(x_k | x_1^{k-1})$ INCREASES WITH k

P.11 DO $\frac{N}{2}$ RASTE NO POTOA OPAĐA !!!

(d) $H(x_1 x_2 \dots x_{52}) = ?$

$$P(x_1 x_2 \dots x_{52}) = 2^{-52} \qquad H(x_1 x_2 \dots x_{52}) = -\sum_{x_1^{52}} P(x_1^{52})\, ld\, P(x_1^{52})$$

$$= \cancel{2^{52}}\cdot\frac{1}{\cancel{2^{52}}}\cdot ld\, 2^{52} = \underline{\underline{52\cdot ld 2 = 52}}$$

OVA NE E VAKA TREBA DA BIDE $ld\binom{52}{26}$ VIDI P.92

$$H(x_1^{52}) = \sum_{i=0}^{52} H(X_{i+1}|X_1^i)$$

SIMULACIJA SO 4 KARTI 2 CRVENI 2 CRNI

$$H(x_1) = 1$$

$$H(x_2|x_1) = -\frac{1}{1}\sum_{i=0}^{1}\binom{1}{i} ld\left(\frac{2-i}{4-1}\right) = -\binom{1}{0} ld\,\frac{2}{3} - \binom{1}{1} ld\,\frac{1}{3}$$

$$= -\, ld\,\frac{2}{3} = 2.16993$$

$$H(x_2|x_1) = P(0)\, ld\, \underline{\underline{P(0|0)}} + P(0)\, ld\, P(1|0) + P(1)\cdot ld(0|1) + P(1)\, ld(1|1)$$

$$= \frac{1}{2} ld\,\frac{1}{4} + \frac{1}{2} ld\,\frac{2}{4} + \frac{1}{2}\cdot ld\,\frac{2}{4} + \frac{1}{2} ld\,\frac{1}{4}$$

$$H(x_3|x_1x_2) = 2\left[\underbrace{P(00)\, ld\, P(0|00)}_{\text{NEVOZMOŽNO}} + P(01)\, ld\, P(0|01) + P(10)\, ld\, P(0|10) + P(11)\, ld\, P(0|11)\right]$$

$$H(X_4|x_1x_2x_3) = 2\Big[\underbrace{P(000)}_{\text{NEVOZMOŽNO}}\, ld\, P(0|000) + \underbrace{P(001)\, ld\, P(0|001)}_{\text{NEVOZMOŽNO}} +$$

$$+ P(010)\, ld\, \underbrace{P(0|010)}_{\text{NEVOZMOŽNO}} + P(011)\, ld\, P(0|011) + \underbrace{P(100)\, ld\, P(0|100)}_{\text{NEVOZMOŽNO}} +$$

$$+ P(101)\, ld\, P(0|101) + P(110)\, ld\, \underbrace{P(0|110)}_{1} + \underbrace{P(111)\cdot ld\, P(0|111)}_{\text{NEVOZMOŽNO}}$$

→ DOKOLKU BI BILO VOZMOŽNO $\frac{1}{8}\cdot ld\,\frac{2-3}{4}$

- NE MOŽE SAMO SO ⊕ MORA DA IMA I VTORA SUMA KOJA ĆE VAŽI ZA $K>26$!!!

$$P(1|00000000000000000000000000) = 1$$

$$P(1|00000000000000000000000001) = 1$$

- SIMULACIJA SO 4 KARTI (revisited)

$$\boxed{H(X_k|X_1^{k-1}) = \frac{1}{2^{k-2}} \sum_{i=0}^{k-1} \binom{k-1}{i} \operatorname{ld} \frac{4-i}{4-k+1}}$$

$$H(X_3|X_1X_2) = \frac{1}{2} \sum_{i=0}^{2} \binom{2}{i} \operatorname{ld} \frac{4-i}{4-k+1} =$$

$$\frac{1}{2}\binom{2}{0} \operatorname{ld} \frac{4}{2} + \frac{1}{2}\binom{2}{1} \operatorname{ld} \frac{3}{2} + \frac{1}{2}\binom{2}{2} \operatorname{ld} \frac{2}{2}$$

- $N \geq 6$ (na golemina od 6 karti)

$$H(X_3|X_1X_2) = -\sum P(X_1X_2X_3) \cdot \operatorname{ld} P(X_3|X_1X_2) = [P(000) \operatorname{ld} P(0|00) + \overset{(*)}{P(001) \operatorname{ld} P(0|01)}$$

$$\overset{(\Delta)}{} + \boxed{\overset{x_1x_2x_3}{P(010)} \operatorname{ld} P(0|10)} + \boxed{P(011) \operatorname{ld} P(0|11)} + \boxed{P(100) \operatorname{ld} P(1|00)} +$$

$$+ P(101) \operatorname{ld} P(1|01) + P(110) \operatorname{ld} P(1|10) + P(111) \operatorname{ld} P(1|11)]$$

$$= -\frac{1}{8}\left[\operatorname{ld} \frac{\frac{N}{2}-2}{N-2} + \operatorname{ld} \frac{\frac{N}{2}-1}{N-2} + \operatorname{ld} \frac{\frac{N}{2}-1}{N-2} + \operatorname{ld} \frac{\frac{N}{2}-0}{N-2}\right] 2 =$$

$$= -\frac{1}{8}\left[\sum_{i=0}^{k-1} \binom{k-1}{i} \operatorname{ld} \left(\frac{\frac{N}{2}-i}{N-k+1}\right)\right] \cdot ② \text{ — od } (*) \quad = -\left(\frac{1}{4}\right) \sum_{i=0}^{k-1} \binom{k}{i} \left(\frac{\frac{N}{2}-i}{N-k+1}\right)$$

$$1/2^{k-1}$$

$$\boxed{k=1} \quad i=0 \quad -\frac{1}{4} \cdot 1 \operatorname{ld} \left(\frac{\frac{N}{2}}{N}\right) = \operatorname{ld}\left(\frac{1}{2}\right) = 1$$

$$\binom{2}{1} = 2 \quad \binom{2}{2} = 1 \quad \binom{2}{0} = 1$$

- DA GO ANALIZIRAM SLUČAJOT ZA $N=4$ (#)

$$H(X_3|X_1X_2) = -[\underbrace{P(001)}_{1/8} \cdot \operatorname{ld} \underbrace{P(0|01)}_{1/2} + \underbrace{P(010)}_{1/8} \operatorname{ld} \underbrace{P(0|10)}_{1/2} + \underbrace{P(011)}_{1/8} \operatorname{ld} \underbrace{P(0|11)}_{1/2}]$$

$$2 \cdot$$

$$(*) \quad (\Delta) \quad \frac{x}{N-② \frac{N}{2}} \overset{(\#)}{=} \frac{x}{N-4-k+1} = \frac{x}{4-2} = \frac{x}{2}$$

$$= -\frac{2}{8}[1 - 1 - 0] = +\frac{2}{8} \cdot 2 = +\frac{4}{8} = \frac{1}{2} = 0.5$$

$$H(X_4|X_1X_2X_3) = -\frac{1}{2^4}[P(0011) \cdot \operatorname{ld} \underbrace{P(0|011)}_{1} + P(0101) \operatorname{ld} \underbrace{P(0|101)}_{1} +$$

$$+ P(0110) \operatorname{ld} \underbrace{P(0|110)}_{1} + P(1001) \operatorname{ld} \underbrace{P(1|001)}_{1} + P(1010) \cdot \operatorname{ld} \underbrace{P(1|010)}_{1}$$

$$+ P(1100) \cdot \operatorname{ld} (\underbrace{1|100}_{1})] = 0$$

① $\underbrace{\binom{2}{1} = \frac{2!}{1!\,1!} = 2}_{\$}$ $\underbrace{\binom{2}{2} = 1}_{\#}$ $\binom{2}{1} + \binom{2}{2} = 3$

- Naloga: $\boxed{N=6}$

$H(X_4 | X_1 X_2 X_3) = -[P(0001)\cdot ld\, P(0|001) + P(0010)\, ld(P(0|010)) + P(0011)\, ld\, P(0|011) + P(0100)\, ld\, P(0|100) + P(0101)\, ld\, P(0|101) + P(0110)\, ld\, P(0|110) + P(0111)\, ld\, P(0|111)) = -\Big[\frac{1}{16}\cdot ld \frac{1}{6-4+1} + \frac{1}{16}\, ld \frac{1}{3} + \frac{1}{16}\, ld \frac{2}{3}$

$+ \frac{1}{16}\, ld \frac{1}{3} + \frac{1}{16}\, ld \frac{2}{3} + \frac{1}{16}\, ld \frac{2}{3} + \frac{1}{16}\, ld \frac{3}{3}\Big] \cdot 2$ $\binom{3}{3} ld \left(\frac{3}{3}\right)$

$= -\frac{2}{16} \sum_{1}^{3} \left[\binom{\frac{N}{2}}{i}\, ld \frac{i}{N-k+1}\right] = -\frac{1}{8}\left[\binom{3}{1} ld \frac{1}{3} + \binom{3}{2} ld\left(\frac{2}{3}\right) + \circledast\right.$

$\binom{3}{1} = \frac{3!}{2!\cdot 1} = \frac{6}{2} = 3$ $\binom{3}{2} = \frac{3!}{1!\cdot 2!} = 3$ $\binom{3}{3} = 1$

$k > \frac{N}{2}$

$$\boxed{H(X_k | X_1^{k-1}) = -\frac{1}{2^{k-1}} \sum_{i=1}^{k} \binom{\frac{N}{2}}{i} ld \frac{i}{N-k+1}}$$

Točen izraz samo $P(x_1 x_2 x_3 x_4)$ ne $\in 2^{-4}$

$\circledast = -\frac{1}{8}\left[+3\cdot ld \frac{1}{3} + 3\, ld \frac{2}{3}\right] = -\frac{1}{8}\cdot 3\, ld \frac{2}{9} = 0.81372$

- MAPLE (EITC96.mw)

$$H(k,N) = -\frac{1}{2^{k-1}} \begin{cases} \sum_{i=1}^{k} \binom{N/2}{i} ld \frac{i}{N-k+1} & k > \frac{N}{2} \\ \underbrace{\sum_{i=0}^{k-1} \binom{k-1}{i} ld \frac{\frac{N}{2}-i}{N-k+1}}_{\circledast} & k \le \frac{N}{2} \end{cases}$$

$$\circledast = \sum_{i=1}^{k} \binom{k-i}{i-1} ld \frac{\frac{N}{2}-i+1}{N-k+1}$$

- $\boxed{P(X_K) = ? \quad \text{FOR} \quad K > \frac{N}{2}}$ TUKA MISLAM DEKA IMAM GREŠKA !!!

- ČE PREGLEDUVAM $N=4$ (IGRA SO 4 KARTI)

$$P(X_3|x_1x_2) = P(00)\cdot\underbrace{P(0|00)}_{0} + \underbrace{P(01)}_{1/4}\cdot\underbrace{P(0|01)}_{1/2} +$$

$$+ \underbrace{P(10)}_{1/4}\cdot\underbrace{P(0|10)}_{1/2} + \underbrace{P(11)}_{1/4}\cdot\underbrace{P(0|11)}_{1/4} + \underbrace{P(00)}_{1/4}\underbrace{P(1|00)}_{1} +$$

$$+ \underbrace{\underbrace{P(01)}_{1/4}\cdot\underbrace{P(1|01)}_{1/2}}_{P(0|x_1x_2)} + \underbrace{P(10)}_{1/4}\cdot\underbrace{P(1|10)}_{1/2} + \underbrace{P(11)}_{1/4}\cdot\underbrace{P(1|11)}_{0} =$$

$$= 1/8 + 1/8 + 1/4 + \underbrace{1/4 + 1/8 + 1/8}_{P(1|x_1x_2)} = 4\cdot\frac{1}{8} + 2\cdot\frac{1}{4} =$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$ ZAŠ IZLEGUVA $\frac{1}{2}$!? -

$$P(0|x_1x_2x_3) = \frac{1}{8}\cdot\underbrace{P(0|011)}_{1} + \frac{1}{8}P(\underbrace{0|101}_{1}) + \frac{1}{8}\cdot\underbrace{P(0|110)}_{1}$$

$$= \frac{3}{8}$$

$$P(1/x_1x_2x_3) = \frac{1}{8}\underbrace{P(1|001)}_{1} + \frac{1}{8}P(\underbrace{1|010}_{1}) + \frac{1}{8}P(\underbrace{1|100}_{1})$$

$$= 3/8$$

- ZA $K=N$ NEMA NEIZVESNOST $\underline{H(N,N)=0}$

T.E. $H(X_N|X_1^{N-1}) = 0$

- ŠPIL SO 4-RI KARTI (2R + 2B)

0011	1010
0101	1100
0110	
1001	

(Δ★)

$$\binom{4}{2} = \frac{4!}{2!\,2!} = \frac{1\cdot2\cdot3\cdot4}{4} = 6$$

VKUPNO IMA KOMBINACII E: $\binom{52}{26}$

$$H(x_1^{52}) = -\sum_{x_1^{52}} P(x_1^{52})\,\mathrm{ld}\,P(x_1^{52}) = \cancel{\binom{52}{26}}\cdot\frac{1}{\cancel{\binom{52}{26}}}\cdot\mathrm{ld}\binom{52}{26}$$

$$\boxed{H(x_1^{52}) = \mathrm{ld}\binom{52}{26}} \qquad \boxed{H(x_1^{52}) \doteq 48.88}$$

- SIMULACIJA SO 4 KARTI

KOLKAVA E VEROJATNOSTA PRVITE TRI KARTI DA BIDAT (001). OVA E ALTERNATIVEN NASTAN.

OD (1*) SLEDUVA DEKA NE E $\frac{1}{8}$ TUKU $\frac{1}{6}$

OD OVDE IZLEGUVA DEKA (*) NA P. 90 E:

$$\boxed{2\cdot\frac{1}{\binom{4}{2}} = \frac{2}{6} = \frac{1}{3}}$$

16 vs. 15

- OD P90 (1) DA VIDAM $P(00)\ldots P(11)$ NA ŠTO SE EDNAKVI

$$\binom{2}{1} = \frac{2!}{1!\cdot 1!} = 2$$

$P(00) = \frac{1}{6}$ $P(11) = \frac{1}{6}$

$$H(X_2|X_1) = P(00)\,ld\,P(0|0) + P(10)\,ld\,(0|1) + P(01)\cdot ld(1|0) + P(11)\cdot ld(1|1) = -\left[\frac{1}{6}\cdot ld\,\frac{1}{3} + \frac{1}{6}\cdot ld\,\frac{2}{3} + \frac{1}{6}\cdot ld\,\frac{2}{3} + \frac{1}{6}\cdot ld\,\frac{1}{3}\right] = -\frac{1}{6}\sum_{k=0}^{k-1}\binom{k-1}{i}\cdot ld\,\frac{2-i}{4-k+1} =$$

$$= 2\cdot\frac{1}{6}\left[\binom{1}{0}\,ld\,\frac{2}{3} + 1\cdot\binom{1}{1}\,ld\,\frac{1}{3}\right] =$$

$$= 2\cdot\frac{1}{6}\left[1\,ld\,\frac{2}{3} + 1\cdot ld\,\frac{1}{3}\right]$$

- ZA $N=6$:

$$\binom{6}{3} = \frac{6!}{3!\,3!} = \frac{1\cdot 2\cdots 6}{\cancel{1\cdot 2\cdot 3}\;1\cdot 2\cdot 3} = \frac{4\cdot 5\cdot 6}{1\cdot 2\cdot 3}$$

$$\binom{6}{3} = \frac{4\cdot 5\cdot 6}{6} = 20$$

000111
001011
001101
001110
010|011
010|101
010|110
011|001
011|010
011|100
100|011
100101
100|110
101|001
101|010
101|100
110|001
110|010
110|100
111|000

$\boxed{P(00) = \frac{4}{20} = \frac{1}{5}}$ $P(10) = \frac{6}{20}$

$\boxed{P(001) = \frac{3}{20}}$ $P(01) = \frac{6}{20}$

$P(11) = \frac{4}{20}$

$P(000) = \frac{1}{20}$ $P(010) = \frac{3}{20}$ $P(011) = \frac{3}{20}$ $P(100) = \frac{3}{20}$

$P(101) = \frac{3}{20}$ $P(110) = \frac{3}{20}$ $P(111) = \frac{1}{20}$ $P(001) = \frac{2}{20}$

$$\binom{3}{1} = \frac{3!}{2!\cdot 1!} = \frac{1\cdot 2\cdot 3}{2} = 3$$

• SIMULACIJA 4 KART (2 CRVENI 2 CRNI)

$P(x_1=0, x_2=0) = ?$

0011 1001
0101 1010
0110 1100

$$P(00) = P(0)\cdot P(0|0) = \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6}$$
$$P(01) = P(0)\cdot P(1|0) = \frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3}$$
$$P(11) = P(1)\cdot P(1|1) = \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6}$$
$$P(10) = P(1)\cdot P(0|1) = \frac{1}{2}\cdot\frac{2}{3} = \frac{1}{6}$$

$$H(x_2|x_1) = -\Big[P(00)\cdot ld P(0|0) + P(01)\cdot ld(1|0) +$$
$$+ P(10)\cdot ld(0|1) + P(11)\cdot ld(1|1)\Big] =$$
$$= \frac{1}{\cancel{2}}\left[\frac{1}{N-1}\cdot ld\frac{1}{N-1} + \frac{2}{N-1}\cdot ld\frac{2}{N-1}\right]\cdot\cancel{2}$$

$$H(x_3|x_1x_2) = -\Big[P(000)\cdot ld P(0|00) + P(001)\, ld P(0|01) +$$
$$+ P(010)\cdot ld P(0|10) + P(011)\, ld P(0|11)\Big]\cdot 2 = ?$$

$$P(000) = P(00)\cdot P(0|00) = \frac{1}{6}\cdot\frac{\frac{N}{2}-2}{N-2} = \frac{\frac{N}{2}-0}{N-0}\cdot\frac{\frac{N}{2}-1}{N-1}\cdot$$
$$\cdot\frac{\frac{N}{2}-2}{N-2} = /N=6/ \left(\frac{3}{6}\cdot\frac{2}{5}\cdot\frac{1}{4}\right) = \frac{3!\cdot 3!}{6!}$$

$$H(x_k|x_1^{k-1}) = \sum_{k=0}^{k-1}\frac{(K!)^{2}}{N!}\overset{1/2}{}\; ld\frac{\frac{N}{2}-i}{N-k+1} \quad ⓘ?$$

$$W(b,p) = +\sum_x p(x)\, ld\, b(x)\cdot o(x) = +E\Big[ld\,\overbrace{b(x)\,o(x)}^{S(x)}\Big]$$

• SIMULACIJA 4 KART

$$P(\overset{0}{x_2}|x_1) = P(0)P(0|0) + P(1)\cdot P(0|1) = \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{2}{3} = 1/2$$
$$P(\overset{0}{x_3}|x_1x_2) = P(00)\cdot\underbrace{P(0|00)}_{0} + P(01)\cdot P(0|01) + P(10)\cdot P(0|10) +$$
$$+ P(11)\cdot P(0|11) = P(01)\cdot\frac{1}{2} + P(10)\cdot\frac{1}{2} + P(11)\cdot 1$$

0011	1010
0101	1100
0110	
1001	

$P(x_1=0\ x_2=0) = \frac{1}{6}$ $\quad P(x_1=1, x_2=0) = \frac{2}{6}$

$P(x_1=0, x_2=1) = \frac{2}{6} = \frac{1}{3}$ $\quad P(x_1=1, x_2=1) = \frac{1}{6}$

$$\boxed{P(x_1\ x_{\textcircled{2}}) = \frac{\binom{K}{\textcircled{i}}}{\binom{4}{2}}}$$

K

→ BROJ NA EDINICI

$\binom{4}{2} = \frac{4!}{2!\cdot 2!} = \frac{1\cdot\cancel{2}\cdot 3\cdot \overset{2}{\cancel{4}}}{\cancel{2}\cdot\cancel{2}} = 6$

$i = 0..2$ $\quad \binom{2}{0} = 1 \quad \binom{2}{1} = 2 \quad \binom{2}{2} = 1$

$P(x_3 \overset{0}{=} | x_1 x_2) = \frac{1}{3}\cdot\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{2} + \frac{1}{6} = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$

