3.330.082.816

Classical Fourier Transform for CT Signals

$$S(j\omega) = \int_{-\infty}^{\infty} s(t) \cdot e^{-j\omega t} dt$$

$$f = \frac{\omega}{2\pi}$$

$$s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(j\omega) e^{j\omega t} d\omega$$

$\mathcal{L}\{\ \}$

$\mathcal{F}\{\ \}$

$$\mathcal{F}\{\delta(t-t_0)\} = \int_{-\infty}^{\infty} \delta(t-t_0)\, e^{-j\omega t} dt = e^{-j\omega t_0}$$

$$\mathcal{F}^{-1}\{2\pi\, \delta(\omega-\omega_0)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\, \delta(\omega-\omega_0) \cdot e^{j\omega t} d\omega = e^{j\omega_0 t}$$

Properties of FT

1) Linearity (superposition)

$$\mathcal{F}\{a f_1(t) + b f_2(t)\} = a\mathcal{F}\{f_1(t)\} + b\, \mathcal{F}\{f_2(t)\}$$

2.) Time Shifting

$$\mathcal{F}\{f(t-t_0)\} = e^{-j\omega t_0} \mathcal{F}\{f(t)\}$$

3.) Frequency Shifting

$$e^{j\omega_0 t} f(t) = \mathcal{F}^{-1}\{F(j(\omega-\omega_0))\}$$

4.) Time-Domain Convolution

$$\mathcal{F}\{f_1(t) * f_2(t)\} = \mathcal{F}\{f_1(t)\}\mathcal{F}\{f_2(t)\}$$

5.) Frequency-Domain Convolution

$$\mathcal{F}\{f_1(t) \cdot f_2(t)\} = \frac{1}{2\pi} \mathcal{F}\{f_1(t)\} * \mathcal{F}\{f_2(t)\}$$

6.) Time Differentiation

$$-j\omega F(j\omega) = \mathcal{F}\left\{\frac{d f(t)}{dt}\right\}$$

7) Time Integration

$$\mathcal{F}\left\{\int_{-\infty}^{t} f(t) dt\right\} = \frac{1}{j\omega} F(j\omega) + \pi F($$

Fourier Spectrum of CT Sampled Signal

$$s_a(t) = \sum_{n=-\infty}^{\infty} s_a(t)\,\delta(t-nT) = \sum_{n=-\infty}^{\infty} s_a(nT)\,\delta(t-nT)$$

$$\mathcal{F}\{s_a(t)\} = \mathcal{F}\Big\{\sum_{n=-\infty}^{\infty} s_a(nT)\,\delta(t-nT)\Big] = \sum_{n=-\infty}^{\infty} s_a(nT)\,e^{-j\omega nT}$$

Dirichlet Condition:

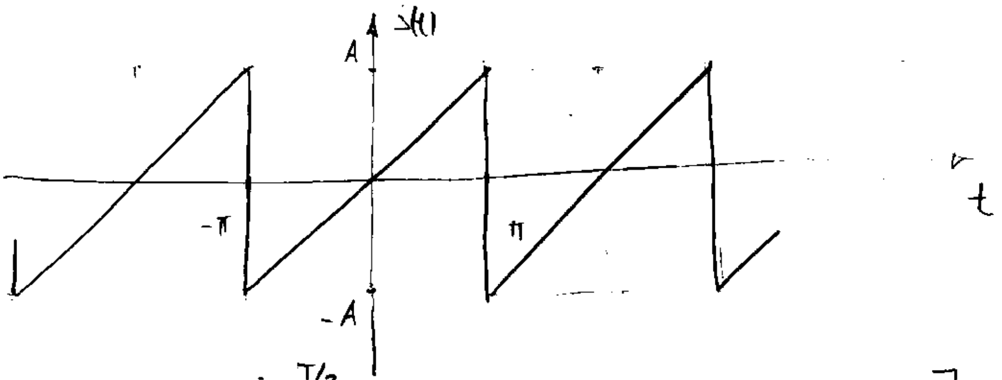
$$\int_{-T/2}^{T/2} |s(t)|\,dt < \infty$$

$$e^{+i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$

Exponential Fourier Series

$$s(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}$$

$$a_n = \frac{1}{T}\int_{-T/2}^{T/2} s(t)\,e^{-jn\omega_0 t}\,dt \qquad -\infty < n < \infty$$



$T = 2\pi$
$\omega_0 = 1$
$a_0 =$

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} A\cdot t\cdot e^{-0}\,dt = \frac{1}{T}\,A\left[\frac{T}{2}+\frac{T}{2}\right] = A$$

$$e^{jn\omega_0 t} = \cos(n\omega_0 t) + j\sin(n\omega_0 t)$$

$$a_1 = \frac{1}{T}\int_{-T/2}^{T/2} A\cdot t\,\left[\cos(\omega_0 t) + j\sin(\omega_0 t)\right]dt$$

$$\int \overset{\nearrow t}{u(t)} \cdot \overset{\nearrow \cos t}{v(t)}\, dt = v'(t) \int u(t)\, dt + u'(t) \int v(t)\, dt$$

$$\int t \cdot \cos t\, dt = \cos(t) \cdot \frac{t^2}{2} + t \cdot \sin(t)$$

$$\int u\, dv = u \cdot v - \int v\, du$$

$\underline{u = t \cdot \cos t}$

$$\int f(x)\, dx = \int f(g(t))\, g'(t)\, dt \qquad x = g(t)$$

пр. $I = \int (x+2) \sin(x^2 + 4x - 6)\, dx$

$x^2 + 4x - 6 = u$; $(2x + 4)dx = du$; $(x + 2)dx = \frac{1}{2} du$ (~~$x = \frac{1}{2} du - 2$~~)

$$I = \frac{1}{2} \int \sin u\, du = -\frac{1}{2} \cos u = -\frac{1}{2} \cos(x^2 + 4x - 6)$$

$f(x) = (x+2) \sin(x^2 + 4x - 6)$

$$\hookrightarrow \quad dF = f(x)\, dx = f(\boxed{g(t)})\, g'(t)\, dt$$

$$I = \frac{1}{2} \int \sin(x^2 + 4x - 6)\, d(x^2 + 4x - 6) = \int \sin u$$

~~$v = \cos t \quad dv = -\sin t\, dt$~~

$u = \cos t \qquad dv = t\, dt \qquad v = \frac{t^2}{2}$

Integration by parts:

$d(u \cdot v) = u\, dv + v\, du \qquad u \cdot v = \int u\, dv + \int v\, du$

$$\int u\, dv = u \cdot v - \int v\, du$$

$\underline{u = \cos t} \qquad \underline{dv = t\, dt} \qquad \boxed{v = \frac{t^2}{2}}$

$$\int t \cos t\, dt = \int \cos u \cdot dv = \cos t \cdot \frac{t^2}{2} - \int \frac{t^2}{2} \cdot \sin t\, dt$$

$$\int \frac{\cot(\ln x)}{x}\, dx = \qquad \ln' x = \frac{1}{x}$$

$$= \left| \ln x = u \quad \frac{1}{x}\, dx = du \right| = \int \cot(u)\, du$$

$$\underset{\int u\, dv}{\int x \ln(x+3)\, dx} = \left| \begin{matrix} u = \ln(x+3) \\ dv = x\, dx \\ v = \frac{x^2}{2} \end{matrix} \right| = \left| du = \frac{1}{x+3}\, dx \right| =$$

$$= \frac{x^2}{2} \cdot \ln(x+3) - \int \frac{x^2}{2} \cdot \frac{1}{x+3}\, dx = \frac{x^2}{2} \ln(x+3) - \frac{1}{2} \underbrace{\int \frac{x^2\, dx}{x+3}}_{*}$$

$$\circledast = \int \left(x - 3 + \frac{9}{x+3} \right) dx \quad \left| \frac{x^2 \cancel{-3x} \cancel{+3x} \cancel{-9} \cancel{+9}}{x+3} = \frac{x^2}{x+3} \right| =$$

$$= \frac{x^2}{2} - 3x + 9 \ln(x+3)$$

$$= \frac{x^2}{2} \ln(x+3) - \frac{1}{2} \left(\frac{x^2}{2} - 3x + 9 \ln(x+3) \right) =$$

$$(x \cos(x))' = \cancel{x} \sin x + \cos(x) \qquad (x^2)' = 2x$$

$$(x \cdot \cos x)' = x' \cdot \cos x + x \cdot \cos' x =$$
$$= 1 \cdot \cos x + x \cdot \sin x$$

$$\int x \cos x\, dx$$

$$u = x \cdot \cos x \qquad du = (\cos x + x \sin x)\, dx$$

$$(\cos t + t \sin(t))' = \cancel{-\sin t} + \cancel{\sin t} + t \cdot \cos t$$
$$= t \cdot \cos t$$

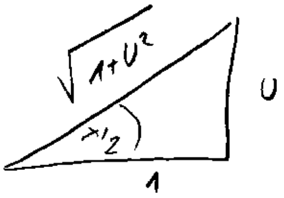
$$\int x \cos x \, \overset{dv}{dx} = \left| \begin{matrix} dv = x\,dx \\ u = \cos x \\ du = -\sin x\,dx \\ v = \frac{x^2}{2} \end{matrix} \right| = \frac{x^2}{2} \cdot \cos x - \int$$

$$\int u \cdot dv = v \cdot u - \int v\,du = \frac{x^2}{2}\cos x + \underbrace{\int \frac{x^2}{2} \sin x \, dx}_{*}$$

$$* = \int \frac{x^2}{2} \sin x \, dx = \left| \begin{matrix} dv = \frac{x^2}{2} dx \\ v = \frac{x^3}{6} \\ u = \sin x \end{matrix} \right.$$

$$\int x \cos x \, dx = \left| \begin{matrix} u = \underline{x \cos x} \\ du = (\cos x - x \sin x)\,dx \end{matrix} \right|$$

$$\int \frac{dx}{5 + 3\cos x} \qquad \tan\frac{x}{2} = u$$



$$\sin\frac{x}{2} = \frac{u}{\sqrt{1+u^2}} \qquad \cos\frac{x}{2} = \frac{1}{\sqrt{1+u^2}}$$

$$\sin(\alpha \pm \beta) = \sin\alpha \cdot \cos\beta \pm \sin\beta \cdot \cos\alpha$$

$$\cos(\alpha \pm \beta) = \cos\alpha \cdot \cos\beta \mp \sin\alpha \cdot \sin\beta$$

$$\cancel{\cos\left(\frac{x}{2} + \frac{x}{2}\right) = \cos\frac{x}{2}}$$

$$\cos(x) = \cos\left(\frac{x}{2} + \frac{x}{2}\right) = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$$

$$\cos x = \frac{1}{1+u^2} - \frac{u^2}{1+u^2} = \frac{1-u^2}{1+u^2}$$

$$du = \left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)' dx = \frac{\sin'\frac{x}{2} \cdot \cos\frac{x}{2} - \sin\frac{x}{2} \cdot \cos'\frac{x}{2}}{\cos^2\frac{x}{2}} =$$

$$= \frac{\frac{1}{2}\cos^2\frac{x}{2} + \frac{1}{2}\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} = \frac{1}{2 \cdot \cos^2\frac{x}{2}} = \underline{\frac{1}{2}\sec^2\frac{x}{2}\,dx}$$

$du = \frac{1}{2\cos^2\frac{x}{2}}\, dx$; $\quad dx = 2\cos^2\frac{x}{2}\, du = \frac{2}{1+u^2}\, du$

$$\int \frac{dx}{5+3\cos x} = \int \frac{dx}{5+3\,\frac{1-u^2}{1+u^2}} = \int \frac{\cancel{1+u^2}}{5+5u^2+3-3u^2}\,\frac{2}{\cancel{1+u^2}}\,du$$

$$= \int \frac{2\cdot du}{8+2u^2} = \int \frac{du}{4+u^2} = \int \frac{du}{2^2+u^2} = \frac{1}{2}\tan^{-1}\frac{u}{2} =$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\tan\frac{x}{2}\right) + c \qquad T\int x\,dx = \frac{x^2}{2} \quad \left(\frac{x^2}{2}\right)' = x$$

$$I = \int \frac{x\sin x}{1+\cos^2 x}\,dx = \left| x = \pi - y \right| = \int \frac{(\pi-y)\,\sin(\pi-y)}{1+\cos^2(\pi-y)} =$$

$$= \left| \begin{matrix} \sin(\pi-y) = \overset{\nearrow 0}{\cancel{\sin\pi}}\cdot\cos y - \overset{\nearrow -1}{\cancel{\cos\pi}}\cdot\sin y \\ = \sin y \end{matrix} \right| = \int \frac{(\pi-y)\,\sin y\,dy}{1+\cos^2 y} =$$

$$= \pi\int\frac{\sin y\,dy}{1+\cos^2 y} - \int\frac{y\sin y\,dy}{1+\cos^2 y} = -\pi\int\frac{d(\cos y)}{1+\cos^2 y} - I$$

$$= \left| \# \int\frac{dx}{1+x^2} = tg^{-1}x \right| = \underline{-\pi\ \mathrm{arctg}(\cos y) - I}$$

$$I = \frac{\pi}{2}\,\mathrm{arctg}(\cos y) = \frac{\pi}{2}\,\mathrm{arctg}(\cos(\pi - x))\Big|_0^{\pi}$$

$$= \frac{\pi}{2}\,\mathrm{arctg}(1) - \frac{\pi}{2}\,\mathrm{arctg}(-1) = \frac{\pi}{2}\cdot\frac{\pi}{4} + \frac{\pi}{2}\cdot\frac{\pi}{4} = \frac{\pi^2}{4}$$

+390592916284

$$I = \int x\cos x\,dx = \left| \begin{matrix} x = \pi - y \\ dx = -dy \end{matrix} \right| = -\int(\pi-y)\cos(\pi-y)\,dy = -\int\pi\cos(\pi-y)\,dy +$$

$$+ \int y\cos(\pi-y)\,dy \left| \begin{matrix} \cos(\pi-y) = \\ = \underset{-1}{\cos\pi}\cdot\cos y + \sin y\cdot\underset{0}{\cancel{\sin\pi}} \\ -\cos y \end{matrix} \right| = +\pi\int\cos y\,dy + \cancel{\int y\cos y\,dy} \nearrow I$$

$$2I = \pi \int \cos y \, dy = \sin y \qquad \boxed{I = \frac{\pi}{2} \sin y}$$

$$I = \int x \cos x \, dx = \left| \begin{array}{l} x = \pi - y \\ dx = -dy \\ \cos(\pi - y) = \\ = -\cos y \end{array} \right| = +\int (\pi - y) \cdot \cos y \, dy = \pi \int \cos y \, dy - \int y \cos y \, dy$$

$$x = \pi - y \quad \boxed{y = \pi - x}$$

$$I = \pi \int \cos y \, dy - I \qquad 2I = \pi \sin y = \pi \sin(\pi - x) =$$

$$= \pi \underset{\emptyset}{\cancel{\sin}} \pi \cdot \cos x - \pi \underset{-1}{\cancel{\cos}} \pi \cdot \sin x = \pi \cdot \sin x \qquad I = \frac{\pi}{2} \sin x$$

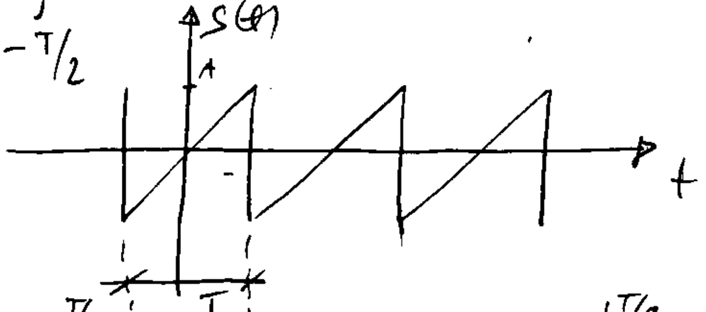
$$\int \underbrace{x}_{u} \cdot \underbrace{\cos x}_{v'} \, dx = u \cdot v' + \int v \cdot du = -x \cdot \sin x + \int \cos x \, dx =$$

$$= -x \sin x + \sin x$$

EXPONENTIAL FOURIER SERIES

$$s(t) = \sum_{n=-\infty}^{\infty} a_n e^{+jn\omega_0 t}$$

$$a_n = \frac{1}{T} \int_{-T/2}^{T/2} s(t) \cdot e^{-jn\omega_0 t} \, dt \qquad -\infty < n < \infty$$



$$s(t) = A \cdot t$$
$$-\frac{T}{2} \leq t \leq \frac{T}{2}$$
$$\boxed{\omega_0 = 1}$$

$$a_0 = \frac{1}{T} \int_{-T/2}^{T/2} A \cdot t \, e^{-0} dt = \frac{1}{T} \cdot \frac{t^2}{2} \Big|_{-T/2}^{T/2} = \frac{1}{2T} \left[\frac{T^2}{4} - \frac{T^2}{4} \right] = \emptyset$$

$$a_1 = \frac{1}{T} \int_{-T/2}^{T/2} A \cdot t \cdot e^{-jn\omega_0 t} dt = \cancel{\frac{1}{2T} A \cdot e^{j\omega t}} t^2 = \frac{1}{T} \int_{-T/2}^{T/2} A \cdot t \cdot e^{-jt} dt$$

$$= A \cdot e^{it} (1 - i \cdot t) \Big|_{-T/2}^{T/2} = A \left[e^{i\frac{T}{2}} (1 - i \tfrac{T}{2}) - e^{-i T/2} (1 + \right.$$

$$A_1 = A\cdot e^{jt}(1-jt) = A\cdot\left(\cos t + j\sin t\right)(1-jt)\Big|_{-T/2}^{\cancel{T/2}\nearrow\pi} =$$

$$= \cos t - jt\cos t + j\sin t + \cancel{j}\,t\cdot\sin t\Big|_{-\pi}^{\pi} =$$

$$= \overset{-1}{\cancel{\cos\pi}} - j\pi\cdot\overset{-1}{\cancel{\cos\pi}} + j\overset{0}{\cancel{\sin\pi}} + \cancel{j}\,\overset{0}{\cancel{\sin\pi}} -$$

$$-\left(\underset{-1}{\cancel{\cos(-\pi)}} + j\pi\underset{-1}{\cancel{\cos(-\pi)}} + j\overset{0}{\cancel{\sin(-\pi)}} + \underset{0}{\cancel{\sin(-\pi)}}\right) =$$

$$= -1 + j\pi - (-1 - j\pi) = \cancel{-1} + j\pi \cancel{+1} + j\pi = \underline{\underline{2j\pi}}$$

$s(t) = \frac{A}{\pi}\cdot t$

$$A_n = \frac{1}{T}\int_{-T/2}^{T/2}\frac{A}{\pi}\cdot t\,e^{-jnt}\,dt = \boxed{\frac{A\cdot e^{int}}{T\cdot\pi}\left(\frac{1}{n^2} - \frac{it}{n}\right)} =$$

$$= \frac{A}{T\cdot\pi}\left(\cos nt + j\sin nt\right)\left(\frac{\cancel{1} - int}{n^2}\right) =$$

$$= \frac{A}{T\cdot n^2\cdot\pi}\left(\cos(nt) + j\sin(nt)\right)\left(1 - j\cdot nt\right) = \frac{A}{T\cdot n^2\cdot\pi}\Big[\cos nt - jnt\cdot\cos(nt) +$$

$$j\underset{0}{\cancel{\sin nt}} + nt\cdot\underset{0}{\cancel{\sin(nt)}}\Big]\Big|_{-\pi}^{\pi}$$

$$\frac{A}{2\pi n^2\pi}\Big[\left(\underset{-1}{\cancel{\cos n\pi}} - jn\pi\cdot\cos n\pi\right) - \left(\underset{-1}{\cancel{\cos(-n\pi)}} + jn\pi\cdot\cos(-n\pi)\right)\Big)$$

$$= \frac{\cancel{2}\cdot A}{\cancel{2\pi}\,n^{\cancel{2}}\cdot\pi}\cdot j\,n\cancel{\pi}\cos(n\pi) = \underline{\underline{\frac{[illegible]\cdot A\cdot\cos(n\pi)}{n\pi j}}}$$

MATHEMATICA: $\dfrac{[illegible]2A\left(-\cancel{n\pi}\cos n\pi\right)}{\pi\cdot n^{2}\cancel{\pi}} = \underline{\underline{\dfrac{[illegible]A\cos(n\pi)}{jn\pi}}}$

$$s(t) = \sum_{n=-\infty}^{\infty} a_n \cdot e^{+jn\omega_0 t} \qquad a_n = \frac{1}{T}\int_{-\infty}^{\infty} s(t)\cdot e^{-jn\omega_0 t}\,dt$$

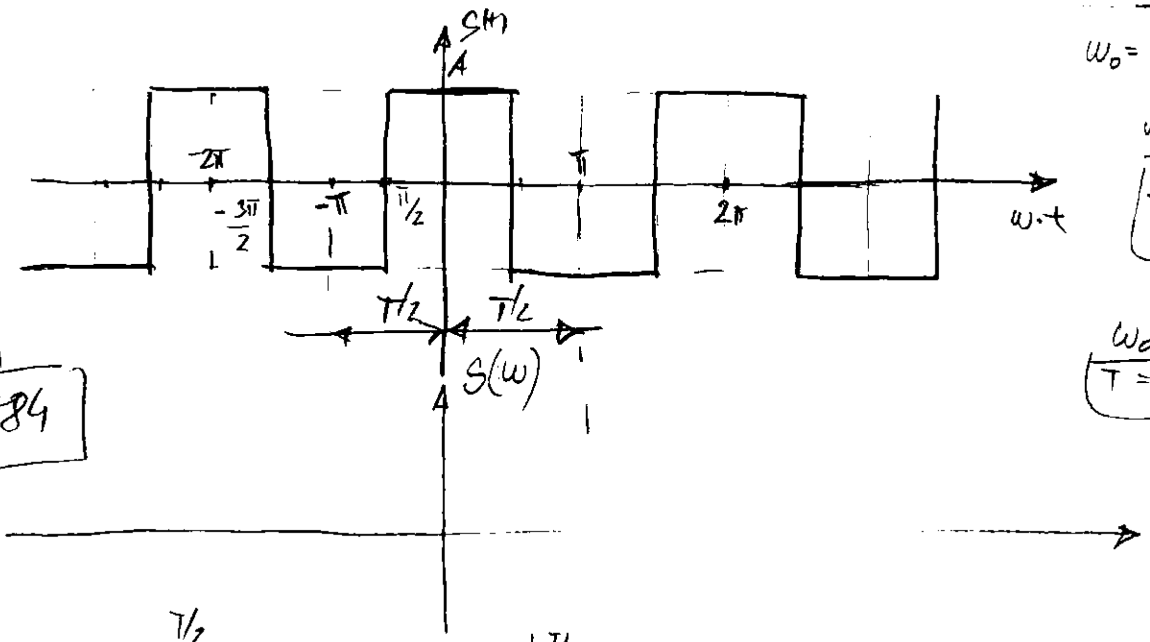
EXPONENTIAL FOURIER SERIES

Trigonometric Fourier Series

$$s(t) = \sum_{n=0}^{\infty} b_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} c_n \sin(n\omega_0)$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} s(t)\cos(n\omega_0 t)\,dt \quad n=1,2,\ldots\infty \;;\quad c_n = \frac{2}{T}\int_{-T/2}^{T/2} s(t)\sin(n\omega_0 t)\,dt \quad n=1,2,\ldots\infty$$

$$b_0 = \frac{2}{T}\int_{-T/2}^{T/2} s(t)\,dt$$



$\omega_0 = \frac{2\pi}{T}$

$\omega = \frac{2\pi}{t}$

$\boxed{t = \frac{2\pi}{\omega}}$

$\omega_0 = 1$

$\boxed{T = 2\pi}$

046
261784

$$\cancel{b_0 = \frac{2}{T}\int_{-T/2}^{T/2} A\cdot dt = \frac{2A}{T}\cdot t\Big|_{-T/2}^{T/2} = \frac{2A}{T}\left[\frac{T}{2}+\frac{T}{2}\right] = \frac{2A}{T}\cdot T}$$

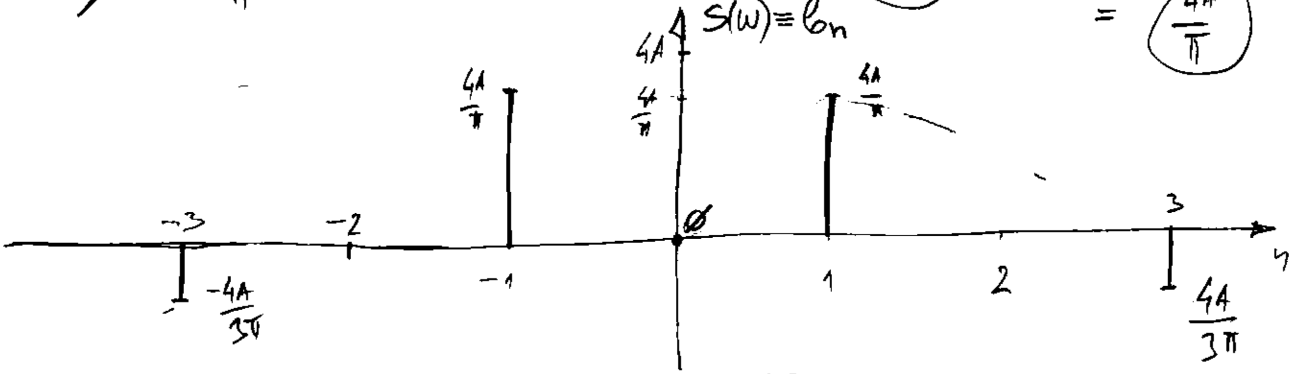
$$b_0 = \frac{2}{T}\int_{-T/2}^{T/2} s(t)\,dt = \frac{2}{T}\int_{-T/2}^{-T/4}(-A)\,dt + \frac{2}{T}\int_{-T/4}^{+T/4} A\,dt + \frac{2}{T}\int_{+T/4}^{T/2}(-A)\,dt$$

$$= \frac{2}{T}\left\{-A\left[\frac{-T}{4}+\frac{T}{2}\right] + A\left[+\frac{T}{4}+\frac{T}{4}\right] - A\left[\frac{T}{2}-\frac{T}{4}\right]\right\} =$$

$$= \frac{2}{T}\left[-\frac{AT}{4} + \frac{AT}{2} - \frac{AT}{4}\right] = 0$$

$$b_n = \frac{2A}{n\pi}\left[2\sin\left(\frac{n\pi}{2}\right) - \overbrace{\sin(n\pi)}^{\text{senosai } 0}\right] \qquad \boxed{b_n = 2A \cdot \frac{\sin\left(\frac{n\pi}{2}\right)}{\frac{n\pi}{2}}}$$

$$b_1 = \frac{2A}{\pi}\left[2 \cdot \cancelto{1}{\sin\frac{\pi}{2}} - \cancelto{0}{\sin\pi}\right] = \frac{4A}{\pi} \qquad b_{-1} = -\frac{2A}{\pi} \cdot 2\,\cancelto{-1}{\sin\left(-\frac{\pi}{2}\right)} = \frac{4A}{\pi}$$



$$b_2 = \frac{2A}{2\pi}\left[2 \cdot \cancelto{0}{\sin\left(\frac{2\pi}{2}\right)} - \cancelto{0}{\sin(2\pi)}\right] = 0 = b_{-2}$$

$$b_3 = \frac{2A}{3\pi}\left[2\,\cancelto{-1}{\sin\left(\frac{3\pi}{2}\right)} - \cancelto{0}{\sin(3\pi)}\right] = -\frac{4A}{3\pi}$$

$$s(t) = s_{even}(t) + s_{odd}(t)$$
$$s_{even}(t) = s_{even}(-t)$$
$$s_{odd}(t) = -s_{odd}(-t)$$

$$s_{even} = \frac{s(t)+s(-t)}{2} \approx \underset{(\cos)}{b_0}$$
$$s_{odd} = \frac{s(t)-s(-t)}{2} \approx \underset{(\sin)}{c_0}$$

$$s(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t} \qquad s(t) = \sum_{n=0}^{\infty} b_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} c_n \sin(n\omega_0 t)$$

$$e^{jn\omega_0 t} = \cos(n\omega_0 t) + j\sin(n\omega_0 t)$$
$$\sin(n\omega_0 t) = j\,\frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}$$

$$c_n = a_n e^{jn\omega_0 t} + a_{-n} e^{-jn\omega_0 t}$$
$$\boxed{a_n = \frac{b_n - jc_n}{2}} \qquad n = 1, 2, \dots n \qquad a_0 = b_0$$

$$a_n = \frac{1}{T}\int_{-T/2}^{T/2} s(t)\, e^{-jn\omega_0 t}\, dt = \frac{1}{T}\int_{-T/2}^{T/2} s(t)\left[\cos(n\omega_0 t) - j\sin(n\omega_0 t)\right] dt =$$

$$= \underbrace{\frac{1}{T}\int_{-T/2}^{T/2} s(t)\cos(n\omega_0 t)\, dt}_{b_n/2} - \underbrace{\frac{j}{T}\int_{-T/2}^{T/2} s(t)\cos(n\omega_0 t)\, dt}_{c_n/2} = \frac{b_n - jc_n}{2}$$

Fourier Transform of Periodic CT Signals (Fourier series)

$$F[s(t)] = F\left\{ \sum_{n=\infty}^{\infty} a_n e^{jn\omega_0 t} \right\} = 2\pi \sum_{n=-\infty}^{\infty} a_n \underline{\delta(\omega - n\omega_0)}$$

$$\mathcal{F}[s(t)] = \int_{-\infty}^{\infty} s(t)\cdot e^{-jn\omega t}\,dt$$

$$\mathcal{F}[\sin(\omega_0 t)] = \int_{-\infty}^{\infty} \sin(\omega_0 t)\cdot[\cos(\omega_0 t) + j\sin(\omega_0 t)]\,dt$$

$$\cos(\alpha+\alpha) = \cos^2\alpha - \sin^2\alpha \qquad \underline{\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha}$$

$$\sin(\alpha+\alpha) = \cos\alpha\cdot\sin\alpha + \sin\alpha\cdot\cos\alpha$$

$$\boxed{\sin(2\alpha) = 2\cdot\cos\alpha\cdot\sin\alpha}$$

$$\int e^{jn\omega_0 t}\,dt = \frac{1}{jn\omega_0}\int e^{jn\omega_0 t}\,d(jn\omega_0 t) = \frac{e^{jn\omega_0 t}}{jn\omega_0}$$

$$\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} a_n e^{jn\omega_0 t} e^{-jn\omega t}\,dt = \sum_{n=-\infty}^{\infty} a_n \int_{-\infty}^{\infty} e^{-jn(\omega-\omega_0)t}\,dt$$

DT Fourier Transform

$$s_a(t) = \sum_{n=-\infty}^{\infty} s_a(t)\,\delta(t-nT) = \sum_{n=-\infty}^{\infty} s_a(nT)\cdot\delta(t-nT)$$

$$\mathcal{F}[s_a(t)] = \mathcal{F}\left[\sum_{n=-\infty}^{\infty} s_a(nT)\,\delta(t-nT)\right] = \sum_{n=-\infty}^{\infty} \underline{s_a(nT)}\, e^{[j\omega T]^{-n}}$$

$\omega' = \omega T$ normalized frequency

$$S(e^{j\omega'}) = \sum_{n=-\infty}^{\infty} s[n]\,e^{-j\omega' n}$$

$$s[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S(e^{j\omega'})\,e^{jn\omega'}\,d\omega$$

$$\mathcal{F}[s_a(t)] = \mathrm{DTFT}\{s[n]\} \qquad \boxed{s[n] = s(t)\big|_{t=nT}}$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a & a & a & a \\ b & b & b & b \\ c & c & c & c \\ d & d & d & d \end{bmatrix}$$

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \begin{bmatrix} x_{11} y_{12} + x_{12} y_{21} ; \; x_{11} x_{12} + x_{12} y_{22} \\ \end{bmatrix}$$

$$\sin(2x) = \sin(x+x) = \sin x \cdot \cos x + \sin x \cos x =$$
$$= 2 \sin x \cdot \cos x$$

$$\sin x \cdot \cos x = \frac{1}{2} \cdot \sin 2x$$

$$y(n) = x(n-n_0)$$
$$m = n - n_0 \qquad \underline{n = m + n_0}$$
$$\underline{y(m+n_0) = x(m)}$$

$$\boxed{P = U_{ef} I_{ef}}$$
$$U_{ef} = \frac{U}{\sqrt{2}}$$
$$P = \frac{1}{2} U \cdot I = /R=1/ = \frac{U^2}{2}$$
$$\approx 0{,}5 \text{ W}$$

$\boxed{\text{Ex } 2.1}$ (a) $x(n) = 2\delta(n+2) - \delta(n-4), \quad -5 \le n \le 5$

(b) $x(n) = n[u(n) - u(n-10)] + 10e^{-0.3(n-10)}[u(n-10) - u(n-20)]$

$$\boxed{0 \le n \le 20}$$

0.57, 0.58, 0.335, 0,338

(c) $x(n) = \cos(0.04\pi n) + 0.2\, w(n), \quad 0 \le n \le 50$

(d) $\tilde{x}(n) = \{\ldots, 5, 4, 3, 2, 1, 5, 4, 3, 2, 1 \ldots\} \quad -5 \le n \le 5$

$\boxed{\text{Ex } 2.2}$ $x(n) = \{1, 2, 3, 4, 5, 6, 7, 6, 5, 4, 3, 2, 1\}$

a.) $x_1(n) = 2x(n-5) - 3x(n+4)$

b.) $x_2(n) = x(3-n) + x(n)x(n-2)$

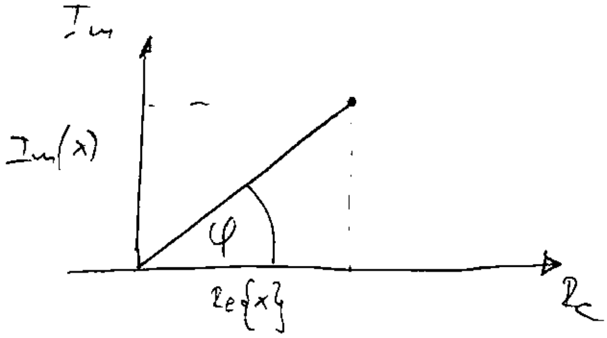
$x_{21}(n) = x(-(n-3))$

$\boxed{\text{Ex 2.3}}$

$$x(n) = e^{(-0,1 + j0.3)n} \qquad -10 \le n \le 10$$

$$e^{jx} = \cos x + j\sin x$$

$$e^{(-0,1 + j0.3)} = e^{-0,1}\cos(0.3n) + j\,e^{-0,1}\sin(0.3n)$$



$$\tan(\varphi) = \frac{\cancel{e^{-0,1}}\sin(0.3n)}{\cancel{e^{-0,1}}\cos(0.3n)}$$

$$\tan(\varphi) = \frac{\sin(0,3n)}{\cos(0,3n)}$$

$$z = x + jy$$

$$z = \sqrt{x^2+y^2}\cdot e^{j\varphi}$$

$$\varphi = \arctan\left(\frac{y}{x}\right)$$

$$\varphi = \arctan\left(\frac{\sin(0,3n)}{\cos(0,3n)}\right)$$

$$\boxed{\varphi = 0,3n}$$

$$z = 2 + j2 = \sqrt{8}\cdot e^{j\frac{\pi}{4}} = \sqrt{8}\left(\cos\frac{\pi}{4} + j\frac{\pi}{4}\right) =$$

$$= \sqrt{8}\left(\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{16}}{2}(1+j) = \frac{4}{2}(1+j) = \underline{2+j2}$$

$$x(n) = \underbrace{e^{-0,1n}}_{abs(x(n))}\cdot\underbrace{e^{j(0.3n)}}_{\varphi}$$

Even and odd synthesis

$$x_e(-n) = x_e(n) \qquad x_o(-n) = -x_o(n)$$

$$x(n) = x_e(n) + x_o(n)$$

$$x_e = \frac{1}{2}\left[x(n) + x(-n)\right] \qquad x_o = \frac{1}{2}\left[x(n) - x(-n)\right]$$

$\boxed{\text{Ex. 2.4}}$ $x(n) = u(n) - u(n-10)$ decompose to $x_o(n); x_e(n)$

$n : -2 \div 10$

$n_1 : -10 \div 2$

$n : -10 \div 10$

$n(1) = -2 \qquad m(1) = -10$ | $\boxed{nm = -2+10 = 8}$

$n1 = 1 : \text{length}(n) = 1 : 13$

$$\boxed{x(n) = x_e + x_o}$$

$$x_e(n) = \frac{1}{2}\left[x(n) + x(-n)\right] \qquad x_o = \frac{1}{2}\left[x(n) - x(-n)\right]$$

$$y(n) = LTI\left[x(n)\right] = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$y(n) \triangleq x(n) * h(n)$$

Ex. 2.5 $x(n) = u(n) - u(n-10)$

$h(n) = (0.9)^n u(n)$

$$y(n) = \sum_{k=0}^{9} (1)\cdot(0.9)^{n-k} u(n-k) = \boxed{(0.9)^n \sum_{k=0}^{9} (0.9)^{-k} u(n-k)}$$

CASE i $\underline{n<0}$: $u(n-k) = 0$, $\underline{0 \le k \le 9} \Rightarrow \underline{y(n) = 0}$

CASE ii $\underline{0 \le n < 9}$: $u(n-k) = 1$, $\underline{0 \le k \le n}$

$$y(n) = (0.9)^n \sum_{k=0}^{n} (0.9)^{-k} = (0.9)^n \sum_{k=0}^{n} (0.9^{-1})^k = (0.9)^n \frac{1-(0.9)^{-(n+1)}}{1-(0.9)^{-1}}$$

$$= \frac{(0.9)^n - (0.9)^{n-n-1}}{1 - \frac{10}{9}} = \frac{(0.9)^n - (0.9)^{-1}}{1-(0.9)^{-1}}\Bigg|\cdot(0.1) = \frac{(0.9)^{n+1} - 1}{(0.9) - 1} =$$

$$= \frac{(0.9)^{n+1} - 1}{\frac{9-10}{10}} = \underline{10\left[1-(0.9)^{n+1}\right]} \qquad 0 \le n < 9$$

case iii $n \ge 9 \quad u(n-k) = 1 \quad \underline{0 \le k \le 9}$

$$y(n) = (0.9)^n \sum_{k=0}^{9} (0.9)^{-k} = (0.9)^n \frac{1-(0.9)^{-10}}{1-(0.9)^{-1}} = (0.9)^n \frac{1-(0.9)^{-10}}{9-10}$$

$$= (0.9)^n \cdot 9\left[(0.9)^{-10} - 1\right] = (0.9)^n\cdot(0.9)\cdot 10\cdot(0.9)^{-10}\left[1-(0.9)^{10}\right]^9$$

$$\boxed{y(n) = (0.9)^{n-9} \cdot 10\left[1-(0.9)^{10}\right]} \quad \boxed{n \geq 9}$$

5.1.2250.2

$$x(n) = u(n) - u(n-10)$$
$$h(n) = (0.9)^n\, u(n)$$
$$y(n) \triangleq x(n) * h(n)$$

$$y(n) = \begin{cases} \emptyset ; & n<0 ;\ 0 \leq k \leq 9 \\ 10\left[1-(0.9)^{n+1}\right] ; & 0 \leq n < 9 ;\ 0 \leq k \leq n \\ (0.9)^{n-9} \cdot 10\left[1-(0.9)^{10}\right] ; & n \geq 9 ;\ 0 \leq k \leq 9 \end{cases}$$

ex 2.6

MMV

$$x(n) = [3, 11, 7, \overset{\downarrow}{0}, -1, 4, 2] \quad -3 \leq n \leq 3$$
$$h(n) = [2, \overset{\downarrow}{3}, 0, -5, 2, 1] \quad \boxed{-1 \leq n \leq 4}$$

$$y(n) \triangleq x(n) * h(n)$$

$$\boxed{y(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k)} \qquad h(-k) = [1, 2, -5, 0, \overset{\downarrow}{3}, 2]$$

$$y(-1) = \sum_{k} x(k)\, h(-1-k) = 3 \cdot (-5) + 7 \cdot 3 = -15 + 21 = 6$$

$$y(2) = \sum_{k} x(k)\, h(2-k) = 1 \cdot 11 + 2 \cdot 7 + 4 \cdot 3 + 2 \cdot 2 =$$
$$= 11 + 14 + 12 + 4 = 41$$

$$n1 = -3-1 = -4 \qquad n2 = 3+4 = 7$$

OVA MI 6 ~~SOMMITERMO~~
BI REKOL ~~TOKA~~ OBSEGOT 6
~~∞~~ -7 ÷ 4 !!

n =	-4	-3	-2	-1	0	1	2	3	4	5	6	7
x		3	11	7	0	-1	4	2				
$h(-k)$	1	2	-5	0	3	2						
$h(-k-4)$	3	2										
$h(-k-3)$	0	3	2									
$h(-k-2)$	-5	0	3	2								
$h(-k-1)$	2	-5	0	3	2							
$h(-k-0)$	1	2	-5	0	3	2						
$h(-k+1)$		1	2	-5	0	3	2					
$h(-k+2)$			1	2	-5	0	3	2				
$h(-k+3)$				1	2	-5	0	3	2			
$h(-k+4)$					1	2	-5	0	3	2		
$h(-k+5)$						1	2	-5	0	3	2	
$h(-k+6)$							1	2	-5	0	3	2
$h(-k+7)$								1	2	-5	0	3

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(-(k-n)) =$$
$$= \sum_{k=-\infty}^{\infty} x(k)\, h\_fold(k-n)$$

$$(*)\ h(-k-4) = h(-(k+4)) = h\_fold(k+4)$$
$$(**)\ h(-k+3) = h(-(k-3)) = h\_fold(k-3)$$

(*) = sigshift(h-fold, nh-fold, -4)

(**) = sigshift(h-fold, nh-fold, 3)

Sequence Correlation Revisited

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)\,y(n-l);$$ l - shif (lag) PARAMETER

AUTOCORELATION: $$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)\,x(n-l)$$

CONVOLUTION: $$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k); \quad y(n) \overset{\Delta}{=} x(n) * h(n)$$

$$\boxed{r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n)\,x(n-l)}$$

MENE MI
IZGLEDA MNOGO
SLIČNO SAMO
ŠTO NEMA
FOLD-NJE

$m = l-n; \quad n = l-m;$

$$r_{yx}(l) = \sum_{m=-\infty}^{\infty} y(l-m)\,x(-m) = \underline{\underline{y(l) * x(-l)}}$$

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)\,x(n-l) = \left| \begin{matrix} m = l-n \\ n-l = -m \\ n = l-m \end{matrix} \right| = \sum_{m=-\infty}^{\infty} x(l-m) \cdot x(-m)$$

$$= \underline{\underline{x(l) * x(-l)}}$$

$\boxed{\text{ex } 2.8}$ $\quad x(n) = [3, 11, 7, \underset{\uparrow}{0}, -1, 4, 2]$

$y(n) = x(n-2) + w(n)$ → nace corrupted and shifted version of $x(n)$

$w(n)$ – GAUSIAN SEQUENCE with $\overline{w} = 0 \quad \sigma_w = 1$

$r_{xy} \overset{\Delta}{=} x(l) * y(l)$

$$r_{xy} = \sum_{n=-\infty}^{\infty} x(n)\,y(n-l) = \left| \begin{matrix} m = l-n \\ n = l-m \end{matrix} \right| = \sum_{m=-\infty}^{\infty} x(l-m) * y(-m) = x(l) * y(l);$$

DIFERENCE EQUATIONS

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{m=0}^{M} b_m x(n-m), \quad \forall n$$

$$ⓧ \quad \boxed{y(n) = \sum_{m=0}^{M} b_m x(n-m) - \sum_{k=1}^{N} a_k y(n-k)}$$

⊙ $y(n) = y_H(n) + y_P(n)$ ⇒ - SOLUTION OF THE EQUATION

- $y_H(n) = \sum_{k=1}^{N} C_k z_k^n$ - HOMOGENOUS PART OF SOLUTION

$z_k, \; k = 1, 2 \dots N$ N roots (natural frequencies)

$$\sum_{0}^{N} a_k z^k = 0$$

$$\boxed{\text{If}: \; |z_k| < 1, \; k = 1, 2, \dots N \Rightarrow \text{system is stable}}$$

$y = \text{filter}(b, a, x)$

$b = [b_0, b_1, \dots, b_M]$; $\quad a = [a_0, a_1, \dots, a_N]$

$$y(n) - y(n-1) + 0.9\, y(n-2) = x(n); \quad \forall n$$

a.) $h(n) = ?$ (calculate & plot) at: $n = -20, \dots, 100$

b.) $s(n) = ?$ (unit step response) at: $n = -20, \dots, 100$

c.) Is the system specified with $h(n)$ STABLE ?

$$\overset{1}{a_0} \cdot y(n) + \overset{-1}{a_1} y(n-1) + \overset{0.9}{a_2} y(n-2) \dots a_N \cdot y(n-N) = \overset{①}{b_0} x(n) + b_1 x(n-1) + \dots + b_M x(n-M)$$

$a = [1, -1, 0.9]$; $\quad b = [1, 0, 0]$

---

c.) $a_0 \cdot z^0 + a_1 \cdot z^1 + a_2 z^2 = 0$

$1 - z + 0.9 z^2 = 0 \qquad \overset{a}{0{,}9 z^2} \overset{b}{-z} \overset{c}{+1} = 0$

$$z_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{+1 \pm \sqrt{1 - 4 \cdot 0.9 \cdot 1}}{2 \cdot 0.9} =$$

$$= \frac{1}{1.8} \pm \frac{1}{1.8} \sqrt{1 - 3{,}6} = \frac{1}{1.8} \pm i \frac{\sqrt{2.6}}{1.8} = 0.556 \pm i\, 0.8958$$

$x^2 + 2x + 1 = 0 \qquad x_{1,2} = \frac{-2 \pm \sqrt{4 - 4}}{2} = -1$

$r = \text{root}(c$

$c_1 s^n + \dots c_n s + c_{n+1} = 0$

$a_0 z^n + a_1 z^{n-1} + \dots a_n z^0 = 0$

$z_{1,2} = \frac{1 \pm \sqrt{1 - 4 \cdot 0{,}9}}{2} = \frac{1}{2} \pm i \frac{\sqrt{2.6}}{2} = 0{,}5 \pm i\, 0{,}8062$

$\boxed{\text{EXAMPLE 2.10}}$

$x(n) = u(n) - u(n-10)$ FINITE DURATION

$h(n) = (0.9)^n \cdot u(n)$ INFINITE DURATION

$y(n) = x(n) * h(n) = ?$

$$0.9\, h(n-1) = 0{,}9 \cdot (0{,}9)^{n-1} u(n-1) = (0{,}9)^n u(n-1)$$

$$h(n) - 0.9\, h(n-1) = (0{,}9)^n \cdot u(n) - (0{,}9)^n u(n-1) = (0{,}9)^n \big(u(n) - u(n-1)\big)$$

$$= (0{,}9)^n \cdot \delta(n) = \delta(n)$$

$y(n) - 0.9\, y(n) = x(n)$, $a = [1, -0.9]$; $b = [1]$

$y = \text{filter}\,[b, a, x]$

Digital Filters

1.) FIR (Finite Duration Impulse Response)

$$\boxed{h(n) = 0 \quad \begin{matrix} n < n_1 \\ n > n_2 \end{matrix}}$$

$$\boxed{y(n) = \sum_{m=0}^{M} b_m x(n-m)}$$

→ MA (Moving Average Filter)

PART OF THE DIFERENCE EQUATION DISCRIBING FIR FILTER.

$h(0) = b_0$; $h(1) = b_1$; .... $h(M) = b_M$

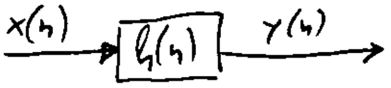
$$\sum_{m=0}^{M} h(m) \cdot x(n-m) \cancel{\begin{matrix} n-m=k \\ m=k-n \end{matrix} = \sum_{k} h(k-n)\, x(k)} = h(n) * x(n)$$

(Y) MOZE DA GO DOBIEŠ: a.) conv.m(x,h) b.) filter(b, 1, x)

2) IIR (Infinite duration Impulse Response)

$$\left.\sum_{k=0}^{N} a_k\, y(n-k) = x(n)\right\}$$ AUTOREGRESIVE FILTER

THE DISCRETE TIME FOURIER ANALYSIS



$$\boxed{y(n) = h(n) * x(n)}$$

DISCRETE TIME FOURIER TRANSFORM

$$\boxed{\begin{array}{l}
\bullet \text{ If: } \sum_{n=-\infty}^{\infty} |x(n)| < \infty \\
X(e^{j\omega}) \overset{\Delta}{=} \mathcal{F}[x(n)] = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \quad \boxed{\text{DTFT}} \ldots \; 3.1 \\
\bullet \; x(n) \overset{\Delta}{=} \mathcal{F}^{-1}[X(e^{j\omega})] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \quad \boxed{\text{IDTFT}} \ldots \; 3.2
\end{array}}$$

$\boxed{\text{EX. 3.1}}$ $x(n) = (0.5)^n u(n)$

$$X(e^{j\omega}) = \sum_{-\infty}^{\infty} x(n) e^{-j\omega n} = \sum_{0}^{\infty} (0,5)^n \cdot e^{-j\omega n} = \sum_{0}^{\infty} (0,5 \cdot e^{-j\omega})^n =$$

$$= \frac{1}{1 - 0.5 e^{-j\omega}} = \frac{e^{j\omega}}{e^{j\omega} - 0.5}$$

$\boxed{\text{ex. 3.2}}$ DTFT = ? $\quad x(n) = \{\overset{-1}{1}, \underset{\uparrow}{\overset{0}{2}}, \overset{1}{3}, \overset{2}{4}, \overset{3}{5}\}$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = \sum_{n=-1}^{3} x(n) e^{-j\omega n} = e^{j\omega} + \mathbf{2} + 3 \cdot e^{-j\omega} + 4 e^{-j2\omega} + 5 e^{-j3\omega}$$

- PROPERTIES OF DTFT

1.) PERIODICITY: $X(e^{j\omega})$ IS PERIODIC IN ω with $T = 2\pi$

$$X(e^{j\omega}) = X(e^{j(\omega + 2\pi)})$$

WE NEED ONLY PERIOD OF $X(e^{j\omega})$ $[\omega \in [0, 2\pi]$, OR $\omega \in [-\pi, \pi]]$

2.) SYMETERY: For REAL $x(n)$ $X(e^{j\omega})$ is conjugate SYMETRIC

$$X(e^{-j\omega}) = X^*(e^{j\omega})$$

$$\mathrm{Re}[X(e^{-j\omega})] = \mathrm{Re}[X(e^{j\omega})]; \quad \mathrm{Im}[X(e^{-j\omega})] = -\mathrm{Im}[X(e^{j\omega})];$$

$$|X(e^{-j\omega})| = |X(e^{j\omega})|; \quad \angle X(e^{-j\omega}) = -\angle X(e^{j\omega})$$

To PLOT $X(e^{j\omega})$ WE need only half period usualy $[0, \pi]$

$\boxed{\text{EX. 3.3}}$ EVALUATE $X(e^{j\omega})$ from ex3.1 in 501 equidistant points between $[0, \pi]$

$\boxed{n_1 \leq n \leq n_2}$ evaluate $X(e^{j\omega})$.

$$\boxed{\begin{array}{l}
\omega_k \overset{\Delta}{=} \frac{\pi}{M} k, \quad k = 0, 1, \ldots M \\
X(e^{j\omega_k}) = \sum_{\ell=1}^{N} e^{-j(\pi/M) \cdot k \cdot n_\ell} \cdot x(n_\ell) \quad k = 0, 1, \ldots, M
\end{array}}$$

$x(n_k) \rightarrow x$

$X(e^{j\omega_k}) \rightarrow \mathcal{X}$

$$\boxed{\mathcal{X} = W x}$$

$W - (M+1) \times N$ MATRIX

$$W \triangleq \left\{ e^{-j\left(\frac{\pi}{M}\right) k \cdot n_k} ;\ n_1 \leq n \leq n_N ;\ k = 0, 1, \ldots M \right\}$$

$$\boxed{W = \left[\exp\left(-j \frac{\pi}{M} k^T n\right) \right]} \qquad \boxed{\mathcal{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n}}$$

PR.: $\left.\begin{matrix} k = 0, 1, \ldots, M \\ n = 0, 1, \ldots N \end{matrix}\right\}$ $k^T \cdot n = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ M \end{bmatrix} [0, 1, \ldots N] = \begin{bmatrix} 0, 0, & \ldots & , 0 \\ 0, 1, & \ldots & , N \\ \cdots \\ 0, M \cdot 1, & \ldots & MN \end{bmatrix}_{M \times N}$

$$\begin{bmatrix} k_0 \\ k_1 \\ \vdots \\ k_M \end{bmatrix} [n_1 \ldots n_N] = \begin{bmatrix} k_0 n_1, & k_0 n_2 \ldots, & k_0 n_M \\ k_1 n_1, & k_1 n_2, \ldots, & k_1 n_N \\ - & - & - \\ k_M \cdot n_1, & k_M \cdot n_2 \ldots & k_M \cdot n_N \end{bmatrix}_{(M+1) \times N}$$

$$\mathcal{X}(e^{j\omega}) = \left[\exp\left(-j \frac{\pi}{M} \cdot k^T \cdot n\right) \right] \cdot x \qquad \begin{matrix} k = 0, 1, 2 \ldots M \\ n = n_1, n_2, \ldots n_N \end{matrix}$$

$$\begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \\ \vdots \\ \mathcal{X}_N \end{bmatrix} = \begin{bmatrix} k_0 \\ k_1 \\ \vdots \\ k_M \end{bmatrix} [n_1, n_2, \ldots n_N] [x_1, x_2, \ldots x_N] =$$

$$= \frac{-j\pi}{M} \begin{bmatrix} k_0 n_1, k_0 n_2, & \ldots & , k_0 n_N \\ k_1 n_1, k_1 n_2, & \ldots & , k_1 n_N \\ - - - & & - \\ k_M n_1, k_M n_2, & \ldots & , k_M n_N \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \frac{-j\pi}{M} \begin{bmatrix} k_0 n_1 x_1 + k_0 n_2 x_2 + \ldots & + k_0 n_N x_N \\ k_1 n_1 x_1 + k_1 n_2 x_2 + \ldots & + k_1 n_N x_N \\ - - - & - \\ k_M n_1 x_1 + k_M n_2 x_2 + \ldots & k_M n_N x_N \end{bmatrix}$$

$$\boxed{\mathcal{X}^T = x^T \left[\exp\left(-j \frac{\pi}{M} n^T \cdot k\right) \right]} = -[x_1, x_2, \ldots x_N] \frac{j\pi}{M} \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_N \end{bmatrix} [k_0, k_1, \ldots k_M] =$$

$$[x_1, x_2, \ldots x_N] \frac{-j\pi}{M} \cdot \begin{bmatrix} k_0 n_1, n_1 k_1, & \ldots & n_1 k_M \\ k_0 n_2, n_2 k_1, & \ldots & n_2 k_M \\ - - - & & \\ k_0 n_N, n_N k_1, & \ldots & n_N k_M \end{bmatrix} = \frac{-j\pi}{M} \left[\begin{matrix} k_0 n_1 x_1 + k_0 n_2 x_2 + \ldots k_0 n_N x_N, \\ k_1 n_1 x_1 + k_1 n_2 x_2 + \ldots k_1 n_N x_N, \\ - - - - \\ k_M n_1 x_1 + k_M n_2 x_2 + \ldots + k_M n_N x_N \end{matrix} \right]$$

ex. 3.4 Numericaly compute DTFT for sequence $x(n)$ given in example 3.2. 501 equidistant frequencies $[0, \pi]$.

ex. 3.5 $x(n) = (0.9 \exp(j\pi/3))^n; \quad 0 \leq n \leq 10;$

$X(e^{j\omega}) = ?$ investigate the periodicity

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \qquad \mathcal{X} = x^T\left(-j\frac{\pi}{M}\cdot n^T \cdot k\right)$$

- PERIODIC IN ω
- NOT CONJUGATE SYMETRIC

$X(e^{j\omega})$

ex. 3.6

$x(n) = 2^n \qquad -10 \leq n \leq 10$

$x(n) = (-0.9)^n \qquad -5 \leq n \leq 5$

PERIODIC IN ω AND CONJUGATE SYMETRIC

The properties of DTFT

1.) Linearity

$$\mathcal{F}[\alpha x_1(n) + \beta x_2(n)] = \alpha\, \mathcal{F}[x_1(n)] + \beta\, \mathcal{F}[x_2(n)]$$

2.) Time shifting ~~[illegible]~~ coresponds in phase shift in f domain

$$\mathcal{F}[x(n-k)] = \mathcal{X}(e^{j\omega})\, e^{-j\omega k}$$

3.) Frequency shifting: $\mathcal{F}[x(n)\cdot e^{j\omega_0 n}] = \mathcal{X}(e^{j(\omega-\omega_0)})$