$P(x_3 = 1 | x_1 x_2) = \frac{1}{2}$

$x_1 x_2 .. x_3$ SE STATISTIČKI NEZAVISNI

$P(x_4 \overset{0}{=} | x_1^3) = P(0|011)\cdot P(011) + P(0|101)\cdot P(101) +$
$P(0|110)\cdot P(110) = \underset{1/6}{P(011)} + \underset{1/6}{P(101)} + \underset{1/6}{P(110)} = \frac{3}{6} = \frac{1}{2}$

DEFINITIVNO $x_1 x_2 x_3 x_4$ SE NEZAVISNI !!!

$$H(Y|X) = \sum_{x} p(x)\cdot H(Y|X=x) = -\sum_{x\in X} p(x) \sum_{y\in Y} P(Y|X=x)\ \mathrm{ld}\ P(\ldots$$

$$= -\sum p(x,y)\ \mathrm{ld}\ P(Y|x)$$

$\binom{3}{2} = \frac{1\cdot 2\cdot 3}{1\cdot 2} = 3$

$P(x_K^{52}) = \frac{\binom{K}{\textcircled{i}}}{\binom{52}{26}}$ · DA SI VRATI SITE EDINICI

VEROJATNOST VO PRVITE TRI KOLONI DA IMA EDNA [illegible]

$P(X_3^4) = \frac{\binom{3}{i}}{\binom{4}{2}}$

$$\boxed{P(X_1X_2X_3) = \frac{\binom{3}{1}}{\binom{4}{2}} = \frac{3}{6} = \frac{1}{2}}$$

$\binom{3}{1} = \frac{3!}{2!} = 3$

$H(X_3|x_1x_2) = -[\underbrace{P(000)}_{0}\ \mathrm{ld}\ \underbrace{P(0|00)}_{0} + \underbrace{P(001)}\cdot \mathrm{ld}\ P(0|\underline{01}) + P(010)\ \mathrm{ld}\ P(0|10)$
$+ P(011)\ \mathrm{ld}\ P(0|11)]\cdot 2 = 2\left[\frac{1}{6}\cdot \cancel{\mathrm{ld}\ 2}^{1} + \frac{1}{6}\cdot\cancel{\mathrm{ld}\ 2}^{1} + \frac{1}{6}\cdot\cancel{\mathrm{ld}\ 1}^{0}\right]$
$= 2\cdot\frac{2}{6} = \frac{2}{3}$

$H(x_2|x_1) = -[P(00)\ \mathrm{ld}\ P(0|0) + P(01)\ \mathrm{ld}\ P(0|1) + P(10)\ \mathrm{ld}\ P(0|0) +$
$+ P(11)\ \mathrm{ld}\ P(1|1)] = \frac{1}{6}\mathrm{ld}\,2 + \frac{1}{3}\underset{1}{\mathrm{ld}\,2} + \frac{1}{3}\underset{1}{\mathrm{ld}\,2} + \frac{1}{6}\mathrm{ld}\,2 = \frac{2}{3} + \frac{1}{3} = 1$

- ОД ПРИМЕРОВ ЛОГИЧЕ СЕ СЛЕДИ ДЕКА $H(X_k|X_1^k)$ ОПАЃА СО k? . ЗА ПОВЕЌЕ ПА ГО МОРЕАМ!

$$H(X_4|X_1X_2X_3) = 1$$

$$H(X_4|X_1X_2X_3) = -\left[P(0011)\,\underline{ld\,P(0|001)} + P(0101)\,ld\,P(0|101) + P(0110)\,ld\,P(0|110)\right.$$
$$\left.+ P(1010)\,ld\,P(1|010) + P(1100)\,ld\,1(1|100) + 1(1001)\,ld\,P(1|001)\right]$$
$$= -\left[\frac{1}{6}\,ld\,1 + \frac{1}{6}\,ld\,1 + \dots + \frac{1}{6}\,ld\,1\right] = 0$$

$\underbrace{H(X_2|X_1) = 1}_{\text{НЕЕ ТОЧНО пр.97}}$; $H(X_3|X_1X_2) = \frac{2}{3}$; $H(X_4|X_1X_2X_3) = 0$

$H(X_{52}|X_1^{51}) = 0$ } ЛОГИЧНО Е ДЕКА $H(X_k|X_1^k)$ ОПАЃА !!!

- EDITION 2 SOLUTIONS:

02[illegible]3060202 DIMITR. JANAK.

$$P(X_{k+1}|X_1^k) \qquad P(Z|XY) \qquad P(XYZ) = P(X)\cdot P(YZ|X)$$

75

$$= P(X)\cdot P(Y|X)\cdot P(Z|XY)$$

$$H(X_k|X_1^{k-1}) = H(X_{k+1}|X_1^{k-1}) \geq H(X_{k+1}|X_1^k)$$

CONDITIONING REDUCES ENTROPY

$$H(X,Y) = H(X) + H(Y|X) = H(Y) + H(X|Y)$$

$$I(X,Y) = H(X) - H(X|Y) \geq 0 \qquad \underline{H(X) \geq H(X|Y)}$$

$$H(X_1^4) = H(X_1) + H(X_2|X_1) + H(X_3|X_1X_2) + H(X_4|X_1X_2X_3) =$$

$$\underbrace{= 1 + 1 + 2/3 + 0 = 2 + 2/3 = \frac{8}{3} = 2.67}_{\text{STEP BY STEP APPROACH}}$$

ЗОШТО ВО РАЗЛИКА?

- DIRECT APPROACH

$$H(X_1^4) = \sum_{X_1^4} P(X_1^4)\,ld\,P(X_1^4) = \cancel{\binom{4}{2}}\cdot\frac{1}{\cancel{\binom{4}{2}}}\,ld\binom{4}{2} = ld\,6 =$$

$$\underline{= 2.585}$$

'ZGLEDA IMAM GREŠKA VO $H(x_2|x_1)$

$H(x_2|x_1) = -[P(00)\,ld\,P(\underline{0|0}) + P(01)\,ld\,P(0|1) + P(10)\,ld\,P(1|0) + P(11)\,ld\,P(1|1)] = -[\frac{1}{6}\cdot ld\frac{1}{3} + \frac{2}{6}\cdot ld\frac{2}{3} + \frac{2}{6}\cdot ld\frac{2}{3} + \frac{1}{6}\cdot ld\frac{1}{3}]$

$= \left|\begin{matrix} 0011 & 1001 \\ 0101 & 1010 \\ 0110 & 1100 \end{matrix}\right| = -[\frac{4}{6}\,ld\frac{2}{3} + \frac{2}{6}\,ld\frac{1}{3}] = \rightarrow 0.91830$

$\boxed{H(x_1^4) = 1 + 0.91830 + \frac{2}{3}} = 2.58496$

SEGA ODGOVARA NA ⊛ $P(011)\,ld\,P(1|1)$

$\cdot\ H(x_3|x_1x_2) = -[P(001)\,ld(0|01) + P(010)\,ld\,P(0|10) + P(100)\,ld\,P(1|00) + P(011) + P(101)\,ld\,P(1|01) + P(110)\,ld\,P(1|10)] =$

$+[P(001) + P(010) + P(101) + P(110)] =$

$= +[\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}] = \frac{4}{6} = \frac{2}{3}$ //

$\binom{N}{N/2} = \frac{N}{N/2!\ N/2!} =$

• SAKAM ISTOVO DA GO [illegible] ZA $N=6$!!!

$H(x_1) = 1$

$H(x_2|x_1) = -[\underbrace{P(00)}_{4/20}\cdot ld\,\underbrace{P(0|0)}_{\frac{2}{5}} + \underbrace{P(01)}_{\frac{6}{20}}\,ld\,\underbrace{P(0|1)}_{\frac{3}{5}} + \underbrace{P(11)}_{\frac{4}{20}}\,ld\,\underbrace{P(1|1)}_{\frac{2}{5}} + \underbrace{P(10)\,ld\,P(1|0)}_{\frac{6}{20}\cdot ld\frac{3}{5}}]$

$= -[\frac{8}{20}\,ld\frac{2}{5} + \frac{12}{20}\,ld\frac{3}{5}] = -\ \underline{0.97095}$

$H(x_3|x_1x_2) = [\overset{}{\underset{1/20}{P(000)}}\cdot ld\,P(0|00) + \underset{3/20}{P(010)}\,ld\,P(0|01) + \underset{3/20}{P(100)}\,ld(0|10) + P(110)\,ld\,P(0|11) + \underset{3/20}{P(001)}\cdot ld\,P(1|00) + \underset{3/20}{P(011)}\cdot ld\,P(1|01) + \overset{3/20}{P(101)}\cdot ld\,P(1|10) + \underset{1/20}{P(111)}\cdot ld\,P(\underline{1|11})] =$

$\underset{3/20}{}$

$= \frac{1}{20}\cdot ld\frac{1}{4} + \frac{3}{20}\cdot ld\frac{2}{4} + \frac{3}{20}\cdot ld\frac{2}{4} + \frac{3}{20}\cdot ld\frac{3}{4} + \frac{\frac{3}{20}\,ld\frac{3}{4}}{8} +$

$+ \frac{3}{20}\,ld\frac{2}{4} + \frac{3}{20}\cdot ld\frac{2}{4} + \frac{1}{20}\,ld\frac{1}{4} =$

$-\frac{1}{10}[ld\frac{1}{4} + 2\cdot 3\,ld\frac{2}{4} + 3\,ld\frac{3}{4}] = +\frac{1}{10}[2 + 2\cdot 3 + 3\,ld\frac{4}{3}] = 0.92$

$$H(X_4|X_3X_2X_1) = -2\Big[\underset{0}{P(0000)}\cdot ld\, P(0|000) + \underset{1/20}{P(0010)}\, ld\, \underset{1/3}{P(0|001)} + \underset{1/20}{P(0100)}\cdot$$

$$\underset{1/3}{ld\, P(0|010)} + \underset{2/20}{P(0110)}\, \overset{2/3}{\underbrace{ld\, P(0|011)}} + \underset{1/20}{P(1000)}\, ld\, \underset{1/3}{P(0|100)} +$$

$$\underset{2/20}{P(1010)}\cdot \underset{2/3}{\underbrace{ld\, P(0|101)}} + \underset{2/20}{P(1100)}\, \underset{2/3}{\underbrace{ld\, P(0|110)}} + \underset{1/20}{P(1110)}\cdot ld\, P(0|111)\Big]$$

$$= -2\Big[\frac{1}{20}\cdot ld\frac{1}{3} + \frac{1}{20}\cdot ld\frac{1}{3} + \frac{2}{20}\, ld\frac{2}{3} + \frac{1}{20}\, ld\frac{1}{3} +$$

$$+ \frac{2}{20}\, ld\frac{2}{3} + \frac{2}{20}\, ld\frac{2}{3} + \frac{1}{20}\cdot ld\frac{3}{3}\Big] = -2\Big[\frac{3}{20} ld\frac{1}{3} + \frac{6}{20} ld\frac{2}{3}\Big]$$

WAITING TIME

$$Pr(S_n = K) = \binom{K-1}{n-1}\frac{1}{2^K}$$

NUMBER OF ZEROS

$$\binom{3}{2} = \frac{3!}{1!\cdot 2!} = 3$$

$$\binom{3}{1} = \frac{3!}{2!\cdot 1!} = 3$$

$$P(0010) = ? \quad Pr(S_3 = 4) = \binom{4-1}{3-1}\frac{1}{2^4} = \binom{3}{2}\frac{1}{\boxed{16}} = \frac{3}{16}$$

$$P(0110) = ? \quad Pr(S_2 = 4) = \binom{4-1}{2-1}\frac{1}{2^4} = \binom{3}{1}\frac{1}{16} = \frac{3}{16}$$

$$P(1110) = ? \quad Pr(S_1 = 4) = \binom{4-1}{1-1} = \binom{3}{0} = 1$$

$$H(X_4|X_3X_2X_1) = -2\,\frac{3}{20}\Big[ld\frac{1}{3} + ld\frac{4}{9}\Big] = -\frac{6}{20}\, ld\frac{4}{27} = \underline{\underline{0.8265}}$$

- DEL SO 3 NULI (3 EDINICI)

$$H_{\underset{0}{3}/\underset{1}{1}} = P(0010)\, ld\, P(0|001) + P(1000)\cdot ld\, P(0|100) + P(0100)\cdot ld\, P(0|010)\Big\} = \frac{3}{20} ld\frac{1}{3} = -0.238$$

- DEL SO 2 NULI (2 EDINICI

$$H_{2|2} = P(0110)\cdot ld\, P(0|011) + P(1010)\cdot ld\, P(0|101) + P(1100)\, ld\, P(0|110)$$

$$= \frac{\textcircled{6}}{20}\, ld\frac{2}{3} = -0.1755$$ OVA E $\underline{\underline{+2}}$

$$H_{\underset{0}{1}/\underset{1}{3}} = P(1110)\, ld\, P(0|111) = \frac{1}{20}\, ld\frac{1}{3} = \underline{-0.0793}$$

- OVA USTVARI KAŽUVA KOLKU MOŽAM ~~KOMBINACII~~ PERMUTACII IMAŠ KADE NA 4 MESTA IMAŠ TRI NULI PRI ŠTO ČETVRTOTO MESTO E „0"
- SEGA MI OSTANUVA DA SKOTAM UŠTE KAKO ANALITIČKI GI PRESTAVAM $\frac{1}{20}$, $\frac{2}{20}$, $\frac{1}{20}$

n - BROJ NA „0"
m - BROJ NA „1"

KOGE „0010" MOŽE ~~DA~~ BUDE SAMO 11

$$K=4 \qquad \overset{K-1}{\binom{\textcircled{3}}{2,1}}_{(i)\;m}\cdot\binom{\overset{N-K}{\textcircled{2}}}{\underset{3-2-1}{K-1-n}\;\;\underset{3-1}{K-1-m}}_{N-K-i\ \text{FIX}} = \binom{3}{2,1}\cdot\binom{2}{0\;2} = 3\cdot 1 = 3$$

$$\binom{3}{1\ 2} \cdot \binom{2}{3-2,\ 3-2} = 3 \cdot \binom{2}{1,1} = \underline{3 \cdot 2 = 6}$$

POSLE 0110 = MOZ
DA DOJDE 10, 01

$\boxed{K=3}$

$$\binom{K-1}{i,\ K-1-i} \cdot \binom{\underline{N-K}}{K-i-2,\ i} \overset{i=0}{=} \binom{2}{0,\ 2} \underbrace{\binom{3}{1,\ 0}}$$

$$i=1 \quad \binom{2}{1,\ 1} \cdot \binom{3}{0,1}$$

? TREBA $\boxed{\binom{N-K}{N-K-i,\ i}}$

- ZA $\boxed{K=4}$ FUNKCIONIRA:

$$\boxed{H(K,N) = -\frac{2}{\binom{N}{N/2}} \sum_{i=0}^{K-2} \binom{K-1}{i,\ K-1-i} \cdot \binom{N-K}{K-i-2,\ i} ld\left(\frac{\frac{N}{2}-i}{N-K+1}\right)}$$

NO NE FUNKCIONIRA ZA DRUGITE „K"!

- TESTIRANJE ZA $\boxed{K=3}$

$N-(K-1)+1 = 6-3+1 = 2$

E.G. $\underline{010{*}{*}{*}}$ $i=1$ $\binom{2}{1\ 1} = 2$

MOZE DA SLEDI SE STO IMA 1,0 , 2x1 : $\underline{011, 101, 110}$

$$\binom{3}{3-1,\ 1} = \binom{3}{2,\ 1} = 3$$

$\frac{2 \cdot 3}{20}$ } GI POKRIVA ! $\underline{P(010) \cdot ld\, P(0|01) + P(100) \cdot ld\, P(0|10)}$

$$\binom{2}{2\ 0} = 1 \qquad \binom{3}{\underbrace{3-2}_{1},\ 2} = \underset{0}{\overset{3}{?}}$$

$$\boxed{ld\ \frac{3-2}{4} = ld\ \frac{1}{4}}$$

$$\binom{K-1}{i,\ K-1-i} \cdot \binom{N-K}{N-K-\underline{i-1},\ \underline{i+1}}$$

$K=4$
$\underline{i=2}$ $\binom{3}{2,\ 1} \cdot \binom{2}{2-3,\ 3}$

$$P_r(S_2=3) = \binom{3-1}{2-1}\frac{1}{2^3} = \binom{2}{1} = \frac{2}{2^3}$$

$\boxed{\begin{matrix}010\\100\end{matrix}}$ $N_* = 2$ $\boxed{P_r = \frac{N_*}{2^3}}$

$$\binom{K-1}{i,\ K-1-i} \cdot \binom{N-K}{\frac{N}{2}-i-1,\ N-K-\frac{N}{2}+i+1} = \binom{K-1}{i,\ K-1-i}\binom{N-K}{\underbrace{\frac{N}{2}-i-1}_{a},\ \underbrace{\frac{N}{2}-K+i+1}_{b}}$$

$a+b = \frac{N}{2} - \cancel{i} - \cancel{1} + \frac{N}{2} - K + \cancel{i} + \cancel{1} = N-K$

VO MATCE POKRIVTE SO ...

$\boxed{MMV}$

$$\boxed{H(K,N) = \frac{2}{\binom{N}{N/2}} \sum_{i=0}^{K-1} \binom{K-1}{i,\ K-1-i}\binom{N-K}{\frac{N}{2}-i-1,\ \frac{N}{2}-K+i+1} ld\left(\frac{\frac{N}{2}-i}{N-K+1}\right)}$$

$i < \frac{N}{2}$
$K-i \leq \frac{N}{2}+1$

00:24:fe:93:6c:cf

(1°) $i < \frac{N}{2}$ (2°) $k - i \le \frac{N}{2} + 1$ $k - \frac{N}{2} < \frac{N}{2} + 1$

$\boxed{k < N + 1}$

$\frac{N}{2} > i$

$i \geq k - \frac{N}{2} + 1$ $i \overset{?}{\geq} k - i + 1$ $2i > k + 1$

$i > \boxed{\frac{k+1}{2}}$

$k + 1 < 2i$ $\boxed{k < 2i - 1}$

$i < k - i + 1$ $2i < k + 1$ $\boxed{i < \frac{k+1}{2}}$

1°) $i < \frac{N}{2}$ 2°) $k - i \le \frac{N}{2} + 1$ $k - i - 1 \le \frac{N}{2}$

$i \geq -\frac{N}{2} + k - 1$ $\boxed{i \geq k - \frac{N}{2} - 1}$

$\boxed{i < \frac{N}{2}}$

PROBLEM 6.4 SUPPOSE THAT ONE GAMBLES SEQUENTIALLY ON THE CARD OUTCOMES IN PROBLEM 6.6.3 EVEN ODDS OF 2-TO-1 ARE PAID. THUS THE WEALTH S_n AT TIME n IS: $S_n = 2^n b(x_1^n)$ WHERE $b(x_1, x_2, \ldots x_n)$ IS PROPORTION OF WEALTH BET ON $x_1 x_2 \ldots x_n$. FIND:

$$\max_{b(x_1^n)} \left\{ E[\mathrm{ld}\, S_{52}] \right\}$$

$$\boxed{W(p, b) = \sum_{x \in X} p(x)\, \mathrm{ld}\, o(x) \cdot b(x)} = E[\mathrm{ld}\, S(x)]$$

$$S(x) = b(x) \cdot o(x) \qquad \boxed{S_n = 2^{+nW(p,b)}}$$

$$\frac{1}{n} \mathrm{ld}\, S_n = \Big| S_n = \prod_{i=1}^{n} S(x_i) \Big| = \frac{1}{n} \sum_{i=1}^{n} \mathrm{ld}\, S(x_i) \to$$