4.) Conjugation: $\mathcal{F}[x^*(n)] = \mathcal{X}^*(e^{-j\omega})$

5.) Folding: $\mathcal{F}[x(-n)] = \mathcal{X}(e^{-j\omega})$

6.) Symetries in real sequencies:

$x(n) = x_e(n) + x_o(n)$

$$\mathcal{F}\{x_e(n)] = Re[\mathcal{X}(e^{j\omega})] \quad \mathcal{F}[x_o(n)] = j\,Im[\mathcal{X}(e^{j\omega})]$$

7.) $\boxed{\text{Convolution: } \mathcal{F}[x_1(n) * x_2(n)] = \mathcal{F}[x_1(n)]\,\mathcal{F}[x_2(n)] = \mathcal{X}_1(e^{j\omega})\mathcal{X}_2(e^{j\omega})}$

8.) Multiplication:

$$\mathcal{F}[x_1(n)\cdot x_2(n)] = \mathcal{F}[x_1(n) \circledast x_2(n)] \triangleq \frac{1}{2\pi}\int \mathcal{X}_1(e^{j\theta})\,\mathcal{X}_2(e^{j(\omega-\theta)})\,d\theta$$

9.) Energy:

$$\varepsilon_x = \sum_{-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\mathcal{X}(e^{j\omega})|^2\, d\omega = \left|\begin{matrix}\text{for}\\ \text{real}\\ x(n)\end{matrix}\right| = \int_0^{\pi} \frac{|\mathcal{X}(e^{j\omega})|^2}{\pi}\, d\omega$$

ENERGY DENSITY SPEKTRUM

- ENERGY OF $x(n)$ in $[\omega_1, \omega_2]$ is:

$$\int_{\omega_1}^{\omega_2} \frac{|\mathcal{X}(e^{j\omega})|^2}{\pi}\, d\omega \qquad 0 \leq \omega_1 < \omega_2 \leq \pi$$

$\boxed{\text{EXAMPLE 3.7}}$ $x_1(n)$ $x_2(n)$ ⇒ RANDOM SEQUENCES $0 \le n \le 10$ } PROVE LINEARITY PROPERTIE

$\boxed{\text{EXAMPLE 3.8}}$ $x(n)$ rand sequence $[0,1]$; $0 \le n \le 10$

$y(n) = x(n-2)$; Verify sample shift property

$$\mathcal{F}[x(n-2)] = \mathcal{F}[y(n)] = X(j\omega)\cdot e^{-j2\cdot\omega}$$

$$X(j\omega) = \sum_{n=-\infty}^{\infty} x(n)\cdot e^{-j\omega n}; \quad \mathcal{F}[x(n-2)] = \sum_{n=-\infty}^{\infty} x(n)\cdot e^{-j2\omega}\cdot e^{-j\omega n} =$$

$$= \sum_{n=-\infty}^{\infty} x(n)\cdot e^{-j\omega(n+2)}$$

$\boxed{\text{ex. 3.9}}$ VERIFY FREQUENCY SHIFT PROPERTY

$x(n) = \cos(\pi n/2)$ $\quad 0 \le n \le 100$; $\quad y(n) = e^{j\pi n/4}\, x(n)$

$$\mathcal{F}[x(n)\, e^{j\underbrace{(\pi n/4)}_{n\omega_0}}] = X\left(e^{j(\omega-\omega_0)}\right) = X(\omega-\omega_0) = X(\omega - \pi/4)$$

$$\omega = \frac{\pi}{500}\cdot k \qquad \omega - \omega_0 = \frac{\pi}{500}\cdot k - \frac{\pi}{4} = \frac{\pi}{500}(k-125)$$

$x = \cos\left(\frac{n\pi}{50}\right)$ $\quad n = 0 - 1000$

$x = \cos(\omega t) = \cos\left(\frac{\pi}{50}\cdot t\right) = \left| \begin{array}{l} \omega = \frac{\pi}{50} \\ \omega = 2\pi f \\ f = \frac{\omega}{2\pi} \end{array} \right|$ $\quad \boxed{f = \frac{1}{100}} = \underline{0{,}01} = \underline{10^{-2}\,Hz}$

$\frac{\pi}{2} = 1{,}571$ $\quad k = [-100:100]$ $\quad \omega = \frac{\pi}{100}\cdot k$ $\quad \boxed{[-\pi, \pi]}$

$\boxed{\text{ex. 3.10}}$ Verify conjugation property

$x(n)$ - COMPLEX VALUED RANDOM SEQUENCE $-5 \le n \le 10$

- REAL AND IMAG. PARTS RAND. DIST $[0,1]$

$$\mathcal{F}[x^*(n)] = X^*(e^{-j\omega})$$

$\boxed{\text{ex. 3.11}}$ Verify folding property

$x(n) = $ randn$(1,16)$; $\quad -5 \le n \le 10$

$$\mathcal{F}[x(-n)] = X(e^{-j\omega})$$

$\boxed{\text{ex. 3.12}}$ Symmetry properties $\quad x(n) = x_e(n) + x_o(n)$

$$\boxed{X_e(e^{j\omega}) = \mathrm{Re}[X(e^{j\omega})], \quad X_o(e^{j\omega}) = j\,\mathrm{Im}[X(e^{j\omega})]}$$

$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$

$x_o(n) = \frac{1}{2}[x(n) - x(-n)]$

$x(n) = \sin(n\pi/2)$ $\quad -5 \le n \le 10$

The Frequency Domain Representation of LTI sist.

$$x(n) = e^{j\omega_0 n}$$

$$e^{j\omega_0 n} \rightarrow \boxed{h(n)} \rightarrow h(n) * e^{j\omega_0 n}$$

SO OVA IMPLICITNO E DOKAZANO SVOJSTVOTO ZA KONVOLUCIJA !

$$y(n) = h(n) * e^{j\omega_0 n} = \sum_{k=-\infty}^{\infty} h(k) \cdot e^{j\omega_0 (n-k)} = \left[\sum_{k=-\infty}^{\infty} h(k)\, e^{-j\omega_0 k} \right] e^{+j\omega_0 n} =$$

$$= \left[\mathcal{F}[h(n)] \Big|_{\omega=\omega_0} \right] e^{j\omega_0 n} \quad \ldots\ldots \quad 3.15$$

DEF. 1 Frequency Response

$$\boxed{H(e^{j\omega}) \triangleq \sum_{n=-\infty}^{\infty} h(n)\, e^{-j\omega n} \qquad \boxed{3.16}}$$

$$\boxed{\boxed{3.17} \quad x(n) = e^{j\omega_0 n} \rightarrow \boxed{H(e^{j\omega})} \longrightarrow y(n) = H(e^{j\omega_0}) \times e^{j\omega_0 n}}$$

$$\sum_k A_k e^{j\omega_k n} \rightarrow \boxed{h(n)} \rightarrow \sum_k A_k H(e^{j\omega_k}) \cdot e^{j\omega_k n}$$

Response to sinusoidal sequences

$$x(n) = A \cos(\omega_0 n + \theta_0) \rightarrow \boxed{3.17} \rightarrow \boxed{y(n) = A\left|H(e^{j\omega_0})\right| \cdot \cos\left(\omega_0 n + \theta_0 + \angle H(e^{j\omega_0})\right)} \quad \boxed{3.18}$$

proof: $\cos x = \frac{1}{2}\left(e^{jx} + e^{-jx}\right)$; $\quad y(n) = H(e^{j\omega_0}) \cdot \frac{A}{2} \cdot \left(e^{j(\omega_0 n + \theta_0)} + e^{-j(\omega_0 n + \theta_0)}\right) =$

$$= \left|H(e^{j\omega_0})\right| \cdot e^{\angle H(e^{j\omega_0})} \cdot \frac{A}{2}\left(e^{j(\omega_0 n + \theta_0)} + e^{-j(\omega_0 n + \theta_0)}\right) = A \cdot \left|H(e^{j\omega_0})\right| \cos\left(\omega_0 n + \theta_0 + \angle H(e^{j\omega_0})\right)$$

$$\sum_k A_k \cos(\omega_k n + \theta_k) \rightarrow \boxed{h(n)} \rightarrow \sum_k A_k \left|H(e^{j\omega_k})\right| \cdot \cos\left(\omega_k n + \theta_k + \angle H(e^{j\omega_k})\right)$$

KONVOLUCIJA VO VREMENSKI DOMEN TREBA VO PROIZVOD VO FREKVENTEN

Response to arbitrary sequences

$$X(e^{j\omega}) = \mathcal{F}[x(n)]\ ; \quad Y(e^{j\omega}) = \mathcal{F}[y(n)]\ ; \qquad \boxed{Y(e^{j\omega}) = H(e^{j\omega}) \cdot X(e^{j\omega})} \quad \boxed{3.19}$$

$$y(n) = \mathcal{F}^{-1}[Y(e^{j\omega})] = \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(e^{j\omega}) \cdot e^{j\omega n} d\omega$$

---

ex. 3.13 $\quad H(e^{j\omega}) = ? \qquad h(n) = (0.9)^n u(n)\ ; \qquad \boxed{H(j\omega) = \sum_{n=-\infty}^{\infty} h(n) \cdot e^{-j\omega n}}$

$$\boxed{H(j\omega) = h * (-j\tfrac{\pi}{M}) \cdot n^T \cdot k} \qquad \boxed{\omega = \frac{\pi}{M} k} \qquad k = [-M : M] \Rightarrow \omega = [-\pi : \pi]$$

$$H(j\omega) = \sum_{n=0}^{N} (0.9)^n \cdot e^{-j\omega n} = \sum_{n=0}^{N} \left(0.9 \cdot e^{-j\omega}\right)^n = \frac{1 - q^{N+1}}{1 - q} = \frac{1 - 0.9^{N+1} \cdot e^{-j\omega(N+1)}}{1 - 0.9 \cdot e^{-j\omega}}$$

$$N \to \infty \qquad H(j\omega) = \frac{1}{1 - 0.9 \cdot e^{-j\omega}} \qquad |H(j\omega)| = \left| \frac{e^{j\omega}}{e^{j\omega} - 0.9} \right|$$

$$|H(j\omega)| = \left| \frac{1}{1 - 0.9\cos\omega + j\,0.9 \cdot \sin\omega} \right| = \frac{1}{\sqrt{(1 - 0.9 \cdot \cos\omega)^2 + (0.9 \cdot \sin\omega)^2}} =$$

$$= \frac{1}{\sqrt{1 - 2 \cdot 0.9 \cdot \cos\omega + \underbrace{0.81 \cdot \cos^2\omega + 0.81 \cdot \cos^2\omega}_{0.81}}} = \frac{1}{\sqrt{1.81 - 1.8\cos\omega}}.$$

$$H(j\omega) = \frac{1}{(1 - 0.9 \cdot \cos\omega) + j\,0.9 \cdot \sin\omega} \cdot \frac{(1 - 0.9 \cdot \cos\omega) - j\,0.9 \cdot \sin\omega}{(1 - 0.9 \cdot \cos\omega) - j\,0.9 \cdot \sin\omega}$$

$$H(j\omega) = \frac{1-0.9\cos\omega - j\,0.9\cdot\sin\omega}{(1-0.9\cdot\cos\omega)^2 + 0.81\sin^2\omega} = |H(j\omega)|\cdot e^{j\varphi}$$

$$= \left| \varphi = -\operatorname{arctg}\frac{0.9\sin\omega}{1-0.9\cos\omega} \right| = \sqrt{\frac{\cancel{(1-0.9\cos\omega)^2 + 0.81\sin^2\omega}}{\left((1-0.9\cdot\cos\omega)^2 + 0.81\cdot\sin^2\omega\right)^{\cancel{2}}}}\; e^{j\omega} =$$

$$\boxed{= \frac{1}{\sqrt{1{,}81 - 1{,}8\cos\omega}} \cdot e^{-j\operatorname{arctg}\frac{0.9\sin\omega}{1-0.9\cos\omega}}}$$

$\boxed{ex. 3.14}$ $\boxed{h(n) = (0{,}9)^n\cdot u(n)}$; Calculate the steady state response $\boxed{y_{ss} = ?}$ ...

$\boxed{x(n) = 0{,}1\cdot u(n)}$

$$y_{ss} = A\cdot|H(j\omega_0)|\cdot\cos(\omega_0 n + \theta + \angle(H(j\omega_0)))$$

→ The input is not apsolutly sumable !!

$$x(n) = 0.1\cdot\cos(\underbrace{\overset{0}{\cancel{\omega_0 n}} + \overset{0}{\cancel{\theta}}}) \cdot u(n) = 0.1\cdot u(n) \Rightarrow \text{constant sequence}$$

$$y_{ss} = 0.1\cdot|H(j\overset{1}{\underset{0}{\cancel{\omega_0}}})|\cdot\cos(\angle(H(j\overset{\cancel{1}0}{\cancel{\omega_0}})))$$

$$y_{ss} = \left| |H(j\omega_0)| = \frac{1}{\sqrt{1{,}81 - 1{,}8\cdot 1}} = 10 \right| = 0.1\cdot\underbrace{10\cdot\cos(\varphi)}_{H(j\omega)} = \left| \begin{matrix}\varphi = atan(0)\\ = 1\end{matrix} \right| = \underline{\underline{1}}$$

MOZE
I VRCE

$$\boxed{H(j\omega) = \frac{1}{1-0.9\cdot e^{j0}} = 10}$$

Frequency response function from difference equation

$$y(n) + \sum_{\ell=1}^{N} a_\ell\, y(n-\ell) = \sum_{m=0}^{M} a_m\, x(n-m); \quad H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-j\omega n}$$

$$x(n) = e^{j\omega_0 n} \Rightarrow y(n) = H(e^{j\omega_0})\cdot e^{j\omega_0 n}$$

$$H(e^{j\omega_0})\cdot e^{j\omega_0 n} + \sum_{\ell=1}^{N} a_\ell\, H(e^{j\omega_0})\, e^{j\omega_0(n-\ell)} = \sum_{m=1}^{M} a_m\cdot e^{j\omega_0(n-m)}$$

$$\cancel{e^{j\omega_0 n}}\left[1 + \sum_{\ell=1}^{N} a_\ell\, e^{-j\omega_0\ell}\right] H(e^{j\omega}) = \cancel{e^{j\omega_0}} \sum_{m=1}^{M} a_m\, e^{-j\omega_0 m}$$

$$\boxed{H(e^{j\omega_0}) = \frac{\sum_{m=0}^{M} b_m\, e^{-j\omega_0 m}}{1 + \sum_{\ell=1}^{N} a_k\, e^{-j\omega_0\ell}}}$$

$\boxed{ex. 3.15}$ $y(n) = 0.8\,y(n-1) + x(n)$

$y(n) - 0.8\,y(n-1) = x(n)$

a) $H(e^{j\omega}) = ?$

b) $y_{ss} = ?$ for $x(n) = \cos(0.05\pi n)u(n)$

5.1.2600.2180

$y(n) = 0.8\, y(n-1) + x(n)$; a.) $H(e^{j\omega}) = ?$ b.) $y_{ss} = ?$ for $x(n) = \cos(0{,}05\pi n)\, u(n)$

ⓐ $y(n) - 0{,}8\, y(n-1) = x(n)$; $H(j\omega) = \dfrac{1}{1-0.8e^{-j\omega}} = \dfrac{1}{1-0{,}8\cos\omega + j\overset{0,8}{\cancel{}}\sin\omega} =$

$$= \frac{(1-0{,}8\cos\omega) - j0{,}8\sin\omega}{(1-0{,}8\cos\omega)^2 + 0{,}8^2\sin^2\omega} = \frac{1-0{,}8\cos\omega - j0{,}8\sin\omega}{1-1{,}6\cos\omega + \underbrace{0{,}64\cos^2\omega + 0{,}8^2\sin^2\omega}_{0{,}64}} = \frac{1-0{,}8\cos\omega - j0{,}8\sin\omega}{1{,}64 - 1{,}6\cos\omega}$$

$$= \sqrt{\frac{(1-0{,}8\cos\omega)^2 + 0{,}8^2\sin^2\omega}{(1{,}64-1{,}6\cos\omega)^2}}\; e^{-j\,\mathrm{atan}\frac{0{,}8\sin\omega}{1-0{,}8\cos\omega}} = \frac{1}{\sqrt{1{,}64-1{,}6\cos\omega}} \cdot e^{-j\,\mathrm{atg}\frac{0{,}8\sin\omega}{1-0{,}8\cos\omega}}$$

ⓑ $x(n) = \cos(0{,}05\pi n)$ $\boxed{\omega_0 = 0{,}05\pi}$ $H(j\omega_0) = \dfrac{\overset{\boxed{4{,}0928}}{1}}{\sqrt{1{,}64-1{,}6\cos(0{,}05\pi)}} \cdot e^{-j\,\mathrm{atan}\frac{0{,}8\sin\omega_0}{1-0{,}8\cos\omega_0}}$

$\omega = 2\pi f$

$$H(j\omega_0) = 4{,}0928 \cdot e^{-j0{,}5377}$$

$$y_{ss} = 4{,}0928 \cdot \cos(0{,}05\pi n - 0{,}5377) = 4{,}0928 \cdot \cos(0{,}05\pi(n-3{,}42))$$

SAMPLING AND RECONSTRUCTION OF ANALOG SIGNALS

$$X_a(j\Omega) \triangleq \int_{-\infty}^{\infty} x_a(t) \cdot e^{-j\Omega t}\, dt$$

$$x_a(t) \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(j\Omega) \cdot e^{-j\Omega t}\, d\Omega$$

Ω - ANALOG FREQUENCY IN RAD/SEC

$x(n) \triangleq x_a(nT_s)$

$X(e^{j\omega})$ - DTFT OF $x(n)$ $\boxed{\omega \text{ - DIGITAL FREQ IN RAD}}$

$$\boxed{X(e^{j\omega}) = \frac{1}{T_s} \sum_{\ell=-\infty}^{\infty} X_a\left(j\frac{\omega}{T_s} - \frac{2\pi}{T_s}\ell\right)} \quad \boxed{3.26}$$

ALIASING FORMULA

$\boxed{\omega = \Omega \cdot T_s}$ $\boxed{F_s = \frac{1}{T_s} \text{ SAMPLING FREQUENCY}}$

$T_s < \dfrac{\pi}{\Omega_0}$ (NEMA PREKLOPUVANJE NA REPLIKITE OD $X(e^{j\omega})$)

$T_s < \dfrac{\pi}{2\pi f_0} = \dfrac{1}{2 f_0} = \dfrac{1}{2 \cdot f_{max}} \Rightarrow 2 f_{max} < \dfrac{1}{T_s}$ $\boxed{f_s > 2 f_{max}}$

$\boxed{Ex.\ 3.17}$ $\boxed{x_a(t) = e^{-1000|t|}}$, Determine and Plot Fourier Transform

$$X_a(j\Omega) = \int_{-\infty}^{\infty} e^{-1000t} e^{j\Omega t} dt = \int_{-\infty}^{0} e^{1000t} e^{j\Omega t} dt + \int_{0}^{\infty} e^{-1000t} e^{j\Omega t} dt =$$

$$= \frac{0.002}{1 + \left(\frac{\Omega}{1000}\right)^2}$$

$e^{-5} = 0{,}0067 \approx 0$

$x_a(t)$ APPROKSIMATIVNO VO INTERVAL

$0.005 < t < 0.005$ [-5ms : 5ms]

$X_a(j\Omega)\Big|_{\Omega=2\pi} = X_a(j2\pi) = 0.0019 \approx 0$ $(5 \cdot 10^{-3}) = \frac{1}{5} \cdot 10^3 = 200$

$\Delta t = 5\, 10^{-5}$; $\dfrac{\Omega}{1000} = 2\pi \Rightarrow X_a = \dfrac{0.002}{1 + (2\pi)^2} = 0.0005$

$\Omega_0 = 2\pi \cdot 1000$

$F_s > 2F_0$; $\dfrac{1}{T_s} > 2 \cdot \dfrac{\Omega_0}{2\pi}$; $T_s < \dfrac{2\pi}{2\Omega_0}$

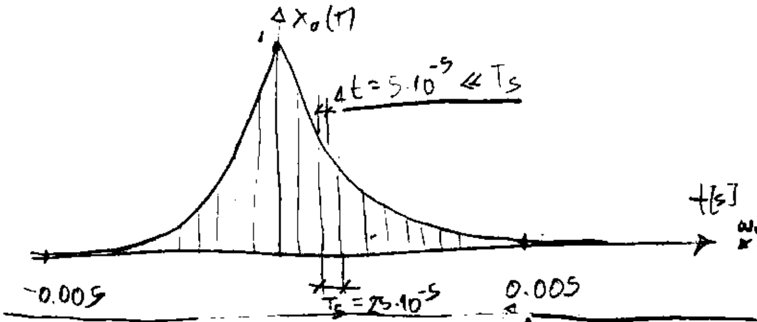
$$T_S < \frac{2\pi}{2\Omega_0} = \frac{\cancel{2\pi}}{2\cdot\cancel{2\pi}\cdot 1000} = \frac{1}{2.1000}$$

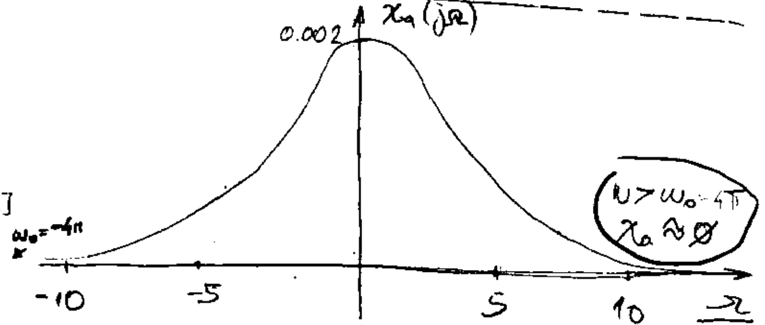
$$\boxed{x_a(t) = e^{-1000|t|}} \quad X_a(j\Omega) = \frac{0.002}{1+\left(\frac{\Omega}{1000}\right)^2} \;;\; X_a(j\Omega) \approx \emptyset \text{ ⓩⓐ: } \Omega > \boxed{\Omega_0 = 2\pi\cdot 2000}$$

$$X_0(j\Omega_0) = \frac{0.002}{1+\left(\frac{2\pi\cdot 200\cancel{0}}{100\cancel{0}}\right)^2} = \frac{0.002}{1+(4\pi)^2} \approx \underline{\underline{0.0000125}}$$

$$T_S = \frac{2\pi}{2\Omega_0} = \frac{2\pi}{2\cdot\cancel{2\pi}\cdot 2000} = \frac{1}{2.(2000)} = \frac{1}{4}\,10^{-3} = 0{,}25\cdot 10^{-3} = 25\cdot 10^{-5}$$

$$\Delta t = 5.10^{-5} \ll \frac{1}{2.2000} = T_s = 25.10^{-5} \quad \boxed{F_0 = \frac{\Omega_0}{2\pi} = 2000\,Hz = \underline{\underline{2\,KHz}}}$$





$$x_q(n) \overset{\Delta}{=} x_a(n\cdot\Delta t) \Rightarrow \boxed{X_a(j\Omega) = \Delta t \sum_n x_q(n)\cdot e^{-j\Omega n\Delta t} \;\boxed{n\Delta t = t}} \quad \boxed{\omega = \Omega\, T_S}$$

APROXIMATION OF CONTINUOUS FOURIER TRANSFORM

$$100\cdot 5\cdot 10^{-5} = \textcircled{5\cdot 10^{-3}} \Rightarrow \quad t = [-100:100]\,\Delta t \qquad \Omega = [-4\pi : 4\pi] * 200\cancel{0}/10\cancel{0}$$

$$\Omega = \underbrace{[-100:100]}_{K}\, 4\pi * 20. \qquad \boxed{\Omega = K * 4\pi * 2000/100}$$

$$\omega = \Omega\cdot\Delta t = K * 4\pi * 2000/100 * 5.10^{-5}$$
$$= K * 4\pi * 2.10 * 5.10^{-5} = K * 4\pi * 10^{-3}$$

$$\boxed{n.\Delta t = t}$$

$$k = [100\cancel{0}\;100] \;;\; \omega = K*\pi/100 \quad \omega \in [-\pi : \pi]$$

$\boxed{Ex.\,3.18}$ $x_a(t) = e^{-1000|t|}$ $X_a(j\Omega) = ?$

a) $F_S = 5000$ sam/sec $\rightarrow x_1(n)$

b) $F_S = 1000$ sam/sec $\rightarrow x_2(n)$

ⓐ $\Delta t = 5.10^{-5} * 0.\cancel{2}\; 10^5 = 2\cdot 10^4 = 20.000{,}00$ KHz

$$T_S = \frac{1}{5.10^3} = 0.2.10^{-3} = 2.10^{-4}\,s = 20\,msec$$

$$F_0 = \frac{\Omega_0}{2\pi} = \frac{\cancel{2\pi}\cdot 2000}{\cancel{2\pi}} = 2000\,Hz = 2\,KHz$$

$$\boxed{F_S \geq 2F_0 = 4\,kHz}$$

ⓑ $\boxed{T_S = 10^{-3} = 1\,msec}$

$$\Delta t = 5\,10^{-5} \ll T_s \leq \frac{T_0}{2} = \frac{1}{2\cdot f_0} = \frac{1}{2\cdot 2000}$$

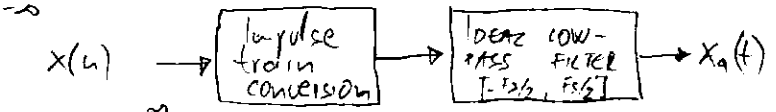
$$= \frac{1}{4}\cdot 10^{-3} = \underline{0.25\cdot 10^{-3}}$$

$$\boxed{5\cdot 10^{-5} \ll 25\cdot 10^{-5}}$$

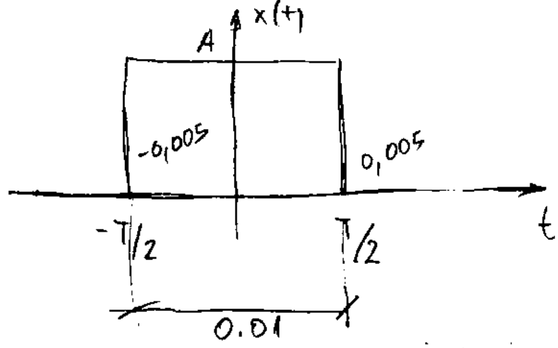
RECONSTRUCTION

IMPULSE TRAIN

$$\sum_{n=-\infty}^{\infty} x(n)\,\delta(t-nT_s) = \ldots + x(-1)\,\delta(n+T_s) + x(0)\,\delta(n) + x(1)\,\delta(n-T_s) +$$



$$x_a(t) = \sum_{n=-\infty}^{\infty} x(n)\,\mathrm{sinc}\left[F_s(t-nT_s)\right]$$



$$X(j\omega) = \int_{-T/2}^{T/2} x(t)\,e^{-j\omega t}\,dt =$$

$$= A\,\frac{e^{-j\omega t}}{j\omega}\Bigg|_{-T/2}^{T/2} = \frac{A}{-j\omega}\left(e^{-j\omega T/2} - e^{j\omega T/2}\right)$$

$$= +\frac{2A}{\omega}\cdot\underbrace{\frac{1}{2j}\left(e^{+j\omega T/2} - e^{-j\omega T/2}\right)}_{} \rightarrow \sin\left(\frac{\omega T}{2}\right)$$

$$= 2A\cdot\frac{\sin(\omega T/2)}{\omega} = \boxed{A\cdot T\cdot\frac{\sin(\omega T/2)}{\left(\frac{\omega T}{2}\right)}}$$

$$\cos x = \frac{1}{2}\left(e^{jx}+e^{-jx}\right)$$

$$e^{jx} = \cos x + j\sin x$$

$$e^{-jx} = \cos x - j\sin x$$

$$2j\sin x = e^{jx} - e^{-jx}$$

$$\sin x = \frac{1}{2j}\left(e^{jx} - e^{-jx}\right)$$

$$\sin x = +\frac{j}{2}\left(e^{-jx} - e^{jx}\right)$$

$A = 1$

$T = 0.01 \Rightarrow$

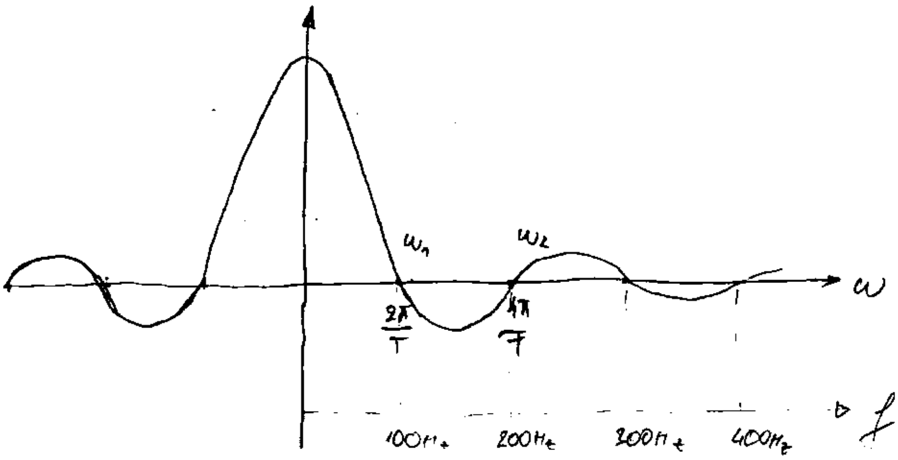
$$\boxed{X(j\omega) = 2\cdot\frac{\sin(0.005\,\omega)}{\omega}}$$