$\to E[\mathrm{ld}\, S(x_i)]$ IN PROBABILITY

$$\Rightarrow S_n = 2^{n E[\mathrm{ld}\, S(x_i)]} = 2^{nW(p,b)}$$

– OVDE PRAKTIČNO IMAŠ **DEPENDANT** HORSE RACE

$$W^*(X_k \mid X_{k-1}\cdots X_1) = E\left[\max_{b(x_1^k)} E\left[\operatorname{ld} S(X_k|X_1^{k-1})\right]\right] =$$

$$= \max_{b(x_1^k)} E\left[\operatorname{ld} b(x_k|x_1^{k-1})\cdot o(x_1^k)\right] = E\left[\operatorname{ld} p(x_k|x_1^{k-1})\cdot m\right]$$

$$= \operatorname{ld} m + \sum_{x_1^k} p(x_1^k)\operatorname{ld} p(x_k|x_1^{k-1}) = \operatorname{ld} m - H(X_k|X_1^{k-1})$$

$$\frac{1}{n} E[\operatorname{ld} S_n] = \frac{1}{n}\sum E[S(x_1^k)] = \frac{1}{n}\sum\left[\operatorname{ld} m - H(X_k|X_1^{k-1})\right] =$$

$$= ① \boxed{\operatorname{ld} m - \frac{H(X_1^k)}{n}} \xrightarrow[n\to\infty]{} \operatorname{ld} m - H(\mathcal{X})$$

$$\boxed{\lim_{n\to\infty}\frac{1}{n} E[\operatorname{ld} S_n] + H(\mathcal{X}) = \operatorname{ld} m}$$

$$H(\mathcal{X}) = \lim_{n\to\infty} H(X_n|X_1^{n-1}) = \frac{1}{n}\lim_{n\to\infty} H(X_1^n)$$

$$H(X_1^n) = \operatorname{ld}\binom{52}{26} = 48.8$$

$$① \rightarrow$$

$$\frac{1}{n}\max_{b(x_1^n)}\left\{E[\operatorname{ld} S_{52}]\right\} = \frac{1}{n}\left\{\operatorname{ld} ⓜ - H(X)\right\} = \left\{1 - H(X)\right\}\cdot\frac{1}{n}$$

$$= \frac{1}{n}\left\{1 - \frac{H(x_1^n)}{n}\right\} = \left\{1 - \frac{H(x_1^{52})}{52}\right\} = \frac{1}{n}\left\{1 - \frac{48.8}{52}\right\} = 1 - 0.939 = 0.061/n$$

$$\max_{b(x_1^n)}\left\{E[\operatorname{ld} S_{52}]\right\} = n\cdot\frac{1}{n}\cdot 0.061 = 0.061$$

$$S_{52}^* = \frac{2^{52}}{\binom{52}{26}} = 9.08$$

$$\boxed{\lim_{n\to\infty}\frac{\operatorname{ld}\binom{n}{n/2}}{n} = 1}$$

$$\operatorname{ld} S_{52} = \max_{b(x)} E[\operatorname{ld} S_{52}] = \operatorname{ld} 9.08 = 3.2$$

- ALTERNATIVE SOLUTION (EDITION 2 SOLUTION?)

$$E[\operatorname{ld} S_n] = E[\operatorname{ld} 2^n b(x_1^n)] = \operatorname{ld} 2^n + E[\operatorname{ld} b(x_1^n)] =$$

$$= n + \sum p(x)\operatorname{ld}\frac{b(x)}{p(x)}\cdot p(x) = n - D(p\|b) - H(X)$$

$$p = b \Rightarrow D(p\|b) = 0 \Rightarrow \max_{b(x)} E[\operatorname{ld} S_n] = n - H(X)$$

$$\max_{b(x)} S_n = n - H(x) = n - H(x_1^n) = n - \mathrm{ld}\binom{52}{26} =$$

$$= 52 - \frac{52!}{26!\,26!} = 52 - 48{,}82 = \left|\begin{array}{r} 52{,}00 \\ 48.82 \\ \hline 3{,}18 \end{array}\right| \doteq \underline{\underline{3.2}}$$

PROBLEM 6.5 CONSIDER A THREE-HORSE RACE WITH WIN PROBABILITIES:

$$(p_1\ p_2\ p_3) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$$

AND FAIR ODDS WITH RESPECT TO (FALSE) DISTRIBUT.

$$(r_1\ r_2\ r_3) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right) \text{ i.e } \underline{(o_1, o_2, o_3) = (4, 4, 2)}$$

(a) WHAT IS THE ENTROPY OF THE RACE?

(b) FIND THE SET OF BETS (b_1, b_2, b_3) SUCH THAT THE COMPOUNDED WEALTH IN REPEATED PLAYS WILL GROW TO INFINITY.

$$\boxed{W(b,p) = \sum_{x\in X} p(x)\cdot \mathrm{ld}\, b(x)\cdot o(x)}$$

$$\boxed{S_n = 2^{-nW}}$$

$$S_n = \prod_{x\in X} S(x) \qquad \underline{S(x) \doteq b(x)\cdot o(x)}$$

$$\frac{1}{n} E[\mathrm{ld}\, S_n] = \frac{1}{n} E\left\{\mathrm{ld}\left[\prod_{i=1}^{n}(S(x))\right]\right\} = \frac{1}{n} E\left[\sum_{i=1}^{n} \mathrm{ld}\, S(x_i)\right]$$

$$= \frac{1}{n} \sum_{i=1}^{n} \underline{E[\mathrm{ld}\, S(x_i)]} \xrightarrow[w.p.]{} \sum_{i=1}^{n} p(x)\, \mathrm{ld}\, S(x) = \underline{W(p,b)}$$

$$\mathrm{ld}\, S_n = n \cdot W(p,b) \qquad \boxed{S_n = 2^{n\,\underline{W(p,b)}}}$$

(a)

$$H(x) = \frac{1}{2}\mathrm{ld}\, 2 + \frac{1}{4}\mathrm{ld}\, 4 + \frac{1}{4}\mathrm{ld}\, 4 = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$$

$$W(p,b) = \sum_{x\in X} p(x)\cdot \mathrm{ld}\, b(x)\cdot o(x) = \sum_{x} p(x)\cdot \mathrm{ld}\, \frac{b(x)}{p(x)}\cdot p(x)\, o(x)$$

$$= \underbrace{\sum_{x} p(x)\cdot \mathrm{ld}\, o(x)}_{\textcircled{*}} - \underbrace{D(p\|b)}_{p=b\ \Rightarrow} - \underline{H(X)} \qquad W(p,b) = \underset{\text{MAXIMUM}}{\underline{W^*(p,b)}}$$

$$\textcircled{*} = \frac{1}{2}\cdot \mathrm{ld}\, 4 + \frac{1}{4}\cdot \mathrm{ld}\, 2 + \frac{1}{4}\mathrm{ld}\, 2 = 1 + \frac{1}{2} = \frac{3}{2}$$

$$W(p,b) = \frac{3}{2} - \frac{3}{2} = 0 \quad ?$$

- DOUBLING RATE FOR NON-UNIFORM FAIR ODDS $\left(\substack{\text{VIDI}\\ 6.31}\right)$

$$W^*(X_k|X_1^{k-1}) = E[\text{ld}\, o(x_1^k)] - \underline{H(X_k|X_1^{k-1})}$$

$$\frac{1}{n}E[\text{ld}\, S_n] = \frac{1}{n}\sum p(x)\,\text{ld}\, o(x_1^k) - \underline{\frac{1}{n}H(X_k|X_1^{k-1})}$$

$$H(\mathcal{X}) = \underline{\lim_{n\to\infty}\frac{1}{n}H(X_n|X_1^{n-1})}$$

$$H(X,Y) = \sum_{x,y} p(x,y)\cdot \text{ld}\frac{1}{p(x,y)} = \sum_x\sum_y p(x)p(y|x)\,\text{ld}\frac{1}{p(x)p(y|x)}$$

$$= -\sum_x p(x)\left(\sum_y p(y|x)\,\text{ld}\, p(x) + \sum_y p(y|x)\,\text{ld}\, p(y|x)\right)$$

$$\boxed{H(X|Y) = \sum_{y\in Y} p(y)\cdot H(\mathcal{X}|Y=y)}$$

$$H(X,Y) = H(X) + H(Y|X)$$

$$\Rightarrow \sum_x\sum_y p(x,y)\cdot\text{ld}\frac{1}{p(x)\cdot p(y|x)} = \sum_x\sum_y p(x,y)\left[\text{ld}\frac{1}{p(x)} + \text{ld}\frac{1}{p(y|x)}\right.$$

$$= \sum_x p(x)\,\text{ld}\frac{1}{p(x)} + \sum_{x,y} p(x,y)\,\text{ld}\frac{1}{p(y|x)} = H(X) + H(Y|X)$$

$$\frac{W^*(X_k|X_1^{k-1})}{n} = \frac{1}{n}E\left[\sum_x p(x)\cdot\text{ld}\, o(x)\right] - \frac{H(X_1 X_2 \dots X_k)}{n}$$

CONST $= 3/2 \cdot n$

$$\frac{1}{n}E[\text{ld}\, S_n] = \frac{3}{2} - \frac{H(X_1^k)}{n} = \frac{3}{2} - \frac{\cancel{n}}{\cancel{n}}\cdot\frac{3}{2} = 0 = \text{LE}$$

$$H(X_1^k) = n\cdot H(X) = n\cdot\frac{3}{2} \qquad \boxed{H(\mathcal{X}) = \lim_{n\to\infty}\frac{nH(X)}{n} = H(X) = \frac{3}{2}}$$

- MISLAM DEKA OVOJ PROBLEM TREBA DA SE POSTAVI KAKO ONOJ ZA KLADENJE NA UKO BEZA KONJA (EXAMPLE 6.3.1 IMPLEM. VO MAPLE

$x \in \{1,2,3\}$ $\quad n = 4 \quad [1\,1\,2\,3]$

$p(x) = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\}$

DA NE ODI SO CONSTAN. OPKLADI TUKU DA SI MENUVAS

1123
1132
1213
1231
1312
1321
2113
2131
2311
3112
3121
3211

$$\boxed{\binom{4}{2,1,1} = 12}$$

- LINEARNA APROKSIMACIJA

$$Y = Kx + a$$

$$\boxed{x = 10^{\frac{Y-a}{10K}}}$$

$$\boxed{x \cdot 10^{\frac{Y-a}{10K}} = 0}$$

$$Y = 2x - 2$$

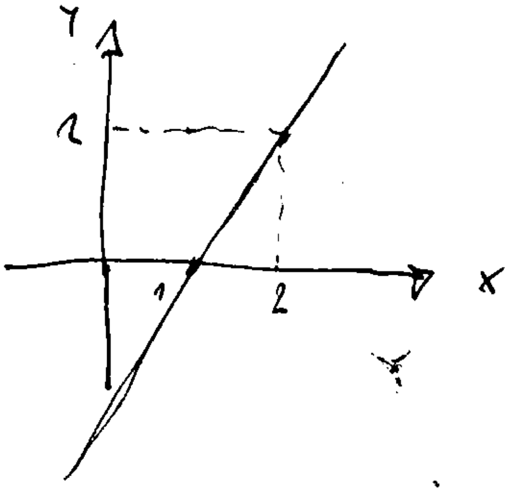
$$Y = K \cdot 10 \log X + a$$

$$Y = K \cdot t + a$$

$$t = 10\, ld\, X$$

$$\frac{Y-a}{10K} = \log X$$

E.G. $Y = 2x - 1 = 2\left(x - \frac{1}{2}\right)$



$$\bar{Y} = 10^{\log \bar{Y}}$$

- PROBLEM 6.5. CONTINUE

$$\boxed{N = \binom{n}{n/2,\ n/4,\ n/4}}$$ LAW OF LARGE NUMBERS

$$H(X_1^n) = \sum_{i=1}^{N} \frac{1}{N} \cdot ld\, N = N \cdot \frac{1}{N}\, ld \binom{n}{n/2,\ n/4,\ n/4}$$

$$\boxed{H(X) = \lim_{n \to \infty} \frac{H(X_1^n)}{n} = \lim_{n \to \infty} \frac{ld \binom{n}{n/2,\ n/4,\ n/4}}{n}}$$

$$\boxed{S_n = \frac{3^n}{\binom{n}{n/2,\ n/4,\ n/4}}}$$

AND $n \to \infty$ $\Rightarrow$ $S_n \to \infty$ VIDI MAPLE EDIT CALC.mw

$$(b_1\, b_2\, b_3)_1 = \left[\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right] = B_1$$

$$B_2 = (b_1\, b_2\, b_3)_2 = \left(\frac{n/2 - 1}{n-1}, \frac{n/4}{n-1}, \frac{n/4}{n-1}\right)$$

IF IN $t = 1$ WON HORSE 1

IF IN TIME „t" THE ~~VECTOR~~ VECTOR OF WINS IS

$W = [a, b, c]$ THEN:

SUCH $a + b + c = t$

$$\boxed{B_{t+1} = \left(\frac{\frac{n}{2} - a}{n-t-1}, \frac{n/4 - b}{n-t-1}, \frac{n/4 - t + a + b}{n-t-1}\right)}$$

$\boxed{\text{EDITION 2 SOLUTION}}$ $p = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$ $r = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$

$$W(b,p) = \sum_{x \in X} p(x)\, \mathrm{ld}\, b(x)\cdot o(x) = \sum_x p(x)\, \mathrm{ld} \frac{b(x)}{r(x)}\frac{p}{p} =$$

$$= \sum_x p(x)\cdot \mathrm{ld}\frac{p(x)}{r(x)} - \sum_x p(x)\, \mathrm{ld}\frac{p(x)}{b(x)} = \underline{D(p\|r) - D(p\|b)}$$

$$D(p\|r) = \frac{1}{2}\cdot \overset{\nearrow 1}{\cancel{\mathrm{ld}\, 2}} + \frac{1}{4}\mathrm{ld}\, 1 + \frac{1}{4}\mathrm{ld}\frac{1}{2} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$W(b,p) > 0 \Rightarrow W(b,p) = \frac{1}{4} - D(p\|b) > 0 \Rightarrow$$

$$\boxed{D(p\|b) < \frac{1}{4}}$$

IZBOROT NA „b" KOJ DAVA POMALA RELATIVNA ENTROPIJA OD 0.25 VODI KON BESKONEČNA DOBIVKA

27.02.13

$\boxed{\text{PROBLEM 6.6}}$ · HORSE RACE.

A THREE HORSE RACE HAS WIN PROBABILITIES: $p = (p_1, p_2, p_3)$ AND ODDS: $o = (1,1,1)$. THE GAMBLER PLACES BETS (b_1, b_2, b_3) $b_i \geq 0$ $\sum b_i = 1$, WHERE b_i DENOTES PROPORTION OF WEALTH BET ON HORSE „i". THIS ODDS ARE VERY BAD. THE GAMBLER GETS HIS MONEY BACK ON THE WINNING HORSE AND LOSES OTHER BETS. THUS, THE WEALTH S_n AT TIME „n" RESULTING FROM INDEPENDENT GAMBLES GOES EXPONENTIALLY TO ZERO

(a) FIND THE EXPONENT

(b) FIND THE OPTIMAL GAMBLING SCHEME b (I.E. THE BET b^* THAT MAXIMIZES THE EXPONENT).

(c) ASSUMING THAT b IS CHOSEN AS IN PART (b), WHAT DISTRIBUTION p CAUSES S_n TO GO TO ZERO AT FASTEST RATE.

$$W(b,p) = \sum_x p(x)\, \mathrm{ld}\, b(x)\cdot o(x) \cdot \frac{p(x)}{p(x)} = D(p\|r) - D(p\|b)$$

$$r = 1/o(x)$$

$$S_n = \prod_{i=1}^{n} S(x) \qquad S(x) = b(x)\cdot o(x)$$

$$\frac{1}{n}\, \mathrm{ld}\, S_n = \frac{1}{n}\, \mathrm{ld} \prod_{i=1}^{n} S(x_i) = \frac{1}{n}\sum_{i=1}^{n} \mathrm{ld}(S(x_i)) \xrightarrow{n \to \infty}$$

$$\sum_{i=1}^{n} p(x_i)\, \mathrm{ld}\, b(x_i)\cdot o(x_i) = W(b,p) \Rightarrow S_n = 2^{nW(b,p)}$$

· THE GAMBLER SHOULD ~~KEEP~~ SOME MONEY

$$S(x) = b_0 + b(x) \cdot o(x)$$

$$\frac{2^4}{\binom{4}{2}} = \frac{16}{\frac{4!}{2!2!}} = \frac{16}{\frac{1\cdot 2\cdot 3\cdot 4}{4}} = \frac{8}{3}$$

(a)

$$W(b,p) = \sum_x p(x) \cdot \mathrm{ld}\, b(x) \cdot o(x) = \sum_{x \in X} p(x) \cdot \mathrm{ld}\, \underline{b(x)}$$

$$W(b,p) = \sum_x p(x) \cdot \mathrm{ld}\, o(x) p(x) + \sum p(x) \cdot \mathrm{ld} \frac{b(x)}{p(x)} =$$

$$= D(p \| r) - D(p \| b)$$

$$\underbrace{\sum p(x) \cdot \mathrm{ld}\, o(x)}_{0} - \sum_x p(x)\, \mathrm{ld} \frac{1}{b(x)}$$

$$S_n = 2^{-n \sum_x p(x)\, \mathrm{ld} \frac{1}{b(x)}}$$

(b)

$$W(b,p) = - \sum_x p(x)\, \mathrm{ld} \frac{1}{b(x)} \qquad \sum_x b(x) = 1$$

$$W(b,p) = - \sum_i p_i\, \mathrm{ld} \frac{1}{b_i} \qquad \sum_i b_i = 1$$

$$\nabla W(b,p) = \lambda \nabla \Big(\sum_i b_i \Big)$$

$$\frac{\partial}{\partial b_i} \Big[\sum_{i=1}^{n} p_i\, \mathrm{ld}\, b_i \Big] = \lambda \frac{\partial}{\partial b_i} \Big[\sum_i b_i \Big] \qquad i = 1 \ldots n$$

$$p_i \frac{1}{b_i} = \lambda \qquad \boxed{b_i = \frac{p_i}{\lambda}}$$

(c)

$$S_n = 2^{-n \sum_x p_{\cancel{x}}\, \mathrm{ld} \frac{\lambda}{p_i}} = 2^{-n \underbrace{\left[\sum_i p_i\, \mathrm{ld}\, \lambda + H(p) \right]}_{f}}$$

$$f = \sum_i p_i\, \mathrm{ld}\, \lambda - H(p) = \underline{\mathrm{ld}\, \lambda + H(p)} \qquad \boxed{\begin{aligned} f_{max} &= \mathrm{ld}\, \lambda + \mathrm{ld}\, 3 \\ &= \mathrm{ld}\, 3\lambda \end{aligned}}$$

$$\max\{H(p)\} = \left| \begin{matrix} \text{UNIFORM DISTRO} \\ \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\} \end{matrix} \right| = \mathrm{ld}\, 3$$

$p = \{p_1 \dots p_n\}$ $\quad q = \{\frac{1}{n}, \frac{1}{n} \dots \frac{1}{n}\}$

$$D(p\|q) \geq 0 \qquad \sum_{i=1}^{n} p \, ld \frac{p}{q} = -\sum_{i=1}^{n} p \, ld \frac{1}{p} + \sum_{i=1}^{n} p \, ld \frac{1}{q}$$

$$= -H(p) + ld(n) \geq 0 \qquad \boxed{H(p) \leq ld\, n}$$

$$\boxed{\max\{H(p)\} = ld\, n}$$

- FOR $\boxed{\lambda = 1}$ $\quad S_n = 2^{-n \cdot ld3\lambda} = 2^{-n\, ld3} = \left(2^{ld3}\right)^{-n}$

$$\boxed{S_n = 3^{-n}}$$

DESPITE THE BAD ODDS OPTIMAL STRATEGY IS STILL PROPORTIONAL BETING.

$$-\sum_x p(x)\, ld \frac{1}{b(x)} \frac{p}{p} = -\left[\sum_x p(x) \cdot ld \frac{1}{p(x)} + D(p\|b)\right]$$

$$W(b,p) = -D(p\|b) - \underline{H(p)} = -[\underbrace{D(p\|b)}_{} + H(p)]$$

OVAA SUMA E MINIMALNA AKO $D(p\|b) = 0$ T.E $\underline{p = b}$
T.e. $\lambda = 1$
↳ MAKSIMALNA NEGATIVNA
T.E SE PRIBLIZUVA OD $-\infty$ KON 0

↳ SUM MOZEL VO STARTOT OVA DA GO RESAVAM BEZ DA RESAVAM LAGRANGE MULTIPLIERS. TAKA PISUVA VO EDITION 2 SOLUTIONS.