$\omega \neq \omega_0 = 400\pi \Rightarrow X(j\omega) \approx 0$

$\boxed{\omega_0 = 400\pi}$

$\boxed{f_0 = \frac{\omega_0}{2\pi} = 200Hz}$

$\boxed{\frac{\omega_1 T}{2} = \pi \quad \omega_1 = \frac{2\pi}{T}}$. $\boxed{\frac{\omega_2 T}{2} = 2\pi \Rightarrow \omega_2 = \frac{4\pi}{T}}$



$$\omega_1 = \frac{2\pi}{T} = \frac{2\pi}{0.01} = 2\pi\cdot 10^{+2}$$

$$f_1 = \frac{\omega_1}{2\pi} = 100Hz$$

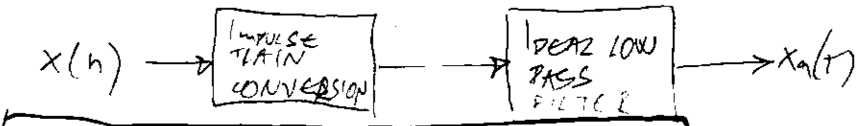
$$f_2 = \frac{\omega_2}{2\pi} = \frac{4\pi\cdot 10^{2}}{2\pi} = 200Hz$$

$$X(j\omega) = A\cdot T\,\frac{\sin(\omega T/2)}{\left(\frac{\omega T}{2}\right)} = AT\,\frac{\sin\left(2\pi f\cdot\frac{1}{f_0}\cdot\frac{1}{2}\right)}{2\pi f\cdot\frac{1}{2f_0}} = AT\,\frac{\sin\left(\frac{\pi f}{f_0}\right)}{\frac{\pi f}{f_0}}$$

$$\boxed{= AT\,\mathrm{sinc}\left(f/f_0\right)}$$

RECONSTRUCTION CONTINUED:

$$\sum_{n=-\infty}^{\infty} x(n)\delta(t-nT_s) = \dots + x(-1)\delta(t+T_s) + x(0)\delta(t) + x(1)\delta(t-T_s) + \dots$$



$$\boxed{x_a(t) = \sum_{n=-\infty} x(n)\,\mathrm{sinc}\left[F_s\cdot(t-nT_s)\right]}$$

$\frac{\sin \pi x}{\pi x} \Rightarrow$ INTERPOLATION FUNCTION

→ INFINITE ORDER INTERPOLATION

- APPROACHES FOR FINITE (LOW) ORDER INTERPOLATION:

a.) ZOH (Zero Order Hold)

$\hat{x}_a(t) = x(n), \quad nT_s < n < (n+1)T_s$

Filtering the impulse train through interpolating filter of the form:

$$\boxed{h_0(t) = \begin{cases} 1 & 0 \le t \le T_s \\ 0 & \text{otherwise} \end{cases}}$$

b.) FOH: (First Order Hold) interpolation

$$\boxed{h_1(t) = \begin{cases} 1 + \frac{t}{T_s}, & 0 \le t < T_s \\ 1 - \frac{t}{T_s}, & T_s \le t \le 2T_s \\ 0, & \text{otherwise} \end{cases}}$$

c.) Cubic spline interpolation (MATLAB)

$$\boxed{x_a(t) = a_0(n) + a_1(n)(t-nT_s) + a_2(n)(t-nT_s)^2 + a_3(n)(t-nT_s)^3} \quad \boxed{nT_s \le n \le (n+1)T_s}$$

$(a_i(n) \; - \; 0 \le i \le 3 \Rightarrow$ POLYNOMIAL COEFFICIENTS$)$

MATLAB IMPLEMENTATION

$$\boxed{x_a(m\Delta t) = \sum_{n=-\infty}^{\infty} x(n)\,\mathrm{sinc}\left[F_s(m\Delta t - nT_s)\right]}$$

$x(n); \quad n_1 \le n \le n_2$

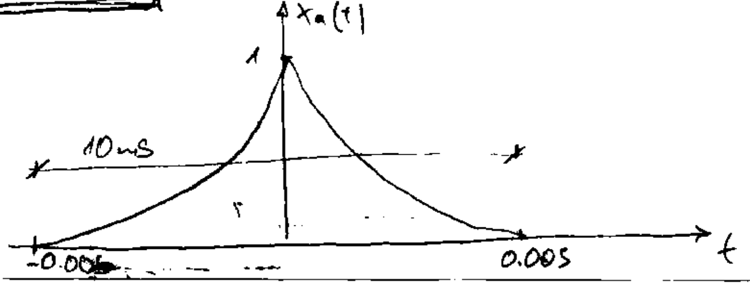
$t_1 \le m\Delta t \le t_2$

```
n = n1:n2;   t = t1:t2;   Fs = 1/Ts   nTs = n * Ts
xa = x * sinc (Fs*(ones(length(n),1)*t - nTs'*ones(1,length(t))))
```

ex 3.19 $x_a = e^{-1000|t|}$ $\quad TS_1 = 0.2\,ms = 2\cdot10^{-4}s \quad n1 = -25:25$

$$\boxed{x_1(n) = x_a(nTS_1)}$$

$$\boxed{t_1 = n1\cdot TS_1}$$



$$\begin{bmatrix}F_s\\F_s\\ \vdots\\ F_s\end{bmatrix}[t_1,t_2\cdots t_n] = \begin{bmatrix}F_st_1, F_st_2, \cdots F_st_n\\ F_st_1, F_st_2, \cdots F_st_n\\ \\ F_st_1, F_st_2, \quad F_st_n\end{bmatrix}$$

$\boxed{P3.2}$ ⓐ $x(n) = 3(0.9)^3 u(n)$; $\quad X(e^{j\omega}) = \sum_{n=\infty}^{\infty} x(n)\cdot e^{-j\omega n} = \sum_{n=-\infty}^{\infty} 3(0.9)^3 e^{-j\omega n}$

$$= 3\cdot(0.9)^3 \sum_{n=-\infty}^{\infty} e^{-j\omega n} = 3\cdot(0.9)^3 \sum_{n=0}^{\infty} e^{-j\omega n} = 3(0.9)^3[1+e^{-j\omega}+e^{-2j\omega}+\dots]$$

$$S = 1 + q + q^2 + q^3 + \dots q^N$$
$$q\cdot S = q + q^2 + q^3 + q^4 + \dots q^{N+1}$$
$$S(1-q) = 1 - q^{N+1}$$

$$S = \frac{1-q^{N+1}}{1-q}$$

$q<1$; $N\to\infty$;

$$S = \frac{1}{1-q}$$

$$X(e^{j\omega}) = 3(0.9)^3 \cdot \frac{1}{1-e^{-j\omega}}$$

ⓑ $x(n) = 2(0.8)^{n+2} u(n-2)$ $\quad X = \sum_{n=2}^{\infty} 2(0.8)^{n+2} e^{-j\omega n} =$

$$= \underbrace{2\sum_{n=0}^{\infty}(0.8)^{n+2}e^{-j\omega n}}_{*} - 2(0.8)^2 - 2(0.8)^3 e^{-j\omega}$$

$$\circledast = 2(0.8)^2\sum_{n=0}^{\infty}\left(0.8\cdot e^{-j\omega}\right)^n = \frac{2(0.8)^2}{1-0.8\cdot e^{-j\omega}}$$

$$X = \frac{2(0.8)^2}{1-0.8\cdot e^{-j\omega}} - \frac{2(0.8)^2(1+0.8e^{-j\omega})(1-0.8e^{-j\omega})}{(1-0.8\cdot e^{-j\omega})} =$$

$$= \frac{2(0.8)^2 - 2(0.8)^2(1-0.64e^{-j2\omega})}{(1-0.8e^{-j\omega})} = \frac{2(0.8)^2 0.64\cdot e^{-j2\omega}}{1-0.8e^{-j\omega}} =$$

$$= \frac{0.8192\cdot e^{-j2\omega}}{(e^{j\omega}-0.8)\,e^{-j\omega}} = \frac{0.8192\, e^{-j\omega}}{e^{j\omega}-0.8}$$

ⓒ $x(n) = n(0.5)^n u(n)$ $\quad X(e^{j\omega}) = \sum_{n=0}^{\infty} n\cdot(0.5)^n e^{-j\omega n} = \frac{0.5e^{j\omega}}{(-0.5+e^{j\omega})^2}$

$$S_n = \frac{n}{2}(1+n)$$

$$S_n = 1+2+3+\dots+n$$
$$S_n = a+(a+d)+(a+2d)+\dots+(a+(n-1)d)$$
$$= n\cdot a + d + 2d + \dots + (n-1)d$$

$$a_n = a_1 + (n-1)d$$

$$S_n = a_1 + a_2 + \dots + a_{n-1} + a_n$$
$$S_n = a_n + a_{n-1} + \dots + a_2 + a_1$$
$$2S_n = n\cdot(a_1+a_n)$$
$$S_n = \frac{n}{2}(a_1+a_n)$$

$$a_2 = a_1 + d$$
$$a_{n-1} = a_1 + (n-2)d$$
$$= 2[a_1+(n-1)d]$$
$$= 2[a_1+a_n]$$

$$S_n = \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n} \qquad S_n = \frac{1 - 2^{n+1}}{1-2} = \frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} =$$

$$= \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots \frac{1}{2^{n-1}}\right) = \cancel{\frac{1}{2}} \cdot \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} = 1 - \frac{1}{2^n}$$

$$x(n) = n \cdot (0.5)^n \cdot u(n) \qquad X = \sum_{n=0}^{\infty} n \cdot (0.5)^n e^{-j\omega n}$$

$$\boxed{(x^\alpha)' = \alpha \cdot x^{\alpha-1}}$$

MANA E
STO SI U-
ČUNAT DO
∞ REKAO
SE DO N

$$\boxed{S(x) = \sum_{n=1}^{\infty} n x^n} = x \sum_{n=1}^{\infty} n \cdot x^{n-1} \qquad \int \frac{S(x)}{x} = \int \sum_{n=1}^{\infty} n \cdot x^{n-1}$$

$$\int \frac{S(x)}{x} dx = \sum_{n=1}^{\infty} n \int x^{n-1} dx = \sum_{n=1}^{\infty} n \frac{x^n}{\cancel{n}} + c = \sum_{n=1}^{\infty} x^n + c = \frac{1}{1-x} + c$$

$$\frac{S(x)}{x} = \left(\frac{1}{1-x}\right)' = \left| U = 1-x \right| = (U^{-1})' dU = -1\, U^{-2} \cdot d(1-x) = -\frac{1}{(1-x)^2}(-1) = \frac{1}{(1-x)^2}$$

$$S(x) = \frac{x}{(1-x)^2}$$

$$X = \sum_{n=0}^{\infty} n \cdot (\underbrace{0.5 \cdot e^{-j\omega}}_{Y})^n = \sum_{n=0}^{\infty} n \cdot Y^n =$$

$$= \sum_{n=1}^{\infty} n \cdot Y^n = \frac{Y}{(1-Y)^2} = \frac{0.5 \cdot e^{-j\omega}}{(1 - 0.5 \cdot e^{-j\omega})^2} = \frac{0.5\, e^{-j\omega}}{\left(e^{-j\omega}(e^{j\omega} - 0.5)\right)^2} = \frac{e^{+j2\omega}\, 0.5\, e^{-j\omega}}{(e^{j\omega} - 0.5)}$$

$$\boxed{= \frac{0.5 \cdot e^{j\omega}}{(e^{j\omega} - 0.5)^2}}$$

ⓓ $x(n) = (n+2)(-0.7)^{n-1} u(n-2)$; $\quad X(e^{-j\omega}) = \sum_{n=2}^{\infty} (n+2)(-0.7)^{n-1} e^{-j\omega n} =$

$$= \underbrace{\sum_{n=2}^{\infty} n \cdot (-0.7)^{n-1} e^{-j\omega n}}_{(*)} + \underbrace{\sum_{n=2}^{\infty} 2(-0.7)^{n-1} e^{-j\omega n}}_{(**)}$$

$$(*) = (-0.7)^{-1} \sum_{n=2}^{\infty} n \cdot (-0.7 e^{-j\omega})^n = (-0.7)^{-1}\left[\sum_{n=1}^{\infty} n \cdot (-0.7 \cdot e^{-j\omega})^n - (-0.7) e^{-j\omega}\right] =$$

$$= (-0.7)^{-1}\left[\frac{(-0.7) e^{-j\omega}}{(1 + 0.7 e^{-j\omega})^2} - (-0.7) e^{-j\omega}\right] = \frac{e^{-j\omega}}{(1 - (-0.7) e^{-j\omega})^2} - e^{-j\omega} = \frac{e^{-j\omega} - e^{-j\omega}(1 + 1.4 e^{-j\omega} + (0.7)^2 e^{-j2\omega})}{(1 + 0.7 e^{-j\omega})^2}$$

$$(**) = \frac{1}{\cancel{(-0.7)^3}} \frac{2 \cdot (-0.7)^{\cancel{4}\,1} \cdot e^{-j2\omega}}{1 + 0.7 e^{-j\omega}}$$

$$= \frac{\cancel{e^{-j\omega}} - \cancel{e^{-j\omega}} - 2(-0.7) e^{-j2\omega} - (-0.7)^2 \cdot e^{-j3\omega}}{(1 - (-0.7) e^{-j\omega})^2} =$$

$$X = \frac{1.4\, e^{-j2\omega} - (-0.7)^2 \cdot e^{-j3\omega}}{(1 - (-0.7) e^{-j\omega})^2} + \frac{2 \cdot (-0.7)^1 e^{-j2\omega} \cdot (1 - (-0.7) e^{-j\omega})}{(1 - (-0.7) e^{-j\omega})^2} = \cancel{\frac{1.4 e^{-j2\omega} - e^{-j3\omega} + 0.1802 e^{-j2\omega} + 0.3364 e^{-j3\omega}}{(1 + 0.7 e^{-j\omega})^2}}$$

$$\cancel{\mathcal{X} = \frac{(1.4+0.4802)e^{-j2\omega} - (0.49-0.33614)e^{-j3\omega}}{(1+0.7e^{-j\omega})^2}} = \cancel{\frac{1.8802\cdot e^{-j2\omega} - 0.15386\cdot e^{-j3\omega}}{(1+0.7e^{-j\omega})^2}} =$$

$$\cancel{\neq \frac{1.8802 - 0.15386\cdot e^{-j\omega}}{(e^{j\omega}+0.7)}}$$

$$\mathcal{X} = \frac{\cancel{1.4e^{-j2\omega}} - 0.49e^{-j3\omega} - \cancel{1.4}\cdot e^{-j2\omega} - 2(-0.7)^2 e^{-j3\omega}}{(1+0.7e^{-j\omega})^2} = \frac{-0.49e^{-j3\omega} - 2\cdot 0.49e^{-j3\omega}}{e^{-j2\omega}(e^{-j\omega}+0.7)^2} =$$

$$\boxed{\mathcal{X} = -\frac{0.49e^{-j\omega}}{(e^{j\omega}+0.7)^2}}$$

ⓔ $x(n) = 5(-0.9)^n \cos(0.1\pi n)\, u(n)$; $\mathcal{X} = \sum_{n=0}^{\infty} 5(-0.9)^n \cos(0.1\pi n)\cdot e^{-j\omega n}$

$e^{jx} = \cos x + j\sin x$
$e^{-jx} = \cos x - j\sin x$

$\cos x = \frac{1}{2}(e^{x} + e^{-x})$
$\cos(0.1\pi n) = \frac{1}{2}(e^{j0.1\pi n} + e^{-j0.1\pi n})$

$$\mathcal{X} = \frac{5}{2}\left[\sum_{n=0}^{\infty}\left(-0.9\cdot e^{-j(\omega-0.1\pi)}\right)^n + \sum_{n=0}^{\infty}\left(-0.9e^{-j(\omega+0.1\pi)}\right)^n\right] = \frac{5}{2}\left[\frac{1}{1+0.9e^{-j(\omega-0.1\pi)}} + \frac{1}{1+0.9e^{-j(\omega+0.1\pi)}}\right]$$

$$\mathcal{X}(e^{j\omega}) = \frac{5}{2}\;\frac{e^{j\omega}}{e^{j\omega}+0.9e^{j0.1\pi}} + \frac{e^{j\omega}}{e^{j\omega}+0.9e^{-j0.1\pi}}$$

$$f_1 = \frac{1}{M}\sum_{i=1}^{M}\left(g_i^k + \mu\cdot\max\{0, g_i^k - g_{max}\}\right)$$

$$f_2 = \max\left(g_i^k + \mu\max\{0, g_i^k - g_{max}\}\right)$$

$$\boxed{g_i^k = \min_{j=1,2..k}\{g_i^j\}}$$

$$OF = f_1 + \psi f_2$$

$$\sigma^2 = \overline{(\xi - \overline{\xi})^2}$$

$$\boxed{\sigma^2 = \overline{\xi^2} - \overline{\xi}^2}$$

$$\begin{array}{r} 2.462 \\ 2\ 412 \\ \hline 50\ \text{MHz} \end{array}$$

$\boxed{5\,\text{MHz}}$ Spacing

① 2.412, 2 2.417, 3 2.422, ④ 2.427, 5 2.432, ⑥ 2.437, ⑦ 2.442, 8 2.447, 9 2.452,
2.457 10, 2.462 11

111

$2^{\xi-n}$ - PERMUTACII

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!} = \frac{6}{1\cdot 2)} = 3$$

$$\binom{400}{5} = \frac{400!}{(400-5)!\;5!}$$

111
010
101
000

000
001
010
011
~~100~~
~~101~~
~~110~~
111

$$\binom{n}{k} = \frac{6!}{4.\ 2!} = \frac{6\cdot5\cdot\cancel{4\cdot3\cdot2\cdot1}}{\cancel{4!}\ 2} = \frac{30}{2} = \underline{\underline{15}}$$

11	22	33	44	55	66	
12	23	34	45	56		
13	24	35	46			
14	25	36				
15	26					
16						
6	5	4	3	2	1	$\Sigma = 20$
5	4	3	2	1	0	$\Sigma = 15$

---

1 2 3 4
000

$n = 4$
$k = 3$

$$\frac{4!}{1!\ 3!} = 4$$

123
124
134
234

a b c
a c b
c b a
c a b
b a c
b c a
a a a
b b b
c c c

$3^3 = 27$

$$\frac{3!}{0!} = 6$$

a b c
0 0

a b
a c
b c

a a
b b
c c

$$\binom{n}{k} = \frac{3!}{1!\ 2!} = 3$$

$$per = \frac{3!}{1!} = 6$$

SO POVTORUVANJE

$$\frac{(n+k-1)!}{k!\ (n-1)!} =$$

$$\frac{(3+3-1)!}{3!\ 2!} = \frac{5!}{3!\cdot2!}$$

$$\frac{\cancel{5}\cancel{4}\cdot\cancel{3\cdot2\cdot1}}{\cancel{3!}\ 2!} = \frac{20}{2} = \underline{\underline{10}}$$

$n = 3 \quad k = 2$

$$\frac{(3+2-1)!}{2!\ 2!} = \frac{4!}{4} = 6$$

$$g_{ik} = \min(g_{ak})$$

$$f = \varphi f_1 + (1-\varphi) f_2$$

$$f_1 = \sum_{i=1}^{M} \left(g_i^k + \mu \max(0, g_i^k - g_{max})\right)$$

$$f_2 = \max\left(g_i^k + \mu \max(0, g_i^k - g_{max})\right)$$

- Mazi znakcinja + ETF na MKD

P.3.8 $X(e^{j\omega}) = X_e(e^{j\omega}) + X_o(e^{j\omega})$

$$X_e(e^{j\omega}) = \frac{1}{2}\left[X(e^{j\omega}) + X^*(e^{-j\omega})\right];\ X_o(e^{j\omega}) = \frac{1}{2}\left[X(e^{j\omega}) - X^*(e^{-j\omega})\right]$$

$$\mathcal{F}^{-1}\left[X_e(e^{j\omega})\right] = x_R(n) \qquad \mathcal{F}^{-1}\left[X_o(e^{j\omega})\right] = x_I(n)$$

$$x(n) = e^{j0.1\pi n}\left[u(n) - u(n-20)\right]$$

$$X_e(e^{j\omega}) = \mathcal{F}\left[x_R(n)\right]; \quad X_o(e^{j\omega}) = \mathcal{F}\left[x_I(n)\right]$$

$$X(e^{j\omega}) = X_e(e^{j\omega}) + X_o(e^{j\omega})$$

$\boxed{7.3.9}$ $\quad x(n) = (\cos\omega_0 n)\,R_N(n)$ USING THE SHIFTING PROPERTY SHOW THAT:

$$X(e^{j\omega}) = \frac{1}{2}\left[\frac{\sin\{(\omega-\omega_0)N/2\}}{\sin\{(\omega-\omega_0)/2\}}\right] + \frac{1}{2}\left\{\frac{\sin\{(\omega+\omega_0)N/2\}}{\sin\{(\omega+\omega_0)/2\}}\right\}$$

$\omega_0 = \pi/2 \qquad N = 5, 15, 25, 100$

$$\mathcal{F}\left[x(n)\cdot e^{j\omega_0 n}\right] = X(e^{j(\omega-\omega_0)})$$

$$\cos(\omega_0 n) = \frac{1}{2}\left[e^{j\omega_0 n} + e^{-j\omega_0 n}\right]$$

$$x(n) = \frac{1}{2}R_N(n)\cdot e^{j\omega_0 n} + \frac{1}{2}R_N(n)\cdot e^{-j\omega_0 n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\cdot e^{-j\omega n} \qquad \boxed{\mathcal{F}\left[x(n)e^{j\omega_0 n}\right] = X(e^{j(\omega-\omega_0)})}$$

$$R_N(n) = \begin{cases} 1 & -N \le n \le N \\ 0 & \text{otherwise}\end{cases} \qquad R_N(e^{j\omega}) = \sum_{n=-\infty}^{\infty} R_N(n)\cdot e^{-j\omega n}$$

$$R_N(e^{j\omega}) = \sum_{n=-N}^{N} e^{-j\omega n} = \sum_{n=-N}^{N}\left(e^{-j\omega}\right)^n = \underbrace{\sum_{n=-N}^{-1}\left(e^{-j\omega}\right)^n}_{} + \underbrace{\sum_{n=0}^{N}\left(e^{-j\omega}\right)^n}_{**}$$

$$(**) = \frac{1 - e^{-j\omega(N+1)}}{1 - e^{-j\omega}} = \frac{1 - \left(\cos(\omega(N+1)) + j\sin(\omega(N+1))\right)}{1 - (\cos\omega - j\sin\omega)}$$

$$= \frac{e^{-j\omega(N+1)}\left(e^{j\omega(N+1)} - 1\right)}{e^{-j\omega}\left(e^{j\omega} - 1\right)} = \frac{e^{-j\omega N}\cancel{e^{-j\omega}}\left(e^{j\omega(N-1)} - 1\right)}{\cancel{e^{-j\omega}}\left(e^{j\omega} - 1\right)}$$

$$(*) = \sum_{n=1}^{N}\left(e^{j\omega}\right)^n = \sum_{n=0}^{N}\left(e^{j\omega}\right)^n - 1 = \frac{\cancel{1} - e^{j\omega(N+1)} \cancel{-1} + e^{j\omega}}{1 - e^{j\omega}} =$$

$$= \frac{e^{j\omega}\left(1 - e^{j\omega N}\right)}{\left(1 - e^{j\omega}\right)}$$

$$(*) + (**) = \frac{e^{j\omega}\left(1 - e^{j\omega N}\right) + e^{-j\omega N}\left(1 - e^{j\omega(N+1)}\right)}{\left(1 - e^{j\omega}\right)}$$

$$R_N(e^{j\omega}) = \frac{\cancel{e^{j\omega}} - e^{j\omega(N+1)} + e^{-j\omega N} - \cancel{e^{j\omega}}}{(1-e^{j\omega})} =$$

$$= \frac{e^{-j\omega N} - e^{j\omega N + j\omega}}{1-e^{j\omega}} = \frac{e^{-j\omega N}\left(1 - e^{j\omega N + j\omega N + j\omega}\right)}{-(e^{j\omega}-1)} =$$

$$= \frac{e^{-j\omega N}\left(1 - e^{j\omega + 2j\omega N}\right)}{-(e^{j\omega}-1)} \overset{*}{=} \frac{e^{-j\omega N}\left(e^{j\omega+2j\omega N} - 1\right)}{e^{j\omega} - 1)}$$

$$\boxed{R_N(e^{j\omega}) = \sum_{n=-N}^{N} e^{-j\omega n} = \frac{e^{-j\omega N}\left(e^{j\omega(2N+1)}-1\right)}{e^{j\omega}-1}}$$

(*) KOLKO ŠTO IMA ČLENOVI NIZATA !!!

$$x(n) = \cos(\omega_0 n) \cdot R_N(n)$$

$$\mathcal{F}\left[R_N(n) \cdot e^{j\omega_0 n}\right] = R_N\left(e^{j(\omega-\omega_0)}\right)$$

$$\mathcal{F}\left[R_N(n) \cdot e^{-j\omega_0 n}\right] = R_N\left(e^{j(\omega+\omega_0)}\right)$$

$$x(n) = \frac{R_N(n)}{2} \cdot e^{j\omega_0 n} + \frac{R_N(n)}{2} e^{-j\omega_0 n}$$

MATHEMATICA

$$\boxed{R_N(e^{j\omega}) = \frac{\sin(2N+1)\omega/2}{\sin(\omega/2)}}$$

$$X(e^{j\omega}) = \frac{1}{2} \frac{\sin(\omega-\omega_0)N/2}{\sin(\omega-\omega_0)/2}$$

$$R_N(e^{j\omega}) = \frac{\overset{*}{\left((\cos\omega - j\sin\omega)\left(\cos(2N+1)\omega + j\sin(2N+1)\omega - 1\right)\right)}}{(\cos\omega-1) + j\sin\omega} \cdot \frac{(\cos\omega-1)-j s}{(\cos\omega-1)+j\sin\omega}$$

$$* = \cos\overset{\alpha}{\omega} \cos\overset{\beta}{(2N+1)\omega} + j\sin\overset{\beta}{(2N+1)\omega} \cdot \cos\omega - j\sin\omega \cdot \cos\overset{\beta}{(2N+1)\omega}$$

$$+ \sin\omega \sin\underset{\beta}{(2N+1)\omega} - \cos\omega + j\sin\omega$$

$$= \underline{\cos\alpha \cdot \cos\beta} + j\,\underline{\sin\beta \cos\alpha} - j\,\underline{\sin\alpha \cdot \cos\beta} + \underline{\sin\alpha \cdot \sin\beta} - \cos\alpha + j\sin\alpha$$

$$= \left/ \begin{array}{l} \sin(\alpha-\beta) = \sin\alpha \cdot \cos\beta - \cos\alpha \cdot \sin\beta \\ \cos(\alpha-\beta) = \cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta \end{array} \right/ = \cos(\alpha-\beta) - j\sin(\alpha-\beta) - \cos\alpha + j\sin\alpha$$

$$* = \cos(\cancel{\omega} - 2N\omega - \cancel{\omega}) - j\sin(\omega - 2N\omega - \omega) - \underline{\cos\omega} + j\sin\omega =$$

$$\cos 2N\omega + j\sin 2N\omega - \cos\omega + j\sin\omega$$

$$④ \cdot (\cos\omega - 1) - j\sin\omega = \cos(2N\omega)\cdot\cos\omega - \cos(2N\omega) - j\sin\omega\cdot\cos(2N\omega) + j\sin 2N\omega\cdot\cos\omega - j\sin(2N\omega) + \sin 2N\omega\cdot\sin\omega - \boxed{\cos^2\omega} + \cos\omega + j\sin\omega\cos\omega + j\sin\omega\cdot\cos\omega - j\sin\omega + \sin^2\omega$$

$$\cos(2N\omega - \omega) - j\sin(2N\omega - \omega) - \cos(2N\omega) - j\sin(2N\omega) + \cos\omega + j\sin\omega + \sin^2\omega - \cos^2\omega) \rightarrow \cos 2\omega)$$

$$R_n = \frac{e^{-j(2N-1)\omega} - e^{j2N\omega} + e^{j\omega} + \cos 2\omega}{\underbrace{(\cos\omega - 1)^2 + \sin^2\omega}_{①}}$$

$$① = \cos^2\omega - 2\cdot\cos\omega + 1 + \sin^2\omega = 2 - 2\cos\omega = 2(\underbrace{1 - \cos\omega}_{①1})$$

$$1$$

$$\cos\left(\frac{\omega}{2} + \frac{\omega}{2}\right) = \cos\frac{\omega}{2}\cdot\cos\frac{\omega}{2} - \sin\frac{\omega}{2}\cdot\sin\frac{\omega}{2}$$

$$①1 = \underbrace{1 - \cos^2\frac{\omega}{2}}_{\sin^2\frac{\omega}{2}} + \sin^2\frac{\omega}{2} = 2\cdot\sin^2\frac{\omega}{2}$$

$$\boxed{R_n = \frac{e^{-j(2N-1)\omega} - e^{j2N\omega} + e^{j\omega} + \cos(2\omega)}{4\cdot\sin^2\frac{\omega}{2}}}$$

$$\boxed{\sum_{n=-N}^{N} e^{-j\omega n} = \frac{e^{-jN\omega}\left(e^{j(2N+1)\omega} - 1\right)}{e^{j\omega} - 1} = \frac{\sin(2N+1)\omega/2}{\sin(\omega/2)}}$$

So: FullSimplify 90 DAVA Mathematica

$$\sum_{n=1}^{N} e^{-j\omega n} = \frac{e^{-jN\omega}\left(e^{jN\omega} - 1\right)}{e^{j\omega} - 1} = \frac{\sin N\omega/2}{\sin \omega/2}$$

$$\sum_{n=0}^{N} e^{-j\omega n} = \frac{e^{-jN\omega}\left(e^{j(N+1)\omega} - 1\right)}{e^{j\omega} - 1} = \frac{\sin(N+1)\omega/2}{\sin \omega/2}$$

$$R_N(n) = \begin{cases} 1 & 0 \leq n \leq N \\ 0 & \text{otherwise} \end{cases} \Rightarrow \boxed{R_N(e^{j\omega}) = \frac{\sin N\omega/2}{\sin \omega/2}}$$

$$x(n) = \cos(\omega_0 n) \cdot R_N(e^{j\omega}) = \frac{1}{2} R_N\left(e^{j(\omega-\omega_0)}\right) + \frac{1}{2} R_N\left(e^{j(\omega+\omega_0)}\right)$$

$$= \frac{1}{2} \frac{\sin(\omega-\omega_0)N/2}{\sin(\omega-\omega_0)/2} + \frac{1}{2} \frac{\sin(\omega+\omega_0)N/2}{\sin(\omega+\omega_0)/2}$$

P3.10 $x(n) = T_{10}(n) = \left[1 - \frac{|n|}{N}\right] R_N(n)$ DTFT = ?

a.) $x(n) = T_{10}(-n)$

b.) $x(n) = T_{10}(n) - T_{10}(n-10)$

c) $x(n) = T_{10}(n) * T_{10}(-n)$

d) $x(n) = T_{10}(n) e^{j\pi n}$

e) $x(n) = T_{10}(n) \cdot T_{10}(n)$

$$\sum_{n=-N}^{N} \frac{|n|}{N} \cdot x^n = \frac{-\left(\frac{1}{x}\right)^{-1+N} - N\left(\frac{1}{x}\right)^{-1+N} + N\left(\frac{1}{x}\right)^{N} + 2x - x^{1+N} - Nx^{1+N} + Nx^{2+N}}{N(-1+x)^2} =$$

$$= \frac{\left(\frac{1}{x}\right)^N \left[-\left(\frac{1}{x}\right)^{-1} - N\left(\frac{1}{x}\right)^{-1} + N\right] + 2x + \left[-1 - N + Nx\right] x^{1+N}}{*} =$$

$$= \frac{\frac{1}{x^N}\left[N - Nx - x\right] + 2x + \left[Nx - N - 1\right]x^{N+1}}{N(-1+x)^2} = ✱$$

$$\sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) x^n = \sum_{n=-N}^{N} x^n - \sum_{n=-N}^{N} \frac{|n|}{N} x^n = \frac{x^{-N}\left(x^{2N+1} - 1\right)}{x - 1} - ✱$$

$$\boxed{y(n) = x_1 \circledast x_2 = \sum_{k=-\infty}^{\infty} x_1(k) \cdot x_2(n-k)}$$

P3.11 $H(e^{j\omega}) = ?$; $|H(e^{j\omega})| = ?$ $\angle H(e^{j\omega}) = ?$

ⓐ $h(n) = 0.9^{|n|}$

ⓑ $h(n) = \mathrm{sinc}(0.2n)\left[u(n+20) - u(n-20)\right]$ $\mathrm{sinc}\, 0 = 1$

ⓒ $h(n) = \mathrm{sinc}(0.2n)\left[u(n) - u(n-40)\right]$

ⓓ $h(n) = \left[(0.5)^n + (0.4)^n\right] u(n)$

ⓔ $h(n) = (0.5)^n \cos(0.1\pi n)$

$\boxed{3.12}$ $x(n) = 3\cos(0.5\pi n + 60^\circ) + 2\sin(0.3\pi n)$

$$\boxed{\sin(\alpha) = \cos\left(\frac{\pi}{2} - \alpha\right)}$$

$\cos(-\alpha) = \cos(\alpha)$

$\sin(\alpha) = \cos\left(\alpha - \frac{\pi}{2}\right)$

$\pi = 180^\circ \qquad 1^\circ = \frac{\pi}{180} \qquad 60^\circ = 60 \cdot \frac{\pi}{180} = \frac{\pi}{3}$

$$x(n) = 3 \cdot \cos\left(0.5\pi n + \frac{\pi}{3}\right) + 2 \cdot \sin(0.3\pi n)$$

$$\boxed{\sum_k A_k \cos(\omega_k n + \theta_k) \rightarrow \boxed{H(e^{j\omega})} \longrightarrow \sum_k A_k \left|H(e^{j\omega_k})\right| \cdot \cos\left(\omega_k n + \theta_k + \angle H(e^{j\omega_k})\right)}$$

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k)$$

$$x(n) = e^{+j\omega_0 n} \longrightarrow \boxed{h(n)} \longrightarrow y(n) = h(n) * e^{+j\omega_0 n} = \sum_{k=-\infty}^{\infty} h(k) \cdot e^{+j\omega_0(n-k)} = \left[\sum_{k=-\infty}^{\infty} h(k) \cdot e^{-j\omega_0 k}\right] \cdot e^{j\omega_0 n}$$

$5 \qquad n(6)$

$-10 \; -9 \; -8 \; -7 \; -6 \; -5$

$$\boxed{y(n) = H(e^{+j\omega_0}) \cdot e^{j\omega_0 n}}$$

$$x(n) = 3 \cdot \cos\left(\overset{\omega_1}{0.5\pi} n + \overset{\theta_1}{\frac{\pi}{2}}\right) + 2 \cdot \cos\left(\overset{\omega_2}{0.3\pi} n \overset{\theta_2}{- \frac{\pi}{2}}\right)$$

$$y(n) = 3 \cdot \left|H(e^{j\omega_1})\right| \cos\left(\omega_1 n + \theta_1 + \angle H(e^{j\omega_1})\right) + 2\left|H(e^{j\omega_2})\right| \cos\left(\omega_2 n + \theta_2 + \angle H(e^{j\omega_2})\right)$$

$\boxed{3.13}$

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\alpha\omega} & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| < \pi \end{cases}$$

ω_c - cutoff frequency

α - phase delay

ⓐ $h_d(n) = ?$ $\qquad h_d(n) = \mathcal{F}^{-1}\left[H_d(e^{j\omega})\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$

ⓑ $$h(n) = \begin{cases} h_d(n), & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

$\cos x = \frac{1}{2}\left[e^{+jx} + e^{-jx}\right]$

$\sin x = \frac{1}{2}\left[e^{jx} - e^{-jx}\right]$

$\sin x = -\frac{j}{2}\left[e^{jx} - e^{-jx}\right]$

$N = 41, \quad \alpha = 20, \quad \omega_c = 0.5\pi$

$$h_d(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\alpha\omega} e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(n-\alpha)} d\omega = \left| \begin{matrix} x = j\omega(n-\alpha) \\ dx = j(n-\alpha)d\omega \\ d\omega = \frac{dx}{j(n-\alpha)} \end{matrix} \right| =$$

$$= \frac{-j}{2\pi} \left. \frac{e^{j\omega(n-\alpha)}}{(n-\alpha)} \right|_{-\omega_c}^{\omega_c} = \frac{-j}{2\pi(n-\alpha)}\left(e^{j\omega_c(n-\alpha)} - e^{-j\omega_c(n-\alpha)}\right) = \boxed{\frac{\sin \omega_c(n-\alpha)}{\pi(n-\alpha)}}$$

$\omega_c = 0.5\pi$

$$h_d(n) = \frac{1}{2} \frac{\sin \omega_c(n-\alpha)}{\omega_c(n-\alpha)} = \operatorname{sinc}\left[\omega_c'(n-\alpha)\right]$$

$\sin[\omega_c'(n-\alpha)]$

$$\boxed{\omega_c' = \frac{\omega_c}{\pi}}$$

P3.14

$$H_d(e^{j\omega}) = \begin{cases} 1\cdot e^{-j\alpha\omega} & \omega_c < |\omega| \le \pi \\ 0 & |\omega| \le \omega_c \end{cases}$$

ⓐ $h_d(n) = ?$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\left[\underbrace{\int_{-\pi}^{-\omega_c} e^{j\omega(n-\alpha)}\,d\omega}_{\circledast} + \int_{\omega_c}^{\pi} e^{j\omega(n-\alpha)}\, d\omega\right]$$

$$= \frac{-j}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{(n-\alpha)}\Bigg|_{-\pi}^{-\omega_c} + \frac{e^{j\omega(n-\alpha)}}{(n-\alpha)}\Bigg|_{\omega_c}^{\pi}\right] = \frac{-j}{2\pi(n-\alpha)}\left[e^{-j\omega_c(n-\alpha)} - e^{-j\pi(n-\alpha)} + e^{j\pi(n-\alpha)} - e^{j\omega_c(n-\alpha)}\right]$$

$$= \frac{1}{\pi(n-\alpha)}\left[-\underbrace{\frac{-j}{2}\left(e^{j\omega_c(n-\alpha)} - e^{-j\omega_c(n-\alpha)}\right)}_{-\sin[\omega_c(n-\alpha)]} + \underbrace{\frac{-j}{2}\left(e^{j\pi(n-\alpha)} - e^{-j\pi(n-\alpha)}\right)}_{\sin\pi(n-\alpha)}\right] = \boxed{\frac{\sin[\pi(n-\alpha)] - \sin[\omega_c(n-\alpha)]}{\pi(n-\alpha)}}$$

$$\circledast = \left|\begin{matrix} \omega = -v \\ d\omega = -dv \\ \omega = -\pi \,\} \\ v = \pi \,\} \\ \omega = -\omega_c \\ v = \omega_c \end{matrix}\right| = -\int_{\omega_c}^{\pi} e^{-j\omega(n-\alpha)}\, d\omega = +j\,\frac{e^{-j\omega(n-\alpha)}}{(n-\alpha)}\Bigg|_{\omega_c}^{\pi} = \frac{-j}{(n-\alpha)}\left[e^{-j\omega_c(n-\alpha)} - e^{-j\pi(n-\alpha)}\right]$$

ⓑ
$$h(n) = \begin{cases} h_d(n), & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases} \qquad \begin{matrix} N = 31 \\ \alpha = 15 \\ \omega_c = 0.5\pi \end{matrix}$$

P.3.15

$$y(n) = \sum_{m=0}^{M} b_m\, x(n-m) - \sum_{l=1}^{N} a_l\, y(n-l)$$

$$H(e^{j\omega}) = \frac{\sum_{m=0}^{M} b_m e^{-j\omega m}}{1 + \sum_{l=1}^{N} a_l e^{-j\omega l}}$$

$$x(n) = e^{j\omega n} \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow y(n) = H(e^{j\omega})\cdot e^{j\omega n}$$

$$H(e^{j\omega})e^{j\omega n} + \sum_{l=1}^{N} a_l\, H(e^{j\omega})\cdot e^{j\omega(n-l)} = \sum_{m=0}^{M} b_m\, e^{j\omega(n-m)}$$

$$H(e^{j\omega})\cancel{e^{j\omega n}}\left[1 + \sum_{l=1}^{N} a_l\, e^{-j\omega l}\right] = \cancel{e^{+j\omega n}}\sum_{m=0}^{M} b_m\, e^{-j\omega m}$$

$$H(e^{j\omega}) = \frac{\sum_{m=0}^{M} b_m\, e^{-j\omega m}}{1 + \sum_{l=1}^{N} a_l\, e^{-j\omega l}} = \frac{\sum_{m=0}^{M} b_m\, e^{-j\omega m}}{\sum_{l=0}^{N} a_l\, e^{-j\omega l}} \qquad a_0 = 1$$

$a = [a_0, a_1, \ldots a_l]$

$e^{j\omega n l} = [e^{j\omega n_1}, e^{j\omega n_2}, \ldots e^{j\omega n_l}]$

$$a' \cdot e^{j\omega n l} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_l \end{bmatrix} [e^{j\omega n_1}, e^{j\omega n_2} \ldots e^{j\omega n_l}] = \begin{bmatrix} a_0 e^{j\omega n_1} & a_0 e^{j\omega n_l} \\ a_1 e^{j\omega'} & \ldots \\ \ldots & \end{bmatrix}$$

$$a \cdot (e^{j\omega n l})' = [a_0\, a_1 \ldots a_l] \begin{bmatrix} e^{j\omega n_1} \\ e^{j\omega n_2} \\ \vdots \\ e^{j\omega n_l} \end{bmatrix} = a_0 e^{j\omega n_1} + a_1 e^{j\omega n_2} + \ldots + a_l e^{j\omega n l}$$

$$\boxed{\sigma^2 = \overline{(\xi - \bar{\xi})^2}}$$

$\boxed{3.16}$ $H(e^{j\omega}) = ?$

ⓐ $y(n) = \sum_{m=0}^{6} x(n-m)$

ⓑ $y(n) = x(n) + 2x(n-1) + x(n-2) - 0.5y(n-1) - 0.25y(n-2)$

ⓒ $y(n) = 2x(n) + x(n-1) - 0.25y(n-1) + 0.25y(n-2)$

ⓓ $y(n) = x(n) + x(n-2) - 0.81y(n-2)$

ⓔ $y(n) = x(n) - \sum_{l=1}^{5} y(n-l)$

$\boxed{3.17}$ $y(n) = \sum_{m=0}^{3} x(n-2m) - \sum_{l=1}^{3} (0.81)^l y(n-2l)$

ⓐ $x(n) = 5 + 10(-1)^n$ ✓

ⓑ $x(n) = 1 + \cos(0.5\pi n + \pi/2)$ ✓?

ⓒ $x(n) = 2 \cdot \sin(\pi n/4) + 3\cos(3\pi n/4)$ ✓

ⓓ $x(n) = \sum_{k=0}^{5} (k+1)\cos(\pi k n/4)$ ✓

ⓔ $x(n) = \cos(\pi n)$ ✓

$x(n);\quad 0 \le n \le 200;$

y_{ss}

$\boxed{Y(j\omega)}$

$$\begin{bmatrix} \cos(\pi K_1 \cdot n_1), & \cos(\pi K_1 n_2), & \ldots & \cos(\pi K_1 n_N) \\ \cos(\pi K_2 n_1), & \cos(\pi K_2 n_2), & \ldots & \cos(\pi K_2 n_N) \\ & & & \\ \cos(\pi K_l n_1), & \cos(\pi K_l n_l), & \ldots & \cos(\pi K_l n_N) \end{bmatrix}$$

$\boxed{3.18}$ $x_a(t) = \sin(1000\pi t)$

$f_s \geq 2 f_{max} = 2 \cdot 1000 = 2\,kHz$

ⓐ $T_s = 0.1\,ms$

ⓑ $T_s = 1\,ms$

ⓒ $T_s = 0.01\,sec$

$f_s = \frac{1}{T_s} = \frac{1}{10^{-4}} = 10^4 = 10\,kHz$

$f_s = 10^3 = 1\,kHz$

$f_s = 10^2 = 100\,Hz$

$\boxed{H(e^{j\omega}) = ?}$

$x(n)=\sin(\overset{\omega_0}{0.5\pi}\, n)$ $\qquad$ $x(n)=\sin(\omega_0 n)$

$$X(e^{j\omega})=\sum_{n=-\infty}^{\infty}x(n)\cdot e^{-j\omega n}=\sum_{n=-\infty}^{\infty}\frac{-j}{2}\left[e^{j\omega_0 n}-e^{-j\omega_0 n}\right]e^{-j\omega n}$$

$$=\frac{j}{2}\sum_{n=-\infty}^{\infty}\left[e^{-j(\omega+\omega_0)n}-e^{-j(\omega-\omega_0)}\right]=\frac{j}{2}\left[\frac{1}{1-e^{-j(\omega+\omega_0)}}-\frac{1}{1-e^{j(\omega-\omega_0)}}\right]$$

$\bullet\ \boxed{n=0:N}$

$$X(e^{j\omega})=\frac{j}{2}\left[\frac{1-e^{-j(\omega+\omega_0)(N+1)}}{1-e^{-j(\omega+\omega_0)}}-\frac{1-e^{-j(\omega-\omega_0)(N+1)}}{1-e^{-j(\omega-\omega_0)}}\right]$$

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cdot\cos(n\omega_0 t)+b_n\sin(n\omega_0 t)$$

$$a_n=\frac{2}{T}\int_{-T/2}^{T/2}f(t)\cos(n\omega_0 t)\,dt \qquad n=0,1,2\ldots$$

$$b_n=\frac{2}{T}\int_{-T/2}^{T/2}f(t)\sin(n\omega_0 t)\,dt \qquad n=1,2,\ldots$$

$f(t)=f(-t)\ \Rightarrow$ PARNA funkcija $\Rightarrow b_n=0$

$-f(t)=f(-t)\ \Rightarrow$ NEPARNA $\Rightarrow a_n=0$

$f(t)=\sin(\omega_0 t)$

$$b_n=\frac{2}{T}\int_{-T/2}^{T/2}\sin\omega_0 t\cdot\sin(n\omega_0 t)\,dt$$

$$f(t)=\sum_{n=-\infty}^{\infty}F_n e^{-jn\omega_0 t} \qquad \boxed{F_n=\frac{1}{T}\int_{-T/2}^{T/2}f(t)e^{-jn\omega_0 t}\,dt}$$

$$F_n=\frac{1}{T}\int_{-T/2}^{T/2}\frac{-j}{2}\left[e^{j\omega_0 t}-e^{-j\omega_0 t}\right]\cdot e^{-jn\omega_0 t}\,dt=$$

$$=-\frac{1}{T}\frac{j}{2}\int_{-T/2}^{T/2}\left[e^{-j(n-1)\omega_0 t}-e^{-j(n+1)\omega_0 t}\right]dt=$$

$$=-\frac{1}{T}\frac{j}{2}\left[\frac{1}{-j(n-1)\omega_0}e^{-j(n-1)\omega_0 t}\Big|_{-T/2}^{T/2}-\underbrace{\frac{1}{-j(n+1)\omega_0}e^{-j(n+1)\omega_0 t}\Big|_{-T/2}^{T/2}}_{(**)}\right]$$

$$\circledast = \int_{-T/2}^{T/2} e^{-j(n-1)\omega_0 t}\,dt = \left| \begin{array}{l} u=-j(n-1)\omega_0 t \\ du = -j(n-1)\omega_0\, dt \\ dt = \frac{du}{-j(n-1)\omega_0}\,, \\ t=-T/2\,;\ u=+j(n-1)\omega_0 T/2 \\ t=T/2\,;\ u=-j(n-1)\omega_0 T/2 \end{array}\right| = \frac{1}{-j(n-1)\omega_0}\int_{+j(n-1)\omega_0 T/2}^{-j(n-1)\omega_0 T/2} e^{u}\,du = \left.\frac{e^{u}}{-j(n-1)\omega_0}\right|_{+j(n-1)\omega_0 T/2}^{-j(n-1)\omega_0 T/2}$$

$$= \frac{1}{-j(n-1)\omega_0}\left[\underbrace{e^{-j(n-1)\omega_0 T/2}} - e^{j(n-1)\omega_0 T/2}\right]$$

ω DOBAR E DIREKTNIOT RAZVOJ !!!

$$\circledast = \frac{-j\cancel{2}}{\cancel{2}(n-1)\omega_0}\left(e^{+j(n-1)\omega_0 T/2} - e^{-j(n-1)\omega_0 T/2}\right) = \frac{2\cdot\sin(n-1)\omega_0 T/2}{(n-1)\omega_0}$$

$$(\ast\ast) = \frac{2\cdot\sin(n+1)\omega_0 T/2}{(n+1)\omega_0 T/2}\ ; \qquad F_n = -\frac{1}{T}\,\frac{j}{2}\left[\circledast - (\ast\ast)\right]$$

$$F_n = \frac{j\cdot\cancel{2}}{\cancel{2}T}\left[\frac{\sin(n+1)\omega_0 T/2}{(n+1)\omega_0 T/2} - \frac{\sin(n-1)\omega_0 T/2}{(n-1)\omega_0 T/2}\right]$$

$$= \left| T = \frac{1}{f_0} = \frac{2\pi}{\omega_0}\right| = \frac{j\omega_0}{2\pi}\left[\overset{①}{\frac{\cancel{\sin(n+1)}\pi}{(n+1)\pi}} - \overset{⑪}{\frac{\cancel{\sin}(n-1)\pi}{(n-1)\pi}}\right]$$

$\forall n \notin 1 \Rightarrow$ ① $= 0$ ⑪ $= 0$

$\boxed{n=1}$ ① $= 0$ ⑪ $= 1$ $\Rightarrow$

$$\boxed{F_n = \frac{j\omega_0}{2\pi} \quad \text{za} \quad \boxed{n=1}} \qquad \boxed{F_1 = \frac{j\omega_0}{2\pi}}$$

$$f(t) = \sum_{n=-\infty}^{\infty} F_n\, e^{+jn\omega_0 t} = \left| \begin{array}{l} F_n = 0 \ \text{za} \\ \forall n \neq 1 \end{array}\right| = F_1 \cdot \underline{\underline{e^{-j\omega_0 t}}}$$

?

$$f(t) = \frac{j\omega_0}{2\pi}\left[\cos\omega_0 t - j\sin\omega_0 t\right] = \frac{\omega_0}{2\pi}\left[\sin(\omega_0 t) + j\cos(\omega_0 t)\right]$$

$$\boxed{F_1 = \frac{j\omega_0}{2\pi} = jb_1} \qquad a_n = 0 \qquad F_1 = \overset{0}{a_1} + j\,b_1$$

$$b_n = \frac{j\omega_0}{2\pi}$$

$$f(t) = \cancel{\frac{a_0}{2}} \to 0 + \sum_{n=1}^{\infty} \cancel{a_n} \to 0 \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

$$\boxed{f(t) = \frac{\omega_0}{2\pi}\cdot\sin(n\omega_0 t)}$$

$\boxed{3.18}$ $x_a(t) = \sin(1000\pi t)$

ⓐ $T_s = 0.1ms$ $f_s = 10^4 = 10kHz$

ⓑ $T_s = 1ms$ $f_s = 10^3 = 1kHz$

ⓒ $T_s = 0.01 sec$ $f_s = 10^2 = 100Hz$

$f_s \geq 2 f_{max} = 2\cdot 500 = 1kHz$

$\omega = 2\pi f$

$2\pi f_{max} = 1000\pi$

$f_0 = f_{max} = 500Hz$

$T_0 = \frac{1}{500} = 0.2\cdot 10^{-2}$

$\Delta t = 0.01ms = 10^{-5}s$

$t: [100\ \ 100]\Delta t = [-10^{-3} : 10^{3}]\ sec = -1ms - 1ms$

$T = 2ms$

$\frac{1}{T} = 0.5\cdot 10^{+3} = 500Hz$

$x_a(t(1)) = \sin(1000\pi(-10^{-3})) = \sin(-\pi) = 0$

$x_a(t(end)) = \sin(1000\pi\cdot 10^{-3}) = \sin(\pi) = 0$

$x(n) = x_a(n * T_s) = \sin(1000\pi\cdot n\cdot \underset{10^{-4}}{T_s}) = \sin((\underbrace{0.1\pi}_{\omega_0}\cdot n))$

$T_0 = 2\cdot 10^{-3} = 2ms$ $\frac{1}{T_s} \geq 2f_0 = \frac{2}{T_0}$ $\boxed{T_0 \geq 2T_s}$

$T_s \leq \frac{T_0}{2} = \frac{2ms}{2} = 1ms$

$\cancel{2\pi} f = 1000\pi = 1000$

$f = \frac{1000}{2} = 500 kHz$

$\Delta t = 0.02ms = 2\cdot 10^{-5}$

$t = [-500 : 500] = [-10ms : 10ms]$

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt$$

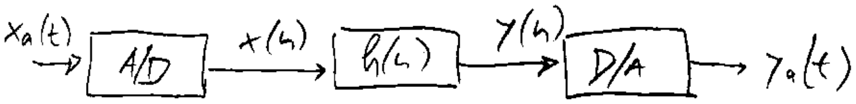
$x_G(m) = x_a(m\Delta t)$

$$X_a(j\Omega) \approx \sum_m x_G(m) e^{-j\Omega m\Delta t}\Delta t = \Delta t \sum_m x_G(m) e^{-j\Omega m\Delta t}$$

$\Omega = -\Omega_{max} \div \Omega_{max}$

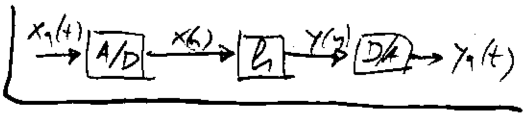
$\boxed{\Omega_{max} = 1000\pi}$

$\boxed{3.19}$



- $f_s = 100Hz$ $T_s = \frac{1}{f_s} = 10^{-2}s = 10ms$
- $h(n) = (0.5)^n u(n)$

$f_s = 100\,Hz \qquad T_s = 10^{-2}\,sec = 10\,mS$



$$\boxed{h(n) = (0.3)^n u(n)}$$

ⓐ $\omega = ?$ $\qquad x_a(t) = 3\cos(20\pi t)$

ⓑ STEADY STATE $y_a(t) = ?$ IF $x_a(t) = 3\cos(20\pi t)$

ⓒ STEADY STATE $y_a(t) = ?$ IF $x_a(t) = 3u(t)$

ⓓ Two other $x_a(t)$ with diferent analog frq with same STADY STATE OUTPUT $y_a(t)$ when $x_a(t) = 3\cos(20\pi t)$

ⓐ $X(n) = x_a(nT_s) = 3\cos(20\pi n T_s) = 3\cos(20\pi n\, 10^{-2}) = 3\cdot\cos(0.2\pi n)$

$\boxed{\omega = 0.2\pi}$ $\quad \pi = 20\pi \quad$ $\boxed{f = \frac{\omega}{2\pi} = 0.1}$ $\qquad \Omega_0 = 10\,Hz \qquad T_0 = 10^{-1}s = 100\,msec$

$\boxed{\text{PERIODICA NA } \cos(x)}$

ⓑ $\Delta t = 0.001 = 1\,msec$

$t = [-100 \cdot 100]\Delta t \nRightarrow \quad \cos(20\pi \cdot \frac{1}{10}) = \cos(-2\pi)$

$t = [-50 : 50]\Delta t \Rightarrow \quad \cos(20\pi \cdot 0.05) = \cos(20\pi \cdot 5\, 10^{-2}) = \cos(\pi)$