PROBLEM 6.7

29.07.13

CONSIDER A HORSE RACE WITH 4 HORSES. ASUME 4-FOR-1 ODDS. LET THE PROBABILITIES OF WINING OF THE HORSES BE:

$$p = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right\}$$

IF YOU STARTED WITH \$100 AND BET OPTIMALY TO MAXIMIZE YOUR LONG TERM GROWTH RATE, WHAT ARE YOUR OPTIMAL BETS ON EACH HORSE? APROXIMATELY HOW MUCH MONEY, YOU WILL HAVE AFTER 20 RACES WITH THIS STRATEGY?

$$W(p,b) = \sum_{x \in X} p(x) \cdot ld\, b(x) \cdot o(x) = \sum_{x \in X} p(x)\, ld\, \underline{\underline{o(x)}} -$$

$$- \sum p(x)\, ld \frac{1}{b(x)} = \left| \begin{matrix} \text{PROPORTIONAL} \\ \text{BETING} \\ \text{FAIR ODDS} \end{matrix} \right| = \sum_x p(x)\, \underbrace{ld\, 4}_{2} - H(p)$$

$W(\gamma, 6) = 2 - H(\gamma) \qquad H(\gamma) = \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 + 2\left(\frac{1}{8} \cdot 3\right)$

$$H(q) = \frac{1}{2} + \frac{1}{2} + \frac{3}{4} = 1 + \frac{3}{4} = \frac{7}{4}$$

$$S_n = 2^{+n W(\gamma, 6)} = 2^{n\left(2 - \frac{7}{4}\right)} = 2^{n\left(\frac{1}{4}\right)}$$

$$S_{20} = 2^{\frac{20}{4}} = 2^5 = \underline{\underline{32}} \Big\}$$ AND SI TRYCAZ SO 1\$

-AND SI TRYCAZ SO 100\$: $S_{20} = 3200$\$

PROBLEM 6.8 THE FOLLOWING ANALYSIS IS CRUDE APPROXIMATION TO THE GAMES OF LOTTO CONDUCTED BY VARIOUS STATES. ASSUME THAT THE PLAYER OF THE GAME IS REQUIRED TO PAY \$1 TO PLAY AND IS ASKED TO CHOSE ONE NUMBER FROM A RANGE 1 TO 8. AT THE END OF EVERY DAY, THE STATE LOTTARY COMMISION PICKS A NUMBER UNIFORMELY OVER THE SAME RANGE. THE JACKPOT (I.E. ALL THE MONEY COLLECTED THAT DAY) IS SPLIT AMONG THE PEOPLE WHO CHOSE THE SAME NUMBER AS THE ONE CHOSEN BY THE STATE. FOR EXAMPLE IF 100 PEOPLE PLAYED TODAY, 10 OF THEM CHOSE NUMBER 2, AND THE DRAWING AT THE END OF THE DAY PICKED 2, THE \$100 IS SPLITED AMONG THE 10 PEOPLE (EACH PERSON WILL PICK 10\$). THE GENERAL POPULATION DOES NOT CHOSE NUMBERS UNIFORMELY, NUMBERS SUCH 3 & 7 ARE SUPPOSEDLY LUCKY AND ARE MORE POPULAR THAN 4 OR 8. ASSUME THAT THE FRACTION OF PEOPLE CHOOSING THE VARIOUS NUMBERS 1, 2 .. 8 IS $(f_1, f_2, \ldots, f_8)$ AND ASSUME THAT n^2 PEOPLE PLAY EVERY DAY. ALSO ASSUME THAT n^2 IS VERY LARGE, SO THAT ANY SINGLE PERSON'S CHOICE DOES NOT CHANGE THE PROPORTION OF PEOPLE BETTING ON ANY NUMBER.

(a) WHAT IS THE OPTIMAL STRATEGY TO DEVIDE YOUR MONEY AMONG VARIOUS POSSIBLE TICKETS SO AS TO MAXIMIZE YOUR LONG-TERM GROWTH RATE? (IGNORE THE FACT THAT YOU CANNOT BUY FRACTIONAL TICKETS)

(b) WHAT IS THE OPTIMAL GROWTH RATE THAT YOU CAN ACHIEVE IN THIS GAME?

(c) IF $(f_1, f_2, \ldots f_8) = \left(\frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{4}, \frac{1}{16}\right)$ AND YOU START WITH 1\$ HOW LONG ~~IS~~ WILL IT BE BEFORE YOU BECOME A MILIONARE?

(a) $\textcircled{w} = n\cdot(f_1, f_2, \dots f_8)$ $\boxed{\sum_{i=1}^{n} f_i = 1}$

E.G $\boxed{n=16}$ ("f" FROM "C")

$\boxed{w = (2,2,4,1,1,1,4,1)}$ $\sum ld\, w = \sum(1,1,2,0,0,0,2,0) = 6$

POTENTIALY WINNING NUMBER OF PEOPLE

$\boxed{O = \frac{n}{w}}$

ODDS

GLAVNA PLATA NA ZADNJOVA.

$\boxed{O = \frac{1}{f} = (8,8,4,16,16,16,4,16)}$

$$W(b,p) = \sum p(x)\, ld\, b(x)\cdot o(x) = \sum p(x)\cdot ld\, o(x) - \underbrace{H(p)}_{}$$

PROPORTIONAL BETTING

$$p(x) = \left[\frac{1}{8}, \frac{1}{8} \dots \frac{1}{8}\right]$$

$$W(b,p) = \frac{1}{8}\sum_x ld\, o(x) - 8\frac{1}{8}\cdot \underline{\underline{ld 8}} = \frac{1}{8}\sum_x ld\, o(x) - \underbrace{ld 8}_{3}$$

$$W(b,p) = -\frac{1}{8}\sum_x ld\, f(x) - ld 8 = -\frac{1}{8}\sum_x ld\, f(x) - 3$$

(b) $S_n = 2^{n\cdot W(b,p)}$

(c) $f(x) = \left(\frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{4}, \frac{1}{16}\right)$

$$\sum ld\, f(x) = -\sum(3,3,2,4,4,4,2,4) = -26$$

$$W(b,p) = \cancel{0} - \frac{1}{8}(-26) - 3 = \frac{26}{8} - 3 = \frac{13}{4} - 3 = \frac{13-12}{4} = 1/4$$

$S_n = 2^{n/4}$ $10^6 = 2^{n/4}$ $\frac{n}{4} = ld\, 10^6$

$\boxed{n = 4\cdot ld\, 10^6}$ $\boxed{n = \lceil 77{,}72 \rceil = 80}$

PROBLEM 6.9 HORSE RACE Suppose one is interested in maximizing the doubling rate for a horse race. Let $p_1, p_2, \dots p_m$ denote win probabilities of the m horses. When the odds $(o_1, o_2, \dots o_m)$ yield higher doubling rate then the odds $(o'_1, o'_2, \dots, o'_m)$?

$$W(b,p) = \sum_{i=1}^{m} p(x_i)\, ld\, o_i\cdot b_i \qquad f(x) = W(b,p) + \lambda\sum_{i=1}^{m}\frac{1}{o_i}$$

$$\frac{d f(o_i)}{d o_i} = 0 \quad i=1..m; \quad p(x_i)\cdot\frac{1}{o_i\cdot \cancel{b_i}}\cdot\cancel{b_i} - \lambda\frac{1}{o_i^2} = 0$$

$$\sum_{i=1}^{m}\frac{1}{o_i} = 1 \qquad o_i\, p(x_i) - \lambda = 0 \qquad \boxed{o_i = \frac{\lambda}{p_i}}$$

$\boxed{\text{OPTION 2 SOLUTION}}$ $W = \sum_{i=1}^{m} p_i \operatorname{ld} o_i - H(p)$

$$W' = \sum_{i=1}^{m} p_i \operatorname{ld} o_i' - H(p)$$

$$W \geq W' \quad \sum_{i=1}^{m} p_i \operatorname{ld} o_i - \cancel{H(p)} > \sum_{i=1}^{m} p_i \operatorname{ld} o_i' - \cancel{H(p)}$$

$$\Rightarrow \sum_{i=1}^{m} p_i \operatorname{ld} o_i \geq \sum_{i=1}^{m} p_i \operatorname{ld} o_i' \Rightarrow \boxed{E[\operatorname{ld} o_i] > E[\operatorname{ld} o_i']}$$

- MNOGO ENOSTAVEN TO KONČEN ZAKLJUČEK !!!

PROBLEM 6.10 HORSE RACE WITH PROBABILITY ESTIMATES

(a) THREE HORSES RACE. THEIR PROBABILITY OF WINNING ARE $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$. THE ODDS ARE 4-FOR-1, 3-FOR-1 AND 3-FOR-1. LET W^* BE THE OPTIMAL DOUBLING RATE. SUPPOSE YOU BELIEVE THAT THE PROBABILITIES ARE $\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$. IF YOU TRY TO MAXIMIZE THE DOUBLING RATE, WHAT DOUBLING RATE W WILL YOU ACHIEVE? BY HOW MUCH HAS YOUR DOUBLING RATE DECREASE DUE TO THE POOR ESTIMATE OF THE PROBABILITIES. (I.E. WHAT IS $\Delta W = W^* - W$)?

(b) NOW LET THE HORSE RACE BE AMONG "m" HORSES WITH PROBABILITIES $p = (p_1, p_2, \dots p_m)$ AND ODDS $o = (o_1, o_2, \dots o_m)$. IF YOU BELIEVE THE TRUE PROBABILITIES TO BE $q = (q_1, q_2, \dots q_m)$, AND TRY TO MAXIMIZE THE DOUBLING RATE W, WHAT IS $W^* - W$?

$o = (4, 3, 3)$ $\qquad \sum_{i=1}^{3} \frac{1}{o_i} = \frac{1}{4} + \frac{2}{3} = \frac{3+8}{12} = \underbrace{\frac{11}{12}}_{\text{SUPER FAIR ODDS}}$

$$W\left[p, b(x_n|x_1^{n-1})\right] = \sum_{x_1^n} p(x_1^n) \operatorname{ld}\left(o(x_1^n) \cdot b(x_n|x_1^{n-1})\right) =$$

$$= /(\text{PROPORTIONAL BETTING})/ = \sum_{x_1^n} p(x_1^n) \operatorname{ld} o(x_1^n) - \sum_{x_1^n} p(x_1^n) \operatorname{ld} \frac{1}{p(x_n|x_1^{n-1})}$$

$$= \sum_{x_1^n} p(x_1^n) \cdot \operatorname{ld} o(x_1^n) - \underline{H(X_n|X_1^{n-1})}$$

$$W^*\left[p, b(x)\right] = \sum p(x) \operatorname{ld} o(x) - H(X)$$

$$W(p, b(x|y)) = \sum_{x} p(x)\cdot ld\, o(x) - \sum_{xy} p(x,y)\, ld \frac{1}{p(x|y)}$$

$$= \sum_{x} p(x)\, ld\, o(x) - H(X|Y)$$

$$\underline{W(p, b(x|y))} - \underline{W(p, b(x))} = -H(X|Y) + H(X) = \underline{I(X;Y)}$$

$27 \cdot 3 = 81$

$$b^*(x|y) = p(x|y)$$

$$W(p,b) = \sum_{x\in X} p(x)\, ld\, b(x)\cdot o(x) \frac{p}{p} = \sum_{x} p(x)\, ld \frac{p}{r(x)} - \sum p(x)\, ld \frac{p}{b}$$

$$= D(p\|r) - D(p\|b) = \left| b = q \right| = \underbrace{D(p\|r)}_{\text{BOOKEE ESTIMATE}} - \underbrace{D(p\|q)}_{\text{MY ESTIMATE}}$$

$$\underline{W(p,b)} = \sum \underline{\left(\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}\right)}\, ld\, (4,3,3)\cdot\left(\tfrac{1}{4}\ \tfrac{1}{2}\ \tfrac{1}{4}\right) =$$

$$= \frac{1}{2}\cdot ld\left(4\cdot\frac{1}{4}\right) + \frac{1}{4}\cdot ld\left(3\cdot\frac{1}{2}\right) + \frac{1}{4}\cdot ld\left(3\cdot\frac{1}{4}\right) =$$

$$= 0 + \frac{1}{4}\, ld \frac{3}{2} + \frac{1}{4}\, ld \frac{3}{4} = \frac{1}{4}\, ld\left(\frac{9}{8}\right) = \underline{0.0425}$$

$$W^*(p,b) = \sum \left(\tfrac{1}{2}\tfrac{1}{4}, \tfrac{1}{4}\right) ld\, (4,3,3)\cdot\left(\tfrac{1}{2}\ \tfrac{1}{4}\ \tfrac{1}{4}\right) =$$

$$= \sum_{x} p(x)\, ld\, o(x) \quad - \underline{H(p)} = \frac{1}{2}\, ld 2 + \left(\frac{1}{4}\, ld \frac{3}{4}\right)\cdot 2 =$$

$$= \frac{1}{2} + \frac{1}{2}\, ld \frac{3}{4} = \frac{1}{2}\left(1 + ld \frac{3}{4}\right) = \underline{0.292} = \underline{\frac{1}{2}\, ld \frac{3}{2}}$$

$$W^* - W = 02.920 - 0.0425 = \underline{02495} \approx \underline{0.25}$$

POPRECENO

$$\frac{1}{2}\, ld \frac{3}{2} - \frac{1}{4}\, ld \frac{9}{8} = \frac{1}{2}\, ld \frac{\cancel{3}}{2}\cdot\frac{\sqrt{8}}{\cancel{3}} = \frac{1}{2}\, ld \sqrt{2} = 1/4$$

$$W^*(p,b) = \underline{\sum_{x} p(x)\, ld\, o(x)} - \underline{H(p)}$$

$$W(p,b) = D(p\|r) \quad - \underline{D(p\|q)}$$

$$\boxed{W^*(p,b) - W(p,b) = W^*(p,b) + D(p\|q) - D(p\|r)}$$

$$H(p) = \frac{1}{2}\cdot ld 2 + \left(\frac{1}{4}\, ld 4\right)\cdot 2 = \frac{1}{2} + \frac{1}{2}\cancel{4}2 = 1 + \frac{1}{2} = \frac{3}{2} \,//$$

$$D(p\|q) = \frac{1}{2}\cdot ld \frac{\frac{1}{2}}{\frac{1}{4}} + \frac{1}{4}\, ld \frac{\frac{1}{4}}{\frac{1}{2}} + \frac{1}{4}\, \cancel{ld \frac{\frac{1}{4}}{\frac{1}{4}}} \xrightarrow{1} 0 =$$

$$= \frac{1}{2} + \frac{1}{4}\, ld \frac{1}{2} + 0 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \,//$$

$$D(p\|r) = \sum p(x)\,\mathrm{ld}\,\frac{p(x)}{r(x)} = \sum_x p(x)\,\mathrm{ld}\,p(x)\cdot o(x) =$$

$$= \underline{\sum p(x)\,\mathrm{ld}\,o(x) - H(p)}$$

⊕ $\Rightarrow W^*(p,b) - W(p,b) = \cancel{\sum p\,\mathrm{ld}\,o - H(p)} + D(p\|q) - \cancel{\sum p\,\mathrm{ld}\,o + H(p)}$

$$\boxed{W^*(p,b) - W(p,b) = D(p\|q) = \frac{1}{4} = 0.25}$$

(b) $W^*(p,b) - W(p,b) = D(p\|q)$

$\max[W(p,b)] = W^*(p,b) \Rightarrow \underline{p = q \Rightarrow D(p\|q) = 0}$

$\boxed{\text{SOLUTION 2}}$ (EDITION)

$$W^* - W = \sum p\,\mathrm{ld}\,p\,o - \sum p\,\mathrm{ld}\,q\cdot o = \sum p\,\mathrm{ld}\,p - \sum p\,\mathrm{ld}\,q$$
$$= D(p\|q)$$

POENOSTAVNO E VAJA!!!

$\boxed{\text{PROBLEM 6.11}}$ $\boxed{30.07.2013}$ <u>TWO ENVELOPE PROBLEM</u>: ONE ENVELOPE CONTAINS b^* DOLARS, THE OTHER $2b$ DOLARS. <u>THE AMOUNT b^* IS UNKNOWN.</u> AN ENVELOPE IS SELECTED AT RANDOM. LET X BE THE AMOUNT OBSERVED IN THE ENVELOPE, AND LET Y BE THE AMOUNT IN THE OTHER ENVELOPE. ADOPT A STRATEGY OF SWITCHING TO OTHER ENVELOPE WITH PROBABILITY $p(x)$ WHERE $\boxed{p(x) = \frac{e^{-x}}{e^{-x}+e^{x}}}$ <u>LET Z BE THE AMOUNT THAT THE PLAYER RECEIVES.</u>

THUS:

$$(X,Y) = \begin{cases}(b, 2b) & \text{WITH } p = 1/2\\ (2b, b) & \text{WITH } p = 1/2\end{cases}$$

$$Z = \begin{cases} X & \text{WITH PROBABILITY : } 1 - p(x)\\ Y & \text{WITH } -\!\!\parallel\!\!- \text{ : } p(x)\end{cases} \qquad \theta = \begin{cases} 1 & 1-p(x)\\ 2 & p(x)\end{cases}$$

(a) SHOW THAT $E[X] = E[Y] = \frac{3b}{2}$ ✓

(b) SHOW THAT $E[Y|X] = 5/4$. SINCE THE EXPECTED RATIO OF THE AMOUNT IN THE OTHER ENVELOPE IS 5/4, IT SEEMS THAT ONE SHOULD ALWAYS SWITCH. (THIS IS ORIGIN OF SWITCHING PARADOX). HOWEVER, OBSERVE THAT $E(Y) \neq E[X]\cdot E[Y|X]$. THUS, ALTHOUGH $E[Y|X] > 1$, IT DOES NOT FOLLOW THAT $E[Y] > E[X]$

(c) LET J BE THE INDEX OF THE ENVELOPE CONTAINING THE MAXIMUM AMOUNT OF MONEY, AND J' BE THE INDEX OF THE ENVELOPE CHOSEN BY THE ALGORITHM. SHOW THAT FOR ANY "b", $I(J; J') > 0$. THUS THE AMOUNT IN THE FIRST ENVELOPE ALWAYS CONTAINS SOME INFORMATION ABOUT WHICH ENVELOPE TO CHOSE.

(d) SHOW THAT $E[Z] > E[X]$. THUS YOU CAN DO BETTER THEN ALWAYS STAYING OR ALWAYS SWITCHING. IN FACT THIS IS TRUE FOR ANY MONOTONIC DECREASING SWITCHING FUNCTION $p(x)$. BY RANDOMLY SWITCHING ACCORDING $p(x)$, YOU ARE MORE LIKELY TO TRADE UP THEN TO TRADE DOWN.

$$W(X|Y) = \sum_{xy} p(x,y)\,\mathrm{ld}\, o(x)\cdot b(x|y) = \underline{\sum_x p(x)\,\mathrm{ld}\, o(x)} +$$

$$+ \underline{\sum_{xy} p(x,y)\,\mathrm{ld}\, b(x|y)} = \left| \begin{array}{c} \text{PROPORTIONAL} \\ \text{BETTING} \end{array} \right| = \sum_x p(x)\,\mathrm{ld}\, \underline{o(x)} -$$

$$- \sum_{xy} p(x,y)\,\mathrm{ld}\,\frac{1}{p(x|y)} = \sum p(x)\,\mathrm{ld}\, o(x) - \underline{\underline{H(X|Y)}} = W^*(X|Y)$$

$$W^*(X|Y) - W(X|Y) = -H(X|Y) + H(X) = \underline{\underline{I(X,Y)}}$$

(a) $E[X] = \sum_x x\cdot p(x) = b\cdot\frac{1}{2} + 2b\cdot\frac{1}{2} = \frac{3b}{2}$

$E[Y] = \sum_y y\, p(y) = 2b\cdot\frac{1}{2} + b\cdot\frac{1}{2} = \frac{3b}{2}$

(b) $E[Y|X=b] = \sum_y y\cdot P(Y|X=b) = 2b\cdot 1 + b\cdot 0 = 2b$

$E[Y|X=2b] = \sum_y y\cdot P(Y|X=2b) = b\cdot 1 + 2b\cdot 0 = b$

TOTAL EXPECTATION

$E[Y] = P(X=b)\cdot E[Y|X=b] + P(X=2b)\cdot E[Y|X=2b] =$

$= \frac{1}{2}\cdot 2b + \frac{1}{2}\cdot b = \frac{3b}{2} \nearrow$

(d) $E[Z] = ?$ $\quad E[Z] = E[Z|X]\cdot P(X) + E[Z|Y]\cdot P(Y) =$

$= E[X]\cdot[1-p(x)] + E[Y]\cdot p(x) = E(X)\cdot[1-p(x)]\cdot$

$E(X)\cdot p(x) = E[X]$

$H(Z|\theta) = p(\theta=1)\cdot H(Z|_{\theta=1}) + p(\theta=2)\cdot H(Z|_{\theta=2})$

$= [1-p(x)]\cdot H(Z|_{\theta=1})_{=H(X)} + p(x)\cdot H(Z|_{\theta=2})_{=H(Y)}$

$H(X) = \sum_{x} p(x)\, ld \frac{1}{p(x)}$ $\qquad$ $H(X|Y) = \sum_{xy} p(x,y)\, ld \frac{1}{p(x|y)}$

$p(x,y) = \left(\frac{1}{2}, \frac{1}{2}\right)$ $\qquad$ $p(x|y) = \frac{p(x,y)}{p(y)}$

$(x,y) = b, 2b$ $\quad$ $(x,y) = 2b, b$

p(x,y)

X\Y	b	2b	p(X)
b	0	1/2	1/2
2b	1/2	0	1/2
p(Y)	1/2	1/2	▨

X\Y	b	2b	
b	0	1	
2b	1	0	

$H(X) = \frac{1}{2}\cdot ld 2 + \frac{1}{2} ld 2 = 1$

$\boxed{H(X|Y) = \frac{1}{2} ld 1 + \frac{1}{2} ld 1 = 0}$

$I(X,Y) = H(X) - H(X|Y) = 1 - 0 = 1$

NE POSTOJI NEZAVISNOST!!!

(d) $\quad Z = \begin{cases} X & p(x) \\ Y & 1-p(x) \end{cases}$

$Z = f(x) \Rightarrow$ JENSEN NEQ

$E[f(x)] = E[Z] \geq f(E[x])$

$Y = g(x)$

$E[Z] = \frac{1}{2}\sum_{x,y} x\cdot p(x) + y(1-p(x)) =$

$= \left[b\, p(x) + 2b[1-p(x)] + 2b\, p(x) + [1-p(x)]\cdot b\right]\cdot\frac{1}{2}$

$= \left[\cancel{bp} + \underline{2b} - \cancel{2bp} + \cancel{2bp} + \underline{b} - \cancel{bp}\right]\cdot\frac{1}{2} = \frac{3b}{2}$

$E[Y|X] = p(x=b)\cdot E[Y|x=b] + p(x=2b)\cdot \underline{E[Y|x=2b]}$

$E[Y|x=b] = \sum_{y\in Y} y\cdot p(\underline{y|x=b}) = b\cdot\underbrace{p(b|x=b)}_{0} + 2b\,\underbrace{p(2b|x=b)}_{1}$

$= 2b$

$E[Y|x=2b] = \sum_{y\in Y} y\cdot p(y|x=2b) = b\cdot\underbrace{p(b|x=2b)}_{1} = b$

$E[Y|X] = \frac{1}{2}\cdot 2b + \frac{1}{2}\cdot b = b + \frac{b}{2} = \frac{3b}{2}$

- JENSEN INEQUALITY

$$E[f(x)] \geq f\{E[x]\}$$

$$Z = f(x) \Rightarrow E[Z] = E[f(x)] \geq f\{E[x]\}$$

$$E[Z] = x\cdot p(x) + y(1-p(x)) = x\cdot\frac{e^{-x}}{e^{-x}+e^{x}} + y\left(1-\frac{e^{-x}}{e^{-x}+e^{x}}\right)$$

$$= x\frac{e^{-x}}{e^{-x}+e^{x}} + y\cdot\frac{e^{x}+\cancel{e^{-x}}-\cancel{e^{-x}}}{e^{-x}+e^{x}} = \frac{1}{e^{-x}+e^{x}}\left[xe^{-x}+ye^{x}\right]$$

$$y = \begin{cases} 2x & p=\frac{1}{2} \\ \frac{x}{2} & p=\frac{1}{2} \end{cases}$$

$$E[z] = \frac{1}{2(e^{-x}+e^{x})}\left[xe^{-x}+2xe^{x}\right] + \frac{1}{2}\frac{1}{e^{-x}+e^{x}}\left[xe^{-x}+\frac{x}{2}e^{x}\right]$$

$$= \frac{x}{2(e^{-x}+e^{x})}\left(e^{-x}+2e^{x}\right) + \frac{x}{2(e^{-x}+e^{x})}\left[e^{-x}+\frac{e^{x}}{2}\right] =$$

$$= \frac{x}{2(e^{-x}+e^{x})}\left[e^{-x}+2e^{x}+e^{-x}+\frac{e^{x}}{2}\right] =$$

$$= \frac{x}{2(e^{-x}+e^{x})}\left[2e^{-x}+\frac{3}{2}e^{x}\right]$$

- M, TREBA DA GO SIMULIRAM KAKO TUKA SO KONDI !!

$$b(x) = \frac{e^{-x}}{e^{-x}+e^{x}}$$ } PROBABILITY OF SWITCHING & VIVS'MOST BETTING STRATEGY

$$W(b,p) = \sum p(x)\, ld\, p(x)\cdot o(x) \qquad o(x) = [1, 2]$$

$$\boxed{S_n} = 2^{n\cdot W(b,p)} = E[Z] \qquad z = \underline{S(x)} = \underline{b(x)\cdot o(x)}$$

$$\frac{1}{n} ld(S_n) = \frac{1}{n} ld \prod S(x) = \frac{1}{n}\sum ld S(x) \rightarrow \sum p(x)\, ld S(x) = W(p,b)$$

$$E[ld \underbrace{S(x)}_{=Z}] \doteq W(p,b) \qquad E[ld Z] = \underline{W(p,b)}$$

$$E[ld Z] \leq \underline{ld E[Z]} = W(p,b) \qquad S_n = 2^{n\cdot W(p,b)}.$$

$$W(Y,b) = \sum_{x} p(x)\, ld\big(\underline{\underline{o(x)}} \cdot b(x)\big) =$$

$$= \frac{1}{2}\, ld\, 1 \cdot \left(\frac{e^{-b}}{e^{-b}+e^{b}}\right) + \frac{1}{2}\, ld\, 2\, \frac{e^{-2b}}{e^{-2b}+e^{2b}} =$$

$$= \frac{1}{2}\, ld\, \frac{2e^{-3b}}{\underbrace{(e^{-b}+e^{b})(e^{-2b}+e^{2b})}_{\circledast}} = \frac{1}{2}\, ld\, \frac{2e^{-3b}}{e^{-3b}+e^{b}+e^{-b}+e^{3b}}$$

$$\circledast = e^{-3b}+e^{b}+e^{-b}+e^{3b}$$

$$(e^{-b}+e^{b})(e^{-b}+e^{b}) = e^{-2b}+1+1+e^{2b} = \underline{e^{-2b}+2+e^{2x}}$$

$$W^*(X|Y) = \sum p(x)\, ld\, o(x) - \underbrace{H(X|Y)}_{0}$$

$$W^*(X|Y) = \sum p(x)\, ld\, o(x) = \frac{1}{2} \cdot ld\, 1 + \frac{1}{2} \cdot ld\, 2 = \frac{1}{2}$$

$$W^*(X) = \sum p(x)\, ld\, o(x) - \underline{H(X)} = \frac{1}{2} - 1 = -\frac{1}{2}$$

$$W^*(X|Y) - W^*(X) = \frac{1}{2} - \left(-\frac{1}{2}\right) = \underbrace{1 = I(X;Y)}_{\text{PROOF OF CONCEPT}}$$

(c) J - INDEX OF ENVELOPE CONTAINING THE MAXIMUM AMOUNT

J' - INDEX OF THE ENVELOPE CHOSEN BY THE ALGORITHM

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t	X (1)	Y (2)
1	b	2b
2	b	2b
3	2b	b
4	b	2b
5	b	2b
6	b	2b
7	2b	b

$$Z_2 = \begin{cases} \textcircled{X} & 1-p(x) \\ Y & p(x) \end{cases}$$

$$p(x) = \frac{e^{-x}}{e^{-x}+e^{x}}$$

$$E[Z_2] = X \cdot \underline{\boxed{1-p(x)}} + Y \cdot p(x) =$$

$$= X\left(1 - \frac{e^{-\textcircled{x}}}{e^{-x}+e^{x}}\right) + Y \cdot \frac{e^{-x}}{e^{x}+e^{x}}$$

$$Z(t) = \frac{Xe^{-x}+Ye^{x}}{e^{-x}+e^{x}} \qquad Z_2 = \frac{Xe^{-b}+Y \cdot e^{x}}{e^{-x}+e^{x}}$$

$$W(b,1) = \sum p(x) \operatorname{ld} o(x) - \underbrace{H(p)}_{1}$$

b - bet $\quad b = p = \frac{1}{2}$

$$W(b,p) = \frac{1}{2}\operatorname{ld} 1 + \frac{1}{2}\overset{1}{\cancel{\operatorname{ld} 2}} - 1 = -\frac{1}{2}$$

- PRETPOSTAVKA AKO SEKOE VREME SE KLADIS NA
OVAA KONJA (TRKA) PODEDNAKVO. AMA TI NE SE
KLADIS TAKA TUKU SE KLADIS SOGLASNO NA VEROJATNOSTA
NA PLIKOTO X VO PRETPOSTAVKATA FUNDA.

NEKA: $b=1$, $t=2$

VO MOMENTOT $t=1$ VO PLIKOTO X IMALO $b=1\$$

A VO PLIKOTO Z IMALO $b=2\$$

$$p(x) = \frac{e^{-b}}{e^{-b}+e^{b}} = \frac{e^{-1}}{e^{-1}+e^{1}} = \frac{\frac{1}{e}}{\frac{1}{e}+e} = \frac{1}{1+e^2} \doteq \underline{\underline{0.12}}$$

$1 - p(x) = 0.88$

$$E[Z_2] = \frac{1}{2}\left[b \cdot 0.88 + 2b \cdot 0.12\right] + \frac{1}{2}\left[\underline{2b \cdot 0.88} + b \cdot 0.12\right]$$

$$= \frac{1}{2}\left[0.88 + 0.24\right] + \frac{1}{2}\left[1{,}76 + 0.12\right] =$$

$$= \frac{1}{2}\,1{,}12 + \frac{1}{2}\,1{,}88 = \frac{1}{2}\,[3] = \frac{3}{2}\ //$$

$$\begin{array}{r} 0.982 \\ 0.982 \\ \hline 1{,}964 \\ 0.018 \\ \hline 1{,}982 \end{array}$$

$\boxed{b=2}$ $\quad \boxed{p(x) = \overset{1{,}000}{0.018}}$ $\quad 1 - p(x) = 0.982 \quad \frac{18}{1000}$ $\quad \boxed{1-p(x) = 0.982}$

$$E[Z_2] = \frac{1}{2}\left[b \cdot 0.182 + 2b \cdot 0.018\right] + \frac{1}{2}\left[2b \cdot \underline{0.982} + b \cdot 0.018\right]$$

$$= \frac{b}{2}\left[1{,}018\right] + \frac{b}{2}\left[\underbrace{1.964 + 0.018}_{1.982}\right] = \frac{b}{2}\left[\underbrace{1.018 + 1.982}_{3}\right] = \frac{3b}{2}$$

(d) $\quad Z = \begin{cases} X & 1-p(x) \\ Z & p(x) \end{cases} \qquad Y = \begin{cases} b & p = 1/2 \\ 2b & p = 1/2 \end{cases}$

$p(Z|X) \qquad p(Z=b \mid X=2b) = 1 \qquad p(Y=2b \mid X=b) = 1$

$$Z = \begin{cases} X & 1-p(x) \\ 2X & p(x)\cdot/2 \\ X/2 & p(x)/2 \end{cases} \qquad p(Z) = \left\{1-p(x);\ p(x)/2;\ p(x)/2\right\}$$

$$Z = \left\{X,\ 2X,\ \frac{X}{2}\right\}?$$

$Z \backslash X,Y$	(X,Y)		(X,Y)		$P(Z)$
	b	$2b$	$2b$	b	▨
b	$(1-p(x))\cdot P(X=b)$		$p(Y=b)\cdot p(x)$		$P(Z=b)$
$2b$	$p(x)\cdot P(Y=2b)$		$[1-p(x)]\cdot P(X=2b)$		$P(Z=2b)$

$$P(Z=b) = \underline{P(X=b)}\,(1-p(x)) + P(X=2b)\cdot p(x)$$

$$P(Z=2b) = P(X=2b)\cdot(1-p(x)) + \underbrace{p(X=b)}_{= P(Y=2b)}\cdot p(x)$$

$Z \backslash X,Y$	$(X,Y)=(b,2b)$	$(X,Y)=(2b,b)$	$P(Z)$
b	$[1-p(x)]\cdot\frac{1}{2}$	$\frac{1}{2}\,p(x)$	$1/2$
$2b$	$p(x)\,\frac{1}{2}$	$[1-p(x)]\cdot\frac{1}{2}$	$1/2$
$P(X,Y)$	$1/2$	$1/2$	▩

$$E[Z] = b\cdot\frac{1}{2} + 2b\cdot\frac{1}{2} = \frac{3b}{2}$$

⇒ VIDI PODOLU ⊛ GO KAZUVA ISTOTO

(c) $J = \begin{cases} 1 & 1/2 \\ 2 & 1/2 \end{cases}$ $\qquad \xi_J = \{X, Y\}$

$$H(J) = \frac{1}{2}\,ld\,2 + \frac{1}{2}\,ld\,2 = 1$$

$J' = \begin{cases} 1 & 1-p(x) \\ 2 & \lambda(x) \end{cases}$

$$H(\xi=2b) = P(J=1)\cdot H(X=2b \mid J=1) + P(J=2)\cdot H(Y=2b \mid J=2)$$

RECALL THEO. 2.7 >

$X_1 \sim p_1 \qquad X_2 \sim p_2$

$\theta = \begin{cases} 1 & prob = \lambda \\ 2 & prob = 1-\lambda \end{cases}$

$Z = X_\theta$

$p(Z) = \lambda\cdot p_1 + (1-\lambda)\,p_2$ ⊛

$$H(Z) \geq H(Z|\theta)$$

$$H(Z|\theta) = P(\theta=1)\cdot H(p_1) + P(\theta=2)\cdot H(p_2)$$

$$H(\lambda p_1 + (1-\lambda)p_2) \geq \lambda\cdot H(p_1) + (1-\lambda)\cdot H(p_2)$$

ENTROPY IS CONCAVE AS A FUNCTION OF DISTRIBUTION.

$X_1 = X = \begin{cases} b & 1/2 \\ 2b & 1/2 \end{cases}$ $\qquad \underline{\underline{X_2 = Y}} = \begin{cases} 2b & 1/2 \\ b & 1/2 \end{cases}$

DA NEMA ZAVISI >G!

$$H(J') = (1-p(x))\,ld\,\frac{1}{1-p(x)} + p(x)\,ld\,\frac{1}{p(x)}$$

$$I(J, J') = H(J) - \underbrace{H(J|J')}_{H(J} = \underbrace{H(J') - \underbrace{H(J'|J)}_{=0}}_{H(J')}$$

$P(J,J')$

$J' \backslash J$	1	2	$P(J')$
1	$(1-\gamma)\frac{1}{2}$	$(1-\gamma)\frac{1}{2}$	$(1-\gamma)$
2	$\gamma\frac{1}{2}$	$\gamma\frac{1}{2}$	γ
$P(J)$	$1/2$	$1/2$	

$P(J|J') = P(J,J')/P(J')$

$J' \backslash J$	1	2	
1	$\frac{1}{2}$	$\frac{1}{2}$	
2	$\frac{1}{2}$	$\frac{1}{2}$	

$$H(J|J') = \sum_{J,J'} P(J,J')\cdot \operatorname{ld}\frac{1}{P(J|J')} = \cancel{2}(1-\gamma)\frac{1}{\cancel{2}} + \cancel{2}\gamma\frac{1}{\cancel{2}} = 1$$

$$I(J,J') = 1 - 1 = 0$$

- ALTERNATIVE

$P(J, J')$

$J' \backslash J$	1	2	$P(J')$
1	$(1-\gamma)$	0	$1-\gamma$
2	0	γ	γ
$P(J)$	$1-\gamma$	γ	▨

$P(J|J')$

$J' \backslash J$	1	2
1	1	0
2	0	1

$H(J|J')$

$$H(J|J') = \underbrace{P(J'=1)}_{(1-\gamma)}\cdot H(J|\underline{J'=1}) + \underbrace{P(J'=2)}_{\gamma} H(J|J') = H(J)$$

$$H(J'|J) = \underbrace{P(J=1)}_{\boxed{1/2}}\cdot H(J'|J=1) + \underbrace{P(J=2)}_{\boxed{1/2}} H(J'|\underline{J=2})$$

$$= H(J')$$

$$(x, y) = (2b, e)$$

$$P(Z) = \Big\{ \overbrace{[1-\gamma(x)]}^{P(Z=x)}\cdot P(\underline{x=b}),\ [1-\gamma(x)]\cdot P(\underline{x=2e}),\ \overbrace{\gamma(x)\cdot P(\underline{z=b})}^{P(z=z)},$$

$$\gamma(x)\cdot P(y=2e) \Big\} = \Big\{ \frac{1}{2}[1-\gamma(x)],\ \frac{1}{2}[1-\gamma(x)],\ \gamma(x)\cdot\frac{1}{2},\ \gamma(x)\frac{1}{2} \Big\}$$

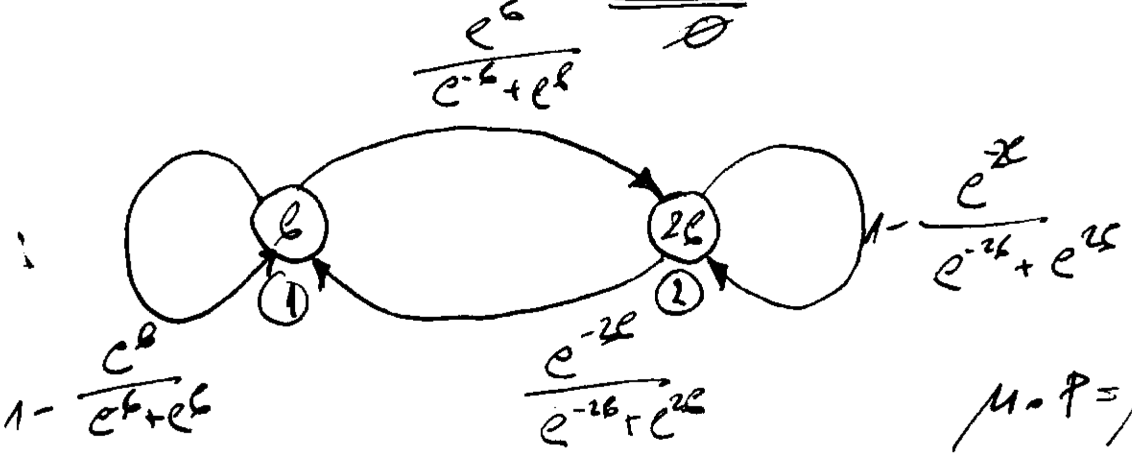
$$Z = \{ b, 2e, b, 2b \}$$

$$E[Z] = \frac{b}{2}(1-\gamma) + b(1-\gamma) + b\cdot\frac{\gamma}{2} + 2b\cdot\gamma = \frac{3e}{2}(1-\gamma) + \frac{3e}{2}\gamma$$

$$= 3b/2$$

$P(X,Y) = \{\frac{1}{2}, \frac{1}{2}\}$ $(X,Y) = \{b, 2b ; 2b, c\}$

$H(X,Y) = \underline{H(X)} + \underbrace{H(Y|X)}_{\emptyset} = \underline{H(X) = H(Y)}$



$\mu \cdot P = \mu$

$$\boxed{P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}}$$

$$[\mu_1\ \mu_2]\begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix} = [\mu_1, \mu_2]$$

$$[(1-\alpha)\mu_1 + \beta\mu_2 ; \alpha\cdot\mu_1 + (1-\beta)\mu_2] = [\mu_1\ \mu_2]$$

$\mu_1 = (1-\alpha)\mu_1 + \beta\mu_2$ $\mu_1(1-1+\alpha) = \beta\mu_2$ $\boxed{\alpha\cdot\mu_1 = \beta\mu_2}$