$t = [-500 : 500]\Delta t \Rightarrow t = -0.5 : 0.5\ sec$

$n = \cancel{[-100 : 100]} \Rightarrow nT_s = [-50 : 50]\cdot 10^{-2}\,sec = [0.5 : 0.5]\,sec$

$f_0 = 10\,Hz \qquad 2f_0 = 20\,Hz = \Omega_{max}$

$\omega_{max} = 0.2\pi \cdot 2 = 0.4\pi$

$$y_a(m\Delta t) = \sum_{n=n_1}^{n_2} y(n)\,\mathrm{sinc}\left[F_s(m\Delta t - nT_s)\right] \qquad \boxed{t_1 \le m\Delta t \le t_2}$$

$50 = \pi \qquad 1 = \frac{\pi}{50} \qquad$ [illegible] $\qquad 0.2\pi = 0.2 \cdot 50 = 10$

ANAL

$T_0 = 100\,msec$

$T_s = 10\,msec$

$\Omega_0 = 20\pi\ rad/sec$

$F_0 = \frac{\Omega}{2\pi} = 10\,Hz$

DIGI

$\omega_0 = \Omega_0 \cdot T_s = 0.2\pi$

$$\boxed{\Omega_0 = \omega_0 \cdot \frac{1}{T_s} = \omega_0 \cdot f_s}$$

$F_0 = \frac{\Omega_0}{2\pi} = \frac{\omega_0 f_s}{2\pi} = \frac{20\pi}{2\pi} = 10\,Hz$

$\frac{2000}{5000} = \frac{2}{5} = 0.4$

$0.4 \quad 0.2$

ⓓ $\underline{0.2\pi} = (40\pi)\cdot \underbrace{5\cdot 10^{-3}}_{T_s} = (\pi)\,0.04 \cdot 5 = \underline{0.2\pi}$

$\Omega_0 = 40\pi \quad \boxed{f_0 = \frac{\Omega_0}{2\pi} = 20\,Hz} \qquad \boxed{T_s = 5\cdot 10^{-3}} \qquad \boxed{f_s = \frac{1}{5}\cdot 10^{3} = 200\,Hz}$

$T_0 = 0.5\cdot 10^{-1} = 50\,msec, \qquad f_s \geq 2f_0 = 40\,Hz$

$40*\pi*n*T_s = 20*\pi n * T_s \rightarrow 10^{-2}$

$$\boxed{T_s = \frac{10^{-2}}{2}} = 0.5\cdot 10^{-1} = \underline{\underline{50\,msec}}$$

$F_s \geq 2F_0 \ ; \quad \frac{1}{T_s} \geq 2\,\frac{1}{T_0} \qquad T_s \leq T_0/2 = \frac{50\,msec}{2} = \underline{25\,msec}$

$T_s = 25 \cdot 10^{-3}$ $F_s = \frac{1}{25} \cdot 10^3 = 40Hz$ $T_0 = 50 msec$ $F_0 = 20Hz$

$T_s \le 25 msec$

$\boxed{T_s = 33 m sec}$ $F_0 = 20Hz$

→ OVA DAVA IST AUTPUT KAKO cos(20πt) PRI VEĆ cos(40πt)

~~$40 * \pi * n_{max1} * T_{s1} = 20 * \pi * n_{max} T_s$~~

~~$40 * \pi * n_{max1} * T_{s1} = 20 * \pi * 50 * 10^{-2}$~~

~~$\boxed{n_{max} T_{s1}} = \frac{20 * \pi * 50 * 10^{-2}}{40\pi \;\; 2} = \boxed{25 \cdot 10^{-2}}$~~

$T_s > 25 msec$

~~$n_{max} 30 \cdot 10^{-3} = 25 \cdot 10^{-2}$ $n_{max} = \frac{25 \cdot 10^{-2}}{30 \cdot 10^{-3}} = \frac{25}{30} \cdot 10 = 8$~~

$$\boxed{n_{max} T_s = 0.5 \; sec \Rightarrow \left| \begin{array}{l} T_s = 0.033 \\ n_{max} = \frac{0.5}{0.033} \end{array} \right| = 15}$$

$F_s \ge 2 f_0$

$\frac{1}{T_s} \ge \frac{2}{T_0}$

$\boxed{T_s \le \frac{T_0}{2}}$

$\boxed{\Omega_0 = 30\pi}$ $\boxed{F_0 = \frac{30\pi}{2\pi} = 15Hz}$ $T_0 = \frac{1}{15} = 66.7 msec$

$\boxed{T_s \le \frac{T_0}{2}}$ $\boxed{T_s \le 33,35} \Rightarrow$ Nyqust limit

$$\boxed{\begin{array}{ll} I_F \;\; \boxed{T_s = 40 msec} & n_{max} \cdot T_s = 500 msec \\ F_s = \frac{1}{40} \cdot 10^3 = 25Hz & \boxed{n_{max} = \frac{500}{40} = 12,5} \end{array}}$$

$T_s = 35 msec$ $n_{max} = \frac{500}{35} = 14$

$F_s = 28 Hz$

$T_s = 38 msec;$ $F_s = 26 Hz;$ $n_{max} = \frac{500}{38} = 13$

~~$T_s = 33 msec$ $F_s = 30Hz$ $n_{max} = 15$~~

OVA DAVA IST AUTPUT KAKO cos(20πt) PRI VEĆ cos(30πt)

ⓔ $\boxed{w_0 = 0.2\pi}$ $\boxed{w_c > 0.2\pi}$

$\Omega_0 = w_0 \cdot f_s = w_0 / T_s = 0.2\pi \cdot \frac{1}{10^{-2}} = \boxed{20\pi}$

$\boxed{\Omega_c \ge 20\pi}$ $\boxed{F_c \ge 10Hz}$

$\boxed{P3.20}$ $x_a(t) = \sin(20\pi t) \quad 0 \le t \le 1$ $\boxed{\Delta t = 0.001}$

$T_s = 0.01;\ 0.05;\ 0.1$ sec

ⓐ For each T_s plot $x(n)$

ⓑ reconstruct $y_a(t)$ using sinc interpolation ($\Delta t = 0.001$)
Determine frequency in $y_a(t)$

ⓒ reconstruct $y_a(t)$ using cubic spline interpolation
Determine F_0

ⓓ Comment on your results

$$\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}}$$
DTFT

$\Omega_0 = 20\pi$ rad/s $\quad F_0 = \frac{\Omega_0}{2\pi} = 10\,Hz$

① $\Omega_0 = \omega_0 \cdot f_s \qquad \omega_0 = \Omega_0 \cdot T_s = 20\pi \cdot 0.01 = 0.2\pi$

② $\omega_0 = \Omega_0 \cdot T_s = 20\pi \cdot 0.05 = \pi$

③ $\omega_0 = \Omega_0 \cdot T_s = 20\pi \cdot 0.1 = 2\pi$

$T_0 = \frac{1}{F_0}$; $\quad T_0 = 0.1s$;

Nyquist
$\boxed{F_s \geq 2 \cdot F_0}$ $\quad \frac{1}{T_s} \geq 2 \cdot \frac{1}{T_0}$ $\quad \boxed{T_s \leq \frac{T_0}{2}}$

$$\boxed{T_s \leq \frac{0.1}{2} = 0.05s}$$

$\boxed{P3.21}$ $x_a(t) = \sin(20\pi t + \pi/4)$; $\quad T_s = 0.05$ sec $\quad \Delta t = 0.001$

ⓐ Plot $x_a(t)$ superimpose $x(n)$ using plot(n, x, 'o')

ⓑ $y_a(t) = ?$ using sinc interpolation ; superimpose $x(n)$

ⓒ $y_a(t) = ?$ using spline – ″ – ; – ″ –

THE Z-TRANSFORM

z-transform of sequence $x(n)$

$$\boxed{X(z) = \mathcal{Z}[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}}$$

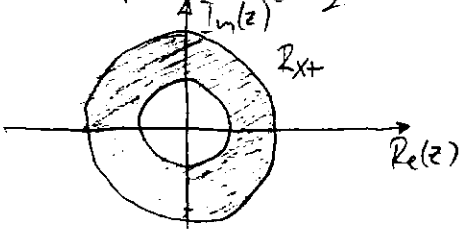
Region of convergence $\quad R_{x-} < |z| < R_{x+}$

$$\boxed{x(n) = \mathcal{Z}^{-1}[X(z)] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz}$$

C - counterclockwise contur lying in ROC

z - complex frequency $\quad z = |z| \cdot e^{j\omega}$

$|z|$ - ATENUATION
ω - REAL FREQUENCY



For $|z| = 1 \quad (z = e^{j\omega})$

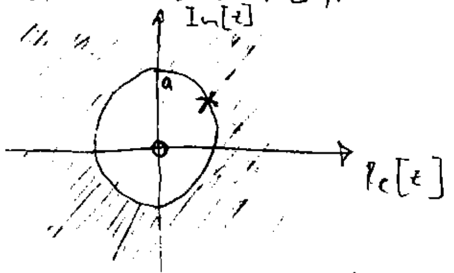
$$X(z)\Big|_{z=e^{j\omega}} = X(e^{j\omega}) = \sum_{n=\infty}^{\infty} x(n) e^{j\omega n}$$
$$= \mathcal{F}[x(n)]$$

$\boxed{ex. 4.1}$ $x_1(n) = a^n u(n)$ $0<|a|<\infty$ POSITIVE TIME SEQUENCE, RIGH-SIDED SEQUENCE

$$\mathcal{X}_1(z) = \sum_{n=0}^{\infty} a^n \cdot z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^{n} = \frac{1}{1-\frac{a}{z}} = \left\| \boxed{\left|\frac{a}{z}\right| < 1} \right\| = \frac{z}{z-a}$$

$B(z) = z$ NUMERATOR POLYNOMIAL $\boxed{NULI}$ ZEROS

$A(z) = z-a$ DENUMERATOR POLYNOMAL $\boxed{POLOVI}$ POLE



$|z| > a \Rightarrow ROC_1$

$$\underbrace{|a|}_{R_{x-}} < |z| < \underbrace{\infty}_{R_{x+}}$$

$\boxed{ex. 4.2}$ $x_2(n) = -b^n u(-n-1)$; $0<|b|<\infty$ NEGATIVE TIME SEQUENCE, LEFT SIDED SEQUENCE

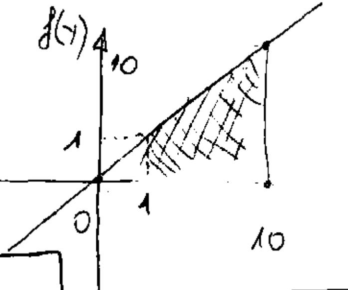
$$\mathcal{X}_2(z) = \sum_{-\infty}^{-1} b^n z^{-n} = \sum_{-\infty}^{-1} \left(\frac{b}{z}\right)^n = -\sum_{1}^{\infty} \left(\frac{b}{z}\right)^{-n} = -\sum_{1}^{\infty} \left(\frac{z}{b}\right)^n$$

$$\int_{-10}^{-1} x\,dx = \left| \begin{array}{l} y=-x \\ dx=-dy \\ x=-10;\ y=10 \\ x=-1;\ y=1 \end{array} \right| = -\int_{10}^{1} -y\,dy = \int_{10}^{1} y\,dy = \frac{y^2}{2}\Big|_{10}^{1} = \frac{1}{2}\left[1^2-10^2\right]$$

$$= \frac{1}{2}\left[1-100\right]$$

$$= \frac{1}{2}(-99) = -49{,}5$$

$$\int_{10}^{1} y\,dy = -\int_{1}^{10} y\,dy$$



$\to y$ $\boxed{P_1 = \frac{10\cdot 10}{2}} = 50$

$P_2 = \frac{1\cdot 1}{2} = 0.5$

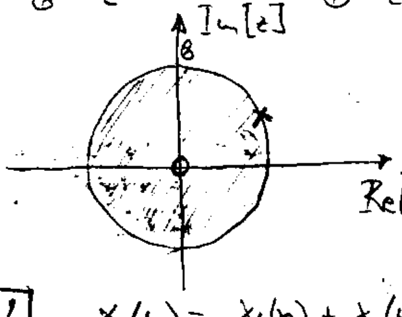
$\boxed{P = P_1 - P_2 = 50 - 0.5 = 44{,}5}$

$$\mathcal{X}_2(z) = 1 - \sum_{n=0}^{\infty}\left(\frac{z}{b}\right)^n = 1 - \frac{1}{1-\frac{z}{b}} =$$

$$= 1 - \frac{b}{b-z} = \frac{b-z-b}{b-z} = \frac{z}{z-b}$$

$\left|\frac{z}{b}\right| < 1$, $|z|<|b|$

$$\underbrace{\boxed{0 <}}_{R_{x-}} \boxed{|z|} \underbrace{\boxed{< b}}_{R_{x+}}$$



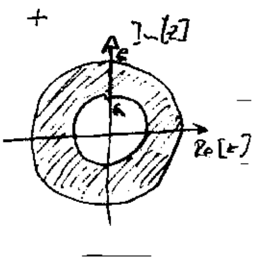
If: $b=a$ $\mathcal{X}_1(z) = \mathcal{X}_2(z)$

$$\boxed{ROC_1 \neq ROC_2}$$

$\boxed{ex. 4.3}$ $x_3(n) = x_1(n) + x_2(n) = a^n u(n) - b^n u(-n-1)$ TWO SIDED SEQUENCE

$$\mathcal{X}_3(z) = \sum_{n=0}^{\infty} a^n z^{-n} - \sum_{-\infty}^{-1} b^n z^{-n} = \left\{\frac{z}{z-a},\ ROC_1: |z|>a\right\} +$$

$$\left\{\frac{z}{z-b},\ ROC_2: |z|<b\right\} = \frac{z}{z-a} + \frac{z}{z-b};\quad ROC_3: ROC1 \cap ROC2$$

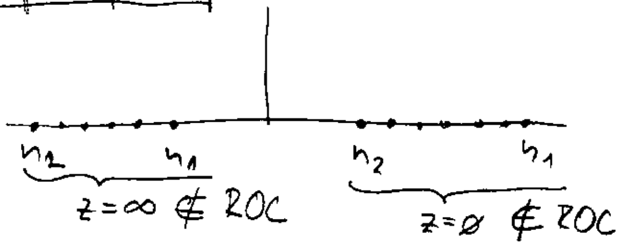


If: $|a| < |b|$ $ROC_3: |a| < |z| < |b|$

$$\int_0^{\pi} \sin x \, dt = -\cos x \Big|_0^{\pi} = -(-1-1) = ②$$

Property 5:

$n = n_2 : n_1$



$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) =$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, h_{fdd}(k-n)$$

$n = nx(1) + nh(1) : nx(end) + nh(end);$

PROPERTIES OF THE Z-TRANSFORM

1) Linearity

$$\mathcal{Z}[a_1 x_1(n) + a_2 x_2(n)] = a_1 X_1(z) + a_2 X_2(z); \quad ROC: ROC_{x_1} \cap ROC_{x_2}$$

2) Semple shifting

$$\mathcal{Z}[x(n-n_0)] = z^{-n_0} X(z); \quad ROC: ROC_x$$

3) Frequency shifting

$$\mathcal{Z}[a^n x(n)] = X\left(\frac{z}{a}\right); \quad ROC: ROC_x \text{ scaled by } |a|$$

4.) Folding $\mathcal{Z}[x(-n)] = X(1/z)$; ROC: Inverted ROC_x

5.) Complex conjugation: $\mathcal{Z}[x^*(n)] = X^*(z^*)$; ROC: ROC_x

6.) Diferentiation in Z domain: $\mathcal{Z}[nx(n)] = -z \dfrac{dX(z)}{dz}$; ROC: ROC_x

MULTIPLICATION by RAMP PROPERTY

7.) MULTIPLICATION; $\mathcal{Z}[x_1(n)\, x_2(n)] = \dfrac{1}{2\pi j} \oint_C X_1(v)\, X_2\left(\dfrac{z}{v}\right) v^{-1}\, dv$

ROC: $ROC_{x_1} \cap$ Inverted ROC_{x_2}

8.) Convolution: $\mathcal{Z}[x_1(n) * x_2(n)] = X_1(z) \cdot X_2(z)$ ROC: $ROC_{x_1} \cap ROC_{x_2}$

Ex. 4.4 $X_1(z) = 2 + 3z^{-1} + 4z^{-2}$ $X_2(z) = 3 + 4z^{-1} + 5z^{-2} + 6z^{-3}$

$X_3(z) = X_1(z)\, X_2(z)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}; \quad X_1(z) = 2 \cdot z^0 + 3 \cdot z^{-1} + 4 \cdot z^{-2}$$

$x_1(n) = \{\underset{\uparrow}{2}, 3, 4\}$ $x_2(n) = \{\underset{\uparrow}{3}, 4, 5, 6\}$

$x_3 = x_1 * x_2 = \text{conv}(x_1, x_2) = [6, 17, 34, 43, 38, 24]$

2, 3, 4

k=0 6, 5, 4, 3 → 6

n=1 6 5 4 3 → 17

n=2 6 5 4 3 → 34

$$X_3(z) = 6 + 17z^{-1} + 34z^{-2} + 43z^{-3} + 38z^{-4} + 24z^{-5}$$

Ex. 4.5 $X_1(z) = z + 2 + 3z^{-1}$ $X_2(z) = 2z^2 + 4z + 3 + 5z^{-1}$

$X_3(z) = X_1(z) * X_2(z)$

$x_1 = \{1, \underset{\uparrow}{2}, 3\}$ $x_2 = \{2, 4, \underset{\uparrow}{3}, 5\}$

$n_1 = -1 : 1$ $n_2 = -2 : 1$

$n_3 = -3 : 2$

$x_2(-k)$ ← 5 3 4 2

			-3	-2	-1	0				
	n				1	2	3			
$x_2(-3-k)$	-3	5	3	4	2					2
$x_2(-2-k)$	-2		5	3	4	2				8
$x_2(-1-k)$	-1			5	3	4	2			17
$x_2(-k)$	0				5	3	4	2		23
	1					5	3	4		
	2						5	3		

$x_3 = \{\overset{-3}{2}, \overset{-2}{8}, \overset{-1}{17}, \underset{\uparrow}{\overset{0}{23}}, \overset{1}{19}, \overset{2}{15}\};$

$$X_3(z) = 2 \cdot z^3 + 8 \cdot z^2 + 17 \cdot z + 23 + 19z^{-1} + 15z^{-2}$$

$$X(z) = \frac{B(z)}{A(z)} \quad ; \quad X_2(z) = \frac{X_3(z)}{X_1(z)}$$

$$e^{jx} = \cos x + j\sin x$$
$$e^{-jx} = \cos x - j\sin x$$
$$\cos x = \frac{1}{2}[e^{jx} + e^{-jx})$$

$\boxed{\text{Ex. } 4.6}$ $$x(n) = (n-2)(0.5)^{(n-2)}\cos\left[\frac{\pi}{3}(n-2)\right]u(n-2)$$

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}}$$

- Using sample shifting property: $\mathcal{Z}[x(n-n_0)] = z^{-n_0}\mathcal{Z}[x(n)]$

$$\mathcal{Z}[x(n)] = z^{-2}\,\mathcal{Z}\left[n\cdot(0.5)^n\cos\left[\frac{\pi}{3}n\right]u(n)\right]$$

- Using multiplication by ramp: $\mathcal{Z}[n\,x(n)] = -z\frac{d}{dz}\mathcal{Z}[x(n)]$

$$① = \mathcal{Z}[x(n)] = -z^{-1}\,\mathcal{Z}\left[(0.5)^n\cos\left[\frac{\pi}{3}n\right]u(n)\right]$$

$$\mathcal{Z}[a^n\cos(\omega_0 n)\,u(n)] = \frac{1 - (a\cos\omega_0)z^{-1}}{1 - (2a\cos\omega_0)z^{-1} + a^2 z^{-2}} \qquad |z| > a$$

MATHEMATICA:
$$\frac{z(-a - a e^{2j\omega_0} + 2e^{j\omega_0}z)}{2(a^2 e^{j\omega_0} - az - a e^{2j\omega_0}z + e^{j\omega_0}z^2)} = \frac{-\cancel{z}\,\cancel{z}e^{j\omega_0}(a\cdot e^{-j\omega_0}z^{-1} + a e^{j\omega_0}z^{-1} - 2)}{2\cdot\cancel{z^2}\,e^{j\omega_0}(a^2 z^{-2} - a z^{-1}\cdot e^{-j\omega_0} - a z^{-1}e^{j\omega_0} + 1)} =$$

$$= \frac{-(a\cdot 2\cdot\cos\omega_0 z^{-1} - 2)}{2\cdot(a^2 z^{-2} - a\cdot 2\cdot\cos\omega_0\cdot z^{-1} + 1)} = \boxed{\frac{1 - a\cdot\cos\omega_0\cdot z^{-1}}{1 - 2a\cdot\cos\omega_0\cdot z^{-1} + a^2 z^{-2}}}$$

MAPLE:
$$\frac{\left(\frac{z}{a} - \cos(\omega_0)\right)z}{a\left(\frac{z^2}{a^2} - \frac{2z\cos\omega_0}{a} + 1\right)} = \frac{(1 - \cos\omega_0\cdot a\cdot z^{-1})\cancel{\frac{z^2}{a}}}{\cancel{a}\,\cancel{\frac{z^2}{a^2}}\left(1 - 2\cancel{z}\cdot\cos\omega_0\cdot\frac{a^2}{z^2}\cdot\frac{1}{\cancel{a}} + a^2\cdot z^{-2}\right)}$$

$$= \frac{1 - a\cdot\cos\omega_0\cdot z^{-1}}{1 - 2\cdot a\cdot\cos\omega_0\cdot z^{-1} + a^2 z^{-2}} \qquad \left[(x^2)' = 2\cdot x^{2-1}\right]$$

$$\mathcal{Z}\left[0.5^n\cos\left(\frac{\pi}{3}n\right)u(n)\right] = \frac{1 - 0.5\cdot\overset{0.5}{\cancel{\cos\left(\frac{\pi}{3}\right)}}\cdot z^{-1}}{1 - \cos\left(\frac{\pi}{3}\right)z^{-1} + 0.25z^{-2}} = \frac{1 - 0.25z^{-1}}{1 - 0.5z^{-1} + 0.25z^{-2}} ⊛ \qquad \underline{|z| > 0.5}$$

$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,du/dx - u\,dv/dx}{v2}}$$

$$⊛ = \frac{(1 - 0.5z^{-1} + 0.25z^{-2})(0.25z^{-2}) - (1 - 0.25z^{-1})(0.5z^{-2} - 0.5z^{-3})}{1 - z^{-1} + 0.75z^{-2} - 0.25\cdot z^{-3} + 0.0625\,z^{-4}}$$

$$= \frac{-0.25z^{-2} + 0.5z^{-3} - 0.0625z^{-4} \ldots}{1 - z^{-1} + 0.75z^{-2} - 0.25z^{-7} + 0.0625z^{-4}}$$

$$\mathcal{Z}[x(n)] = -z^{-1}\cdot ⊛ = \frac{0.25z^{-3} - 0.5z^{-4} + 0.0625z^{-5}}{1 - z^{-1} + 0.75z^{-2} - 0.25z^{-3} + 0.0625z^{-4}}$$

INVERSION OF THE Z-TRANSFORM

$$x(n) = Z^{-1}[X(z)] = \frac{1}{2\pi j} \oint_C X(z)\, z^{n-1}\, dz$$

$$\frac{5x^3 - x^2 + 6}{x-4}$$

$(x-4)\cdot 5x^2$

$$\frac{b}{a} = \frac{6 + 0x - x^2 + 5x^3}{-4 + x}$$

$$\begin{array}{r|l} & 208 \\ \hline 26 & 5430 \\ & 52 \\ \hline & \ \ 23 \\ & \ \ \ 0 \\ \hline & \ 230 \\ & -208 \\ \hline & \ \ 22 \end{array}$$

$5430 = 208 \cdot 26 + 22$

$$\begin{array}{r|l} & 207 \\ \hline ⓶3 & 4770 \\ & 46 \\ \hline & \ 17 \\ & \ \ 0 \\ \hline & \ 170 \\ & \ 161 \\ \hline & \ \ \ 9 \end{array}$$

$4770 = 23 \cdot 207 + 9$

$$\boxed{\frac{x^5 - 2x^4 + 3x + 7}{x^2+1} = (x^3 - 2x^2 - x + 2) + \frac{4x+5}{x^2+1}}$$

$$\begin{array}{r|l} & x^3 - 2x^2 - x + 2 \\ \hline x^2+1 & x^5 - 2x^4 + 0x^3 + 0x^2 + 3x + 7 \\ & x^5 \quad + x^3 \\ \hline & -2x^4 - x^3 + 0x^2 \\ & -2x^4 \quad - 2x^2 \\ \hline & -x^3 + 2x^2 + 3x \\ & -x^3 \quad - x \\ \hline & 2x^2 + 4x + 7 \\ & 2x^2 \quad + 2 \\ \hline & 4x + 5 \end{array}$$

$$x^5 - 2x^4 + 3x + 7 = (x^2+1)(x^3 - 2x^2 - x + 2) + 4x + 5$$

$$\begin{array}{r|l} & 5x^2 + 19x + 76 \\ \hline x-4 & 5x^3 - x^2 + 0x + 6 \\ & 5x^3 - 20x^2 \\ \hline & 19x^2 + 0x \\ & 19x^2 - 76x \\ \hline & +76x + 6 \\ & 76x - 304 \\ \hline & +310 \end{array}$$

$$\boxed{\frac{5x^3 - x^2 + 6}{x-4} = 5x^2 + 19x + 76 + \frac{310}{x-4}}$$

rem (a, b, x, 2) – REMINDER

quo (a, b) – QUOTIENT (KOLIČNIK)

$$X(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}}, \quad R_{x-} < |z| < R_{x+}$$

$$X(z) = \underbrace{\frac{\bar{b}_0 + \bar{b}_1 z^{-1} + \ldots + \bar{b}_{N-1} z^{-N}}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}}}_{\text{PROPER RATIONAL PART}} + \underbrace{\sum_{k=0}^{M-N} C_k z^{-k}}_{\text{POLYNOMIAL PART M } M \geq N}$$

$$\boxed{X(z) = \sum_{k=1}^{N} \frac{R_k}{1 - p_k z^{-1}} + \underbrace{\sum_{k=0}^{M-N} C_k z^{-k}}_{M \geq N}}$$

$\boxed{[R, p, C] = \text{residuez}(b, a)}$

p_k - k-th pole of $X(z)$
R_k - residue at p_k

$$R_k = \frac{\bar{b}_0 + \bar{b}_1 z^{-1} + \ldots + \bar{b}_{N-1} z^{-(N-1)}}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}} (1 - p_k z^{-1}) \Big|_{z = p_k}$$

without this part for repeated poles only!!!

Repeated poles & zeros

$$\sum_{l=1}^{r} \frac{R_{k,l} z^{-(l-1)}}{(1 - p_k z^{-1})^l} = \frac{R_{k,1}}{1 - p_k z^{-1}} + \frac{R_{k,2} z^{-1}}{(1 - p_k z^{-1})^2} + \ldots + \frac{R_{k,r} z^{-(r-1)}}{(1 - p_k z^{-1})^r}$$

$$x(n) = \sum_{k=1}^{N} R_k Z^{-1}\left[\frac{1}{1 - p_k z^{-1}}\right] + \underbrace{\sum_{k=0}^{M-N} C_k \delta(n-k)}_{M \geq N}$$

$$\mathcal{Z}[x(n-k)] = X(z) \cdot z^{-k}$$
$$\mathcal{Z}[\delta(n-k)] = 1 \cdot z^{-k}$$
$$\mathcal{Z}^{-1}[z^{-k}] = \delta(n-k)$$

$$\boxed{\mathcal{Z}^{-1}\left[\frac{z}{z - p_k}\right] = \begin{cases} p_k^n u(n) & |z_k| < R_{x-} \\ -p_k^n u(-n-1) & |z_k| \geq R_{x+} \end{cases}}$$

Ex. 4.7

PROBAJ SO FILTER + $\delta(n)$

$$X(z) = \frac{z}{3z^2 - 4z + 1} = \frac{z}{3z^2(1 - \frac{4}{3} z^{-1} + \frac{1}{3} z^{-2})} = \frac{\frac{1}{3} z^{-1}}{(1 - \frac{4}{3} z^{-1} + \frac{1}{3} z^{-2})} =$$

$$= \Big/ \frac{(3z - 1)(z - 1)}{3z^2} = (z^{-1} - \frac{1}{3} z^{-2})(z - 1) = \cancel{z^{-1}}(1 - \frac{1}{3} z^{-1}) \cancel{z}(1 - z^{-1})$$

$$= (1 - z^{-1})(1 - \frac{1}{3} z^{-1}) \Big/ = \frac{\frac{1}{3} z^{-1}}{(1 - z^{-1})(1 - \frac{1}{3} z^{-1})} = \frac{a}{(1 - z^{-1})} + \frac{b}{(1 - \frac{1}{3} z^{-1})}$$

$$a \cdot (1 - \frac{1}{3} z^{-1}) - a(1 - z^{-1}) = \frac{1}{3} z^{-1} \qquad \boxed{b = -a}$$

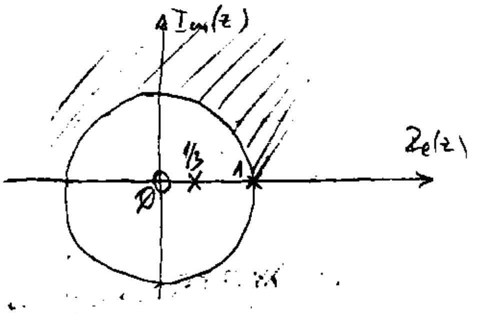
$$x = 3a(1 - \frac{1}{3} x) + 3a(1 - x); \qquad \boxed{a = \frac{1}{2}}$$

→ GO REŠIV VO MAPLE

$$\boxed{X(z) = \frac{\frac{1}{2}}{(1 - z^{-1})} - \frac{\frac{1}{2}}{(1 - \frac{1}{3} z^{-1})} = \frac{1}{2}\left(\frac{1}{1 - z^{-1}}\right) - \frac{1}{2}\left(\frac{1}{1 - \frac{1}{3} z^{-1}}\right)}$$

$$\boxed{z_1 = 1 \qquad z_2 = \frac{1}{3}}$$

(a)



ROC: $\boxed{1 < |z| < \infty}$

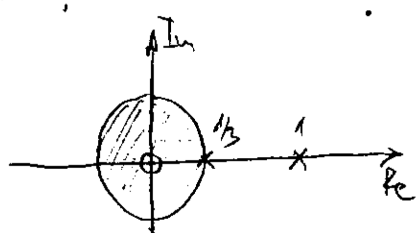
$|z_1| \leq R_{x-} = 1 \qquad |z_2| \leq 1$

$\boxed{\begin{array}{l} z_1 = 1 \\ z_2 = \frac{1}{3} \end{array}}$

$$x_1(n) = \frac{1}{2} u(n) - \frac{1}{2}\left(\frac{1}{3}\right)^n u(n)$$

RIGHT SIDED SEQUENE

(b)

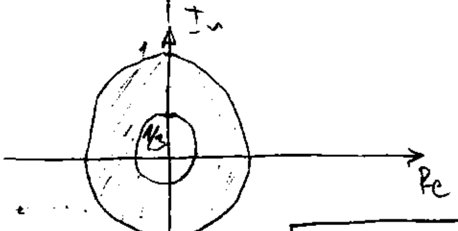


ROC: $\boxed{0 < |z| < \frac{1}{3}}$; $|z_1| \geq R_{x+} = \frac{1}{3} \qquad \cancel{|z_2|} > R_{x+}$ <1

LEFT SIDED SEQUEN

$$x_2(n) = -\frac{1}{2} u(-n-1) + \frac{1}{2}\left(\frac{1}{3}\right)^n u(-n-1) =$$
$$= \frac{1}{2}\left(\frac{1}{3}\right)^n u(-n-1) - \frac{1}{2} u(-n-1)$$

(c)



ROC: $\boxed{\frac{1}{3} < |z| < 1}$ $\qquad |z_1| \leq R_{x-} = \frac{1}{3} \qquad |z_2| \geq R_{x+} = 1$

$$x_3(n) = \frac{1}{2} u(n) - \frac{1}{2}\left(\frac{1}{3}\right)^n u(-n-1)$$

TWO SIDED SEQUENCE

TAYLOR SERIES: $\boxed{f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(c)(x-c)^n}$ n-TI IZVOD

APPROXIMATION OF THE FUNCTION f(x) AROUND VALUE "c"

- MATLAB IMPLEMENTATION

$$X(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \dots + a_N z^{-N}} = \sum_{k=1}^{N} \frac{R_k}{1 - p_k z^{-1}} + \underbrace{\sum_{k=0}^{M-N} C_k z^{-k}}_{M \geq N}$$

$\boxed{[R\ p\ C] = \text{residuez}(b, a) \qquad [b, a] = \text{residuez}(R, p, C)}$

$$\cancel{X(z)} = \frac{z}{1 - 4z + 3z^2} = 1.5 \frac{1}{1 - 3z^{-1}} - \frac{0.5}{1 - z^{-1}} = \frac{0.5}{\frac{1}{3} - z^{-1}} - \frac{0.5}{1 - z^{-1}}$$

MAPLE

$$-\frac{1}{2(-1+x)} + \frac{3}{2(-3+x)} = \left|\begin{array}{l} x = \\ = z^{-1} \end{array}\right| = \frac{0.5}{1 - z^{-1}} - \frac{0.5 \cdot \cancel{3}}{\cancel{3}(1 - \frac{1}{3} z^{-1})} = \frac{0.5}{1 - z^{-1}} - \frac{0.5}{1 - \frac{1}{3} z^{-1}}$$

Ex. 4.8 $\quad X(z) = \frac{z}{1 - 4z + 3z^2} = \frac{z}{z^2(z^{-2} - 4z^{-1} + 3} = \frac{0 + z^{-1}}{3 - 4z^{-1} + z^{-2}}$

$\left.\begin{array}{l} b = [0, 1] \\ a = [3, -4, 1] \end{array}\right\}$ $\quad R = [0.5, -0.5] \qquad C = [\ ]$

$p = [1, 0.33]$

$$\boxed{X(z) = \frac{0.5}{1 - z^{-1}} - \frac{0.5}{1 - \frac{1}{3} z^{-1}}}$$

$[b, a] = \text{residuez}(R, p, C) \qquad b = [0, 0.333] \qquad a = [1, -1.333, 0.333]$

$$X(z) = \frac{0 \cdot z^0 + \frac{1}{3} z^{-1}}{1 - \frac{4}{3} z^{-1} + \frac{1}{3} z^{-2}} = \frac{z^{-1}}{3 - 4z^{-1} + z^{-2}} = \frac{\cancel{z^{-1}}}{z^{-2}(3 \cdot z^2 - \quad 1)}$$

$$X(z) = \frac{z}{1 - 4z + 3z^2}$$

ex. 4.9

$$X(z) = \frac{1}{(1-0.9z^{-1})^2(1+0.9z^{-1})}, \quad |z| > 0.9$$

$$= X(z) = \frac{1 \cdot z^0}{1 - 0.9z^{-1} - 0.81z^{-2} + 0.729z^{-3}}$$

MAPLE !!

expand(f,x)

OR: MATLAB poly(0.9,0.9,-0.9)

b = [1];
a = [1, -0.9, -0.81, 0.729];

~~R = [0.3754 − 3.1728i, 0.3754 [illegible]]~~
~~p = [0.8986 + 0.0708i, 0.8986 − 0.0708i, 0.8972]~~

R = [0.25, 0.25, 0.5]
p = [-0.9, 0.9, 0.9]

$$X(z) = \frac{0.25}{1+0.9z^{-1}} + \frac{0.25}{1-0.9z^{-1}} + \frac{0.5}{1-0.9z^{-1}}$$

SPORED MENE !!
NO AKO NE SE POVTORUVAA POLOVITE !!(8))

PROAKIS

REPEATED POLE !!!

$$X(z) = \overbrace{\frac{0.25}{1-0.9z^{-1}} + \frac{0.5}{(1-0.9z^{-1})^2}} + \frac{0.25}{1+0.9z^{-1}}$$

REPEATED POLE

$$\sum_{l=1}^{r} \frac{R_{k,l}}{(1-p_k z^{-1})^l} = \frac{R_{k,1}}{1-p_k z^{-1}} + \frac{R_{k,2}}{(1-p_k z^{-2})^l}$$

$$= \frac{0.25}{1-0.9z^{-1}} + \frac{(0.9z^{-1})}{(1-0.9z^{-1})^2} \frac{0.5}{0.9} \boxed{z} + \frac{0.25}{1+0.9z^{-1}} \qquad |z| > 0.9$$

$$\mathcal{Z}[x(n-n_0)] = z^{-n_0} X(z);$$

$$\mathcal{Z}[n a^n u(n)] = \frac{a z^{-1}}{(1-0.9z^{-1})^2}, \quad \mathcal{Z}[(n+1)a^{n+1}u(n+1)] = z^{+1} \frac{a z^{-1}}{(1-0.9z^{-1})^2}$$

$$= 0.25(0.9)^n u(n) + (n+1)(0.9)^{\overset{n}{\cancel{n+1}}} u(n+1) \frac{0.5}{\cancel{0.9}} + 0.25(-0.9)^n \cdot u(n) =$$

$$= 0.25(0.9)^n u(n) + 0.5(n+1)(0.9)^n u(n+1) + 0.25(-0.9)^n \cdot u(n) =$$

$$= 0.25(0.9)^n u(n) + 0.5 \cdot n(0.9)^n u(n+1) + (0.5)(0.9)^n \cdot u(n+1) + 0.25(-0.9)^n u(n)$$

$$= 0.75(0.9)^n u(n) + 0.5\, n(0.9)^n u(n) + 0.25(-0.9)^n \cdot u(n)$$

ex 4.10

$$X(z) = \frac{1 + 0.4\sqrt{2}z^{-1}}{1 - 0.8\sqrt{2}z^{-1} + 0.64z^{-2}}$$

b = [1, 0.4√2]
a = [1, -0.8√2, 0.64]

$\mathcal{Z}^{-1}[X(z)] = ?$ so the $x(n)$ is casual and contains no complex numbers

[R, p, C] = residuez[b, a]

R = [0.5 − j, 0.5 + j] ; p = [0.5657 + 0.5657j, 0.5656 − 0.5657j]

p-mag = [0.8, 0.8]   p-ang = [0.7854, −0.7854] = [π/4, −π/4]

$$X(z) = \frac{0.5-j}{1 - 0.8 \cdot e^{+j\frac{\pi}{4}} z^{-1}} + \frac{0.5+j}{1 - 0.8 \cdot e^{-j\frac{\pi}{4}} z^{-1}} \quad \left.|z| > 0.8\right\} \text{— CAUSAL}$$

$$X(z) = \frac{0.5+j}{1-|0.8|e^{-j\frac{\pi}{4}}z^{-1}} + \frac{0.5-j}{1-|0.8|e^{j\frac{\pi}{4}}z^{-1}} \qquad \boxed{|z| > 0.8}$$ causal sequence

$$x(n) = (0.5+j)\,|0.8|^n e^{-j\frac{\pi n}{4}}u(n) + (0.5-j)\,|0.8|^n e^{j\frac{\pi n}{4}}u(n) = \Big/ \begin{matrix} \sin x = -\frac{j}{2}[e^{jx}-e^{-jx}] \\ \cos x = \frac{1}{2}[e^{jx}+e^{-jx}] \end{matrix} \Big/ =$$

$$= |0.8|^n \cdot u(n)\left[0.5\cdot e^{-j\frac{\pi n}{4}} + 0.5\,e^{j\frac{\pi n}{4}} + j\left(e^{-j\frac{\pi n}{4}} - e^{j\frac{\pi n}{4}}\right)\right] =$$

$$= |0.8|^n u(n)\left[\cos\frac{n\pi}{4} - \frac{j}{2}\,2\left(e^{j\frac{\pi n}{4}} - e^{-j\frac{\pi n}{4}}\right)\right] = \underline{|0.8|^n u(n)\left[\cos\frac{n\pi}{4} + 2\sin\frac{n\pi}{4}\right]}$$

Sistem Representation in z-DOMAIN

$$H(z) \triangleq \mathcal{Z}[h(n)] = \sum_{-\infty}^{\infty} h(n)z^{-n}; \quad R_{h-} < |z| < R_{h+}$$

$$Y(z) = H(z)X(z) \quad : \quad ROC_y = ROC_h \cap ROC_x$$

$$y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{l=0}^{M} b_l x(n-l) \quad \Big/ \mathcal{Z}[\;]$$

$$Y(z) + \sum_{k=1}^{N} a_k z^{-k} Y(z) = \sum_{l=0}^{M} b_l z^{-l} X(z)$$

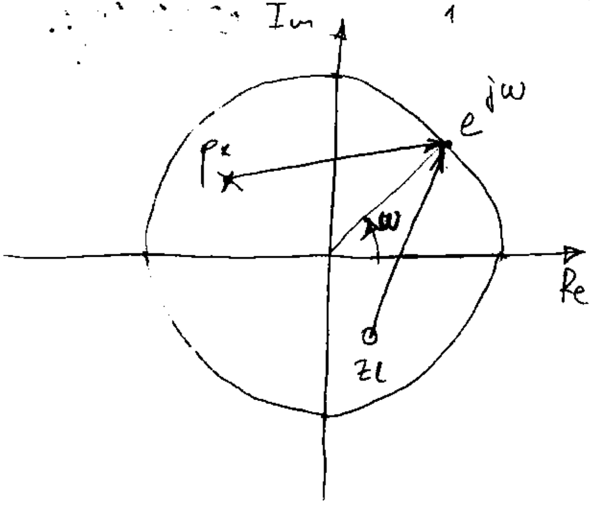
$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{l=0}^{M} b_l z^{-l}}{1+\sum_{k=1}^{N} a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}}$$

$$H(z) = \frac{b_0 z^{-M}\left(z^M + \frac{b_1}{b_0}z^{M-1} + \ldots + \frac{b_M}{b_0}\right)}{z^{-N}\left(z^N + a_1 z^{N-1} + \ldots + a_N\right)} = b_0 z^{N-M}\frac{\prod_{l=1}^{M}(z-z_l)}{\prod_{k=1}^{N}(z-z_k)}$$

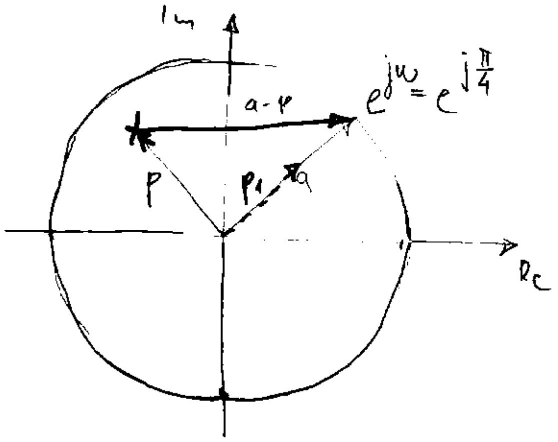
$$\boxed{|e^{jx}| = |\cos x + j\sin x| = \cos^2 x + \sin^2 x = 1}$$

- EVALUATE $H(z)$ AT UNIT CIRCLE $\Rightarrow H(e^{j\omega})$

$$H(e^{j\omega}) = b_0 e^{j(N-M)\omega}\frac{\prod_{1}^{M}(e^{j\omega}-z_l)}{\prod_{1}^{N}(e^{j\omega}-p_k)} \qquad |H(e^{j\omega})| = b_0\frac{|e^{j\omega}-z_1|\ldots|e^{j\omega}-z_M|}{|e^{j\omega}-p_1|\ldots|e^{j\omega}-p_N|}$$



$$\angle H(e^{j\omega}) = \overbrace{[0 \text{ or } \pi]}^{\text{constant}} + \overbrace{[(N-M)\omega]}^{\text{linear}} + \underbrace{\sum_{1}^{M}\angle(e^{j\omega}-z_l) - \sum_{1}^{N}\angle(e^{j\omega}-p_k)}_{\text{nonlinear}}$$



$$p=0.5\cdot e^{j\frac{3\pi}{4}} \qquad a=e^{j\frac{\pi}{4}}$$

$$a-p= e^{j\frac{\pi}{4}}-0.5\cdot e^{j\frac{3\pi}{4}}$$

$$p_1=0.5\cdot e^{j\frac{\pi}{4}}$$

$$a-p= e^{j\frac{\pi}{4}}-0.5\, e^{j\frac{\pi}{4}} = 0.5\cdot e^{j\frac{\pi}{4}}$$

$$=0.5\cos\left(\frac{\pi}{4}\right) + 0.5 j \sin\frac{\pi}{4} =$$

$$=0.5\,\frac{\sqrt{2}}{2} + 0.5\,j\,\frac{\sqrt{2}}{2}$$

$[H,w]=freqz[b,a,N]$; $[H,w]=freqz(b,a,N,$ whole'$)$;

$H = freqz(b,a,w)$

$\boxed{ex.\,4.11}$ $\quad y(n) = 0.9\,y(n-1)+x(n)$

a.) $H(z)=?$ polezero plot

b.) Plot $|H(e^{jw})|$ AND $\angle H(e^{jw})$;

c.) $h(n) = ?$

$$X(z)=\sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$z=e^{jw}$$

$$X(z)=\sum_{n=-\infty}^{\infty} x(n) e^{-jwn}$$

---

a.) $y(n)-0.9\,y(n-1)=x(n)\ /\mathcal{Z}$

$Y(z)-0.9\cdot z^{-1}\,Y(z) = X(z)$; $\quad H(z)=\dfrac{Y(z)}{X(z)}=\dfrac{1}{1-0.9z^{-1}}\quad |z|>0.9$;

$\boxed{b=[1];\quad a=[1,-0.9]}$

b.) $[H,w]=freqz(b,a,100,$ 'whole'$)$

c.) $\mathcal{Z}^{-1}[H(z)]= 0.9^n u(n)$

$\boxed{ex.\,4.12}$ $\quad H(z)=\dfrac{z+1}{z^2-0.9z+0.81} = z^{-1}\dfrac{1+z^{-1}}{1-0.9z^{-1}+0.81z^{-2}}$

$$=\frac{0\cdot z^0+z^{-1}+z^{-2}}{1-0.9z^{-1}+0.81z^{-2}};$$

$b=[0,1,1]$

$a=[1,-0.9,0.81]$

$\boxed{\begin{array}{l}\text{zplane(b,a)}\\ \text{CE TI GI PRIKAZE POLOVITE I}\\ \text{NULITE VO Z-RAMNINA !!}\end{array}}$

$$H(z)=\frac{\prod_{l=1}^{M}(z-z_l)}{\prod_{k=1}^{N}(z-z_k)} = \frac{z+1}{(z-(0.45+0.7794i))(z-(0.45-0.7794i))}$$

roots(...)

$p_1=0.45+0.7794j=0.9\cdot e^{j\frac{\pi}{3}}$; $\quad p_2=0.45-0.7794j = 0.9\cdot e^{-j\frac{\pi}{3}}$

ROC : $|z|>0.9$ $\qquad e^{jw}\in$ ROC $\Rightarrow H(e^{jw})$ exists

ⓐ $$H(e^{jw})=\frac{e^{jw}+1}{(e^{jw}-0.9\cdot e^{j\pi/3})(e^{jw}-0.9\cdot e^{-j\pi/3})}$$

ⓑ $H(z) = \dfrac{0\cdot z^{0} + z^{-1} + z^{-2}}{1 - 0.9z^{-1} + 0.81z^{-2}} = \dfrac{Y(z)}{X(z)}$

$$Y(z) - \overset{0.9}{\cancel{V}}z^{-1}\cdot Y(z) + 0.81\cdot z^{-2}\, Y(z) = z^{-1}X(z) + z^{-2}X(z)$$

$$y(n) - 0.9\,y(n-1) + 0.81\,y(n-2) = x(n-1) + x(n-2)$$

ⓒ $h(n) = ?$ $\qquad b = [0, 1, 1] \qquad a = [1, -0.9, 0.81]$

; h = filter(b, a, impseq(0,0,N)); NUMERIČKI će SE DOBIE IMP. ODZIV!

→ [R, p, c] = residuez(b, a)

$p_1 = 0.45 + 0.7794i$; $\quad p_1 = 0.9\cdot e^{j\frac{\pi}{3}}$; $\quad R_1 = -0.6173 - 0.9979j$; $\quad C = 1.2346$

$p_2 = 0.45 - 0.7794i$ $\quad p_2 = 0.9\cdot e^{-j\frac{\pi}{3}}$; $\quad R_2 = -0.6173 + 0.9979j$

$$\boxed{H(z) = \sum_{k=1}^{N} \frac{R_k}{(1 - p_k z^{-1})} + \sum_{k=0}^{M-N} C_k z^{-k}}$$

$$H(z) = 1.2346 + \frac{-0.6173 - j0.9979}{(1 - |0.9|\cdot e^{j\frac{\pi}{3}}z^{-1})} + \frac{-0.6173 + j0.9979}{1 - |0.9|e^{-j\frac{\pi}{3}}z^{-1}} \qquad |z| > 1$$

$$h(n) = 1.2346\,\delta(n) + \left[(-0.6173 - j0.9979)|0.9|^{n}\cdot e^{jn\pi/3} + (-0.6173 + j0.9979)\overset{|0.9|^{n}}{\vee} e^{-jn\pi/3}\right]u(n)$$

$$= 1.2346\,\delta(n) + |0.9|^{n}\Big[\underbrace{-0.6173\cdot e^{jn\pi/3} - 0.6173\cdot e^{-jn\pi/3}}_{-2\cdot\cos(n\pi/3)} \underbrace{- j0.9979\,e^{jn\pi/3} + j0.9979\cdot e^{-jn\pi/3}}_{+2\,\sin(n\pi/3)}\Big]u(n)$$

$$= 1.2346\,\delta(n) + |0.9|^{n}\left[-1.2346\cdot\cos(n\pi/3) + 1.9958\,\sin(n\pi/2)\right]u(n)$$

$$n = 0 \quad h(0) = 1.2346 + |0.9|^{\overset{0}{\cancel{n}}}\left[-1.2346\cdot 1\right] = 1.2346 - 1.2346 = 0$$

$$\boxed{h(n) = |0.9|^{n}\left[-1.2346\cdot\cos(n\pi/3) + 1.9958\,\sin(n\pi/3)\right]u(n-1)}$$

relationship between system representations

Ex. 4.13 $\qquad y(n) = 0.81\,y(n-2) + x(n) - x(n-2)$

ⓐ $H(z) = ?$	ⓒ unit step response $\sigma(n) = ?$
ⓑ unit impulse response $h(n) = ?$	ⓓ $H(e^{j\omega})$ + plot

ⓐ $y(n) - 0.81\,y(n-2) = x(n) - x(n-2) \quad / \mathcal{Z}$

$$Y(z) - 0.81\,Y(z)\cdot z^{-2} = X(z) - X(z)\cdot z^{-2} \qquad H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-2}}{1 - 0.81\cdot z^{-2}}$$

ⓑ $H(z) = \dfrac{1\cdot z^{0} + 0\cdot z^{-1} - z^{-2}}{1z^{0} + 0\cdot z^{-1} - 0.81z^{-2}}$ $\qquad a = [1, 0, -0.81];$ $\quad b = [1, 0, -1];$

[R, p, C] = residuez(b, a)

$$\boxed{H(z) = \frac{1 \cdot z^0 + 0 \cdot z^{-1} - z^{-2}}{1 \cdot z^0 + 0 \cdot z^{-1} - 0.81 z^{-2}}} ,$$

$R = [-0.1173, -0.1173]$
$p = [-0.9, \; +0.9]$
$C = 1.2346$

$$H(z) = -\frac{0.1173}{1 - 0.9 \cdot z^{-1}} - \frac{0.1173}{1 + 0.9 \cdot z^{-1}} + 1.2344 \quad / \mathcal{Z}^{-1} \qquad |z| > 0.9$$

$$h(n) = -\overset{0.1173}{\gamma} 0.9^n u(n) - \overset{0.1173}{\gamma} (-0.9)^n \cdot u(n) + 1.2344\,\delta(n) = 1.2344\,\delta(n) - 0.1173 \cdot 0.9^n (1 + (-1)^n) u(n) -$$

$$\boxed{h(n) = 1.2344\,\delta(n) - 0.1173 \{1 + (-1)^n\} (0.9)^n \cdot u(n)}$$

Ⓒ unit step response = ?

$$y(n) - 0.81\,y(n-2) = u(n) - u(n-2) \quad / \mathcal{Z}^{-1}$$

$$Y(z) - 0.81 \cdot z^{-2}\, Y(z) = \frac{1}{1 - z^{-1}} - \frac{z^{-2}}{1 - z^{-1}} = \frac{1 - \cancel{z^{-1}} - z^{-2}}{1 - z^{-1}}$$

$$Y(z) = \frac{Y(z)}{X(z)} = \frac{1}{(1 - 0.81 z^{-2})(1 - z^{-1})} - \frac{z^{-2}}{(1 - 0.81 z^{-2})(1 - z^{-1})} =$$

$$= \frac{\cancel{1}}{\cancel{(1 - 0.81 z^{-1})}(1 + 0.81 z^{-1})(1 - z^{-1})} - \frac{z^{-2}}{}$$

$$Y(z) = \frac{1 - \cancel{z^{-1}} - z^{-2}}{(1 + 0.9 z^{-1})(1 - 0.9 z^{-1})(1 - z^{-1})} = \left| \begin{array}{l} a = poly([-0.9, 0.9, +1]) \\ a = [1, -1, -0.81, 0.81] \end{array} \right|$$

zplane(b,a)
3 - POLES
3 - ZEROS

$$Y(z) = \frac{1 \cdot z^0 - \cancel{z^{-1}} - z^{-2}}{1 \cdot z^0 - 1 \cdot z^{-1} - 0.81 z^{-2} + 0.81\, z^{-3}} ;$$

$$\boxed{\begin{array}{l} [R, p, c] = \text{residuez}\,[b, a] \\ b = [1, 0, -1]; \quad a = [1, -1, -0.81, 0.81] \end{array}}$$

$$Y(z) = -\frac{\overset{0}{\cancel{5.2632}}}{1 - z^{-1}} + \frac{\overset{-0.0556}{\cancel{0.2076}}}{1 + 0.9 z^{-1}} + \frac{\overset{1.0536}{\cancel{6.0556}}}{1 - 0.9 z^{-1}} = \sigma(n)$$

~~$y(n) = -\overset{5.2632}{2.9078}\, u(n) + \overset{0.2076}{3.7489} (-0.9)^n \cdot u(n) + \overset{6.0556}{0.1590} (+0.9)^n u(n)$~~

~~[illegible]~~

$$V(z) \equiv H(z) \cdot U(z) = \left| \begin{array}{l} U(z) = \mathcal{Z}^{-1}[u(n)] \\ = \frac{1}{1 - z^{-1}} \end{array} \right| = \frac{1 - z^{-2}}{1 - 0.81 z^{-2}} \cdot \frac{1}{1 - z^{-1}} = \frac{1 - z^{-2}}{(1 + 0.9 z^{-1})(1 - 0.9 z^{-1})(1 - z^{-1})}$$

$$\boxed{y(n) = \{-0.0556 \cdot (-0.9)^n + 1.0556 \cdot (0.9)^n\}\, u(n) = \sigma(n)}$$

$$V(z) = \frac{\cancel{(1 - z^{-1})}(1 + z^{-1})}{(1 + 0.9 z^{-1})(1 - 0.9 z^{-1})\cancel{(1 - z^{-1})}} = \frac{1 + z^{-1}}{1 + 0 z^{-1} - 0.81 z^{-2}} = \left| \begin{array}{l} b = [1, 1] \\ a = [1, 0, -0.81] \end{array} \right|$$

js_ex413.m
ⓒ1

$[R, p, c] = \text{residuez}\,(b, a)$

$R = [-0.0556, 1.0556]$
$p = [-0.9, \; 0.9]$

$$V(z) = \frac{-0.0556}{1 + 0.9 z^{-1}} + \frac{1.0556}{1 - 0.9 z^{-1}} ; \qquad \sigma(n) = \{-0.0556 (-0.9)^n + 1.0556 (0.9)^n\} u(n)$$