$\alpha\cdot\mu_1 + (1-\beta)\mu_2 = \mu_2$

$\alpha\mu_1 + \mu_2 - \beta\mu_2 = \mu_2$ $\boxed{\mu_1 + \mu_2 = 1}$

$\alpha\mu_1 = \beta(1-\mu_1)$ $\alpha\mu_1 = \beta - \beta\mu_1$ $\boxed{\mu_1 = \frac{\beta}{\alpha+\beta}}$

$$\boxed{\mu_2 = 1 - \frac{\beta}{\alpha+\beta} = \frac{\alpha}{\alpha+\beta}}$$

$X_1 \sim \mu = \{\mu_1, \mu_2\}$

$$H(X) = H(X_2|X_1) = P(X_1=1)\cdot H(X_2|X_1=1) + P(X_2=2)\cdot H(X_2|X_1=2)$$

$$= \mu_1 H(X_2|\mu=\mu_1) + \mu_2 H(X_2|\mu=\mu_2)$$

$$H(X_2|\mu=\mu_1) = P_{12}\cdot ld\frac{1}{P_{12}} + P_{11}\, ld\frac{1}{P_{11}} = \alpha\, ld\frac{1}{\alpha} + (1-\alpha)\, ld\frac{1}{1-\alpha}$$

$$H(X_2|X_1=2) = \overset{=\beta}{P_{21}}\, ld\frac{1}{P_{21}} + P_{22}\, ld\frac{1}{P_{22}}$$

$X_1 \backslash X_2$	1	2
1	P_{11}	P_{12}
2	P_{21}	P_{22}

$$\boxed{H(X) = H(X_2|X_1) = \sum_i \mu_i \sum_j P_{ij}\, ld\frac{1}{P_{ij}} = \sum_{i,j} \mu_i P_{ij}\, ld\frac{1}{P_{ij}}} = \underbrace{\frac{\beta}{\alpha+\beta}}_{\mu_1}\cdot H(\alpha) + \underbrace{\frac{\alpha}{\alpha+\beta}}_{\mu_2} H(\beta)$$

[redacted]

$$P_G^{up} = \cancel{2}\cdot\frac{1}{\cancel{2}}\left(\frac{2}{\rho^2+4}\right)^{N^2}\sum_{i=0}^{N^2-1}\frac{(N^2-1+i)!}{i!\,(N^2-1)!}\left(\frac{2}{\rho^2+4}\right)^{i} +$$

$$+\frac{1}{2}\left(\frac{2}{\rho^2+4}\right)^{N^2}\frac{(2N^2-1)!}{N^2!\,(N^2-1)!}\left(\frac{2}{\rho^2+4}\right)^{N^2} = \underline{\left(\frac{2}{\rho^2+4}\right)^{N^2}\sum_{i=0}^{N^2-1}\frac{(N^2-1+i)!}{i!\,(N^2-1)!}\left(\frac{2}{\rho^2+4}\right)^{i}}$$

$$\underline{+\frac{(2N^2-1)!}{2\cdot N^2!\,(N^2-1)!}\left(\frac{2}{\rho^2+4}\right)^{2N^2}}$$

$$P_{bas}^{up} = \frac{1}{2}\left(\frac{\sqrt{2}}{\rho}\right)^{2N^2} + \frac{1}{2}\left(\frac{\sqrt{2}}{\rho}\right)^{2N^2} = \left(\frac{\sqrt{2}}{\rho}\right)^{2N^2}$$

$$\hat{\epsilon} = P\cdot \textcircled{c} = P\cdot\frac{L}{K\cdot N} \qquad \bar{\gamma} = \rho\cdot\textcircled{c} \qquad \bar{\gamma} = \rho\cdot\frac{L}{K\cdot N}$$

MOETO $c =$

$$\textcircled{C} = \frac{\bar{\gamma}\cdot T}{N\cdot K} = \left|(2\times2\times2 - 222)\right| = \frac{\bar{\gamma}}{2} = \rho\left(\frac{T}{K\cdot N}\right)^2 = \frac{\rho}{4}$$

DO LEE GOLEMO $C =$

$$\underbrace{P_{bas}^{up} = \frac{1}{2}\left(\frac{2}{\rho\cdot\frac{\rho}{4}}\right)^{N^2} + \frac{1}{2}\left(\frac{8}{\rho^2}\right)^{N^2} = \left(\frac{8}{\rho^2}\right)^{N^2} = \left(\frac{2\sqrt{2}}{\rho}\right)^{2N^2}}_{\text{SAMO ZA } N=2}$$

- ZA $\boxed{C = \frac{\rho}{4}}$ VO GORNIOT IZRAZ SEKADE KADE IMAM ρ^2 STAVAM $\rho^2/4$:

$$\boxed{\begin{aligned} P_G^{up} &= \left(\frac{8}{\rho^2+16}\right)^{N^2}\sum_{i=0}^{N^2-1}\frac{(N^2-1+i)!}{i!\,(N^2-1)!}\left(\frac{8}{\rho^2+16}\right)^{i} + \frac{(2N^2-1)!}{2\cdot N^2!\,(N^2-1)!}\left(\frac{8}{\rho^2+16}\right)^{2N^2} \\ P_{bas}^{up} &= \left(\frac{2\sqrt{2}}{\rho}\right)^{2N^2} \end{aligned}}$$

$$\boxed{c = \frac{\rho}{N^2}}$$

- ZA OISTA VREDNOST i.e. $N\times N\times N$, $(K=T) \Rightarrow \boxed{C = \frac{\bar{\gamma}}{N}}$

$$P_{bas}^{up} = \left(\frac{2N^2}{\rho^2}\right)^{N^2}; \quad P_G^{up} = \left(\frac{2N^2}{\rho^2+4N^2}\right)^{N^2}\sum_{i=0}^{N^2-1}\frac{(N^2-1+i)!}{i!\,(N^2-1)!}\left(\frac{2N^2}{\rho^2+4N^2}\right)^{i}$$

$$+\frac{(2N^2-1)!}{2\cdot N^2!\cdot(N^2-1)!}\left(\frac{2N^2}{\rho^2+4N^2}\right)^{2N^2}$$

[redacted]

TOČEN IZRAZ KOJ ĆE GO KORISTAM VO ČLANAKOT ~~ZA~~ SPOREDBA NA REZULTATITE

$$C = \frac{\beta \cdot T}{N \cdot K} \; ; \quad \beta = 1 \qquad \boxed{C = \frac{T}{N \cdot K} = \left| \begin{matrix} 2\times2\times2 \\ 222 \text{ code} \end{matrix} \right| = \frac{1}{N}}$$

TOČNO!!!

$$\boxed{P_{bas}^{UP} = \left(\frac{2}{\varrho C}\right)^{n_t^s n_t^k} = \left(\frac{2N}{\varrho}\right)^{N^2}}$$

$$\boxed{P_B^{UP} = \left(\frac{2N}{\varrho + 4N}\right)^{N^2} \sum_{i=0}^{N^2-1} \frac{(N^2-1+i)!}{i!\,(N^2-1)!} \left(\frac{2N}{\varrho+4N}\right)^{i}}$$

OVIE IZRAZI SE KORISTAT VO SIMULACIIE 222 - 2×2×2

- VO GENERALIZEN SLUČAJ ~~ZA~~ BILO KOJ TIP NA OSTBC CODE SE KORISTI: $(T \overset{\Delta}{=} L)$ $\boxed{C = \frac{L}{N \cdot K}}$

$$P_{bas}^{UP} = \left(\frac{2\,N \cdot K}{\varrho \cdot L}\right)^{N^2} ; \quad P_B^{UP} = \left(\frac{2NK}{\varrho L + 4NK}\right)^{N^2} \sum_{i=0}^{N^2-1} \frac{(N^2-1+i)!}{i!\,(N^2-1)!} \left(\frac{2NK}{\varrho L+4NK}\right)^{i}$$

- VO NADGENERALIZEN SLUČAJ ZA BILO KOJA N×[redacted] 1×N KONFIGURACIJA VAŽI:

$$\boxed{P_B^{UP} = \left(\frac{2N \cdot K}{\varrho \cdot T + 4NK}\right)^{N} \sum_{i=0}^{N-1} \frac{(N-1)!}{i!\,(N-1)!} \left(\frac{2N \cdot K}{\varrho T + 4NK}\right)^{i}}$$

$$\boxed{P_{Bas}^{UP} = \left(\frac{2NK}{\varrho T}\right)^{N}}$$

PROBLEM 5.32 BAD WINE ONE IS GIVEN SIX BOTTLES OF WINE. IT IS KNOWN THAT PRECISELY ONE BOTTLE HAS GONE BAD (TASTE TERRIBLE). FROM INSPECTION OF THE BOTTLES IT IS DETERMINED THAT THE PROBABILITY p_i THAT THE i-th BOTTLE IS BAD IS GIVEN BY $(p_1, p_2, \ldots, p_6) = \left(\frac{8}{23}, \frac{6}{23}, \frac{4}{23}, \frac{2}{23}, \frac{2}{23}, \frac{1}{23}\right)$. TASTING WILL DETERMINE THE BAD WINE. SUPOSE THAT YOU TASTE THE WINES ONE AT TIME. CHOSE THE ORDER OF TASTING TO MINIMIZE THE EXPECTED NUMBER OF TASTING REQUIRED

[redacted]

3

(a) WHAT IS THE NUMBER OF TASTING REQUIRED?
(B) WHICH BOTTLE SHOULD BE TASTED FIRST?

NOW YOU GET SMART. FOR THE FIRST SAMPLE, YOU MIX SOME OF THE WINES IN A FRESH GLASS AND SAMPLE THE MIXTURE. YOU PROCEED, MIXING AND TASTING, STOPING WHEN THE BAD BOTTLE HAS BEEN DETERMINED.

(a) WHAT IS THE MINIMUM EXPECTED NUMBER OF TASTING REQUIRED TO DETERMINE THE BAD WINE?
(B) WHAT MIXTURE SHOULD BE TASTED FIRST

X — THE CHOSEN BOTTLE OF WINE IS BAD

p(x)					c(x)	l(x)
8/23	8/23	8/23	9/23	14/23 0	00	2
6/23	6/23	6/23	8/23 0	9/23 1	01	2
4/23	4/23	5/23 0	6/23 1		11	2
2/23	3/23 0	4/23 1			101	3
2/23 0	2/23 1				1000	4
1/23 1					1001	4

$3\cdot\frac{1}{4}+1\cdot\frac{1}{8}+2\cdot\frac{1}{16}$
$=\frac{3}{4}+\frac{1}{4}$
$=1$

THE LENGTH OF BINARY HUFFMAN CODE SHOULD SATISFY KRAFT REGULARITY WITH EQUALITY.

$$E[L(X)] = \sum_{X=1}^{6} l(x)p(x) = 2\left(\frac{8}{23}+\frac{6}{23}+\frac{4}{23}\right)+\frac{6}{23}+\frac{3}{23}\cdot 4$$

$$= 2\cdot\frac{18}{23}+\frac{6}{23}+\frac{12}{23} = \frac{36+6+12}{23} = \frac{42+12}{23} = \frac{54}{23} \doteq \underline{2.35}$$

$$H(X) = \sum_{X} p(x)\,ld\frac{1}{p(x)} = \underline{2.284} \qquad H(X) \leqslant E[L(X)] \leqslant H(X)+1$$

X — NUMBER OF TASTING

- FIRST APPROACH:
  - VEROJATNOST DEKA DO j-TO PROBUVANJE ĆE OTKRIEŠ LOŠO ŠIŠE VINO E:

$$\underbrace{\frac{23-8}{23}}_{\text{PRVOTO E DOBRO}}\cdot\underbrace{\frac{23-6}{23}}_{\text{VTOROTO E OK}}\cdot\frac{23-4}{23}\;\cdots$$

PRVOTO E DOBRO ∟ VTOROTO E OK - - -

(1°) PRVOTO ŠIŠE E LOŠO: $X=1$ $\quad P(X=1) = 8/23 = 0.35$

(2°) VTOROTO ŠIŠE E LOŠO: $X=2$ $\quad P(X=2) = \frac{23-8}{23}\cdot\frac{6}{23} = \frac{90}{23^2}$ $= 90/529 = 0.17$

— OVA NEMA VRSKA TREBA PRVO DA PRESMETAŠ $E[L(X)]$!!!

[redacted]

- VEROJATNOST DEKA i-TO ŠIŠE E LOŠO:

$$p_i = \left\{ \frac{8}{23}, \frac{6}{23}, \frac{4}{23}, \frac{2}{23}, \frac{2}{23}, \frac{1}{23} \right\}$$

$$l = \{1, 2, 3, 4, 5, 6\} \rightarrow \text{SLUČAJNATA PROMENLIVA.}$$

- SREDNATA DOLŽINA NA OTKRIVANJE NA LOŠO VINO E

$$E[l] = \sum_{i=1}^{6} p_i l_i = \frac{8}{23} + 2 \cdot \frac{6}{23} + \cdots \; 6 \cdot \frac{1}{23} = 2.348$$

- PRVO TREBA DA SE PROBA PRVOTO ŠIŠE, PA VTOROTO, PA TRETOTO T.N.

X					$l(x)$	PROBA $l'(x)$	P
1	8/23	I(0)	Ia/0	00	2	1	14/23
2	6/23		Ib(1)	01	2		
3	4/23	II(1)	IIa(0)	100	3	2	6/23
4	2/23			101	3		
5	2/23		IIb(1)	110	3	3	3/23
6	1/23			111	3		

$$E[l'] = 1 \cdot \frac{14}{23} + 2 \cdot \frac{6}{23} + 3 \cdot \frac{3}{23} = \frac{14}{23} + \frac{12}{23} + \frac{9}{23} =$$

$$= \frac{26+9}{23} = \frac{35}{23} = 1.52$$

- EXPECTED NUMBER OF TASTING IS: 1.52
- THE MIXTURE WITH HIGHER PROBABILITY OF BAD WINE SHOULD BE TASTET FIRST

SOLUTION 2 SOLUTIONS

(a) NEMA POTREBA OD POSLEDNOTO PROBUVANJE:

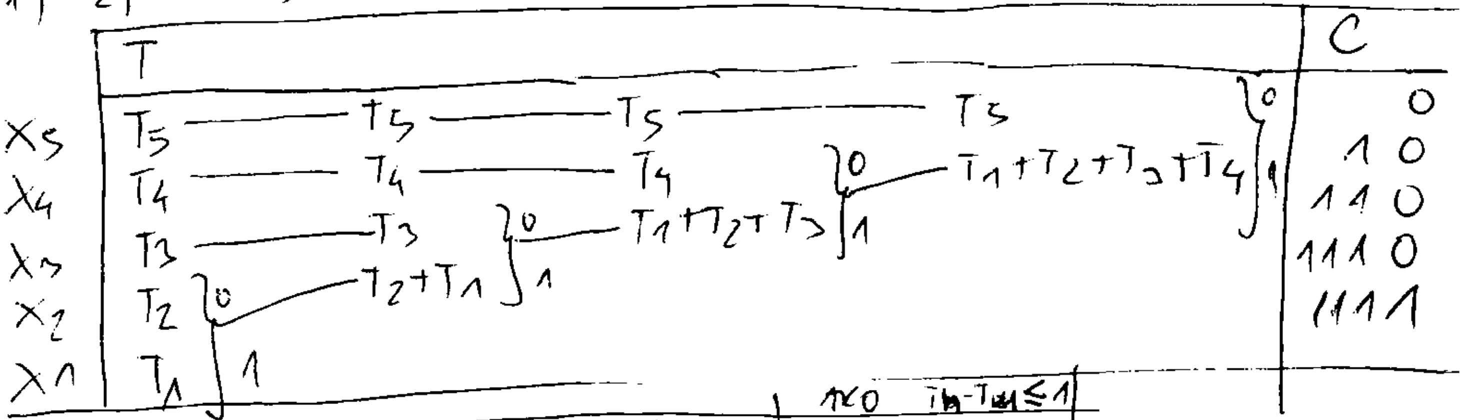
$$E[l] = \sum_{i=1}^{5} p_i l_i + p_6 \cdot 5 = \frac{50}{23} + \frac{5}{23} = \frac{55}{23} = 2.39$$

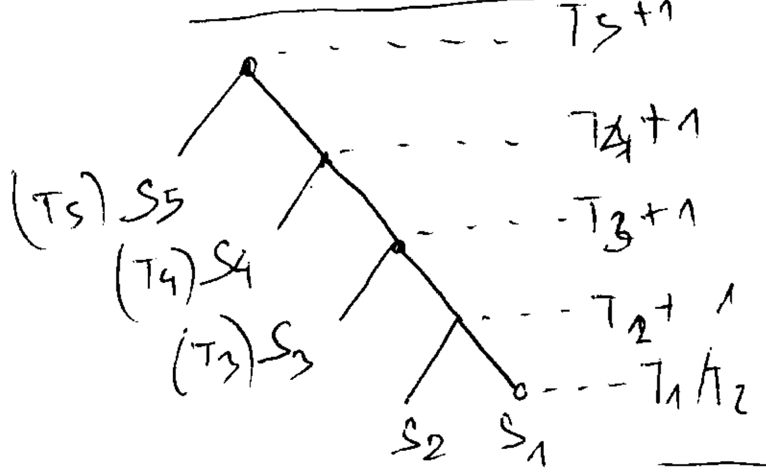
(b) SO KORISTENJE NA HUFFMANOVATA POSTAPKA OD PR.3 SE DOBIVA:

$$E[l] = \frac{54}{23} = 2.35$$

[redacted]

PROBLEM 5.34 (CONT) VO SPOREDBA SO N15a.25 SAKAM $T_1, T_2, \dots T_5$ DA GI PODREDAM VO OBRATEN REDOSLED





AKO $T_n - T_{n-1} \leq 1$
T_2+4
T_2+3
T_2+2
T_2+1

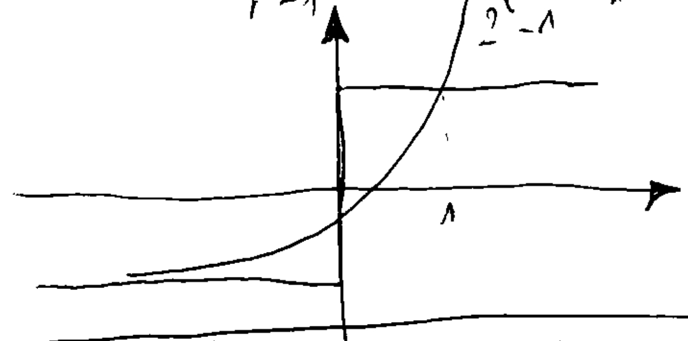
$T_2+(n-1) \leq T_2 \leq T_n+1$

- DA PRETPOSTAVAM DEKA T_i SE VEROJATNOSTI

$$H(X) \leq E[l(X)] \leq H(X)+1$$

$$E[l(X)] = \sum_{i=1}^{5} T_i l_i = T_5 \cdot 1 + T_4 \cdot 2 + T_3 \cdot 3 + (T_2+T_1) \cdot 4$$

$$\boxed{\operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}}$$

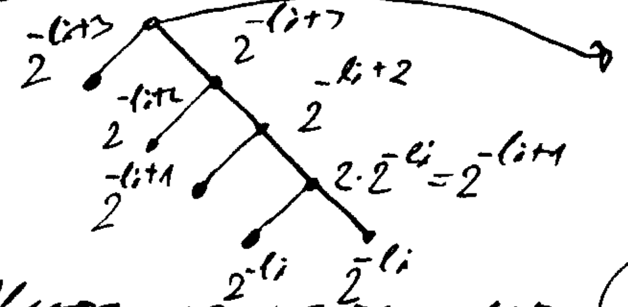


PROBLEM 5.35 EDITION 2 SOLUTIONS: WORD LENGHTS OF A BINARY HUFFMAN CODE MUST SATISFY THE KRAFT INEQUALITY (I PAS DOBOV DO OVAA KONSTATACIJA N15a), WITH EQUALITY. EVERY NODE IN THE TREE HAS A SIBLING (PROPERTY OF OPTIMAL BINARY CODE). IF WE ASIGNE EACH NODE A „WEIGHT", NAMELY, 2^{-l_i}, THEN 2×2^{-l_i} IS THE WEIGHT OF THE FATHER NODE.

[redacted]

THUS COLLAPSING THE TREE BACK WE HAVE:

$$\boxed{\sum 2^{-l_i} = 1}$$



$$2^{-l_i+4} = \sum_{i=1}^{4} 2^{-l_i} = 1$$

(a) (1,2,2) SATISFIES KRAFT EQUALITY, AND (2,2,3,3) DOESN'T. THUS (1,2,2) IS HUFFMAN CODE'S LENGTHS OF CODEWORDS, AND (2,2,3,3) ARE NOT.

PROBLEM 37 WHICH OF THE FOLLOWING CODES ARE
(a) UNIQUELY DECODABLE
(b) INSTANTENOUS

$C_1 = \{00, 01, 0\}$ $\qquad$ $C_2 = \{\overset{2}{\underline{\underline{00}}}, \overset{2}{\underline{\underline{01}}}, \overset{3}{100}, \overset{3}{101}, \overset{2}{11}\}$

$C_3 = \{0, 10, 110, 1110, \dots\}$ $\qquad$ $C_4 = \{0, 00, 000, 0000\}$

(a) SARDINAS-PETERSON ALGORITHM FOR UNIQUE DECODABILITY:

C_2 Seg: 00, 01, 100, 101, 11 } UNIQUELY DECODABLE
Seg: = = =

C_1 Seg 0: 00 01 0 IS NOT UNIQUELY DECODABLE
Seg 1: ⓪

C_3 ⇒ CLASICAL HUFFMAN CODE (UNIQUELY DECODABLE)

(b) $C_3 \in$ INSTANTENOUS (HUFFMAN CODE)

PROOF: C_2 IS ALSO ~~EVEN~~ INSTANTENOUS

100 ⟷ 00
101 ⟷ 01

~~[illegible]~~ POPOLNO SE GLEDA KAKO 100 MTE 800- VI GI 10KKEDIS!

$$\boxed{3 \cdot \frac{1}{4} + 2 \cdot \frac{1}{8} = \frac{3}{4} + \frac{1}{4} = 1}$$

$\{00, 01, 11, 100, 101\}$

PROBLEM 38 FIND THE HUFFMAN D-ARY CODE FOR:
$(p_1, p_2, p_3, p_5, p_6) = \left(\frac{6}{25}, \frac{6}{25}, \frac{4}{25}, \frac{4}{25}, \frac{3}{25}, \frac{2}{25}\right)$

(a) FOR $D=2$ $\qquad$ (b) FOR $D=4$

[redacted]

X	p						c(x)
x_1	p_1	6/25	6/25	6/25	13/25	13/25 } 0	10
x_2	p_2	6/25	6/25	6/25	6/25 } 0	12/25 } 1	11
x_3	p_3	4/25	5/25	8/25 } 0	6/25 } 1		000
x_4	p_4	4/25	4/25 } 0	5/25 } 1			001
x_5	p_5	2/25 } 0	4/25 } 1				010
x_6	p_6	2/25 } 1					011

6

PROBLEM 5.41 ~~[illegible]~~

OPTIMAL CODES: LET $l_1, l_2, \dots, l_{10}$ BE THE BINARY HUFFMAN CODE WORD LENGTHS FOR THE PROBABILITIES $p_1 \geq p_2 \geq \dots \geq p_{10}$. SUPPOSE THAT WE GET A NEW DISTRIBUTION BY SPLITTING THE LAST PROBABILITY MASS. WHAT CAN YOU SAY ABOUT THE OPTIMAL BINARY CODEWORD LENGTHS $\tilde{l}_1, \tilde{l}_2, \dots, \tilde{l}_{11}$ FOR PROBABILITIES $p_1, p_2, \dots p_9, \alpha p_{10}, (1-\alpha) p_{10}$ WHERE $0 \leq \alpha \leq 1$.

: EXAMPLE: $p_1 \geq p_2 \geq p_3$

X			c(x)
x_1	1/2	1/2 } 0	0
x_2	1/4 } 0	1/2 } 1	10
x_3	1/4 } 1		11

X				c(x)
x_1	1/2	1/2	1/2 } 0	0
x_2	1/4	1/4 } 0	1/2 } 1	10
x_3	1/8 } 0	1/4 } 1		110
x_4	1/8 } 1			111

$$E[l_1(x)] = 1\frac{1}{2} + 2\cdot\frac{1}{4} + (2\cdot\frac{1}{4} = \frac{1}{2} + 2\,\frac{1}{2} = \frac{3}{2} = 1.5$$

$$E[l_2(x)] = 1\cdot\frac{1}{2} + 2\cdot\frac{1}{4} + 3\frac{\cdot\alpha}{4} + 3\frac{(1-\alpha)}{4} = 1\cdot\frac{1}{2} + \frac{1}{2} + 3\cdot\frac{1}{4}$$

$$= \frac{1}{4} + \frac{3}{4} = \frac{7}{4}$$

$$= 1.75$$

$$\boxed{E[l_1(x)] \leq E[l_2(x)]}$$

$$\frac{7}{21} + \frac{6}{21} + \frac{5}{21} + \frac{4}{21}$$

$$\frac{1}{21} + \frac{2}{21} + \frac{3}{21} + \frac{4}{21} + \frac{5}{21} + \frac{6}{21} = 1$$

$S = 1 + 2 + \dots\ 4$

$S = n + n-1 + \dots\ 1$

$$\frac{n(n+1)}{2} = S$$

$S = 21$

$42 = n^2 + n$ } $n = 6$

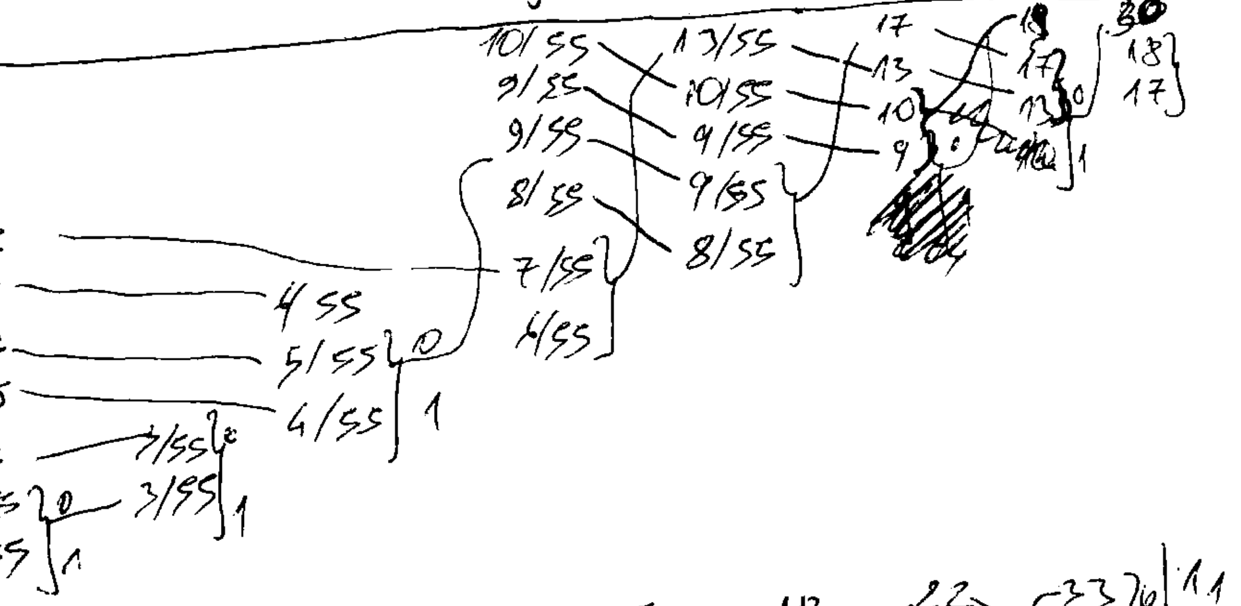
[redacted]

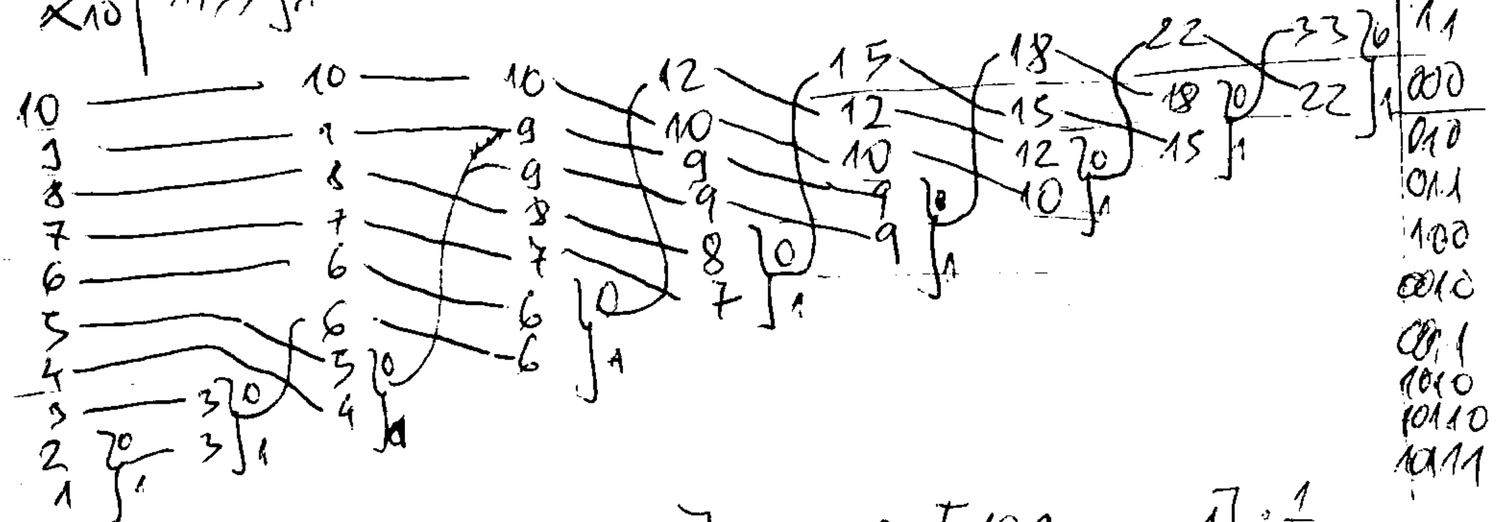
$\frac{n(n+1)}{2} = S$ $\quad n^2+n=2S$ $\quad n=6$, $36+6=2S$ $\boxed{S=21}$

$\boxed{n=10}$ $\quad 100+10=2S$ $\quad \boxed{S=55}$

$p = \left\{\frac{10}{55}, \frac{9}{55}, \frac{8}{55}, \dots \frac{1}{55}\right\}$

X	
x_1	10/55
x_2	9/55
x_3	8/55
x_4	7/55
x_5	6/55
x_6	5/55
x_7	4/55
x_8	3/55
x_9	2/55
x_{10}	1/55





$l = [2,3,3,3,3,4,4,4,5,5]$ $\qquad p=[10,9,\dots 1]\cdot\frac{1}{55}$

$\qquad\qquad\qquad\qquad\qquad\qquad \alpha\ (1-\alpha)$

$l_1 = [2,3,3,3,3,4,4,4,5,\boxed{6}\boxed{6}]$ $\qquad p_1=[10,9,\dots \frac{1}{2}\ \frac{1}{2}]\cdot\frac{1}{55}$

$$E[l_1] = \sum_{i=1}^{9} l(i)\,p(i) + \frac{6\cdot\alpha}{55} + \frac{6(1-\alpha)}{55} =$$

$$= \sum_{i=1}^{9} l(i)\,p(i) + \frac{6}{55} = \underbrace{\sum_{i=1}^{9} l_i p_i + \frac{5}{55}}_{l} + \frac{1}{55} = l + \frac{1}{55}$$

- VO GENERALEN SLUČAJ DOLŽINITE OSTANUVAAT ISTI NO SE MNOŽAT SO GENERALNI VEROJATNOSTI NA VEROJATNOSTITE:

$$E[l_1] = \sum_{i=1}^{9} l(i)\,p(i) + 6\cdot(1-\alpha)p_{10} + 6(\alpha)\cdot p_{10} = E[l] + p_{10}$$

[redacted]

$$E[l_1] = EL_1 = EL + p_{10} \qquad H_1(X) = ?$$

$$H(X) = \sum_{i=1}^{10} p_i \, ld \frac{1}{p_i} \qquad H_1 = \sum_{i=1}^{9} p_i \, ld \frac{1}{p_i} + \alpha p_{10} \, ld \frac{1}{\alpha p_{10}} +$$

$$+ (1-\alpha) p_{10} \, ld \frac{1}{(1-\alpha) p_{10}} = -\sum_{i=1}^{9} p_i \, ld\, p_i + \alpha p_{10} \, ld \frac{1}{\alpha p_{10}} + \underline{\underline{p_{10} \, ld \frac{1}{p_{10}}}} +$$

$$p_{10} \, ld \frac{1}{(1-\alpha)} - \alpha p_{10} \, ld \frac{1}{(1-\alpha) p_{10}} = -\sum_{i=1}^{10} p_i \, ld\, p_i + \underline{\underline{\alpha p_{10}}} \, ld \frac{1}{\alpha p_{10}}$$

$$+ p_{10} \, ld \frac{1}{1-\alpha} + \underline{\underline{\alpha p_{10}}} \, ld (1-\alpha) p_{10} = H(X) + \alpha p_{10} \, ld \frac{1-\alpha}{\alpha} +$$

$$+ p_{10} \, ld \frac{1}{1-\alpha} = H(X) + p_{10} \underbrace{\left[\alpha \, ld \frac{1-\alpha}{\alpha} + ld \frac{1}{(1-\alpha)} \right]}_{f(\alpha)}$$

$$H_1(X) = H(X) + p_{10} f(\alpha)$$

$$\frac{d f(\alpha)}{d \alpha} = 0 \Rightarrow \text{EXTREMA} \quad (\text{MAXIMUM})$$

$$f(\alpha) = \alpha \, ld \frac{1-\alpha}{\alpha} + ld \frac{1}{(1-\alpha)} = \underline{\alpha \, ld\, 1-\alpha} + \alpha \, ld \frac{1}{\alpha} + \underline{ld \frac{1}{(1-\alpha)}}$$

$$= \underline{-\alpha \, ld \frac{1}{(1-\alpha)}} + \alpha \, ld \frac{1}{\alpha} + \underline{ld \frac{1}{1-\alpha}} = (1-\alpha) \, ld \frac{1}{(1-\alpha)} + \alpha \, ld \frac{1}{\alpha} = H(\alpha)$$

$$H_1(X) = H(X) + H(\alpha) \qquad \text{FOR} \quad \alpha = \frac{1}{2} \Rightarrow \underline{H_{1\,max}(X) = H(X) + \frac{1}{2}}$$

- DOKAZ VO GENERALEN SLUČAJ :

$$p = \{ p_1 \; p_2 \; - - \; p_{10} \} \qquad l = \{ l_1 \; l_2 \; - - - \; l_{10} \}$$

$$p' = \{ p_1 \; p_2 \; - - \; \alpha p_{10}, p_{10} \} \qquad p = (1-\alpha)$$

$$l' = \{ l_1 \; l_2 \; - - - \; l_{10}+1, l_{10}+1 \}$$

$$E[l'] = \sum_{i=1}^{11} l_i \cdot p_i' = \sum_{i=1}^{9} l_i \cdot p_i + (l_{10}+1) p_{10} \alpha + \overset{1-\alpha}{(l_{10}+1)} p_{10}$$

$$= \sum_{i=1}^{9} l_i p_i + l_{10} p_{10} \alpha + \underline{p_{10} \alpha} + (1-\alpha) l_{10} p_{10} + \underline{(1-\alpha) p_{10}} =$$

$$= \sum_{1}^{9} l_i p_i + l_{10} p_{10} (\cancel{\alpha} + 1 - \cancel{\alpha}) + p_{10} = E[l] + p_{10}$$

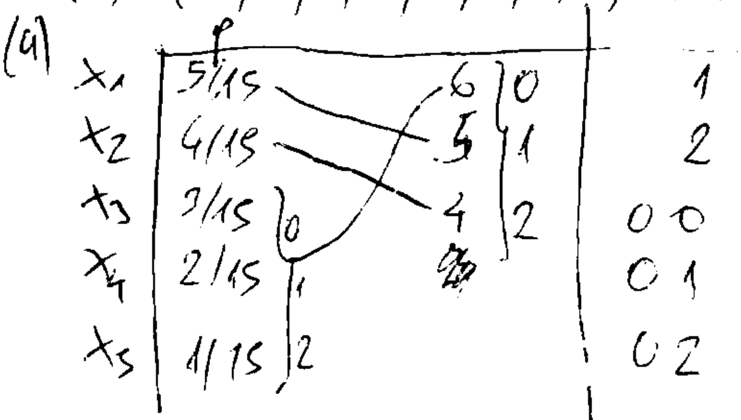
[redacted]

PROBLEM 5.42 WHICH OF THE FOLLOWING CODEWORD LENGTHS CAN BE A WORD LENGTHS OF A 3-ARY HUFFMAN CODE:

(a) (1, 2, 2, 2, 2)   {5, 3}

(b) (2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3)

(a)



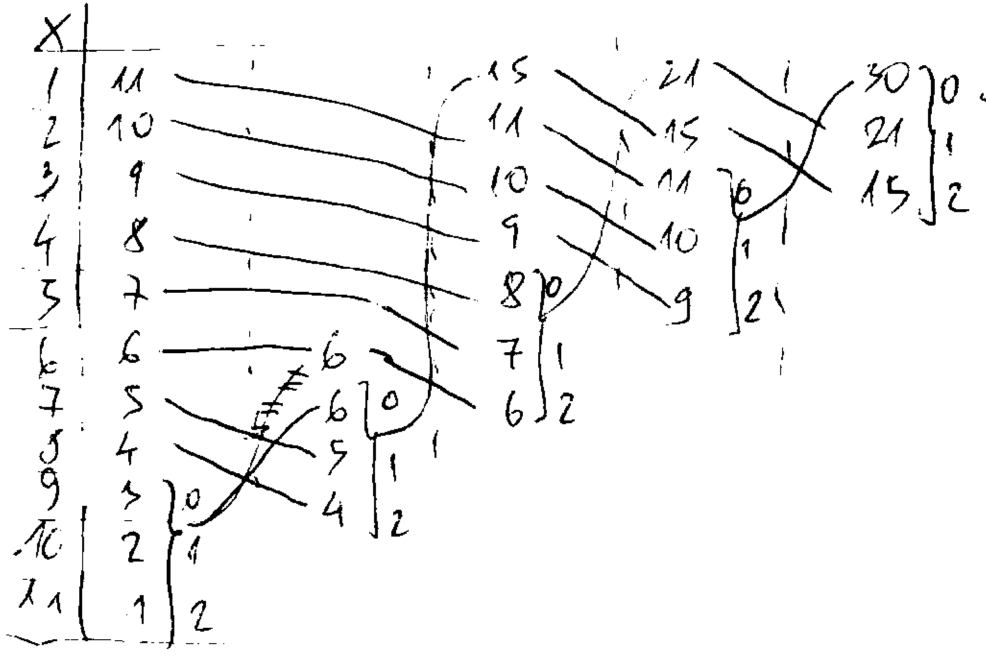
(1, 2, 2, 2, 2)

NE MOZE MERA ~~[illegible]~~ 2 CLENA DA DIDAT SO DOLZINA 1

$$\frac{11\cdot 12}{2} = 66$$

$\frac{11}{66}, \frac{10}{66}, \frac{9}{66}$ -

(b) {11, 9, 7, 5, 3}



	l
00	2
01	2
02	2
10	2
11	2
12	2
21	2
22	2
200	3
201	3
202	3

11 9 7 5

(2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3) — OVIE DOLZINI NA KODNI ZBOROVI MOZE DA PRETSTAVUVAAT DOLZINI NA KODOVI NA 3-AREN HUFMAN-OV KOD.

$\boxed{b}$

· EDITION 2 SOLUTIONS: (b) SATISFIES KRAFT INEQUALITY

$$\sum_i D^{l_i} = 1 \qquad 8\cdot 3^{-2} + 3\cdot 3^{-3} = \frac{8}{9} + \frac{3}{27} = \frac{24+3}{27} = 1$$

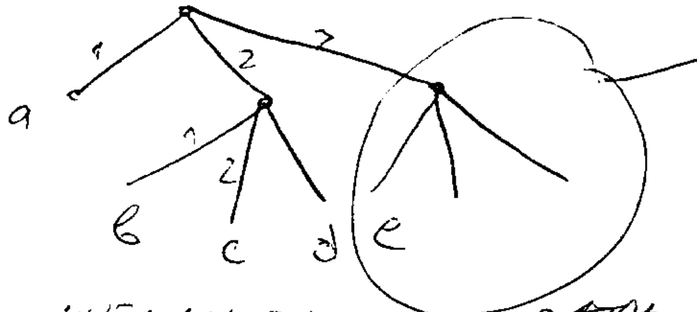
TEST ZA (a)

$$\sum_1 D^{-l_i} = 1\cdot 3^{-1} + 4\cdot 3^{-2} = \frac{1}{3} + \frac{4}{9} = \frac{3+4}{9} = \frac{7}{9}$$

[redacted]

11b

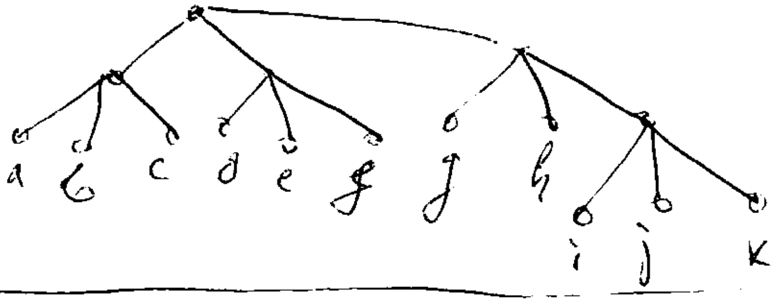
(a) $(1,2,2,2,2)$
a b c d e



OVA OVDE MOZE DA SE REDUCIRA NA VODEN ZBOR SO DOLZINA 1 ⇒ DOLZINITE $(1,2,2,2,2)$ NE SE

DOLZINI NA HUFFMAN-OV KOD ZATOA ŠTO POSTOI KOD $(1,2,2,2,1)$ KOD E POKRATOK

(b) $(2,2,2,2,2,2,2,2,3,3,3)$
a b c d e f g h i j k



PROBLEM 5.43 PIECEWISE HUFFMAN.

b

SUPPOSE THAT THE CODEWORD THAT WE USE TO DESCRIBE A RANDOM VARIABLE $X \sim p(x)$ ALWAYS STARTS WITH SYMBOL CHOSEN FROM THE SET $\{A, B, C\}$, FOLLOWED BY BINARY DIGITS $\{0, 1\}$. THUS WE HAVE TERNARY CODE FOR THE FIRST SYMBOL AND BINARY THEREAFTER. GIVE THE OPTIMAL UNIQUELY DECODABLE CODE (MINIMUM EXPECTED NUMBER OF SYMBOLS) FOR THE PROBABILITY DISTRIBUTION

$$p = \left(\frac{16}{69}, \frac{15}{69}, \frac{12}{69}, \frac{10}{69}, \frac{8}{69}, \frac{8}{69}\right)$$

x	p		$C(x)$	$l(x)$
x_1	16/69	A0	A0	2
x_2	15/69	B0	B0	2
x_3	12/69	C0	C0	2
x_4	10/69	A1	A1	2
x_5	8/69	B1	B1	2
x_6	8/69	C1	C1	2

$$E[l(X)] = 2\left(\sum_{i=1}^{6} p_i\right) = 2$$

$$H(X) = -\sum_{i=1}^{6} p(x)\log_6 p(x) = 0.979$$

$$y = \mathrm{lb}\, x$$

$$2^y = x \quad / \ln \qquad y\ln 2 = \ln x$$

$$\boxed{y = \frac{\ln x}{\ln 2}}$$

[redacted]

X	p					C	l(x)
x_1	16/69	16/69	22/69	31/69	A (B)	C	1
x_2	15/69	16/69	16/69	22/69	B (C)	A1	2
x_3	12/69	15/69	16/69 0	16/69	C (A)	B0	2
x_4	10/69	12/69 0	15/69 1			B1	2
x_5	8/69 0	10/69 1				A0 0	3
x_6	8/69 1					A0 1	3

SOLUTIONS 2

$$E[l(x)] = \sum_{i=1}^{6} p(x)\cdot l(x) = \frac{16}{69} + 2\left(\frac{37}{69}\right) + 3\frac{16}{69} =$$

$$= \frac{16+74+48}{69} = \frac{90+48}{69} = \frac{138}{69} = 2$$

- ALTERNATIVO:

X	p					C(x)
x_1	16	16	22	31	A	A0
x_2	15	16	16	22	B	A1
x_3	12	15	16 0	16	C	B0
x_4	10	12 0	15 1			B1
x_5	8 0	10 1				C0
x_6	8 1					C1

INICIJALNIOT
VID ŠTO GO DOBIV
NA Q77.12 E
DOBAR !!!

$$\boxed{E[l(x)] = 2}$$

$$H_5(X) = -\sum_{i=1}^{6} p(x)\log_5(p(x)) = 1\,09009$$

$$\boxed{1\,09 \le E[l(x)] \le 2.09}$$

- VO EDITION 2 SOLUTIONS ISTO ZA IMAT REŠENO NO TVRDAT DEKA VO GENERALEN SLUČAJ AKO E DADEN UNIQUELY DECODABLE CODE, MOŽE SO ISTA DOLŽINA NA KODNI ZBOROVI DA SE GENERIRA INSTANTENOUS CODE, NO TOA NE VAŽI ZA PIECEWISE HUFFMAN CODOT.

e

- CONTINUE FROM 37a!

$$E[L] = \min\left(\frac{7M + 6(N-M)}{N}\right) \qquad \frac{dE[L]}{dM} = 0$$

$$\frac{1}{100}(7-6) = 0 \quad ?$$

$$M = 2^{\lfloor ld\,100 \rfloor} + \sum_{i=1}^{n} 2^i$$

[redacted]

$$\frac{dE[L]}{dn} = \frac{1}{N}\left\{\frac{d}{dn}\left[7\sum_{i=1}^{n}2^{i}\right] - \frac{d}{dn}\left[6\sum_{i=1}^{n}2^{i}\right]\right\} = 0$$

$$S=\sum_{i=1}^{n}2^{i} = 2^{1}+2^{2}+2^{3}+\dots 2^{n} = 2^{n}\underbrace{\left(\frac{1}{2^{n-1}}+\frac{1}{2^{n-2}}+\dots\frac{1}{}\right)}_{S_0}$$

$$S_0=\sum_{i=0}^{n-1}2^{-i} = \frac{1-\frac{1}{2^{n}}}{1-\frac{1}{2}}$$

$$S_0 = 1+\frac{1}{2}+\dots\frac{1}{2^{n-1}}$$

$$2^{-1}S_0 = \frac{1}{2}+\frac{1}{2^{2}}+\dots\frac{1}{2^{n}} \Big| -$$

$$S_0 = 2\left(1-\frac{1}{2^{n}}\right) \qquad S = 2^{n}\cdot 2\left(1-\frac{1}{2^{n}}\right) = \underline{\underline{2^{n+1}-1}}$$

$$\frac{dE[L]}{dn} = \frac{1}{N}\left\{\frac{d}{dn}\left[7\left(2^{n+1}-1\right)\right] - \frac{d}{dn}\left[6\left(2^{n+1}-1\right)\right]\right\} = 0$$

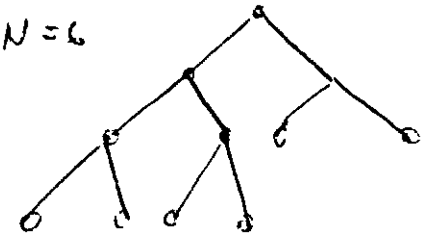
$$7\cdot\frac{d}{dn}\left(2^{n+1}\right) - 6\frac{d}{dn}\left(2^{n+1}\right) = 0 \qquad \frac{d}{dn}\left(2^{n+1}\right) = 0$$

$$\frac{d}{dx}\left(\alpha^{x}\right) = \frac{d}{dx}\left(e^{x\ln\alpha}\right) = \ln\alpha\cdot e^{x\ln\alpha} = \underline{\underline{\ln\alpha\cdot\alpha^{x}}}$$

$$\frac{d}{dn}\left(2^{n+1}\right) = 0 \quad \nRightarrow \quad 2\frac{d}{dn}\left(2^{n}\right) = 2\ln 2\cdot 2^{n} = \underline{\underline{\cancel{0}}}$$

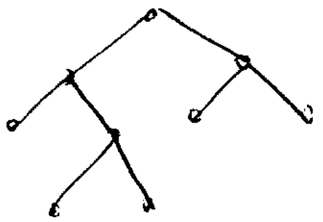
- NE E DOBRO!!! SE NAVRATIAM NA SIMULACIJATA VO MATLE. FORMULATA ZA $E[L]$ MORA DA E GENERIRA A NE SAMO ZA $f(A) = X.Y$ $Y \geq 0.5$!!!

N = 6



$\ell d6 = 2.585$

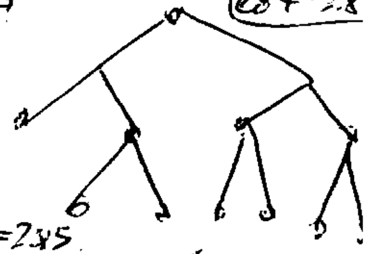
N = 5



$\boxed{\ell d5 = 2.32}$

N = 7

$\boxed{\ell d7 = 2.8}$

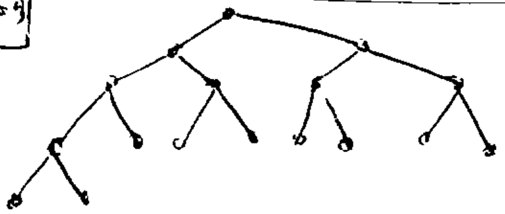


EL = 2.85

$M = 6$

$N-M = 7-6 = 1$

$\boxed{N=9}$



$$E[L] = 2\cdot 4\frac{1}{9} + 7\cdot 3\frac{1}{9} = \frac{8+21}{9} = \frac{29}{9} = \underline{\underline{3.22}}$$

[redacted]

• ЗНАЧИ ДОБИВАМ

(а) АКО ВО ПОСЛЕДНАТА ГРАНКА ОД ПРВОТО ИМА ПОВЕЌЕ ЛИСТА ОД БРОЈОТ НА ЛИСТА ВО ТРЕТОПОСЛЕДНАТА:

$$E[L] = \min \frac{\mathrm{ceil}(H)\cdot M + \lfloor H \rfloor \cdot (N-M)}{N} \qquad M = 2^{\lfloor H \rfloor} + \sum_{i=1}^{n} 2^i$$

(б) ВО ОПШТЕН СЛУЧАЈ:

$$E[L] = \min \frac{\lceil H \rceil \cdot M + \lfloor H \rfloor (N-M)}{N} \qquad M = 2 + \sum_{i=1}^{n} 2^i$$

$2, 4, 6, 8, 10, 12, 14, \ldots \Big\}$ АП

$a_0 = 2 \qquad a_1 = 2 + d = 2 + 2 = 4 \qquad a_n = a_{n-1} + 2$

$a_n = \underset{a_0}{2} + n\cdot d = a_0 + n\cdot d \qquad a_n - a_0 = nd$

$S = a_0 + a_1 + a_2 + \ldots \; a_n$

$S = a_0 + a_0 + d + a_0 + 2d + \ldots + a_0 + nd = (n+1)a_0 + d\,\frac{n(n+1)}{2}$

$$\downarrow$$

$$\frac{2(n+1)a_0 + dn^2 + dn}{2}$$

$a_0 = 2 \quad d = 2$

$$\boxed{S = 2(n+1) + \frac{n(n+1)}{2} \quad (*)}$$

$\Rightarrow a_{n-1} = a_{n-2} + d = a_0 + (n-1)d$

$S = a_0 + a_1 + a_2 + \ldots \; a_n$

$S = a_n + a_{n-1} + a_{n-2} + \ldots \; a_0$

$a_0 + a_n = 2a_0 + nd$

$a_1 + a_{n-1} = a_0 + d + a_0 + (n-1)d = 2a_0 + nd$

$2S = (n+1)(2a_0 + nd)$

$S = \frac{(n+1)(2a_0+nd)}{2} \quad (□)$

$$\downarrow$$

$$\boxed{\frac{2a_0n + n^2d + 2a_0 + nd}{2}}$$

$(*)\ \boxed{n = 10} \qquad S = 2\cdot 10 + \frac{10\cdot 11}{2} = 20 + 55 = 75$

$[2, 4, 6, 8, 10]$
$\;\; a_0\, a_1\, a_2\, a_3\, a_4$

$2\cdot[1, 2, 3, 4, 5] \Rightarrow S = \cancel{2}\cdot\frac{n(n+1)}{\cancel{2}} = 5\cdot 6 = 30$

$(□)\ S = \frac{(n+1)(2a_0 + nd)}{2} = \frac{5(2\cdot 2 + 2\cdot 4)}{2} = \frac{5\cdot(2+4)\cdot 2}{2} = 30$

$\boxed{S = 30} \qquad S = n(n+1) = 100 \qquad n^2 + n = 100$

[redacted]

$$h^2+h=100$$

$$E[\ell] = \min \frac{\lceil H\rceil \textcircled{M} + \lfloor H\rfloor (N-M)}{N}$$

$$M=2h; \qquad \frac{d\,E[\ell]}{dh} = \frac{d}{dh}\left[\frac{\underline{7\cdot 2h} + 6\cdot(100-2h)}{\cancel{N} \rightarrow 100}\right] = 0$$

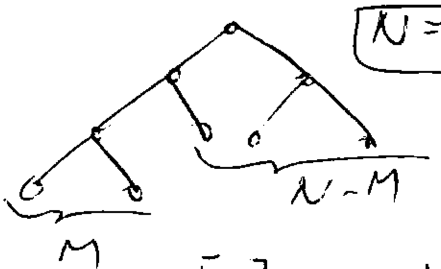
$$14 \; 12 = 0 \qquad ?$$

6

$$h = 1, 2, 3, 4, 5 \qquad M = \frac{h(h+1)}{}$$

$$M = \frac{2}{2}, \frac{6}{2}, \frac{12}{2}, \frac{20}{2}, \frac{30}{2} = \underline{1, 3, 6, 10, 15}$$

$$M = 2, 4, 6, 8, 10 \Rightarrow \text{FOR } \boxed{M=2h}$$



$$\boxed{N=5}$$

$$H = ld(5) = \textcircled{2.32} \qquad \lceil H\rceil = 3 \qquad \lfloor H\rfloor = 2$$

$$\frac{3\cdot 2 + 2\cdot(5-2)}{5} = \frac{6+6}{5} = \frac{12}{5} = 2.4$$

$$F = \frac{\lceil H\rceil M + \lfloor H\rfloor\cdot(N-M)}{N} + \lambda\left[E[\ell]-H\right]$$

$$E[\ell] \geq H(X) \qquad E[\ell]-H \geq 0 \qquad \underline{E[\ell]=H}$$

$$\frac{dF}{dh} = 0 \qquad \frac{d}{dh}\left[\frac{\lceil H\rceil\cdot 2h + \lfloor H\rfloor(N-2h)}{\cancel{N}} + \frac{\lambda\{\lceil H\rceil 2h + \lfloor H\rfloor (N-2h)}{N} \neq 0\right.$$

$$\underline{2\lceil H\rceil} - 2\lfloor H\rfloor + \underline{2\lambda\lceil H\rceil} - 2\lambda\lfloor H\rfloor = 0 \quad \Rightarrow$$

$$\cancel{2}(1+\lambda)\lceil H\rceil - \cancel{2}(1+\lambda)\lfloor H\rfloor = 0$$

$$\cancel{2}\lceil H\rceil - \cancel{2}\lfloor H\rfloor = \cancel{2}\lambda\underbrace{\left(\lfloor H\rfloor - \lceil H\rceil\right)}_{-1} \qquad \frac{\lceil H\rceil - \lfloor H\rfloor = 1}{\boxed{\lambda=-1}}$$

- <u>EDITION 2 SOLUTIONS</u>

→ VO $\lfloor H\rfloor$ NIVO IMA $2^{\lfloor H\rfloor} = 2^2 = 4$ JAZLI OD KOI $k=$ JAZLI NE SE LISTOVI I DOPOLNITELNO SE RAZGRANUVAAT SO 10, $2=$ LISTA.

- ZATOA: $\boxed{2K + 2^{\lfloor H \rfloor} - K = N}$ Ⓐ

ZA: $N = 5$ $H = ld\,5 = 2.32$ $\boxed{2K + 4 - K = 5}$

$K + 4 = 5$ $\boxed{K = 1}$ $\Rightarrow$ $2K = 2$ KOJI ZBOROVI
IMAAT DOLŽINA $\lceil H \rceil = 3$, A OSTANATITE $5 - 2 = 3$
IMAAT DOLŽINA $\lfloor H \rfloor = 2$.

- SLIČNO ZA $\boxed{N = 100}$ $H = ld(100) = 6.644$

Ⓐ $\rightarrow$ $2K + 2^6 - K = 100$ $K = 100 - 64 = 36$

ZNAČI $2K = 72$ KOI ZBOROVI ĆE IMAAT DOLŽINA
$7 = \lceil H \rceil$ A OSTANATITE 28 DOLŽINA $\lfloor H \rfloor = 6$

$$E[L] = \frac{72 \cdot 7 + 28 \cdot 6}{100} = \frac{504 + 168}{100} = \frac{672}{100} = \underline{6.72}$$

$\boxed{\text{PROBLEM 45}}$ RANDOM 20 QUESTIONS LET X BE UNIFORMLY DISTRIBUTED OVER $\{1, 2, \ldots m\}$. ASSUME $m = 2^n$. WE ASK RANDOM QUESTIONS: IS $X \in S_1$? IS $X \in S_2$? UNTIL ONLY ONE INTEGER REMAINS. ALL 2^m SUBSETS ARE EQUALLY LIKELY.

(a) HOW MANY DETERMINISTIC QUESTIONS ARE NEEDED TO DETERMINE X?

6

(b) WHAT IS THE EXPECTED NUMBER OF OBJECTS IN $\{2, 3, \ldots m\}$ THAT HAVE THE SAME ANSWERS TO THE QUESTIONS AS DOES THE CORRECT OBJECT 1

(a) [FROM TEXTBOOK] WITHOUT LOSS OF GENERALITY, SUPPOSE THAT $X = 1$ IS THE RANDOM OBJECT. WHAT IS THE PROBABILI-TY THAT OBJECT 2 YIELDS THE SAME ANSWERS FOR k QUESTIONS AS DOES OBJECT 1?

(c) SUPPOSE THAT WE ASK $n + \sqrt{n}$ RANDOM QUESTIONS. WHAT IS THE EXPECTED NUMBER OF WRONG ~~QUESTIONS~~ OBJECTS AGREEING WITH THE ANSWERS?

(d) USE MARKOV'S INEQUALITY $Pr\{X \geq t\mu\} \leq \frac{1}{t}$, TO SHOW THAT THE PROBABILITY OF ERROR (ONE OR MORE WRONG OBJECTS REMAINING) GOES TO ZERO AS $n \rightarrow \infty$.

(a) NEMA $\boxed{m = 3}$ $\{1, 2, 3\}$ SUBSETS: $2^m = 2^3 = 8$

11 21 31
12 22 32
13 23 33 $\}$ $3^2 = 9$

$\{1\ 2\}$ $\{2\ 1\}$ $\{3\ 1\}$
$\{1\ 3\}$ $\{2\ 3\}$ $\{3\ 2\}$

[redacted]

• $\{1, 2, 3\}$ НА ПРИМЕР $x \in 2$

1° ДАЛИ $x \in \{1, 2\}$? ДА!

2° ДАЛИ $x \in 2$ ДА!

Е

• ИЗГЛЕДА СУБСЕТИ НА $\{1, 2, 3\}$ СЕ

$\{0\}$ $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{1, 2, 3\}$ $2^3 = 8$

S_1 S_2 S_3 S_4 S_5 S_6

$$C^3_2 = \frac{4}{(4-k)!\ 4!} = \frac{3!}{1!\ 2!} = \frac{1 \cdot 2 \cdot 3}{2!} = 3 \Big\}$$ БЕЗ ПОВТОРУВАЊЕ

$$C^3_1 = \frac{4!}{(4-1)!\ 4!} = \frac{3!}{2!} = 3 \qquad C^3_3 = \frac{3!}{0!\ 3!} = 1$$

$$C^3_0 = \frac{3!}{3!} = 1 \qquad \boxed{3+3+1+1 = 8 = 2^3}$$

- БРОЈ НА ПОДМНОЖЕСТВА ВО ЕДНО МНОЖЕСТВО:

$$\boxed{N = \sum_{i=0}^{4} C^4_i}$$

$N(3) = 8 = 2^3$

$N(4) = 16 = 2^4$

$$\boxed{\sum_{i=0}^{4} \binom{4}{i} = 2^4} \rightarrow$$

$\boxed{\text{ММУ}}$ ОВА Е ФОРМУЛАТА ЗА БРОЈОТ НА ПОДМНОЖЕСТВА ВО ДАДЕНО МНОЖЕСТВО