(d)

$$\boxed{H(z) = \frac{1 - z^{-2}}{1 - 0.81 z^{-2}} = \left| z = e^{jw} \right| = \frac{1 - e^{-2jw}}{1 - 0.81 \cdot e^{-2jw}}} \Longrightarrow \begin{array}{l} b, a \\ H = freqz(b, a, w) \end{array}$$

$$V(z) = \frac{1 - z^{-2}}{1 - z^{-1} - 0.81 z^{-2} + 0.81 z^{-3}} \cdot \frac{z^3}{z^3} = \frac{z^3 - z^{+1}}{z^3 - z^2 - 0.81 z + 0.81}$$

$$\boxed{\lim_{z \to 0} V(z) = 0}$$ IMA NULA I VO 0 POKRAJ NULATA VO ±1

SOLUTIONS OF THE DIFFERENCE EQUATIONS

- One sided z-Transform

$$Z^+[x(n)] \triangleq Z[x(n)u(n)] \triangleq X^+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$Z^+[x(n-k)] = Z[x(n-k)u(n)] = \sum_{n=0}^{\infty} x(n-k) \cdot z^{-n} = \left| \begin{array}{l} n-k=m \\ n=m+k \\ n=0 \quad m=-k \end{array} \right| = \sum_{m=-k}^{\infty} x(m) \cdot z^{-(m+k)} =$$

$$= \sum_{m=-k}^{-1} x(m) z^{-(m+k)} + \left[\sum_{m=0}^{\infty} x(m) \cdot z^{-m} \right] z^{-k}$$

$$\boxed{Z^+[x(n-k)] = x(-1) z^{1-k} + x(-2) z^{2-k} + \dots + x(-k) + z^{-k} X^+(z)}$$

$$1 + \sum_{k=1}^{N} a_k y(n-k) = \sum_{m=0}^{M} b_m x(n-m), \quad n > 0$$

INITIAL CONDITIONS: $\{y(i), i = -1, \dots, -N\}$ $\{x(i), i = -1, \dots, -M\}$

Ex 4.14 $y(n) - \frac{3}{2} y(n-1) + \frac{1}{2} y(n-2) = x(n), \quad n \geq 0$

$x(n) = \left(\frac{1}{4}\right)^n u(n)$, $\boxed{s.t. \quad y(-1) = 4 \quad y(-2) = 10}$ — INITIAL CONDITIONS!!

$$Y^+(z) - \frac{3}{2}\left[y(-1) + z^{-1} Y^+(z)\right] + \frac{1}{2}\left[y(-1) z^{-1} + y(-2) + z^{-2} Y^+(z)\right] = \frac{1}{1 - \frac{1}{4} z^{-1}}$$

$$\underline{Y^+(z)} - \frac{3}{\cancel{2}} \cdot \overset{2}{\cancel{4}} - \frac{3}{2} z^{-1} \underline{Y^+(z)} + \frac{1}{\cancel{2}} \cdot \overset{2}{\cancel{4}} \cdot z^{-1} + \frac{1}{\cancel{2}} \overset{5}{\cancel{10}} + \frac{1}{2} z^{-2} \underline{Y^+(z)} = \frac{1}{1 - \frac{1}{4} z^{-1}}$$

$$Y^+(z)\left[1 - \frac{3}{2} z^{-1} + \frac{1}{2} z^{-2}\right] = \frac{1}{1 - \frac{1}{4} z^{-1}} + \left(1 - 2 z^{-1}\right)$$

$$Y^+(z) = \frac{1}{\left(1 - \frac{1}{4} z^{-1}\right)\left(1 - \frac{3}{2} z^{-1} + \frac{1}{2} z^{-2}\right)} + \frac{1 - 2z^{-1}}{\left(1 - \frac{3}{2} z^{-1} + \frac{1}{2} z^{-2}\right)} = \frac{1 + \left(1 - \frac{1}{4} z^{-1}\right)\left(1 - 2z^{-1}\right)}{\left(1 - \frac{1}{4} z^{-1}\right)\left(1 - \frac{3}{2} z^{-1} + \frac{1}{2} z^{-2}\right)}$$

$$= \frac{2 - \frac{9}{4} z^{-1} + \frac{1}{2} z^{-2}}{1 - \frac{7}{4} z^{-1} + \frac{7}{8} z^{-2} - \frac{1}{8} z^{-3}} = \frac{16 - 18 z^{-1} + 4 z^{-2}}{8 - 14 z^{-1} + 7 z^{-2} - z^{-3}} =$$

$$= \left| \begin{array}{l} b = [16, -18, 4] \\ a = [8, -14, 7, -1] \\ \boxed{[R, p, C] = residuez(b, a)} \end{array} \right| = \frac{2/3}{1 - z^{-1}} + \frac{1}{1 - 0.5 z^{-1}} + \frac{1/3}{1 - 0.25 z^{-1}}$$

$$y(n) = \left[\frac{2}{3} + (0.5)^n + \frac{1}{3}(0.25)^n\right] u(n)$$

$$Y^+(z) = \frac{\overbrace{1 + \left(1-\frac{1}{4}z^{-1}\right)\left(1-2z^{-1}\right)}^{roots([2,-9/4,1/2])}}{\left(1-\frac{1}{4}z^{-1}\right)\underbrace{\left(1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}\right)}_{roots([1,-3/2,1/2])}} = \frac{1+\left(1-\frac{1}{4}z^{-1}\right)\left(1-2z^{-1}\right)}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}$$

$$= \frac{2/3}{1-z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1/3}{1-\frac{1}{4}z^{-1}} = \frac{\left(1-0.8202\,z^{-1}\right)\left(1-0.3048\,z^{-1}\right)}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}$$

⊙ $$y(n) = \left[\frac{2}{3} + \left(\frac{1}{2}\right)^n + \frac{1}{3}\left(\frac{1}{4}\right)^n\right]u(n) = \underbrace{\left[\frac{2}{3} + \left(\frac{1}{2}\right)^n\right]u(n)}_{(*)} + \underbrace{\frac{1}{3}\left(\frac{1}{4}\right)^n u(n)}_{(**)}$$

(*) homogenous part (**) particular part

(*) - due to the system poles
(**) - due to the input poles

⊙ $$y(n) = \underbrace{\left[\frac{1}{3}\left(\frac{1}{4}\right)^n + \left(\frac{1}{2}\right)^n\right]u(n)}_{(*)} + \underbrace{\frac{2}{3}u(n)}_{(**)}$$

(*) transient response (**) steady state response

(*) due to the poles inside of unit circle
(**) due to the poles on the unit circle

$$\boxed{Y^+(z) = \frac{\frac{1}{1-\frac{1}{4}z^{-1}}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}} + \frac{1-2z^{-1}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}}$$

→ $H(z)X(z)$

initial Conditions → $H(z)\cdot X_{ic}(z)$

MHV

$$\boxed{Y_{zs}(z) = H(z)\cdot X(z) \qquad (zi)(z) = H(z)\,X_{ic}(z)}$$

→ zero state → zero input

$$Y^+(z) = \frac{1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} + \frac{1-2z^{-1}}{\left(1-z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} =$$

$$= \frac{1}{1-\frac{7}{4}z^{-1}+\frac{7}{8}z^{-2}-\frac{1}{8}z^{-3}} + \frac{1-2z^{-1}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}} = -\frac{2}{1-\frac{1}{2}z^{-1}} + \frac{1}{3}\frac{1}{1-\frac{1}{4}z^{-1}} + \frac{8}{3}\frac{1}{1-z^{-1}} + \frac{3}{1-\frac{1}{2}z^{-1}} - \frac{2}{1-z^{-1}}$$

$$y(n) = \underbrace{\left[-2\left(\frac{1}{2}\right)^n + \frac{1}{3}\left(\frac{1}{4}\right)^n + \frac{8}{3}\right]u(n)}_{\text{zero state response}} + \underbrace{\left[3\cdot\left(\frac{1}{2}\right)^n - 2\right]u(n)}_{\text{zero-input response}}$$

$y(-1) = 3\cdot\left(\frac{1}{2}\right)^{-1} - 2 = 6-2 = 4$

$y(-2) = 3\cdot\left(\frac{1}{2}\right)^{-2} - 2 = 12-2 = 10$

`y = filter(b,a,x,xic)`

$y(n) - \frac{3}{2}y(n-1) + \frac{1}{2}y(n-2) = x(n)$

`a = [1, -3/2, 1/2]; b = [1]; xic = [1, -2]`

$$H(z) = \frac{1}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}} \quad ; \quad X(z) = \frac{1}{1-\frac{1}{4}z^{-1}} \qquad \boxed{X_{ic}(z) = 1-2z^{-1}}$$

$$H(z) = \frac{2}{1-z^{-1}} - \frac{1}{1-\frac{1}{2}z^{-1}}$$

`xic = filtic(b,a,y,x)`

MATLAB IMPLEMENTATION!!!

$\boxed{\text{EX. 4.15}}$ solve the diference equation

$$y(n) = \frac{1}{3}\left[x(n) + x(n-1) + x(n-2)\right] + 0.95\, y(n-1) - 0.9025\, y(n-2) \quad n \geq 0$$

$$x(n) = \cos(\pi n/3)\, u(n); \qquad \boxed{y(-1) = -2;\ \ y(-2) = -3;\ \ x(-1) = 1,\ \ x(-2) = 1}$$

$$\mathcal{Z}^+\left[x(n-k)\right] = x(-1)\cdot z^{1-k} + x(-2)\, z^{2-k} + \dots\ x(-k) + z^{-k} X^+(z)$$

$$y(n) - 0.95\, y(n-1) + 0.9025\, y(n-2) = \frac{1}{3}x(n) + \frac{1}{3}x(n-1) + \frac{1}{3}x(n-2) \ \Big/ \mathcal{Z}^+$$

$$Y^+(z) - 0.95\left[\overset{-2}{\cancel{y(-1)}} + z^{-1} Y^+(z)\right] + 0.9025\left[\overset{-2}{y(-1)}\cdot z^{-1} + \overset{-3}{\cancel{y(-2)}} + z^{-2} Y^+(z)\right] =$$

$$\frac{1}{3} X(z) + \frac{1}{3}\left[\underset{1}{\cancel{x(-1)}} + z^{-1} X^+(z)\right] + \frac{1}{3}\left[x(-1)\, z^{-1} + \overset{1}{\cancel{x(-2)}} + z^{-2} X^+(z)\right] \Rightarrow$$

$$Y^+(z)\left[1 - 0.95 z^{-1} + 0.9025 z^{-2}\right] + \overset{1.9}{0.95\cdot 2} - \overset{1.805}{2\cdot 0.9025\, z^{-1}} - \overset{2.7075}{3\cdot 0.9025} =$$

$$= \frac{1}{3} X^+(z)\left[1 + z^{-1} + z^{-2}\right] + \frac{1}{3} + \frac{1}{3}\,z^{-1} + \frac{1}{3}$$

$$Y^+(z) = \frac{1}{3}\,\frac{(1 + z^{-1} + z^{-2})\, X^+(z)}{1 - 0.95 z^{-1} + 0.9025 z^{-2}} + \overbrace{\frac{0.9025 + 1.805 z^{-1}}{1 - 0.95 z^{-1} + 0.9025 z^{-2}}}^{YIC} + \overbrace{\frac{1}{3}\,\frac{2 + z^{-1}}{1 - 0.95 z^{-1} + 0.9025 z^{-2}}}^{XIC}$$

$$\mathcal{Z}\left[a^n \cos(\omega_0 n)\, u(n)\right] = \frac{1 - (a \cos\omega_0)\, z^{-1}}{1 - (2a\cos\omega_0)\, z^{-1} + a^2 z^{-2}} = \frac{1 - \frac{1}{2} z^{-1}}{1 - z^{-1} + z^{-2}} = \boxed{\frac{2 - z^{-1}}{2 - 2z^{-1} + 2z^{-2}}}$$

MOZE DIREKTNO VO MATLAB !!!

$$\boxed{Y^+(z) = \frac{1}{3}\,\frac{1 + z^{-1} + z^{-2}}{1 - 0.95 z^{-1} + 0.9025 z^{-2}}\,\frac{1 - \frac{1}{2} z^{-1}}{1 - z^{-1} + z^{-2}} + \frac{\overset{\frac{2}{3} + 0.8075 \qquad (1.805 + \frac{1}{3})}{1.4742 + 2.1383 z^{-1}}}{1 - 0.95 z^{-1} + 0.9025 z^{-2}}}$$

$$Y^+(z) = \frac{1}{3}\left[\frac{1 + \frac{1}{2} z^{-1} + \frac{1}{2} z^{-2} - \frac{1}{2} z^{-3}}{1 - 1.95 z^{-1} + 2.8525 z^{-2} - 1.8525 z^{-3} + 0.9025 z^{-4}} + \frac{4.4226 + 6.415 z^{-1}}{1 - 0.95 z^{-1} + 0.9025 z^{-2}}\right]$$

$$XIC(n) = \left[1.4742,\ 2.1383\right]$$

$$\boxed{\begin{array}{ll} XIC = \text{filtic}(b, a, Y, X) & b = [1, 1, 1, 1] \\ & a = [1, -0.95, +0.9025] \\ & X = [1\ 1] \quad Y = [-2, -3] \end{array}}$$

$$Y^+(z) = \frac{1}{3}\,\frac{1 + \frac{1}{2} z^{-1} + \frac{1}{2} z^{-2} - \frac{1}{2} z^{-3} + (4.4226 + 6.415 z^{-1})(1 - z^{-1} + z^{-2})}{1 - 1.95 z^{-1} + 2.8525 z^{-2} - 1.8525 z^{-3} + 0.9025 z^{-4}}$$

$$= \frac{1}{3}\,\frac{5.4226 + 2.4924 z^{-1} - 1.4924 z^{-2} + 5.915 z^{-3}}{1 - 1.95 z^{-1} + 2.8525 z^{-2} - 1.8525 z^{-3} + 0.9025 z^{-4}} \Rightarrow$$

$$\boxed{Y^+(z) = \frac{1.8075 + 0.8308 z^{-1} - 0.4975 z^{-2} + 1.9717 z^{-3}}{1 - 1.95 z^{-1} + 2.8525 z^{-2} - 1.8525 z^{-3} + 0.9025 z^{-4}}}$$

MOZE DA SE DOBIE VO MATLAB SO SIMPLIFY ILI VO MATLAB SO „CONV"

$$z = [0.0584 - 3.9468i,\ 0.0584 + 3.9468i,\ 0.8453 + 2.0311i,\ 0.8453 - 2.0311i]$$

$$p = [\underset{1}{\cancel{3.9473}}\cdot e^{+j\pi/3},\ \underset{1}{\cancel{3.9473}}\cdot e^{-j\pi/3},\ \underset{0.95}{\cancel{2.2}}\cdot e^{j\pi/3},\ \underset{0.95}{\cancel{2.2}}\cdot e^{-j\pi/3}]$$

$$y(n) = \left[R_1\cdot(3.9473)^n e^{j\pi/3 n} + R_2\cdot(3.9473)^n e^{-j\pi/3 n} + R_3\cdot(2.2)^n e^{j\pi/3 n} + R_4\,(2.2)^n e^{-j\pi/3 n}\right] u(n)$$

$$y(n) = \Big\{0.1168\cdot\cos(\pi/3\, n) + 7.8936 \sin(n\pi/3) +$$

$$+ (0.95)^n\left[1.6906 \cos(n\pi/3) - 4.0622 \sin(n\pi/3)\right]\Big\}\, u(n)$$

$\boxed{9.4.1}$ $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

ⓐ $x(n) = \{3, 2, \underset{\uparrow}{1}, -2, -3\}$; $X(z) = ?$

$X(z) = 3 \cdot z^2 + 2z^1 + 1 - 2z^{-1} - 3z^{-2} = z^2 \left(3 + 2z^{-1} + z^{-2} - 2z^{-3} - 3z^{-4}\right)$

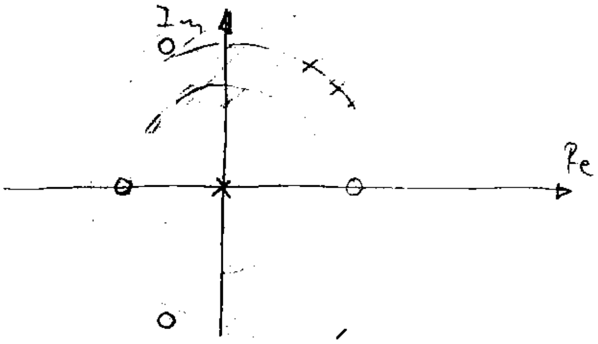
$X(z) = \frac{3 + 2z^{-1} + z^{-2} - 2z^{-3} - 3z^{-4}}{z^{-2}}$

$X(z) = 3z^2 + 2z + 1 - \frac{2}{z} - \frac{3}{z^2} = \frac{3z^4 + 2z^3 + z^2 - 2z - 3}{z^2}$; $|z| > 0$

$n_1 = \text{roots}([3, 2, 1, -2, -3]) = [0{,}9395, -0.3611 + 1.0362i, -0.3611 - 1.0362i, -0.884]$

$\sum_{n=-2}^{2} x(n) z^{-n} < \infty \Rightarrow$ ROC

ROC : $\boxed{0 < z < \infty}$



ⓑ $x(n) = (0.8)^n u(n-2)$

$\mathcal{Z}[a^n u(n)] = \frac{1}{1 - az^{-1}}$; $|z| > a$

$|az^{-1}| < 1$ $z > a$

$\mathcal{Z}[x(n-2)] = z^{-2} X(z)$;

$x(n) = (0.8)^2 (0.8)^{n-2} u(n-2)$

$\mathcal{Z}[x(n)] = 0.8^2 \cdot \frac{z^{-2}}{1 - 0.8z^{-1}} = \frac{0.64z^{-2}}{1 - 0.8z^{-1}}$

$X(z) = \sum_{n=2}^{\infty} (0.8)^n \cdot z^{-n} = -(0.8 \cdot z^{-1} + 0.8) + \sum_{n=0}^{\infty} (0.8)^n \cdot z^{-n} =$

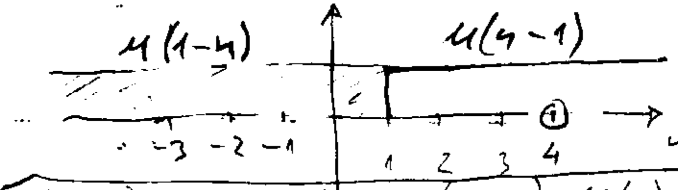
$= -1 - 0.8z^{-1} + \frac{1}{1 - 0.8z^{-1}} = \frac{\cancel{-1} \cancel{-0.8z^{-1}} \cancel{+(0.8)z^{-1}} + (0.8)^2 z^{-2} \cancel{+1}}{1 - 0.8z^{-1}}$

$= \boxed{\frac{0.64z^{-2}}{1 - 0.8z^{-1}} \; ; \quad |z| > 0.8}$

$\frac{3.2}{5z^2 - 4z} = \frac{3.2/5}{z^2 - 0.8z} = \frac{0.64z^{-2}}{1 - 0.8z^{-1}}$

MAPLE ZTRANS CHECK

ⓒ $x(n) = \left(\frac{4}{3}\right)^n u(1-n)$ $\quad u(1-n) = u(-(n-1))$



$u(1-+4) = u(-3) = 0$ | $u(1-0) = u(1) = 1$

$u(1-1) = u(0) = 1$

$X(z) = \sum_{n=-\infty}^{1} \left(\frac{4}{3}\right)^n z^{-n}$

$x(n) = a^n u(-1-n)$

$m = -n$

$n = -m$

$X(z) = \sum_{n=-\infty}^{-1} a^n \cdot z^{-n} = -1 + \sum_{n=-\infty}^{0} a^n z^{-n} = -1 + \sum_{m=\infty}^{0} a^{-m} z^{m} = \left| m = -n \right|$

$= -1 + \sum_{n=0}^{\infty} (a^{-1}z)^n = \left| \begin{array}{c} \left|\frac{z}{a}\right| < 1 \\ |z| < |a| \end{array} \right| = -1 + \frac{1}{1 - \frac{z}{a}} = -1 + \frac{a}{a - z} =$

$$X(z) = -1 + \frac{a}{a-z} = \frac{\cancel{-a}+z+\cancel{a}}{a-z} = \frac{z}{a-z} = -\frac{1}{1-az^{-1}}$$

$$X(z) = -b^n u(-n-1) = -\sum_{-\infty}^{-1} b^n z^{-n} = -\sum_{-\infty}^{-1}\left(\frac{b}{z}\right)^n = -\sum_{1}^{\infty}\left(\frac{z}{b}\right)^n =$$

$$= 1 - \sum_{0}^{\infty}\left(\frac{z}{b}\right)^n = 1 - \frac{1}{1-\frac{z}{b}} = 1 - \frac{b}{b-z} = \frac{\cancel{b}-z-\cancel{b}}{b-z} = \frac{z}{z-b}$$

$$\boxed{X(z) = \frac{1}{1-bz^{-1}}}$$

$$x(n) = \left(\frac{4}{3}\right)^n u(1-n) \qquad \boxed{\mathcal{Z}[x(-n)] = X\left(\frac{1}{z}\right)} \qquad \left|\frac{3}{4z}\right| < 1 \quad \boxed{|z| > \frac{3}{4}}$$

$$x(n) = \frac{4}{3}\left(\frac{3}{4}\right)^{-n+1} u(1-n)$$

$$x_1(n) = \left(\frac{3}{4}\right)^{n-1} \cdot u(n-1) \qquad X_1(z) = \mathcal{Z}[x_1(n)] = z^{-1}\,\mathcal{Z}\left[\left(\frac{3}{4}\right)^n \cdot u(n)\right] = z^{-1}\frac{1}{1-\frac{3}{4}z^{-1}}$$

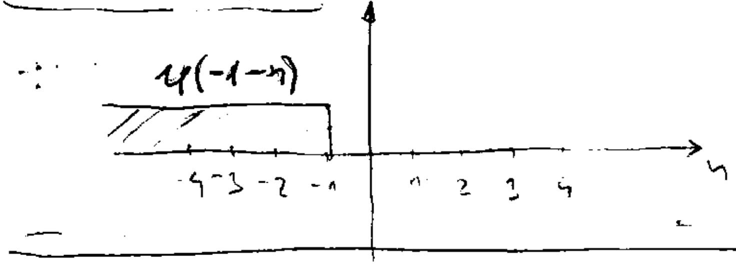
$$\mathcal{Z}[x(n)] = \frac{4}{3}\,\mathcal{Z}[x_1(-n)] = \frac{4}{3}\,X_1\left(\frac{1}{z}\right) = \frac{4}{3}\cdot\frac{z}{1-\frac{3}{4}z} = \frac{4/3}{z^{-1}-\frac{3}{4}}$$

$$= \frac{4/3}{3/4\left(\frac{4}{3}z^{-1}-1\right)} = -\frac{(4/3)^2}{1-\frac{4}{3}z^{-1}}$$

$$X(z) = \sum_{n=-\infty}^{1}\left(\frac{4}{3}\right)^n z^{-n} = \left(\frac{4}{3}\right)\cdot z^{-1} + \sum_{n=-\infty}^{0}\left(\frac{4}{3z}\right)^n = \left(\frac{4}{3}\right)z^{-1} + \sum_{n=0}^{\infty}\left(\frac{3z}{4}\right)^n =$$

$$= \left(\left|\frac{3z}{4}\right| < 1 \quad z < \left|\frac{4}{3}\right|\right) = \frac{4}{3}z^{-1} + \frac{1}{1-\frac{3}{4}z} = \frac{\frac{4}{3}z^{-1}-\cancel{1}+\cancel{1}}{1-\frac{3}{4}z} = \frac{\frac{4}{3}z^{-1}}{1-\frac{3}{4}z}$$

$$= \frac{4/3}{z-\frac{3}{4}z^2} = \frac{4}{3z-\frac{9}{4}z^2} = \frac{16}{12z-9z^2} = -\frac{16}{9z^2-12z}$$



$$u(-1-0) = u(-1) = 0$$
$$u(-1-(-1)) = u(-1+1) = u(0) = 1$$

ⓓ $x(n) = 2^{-|n|} + \left(\frac{1}{3}\right)^{|n|}$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty}\left[2^{-|n|} + \left(\frac{1}{3}\right)^{|n|}\right] z^{-n} = \overbrace{\sum_{n=0}^{\infty}\left[2^{-n} + \left(\frac{1}{3}\right)^{n}\right] z^{-n}}^{*} +$$

$$+ \underbrace{\sum_{n=-\infty}^{-1}\left[2^{n} + \left(\frac{1}{3}\right)^{-n}\right] z^{-n}}_{(**)}$$

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} 2^{-n} z^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n z^{-n} + \sum_{n=-\infty}^{-1} 2^n z^{-n} + \sum_{n=-\infty}^{-1} \left(\frac{1}{3}\right)^{-n} z^{-n} =$$

$$\boxed{x(n) = 2^{-|n|} + \left(\frac{1}{3}\right)^{|n|}}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2z}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{3z}\right)^n + \sum_{n=1}^{\infty} 2^{-n} z^n + \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n z^{+n} =$$

$$= \underbrace{\frac{1}{1-\frac{1}{2}z^{-1}}}_{\left|\frac{1}{2z}\right| < 1;\ |z| > \frac{1}{2}} + \underbrace{\frac{1}{1-\frac{1}{3}z^{-1}}}_{|z| > \frac{1}{3}} + \underbrace{\sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n}_{\left|\frac{z}{2}\right| < 1,\ |z| < 2} + \underbrace{\sum_{n=0}^{\infty} \left(\frac{z}{3}\right)^n}_{\left|\frac{z}{3}\right| < 1;\ |z| < 3} - 2$$

ROC: $|z| > \frac{1}{2}$ ROC: $|z| < 2$

$$\mathcal{X}(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z} + \frac{1}{1-\frac{1}{3}z} - 2; \quad \frac{1}{2} < |z| < 2$$

$$\boxed{\mathcal{X}(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-\frac{1}{3}z^{-1}} - \frac{2z^{-1}}{1-2z^{-1}} - \frac{3z^{-1}}{1-3z^{-1}} - 2; \quad \boxed{\frac{1}{2} < |z| < 2}}$$

MATLAB CHECKED

ⓔ $x(n) = (n+1)\, 3^n u(n)$

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} (n+1)\, 3^n z^{-n};$$

$$\boxed{S(x) = \sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}}$$

$$\left|\frac{3}{z}\right| < 1$$

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} n \cdot \left(\frac{3}{z}\right)^n + \sum_{n=0}^{\infty} \left(\frac{3}{z}\right)^n = \frac{\frac{3}{z}}{\left(1-\frac{3}{z}\right)^2} + \frac{1}{1-\frac{3}{z}}$$

$$\mathcal{X}(z) = \frac{3 \cdot z^{-1}}{(1-3z^{-1})^2} + \frac{1}{1-3z^{-1}} = \frac{3z}{(z-3)^2} + \frac{z}{(z-3)}$$

$$\mathcal{X}(z) = \frac{3z + z(z-3)}{(z-3)^2} = \frac{\cancel{3z} + z^2 - \cancel{3z}}{(z-3)^2} = \frac{z^2}{(z-3)^2}; \quad \overset{\text{ROC}}{\boxed{|z| > 3}}$$

$$\mathcal{X}(z) = \frac{1}{\left(\frac{z-3}{z}\right)^2} = \frac{1}{(1-3z^{-1})^2}, \quad X(z) = \frac{z^2}{z^2-6z+9};$$

ZPLANE

$\boxed{P.4.2}$ $\mathcal{X}(z) = ?$ by USING Z-TRANSFORM TABLE & Z-TRANSFORM PROPERTIES

ⓐ $x(n) = 2\,\delta(n-2) + 3\,u(n-3)$

$$\mathcal{X}(z) = \mathcal{Z}\{x(n)\} = 2 z^{-2} \cdot \mathcal{Z}[\delta(n)] + 3 z^{-3} \mathcal{Z}[u(n)] = 2 \cdot z^{-2} + \frac{3z^{-3}}{1-z^{-1}}$$

$$\mathcal{X}(z) = \frac{2z^{-2} - 2z^{-3} + 3z^{-3}}{1-z^{-1}} = \boxed{\frac{2z^{-2}+z^{-3}}{1-z^{-1}} \quad |z| > 1}$$

$$\mathcal{Z}[u(n-1)] = z^{-1}\, \mathcal{Z}[u(n)] \ldots$$

$$\frac{z}{z-1} - 1 = \frac{z - z + 1}{z-1} = \frac{1 \cdot z^{-1}}{1 - z^{-1}} \quad \frac{1}{z-1}$$

$(z^2)' = 2z^{2-1}$
$(z^1)' = 1 \cdot z^0 = 1$

$$X(z) = \frac{(2z^{-2}+z^{-3}) \cdot z^3}{(1-z^{-1})z^3} = \frac{2z+1}{z^3-z^2}$$

$$\boxed{b=[2,1] \qquad a=[1,-1,0,0]}$$

ⓑ $x(n) = \left(\frac{1}{3}\right)^n u(n-2) + (0.9)^{n-3} u(n) = \left(\frac{3}{1}\right)^2 \cdot \left(\frac{1}{3}\right)^{n-2} u(n-2) + (0.9)^{-3} \cdot (0.9)^n \cdot u(n)$

$$X(z) = \mathcal{Z}[x(n)] = 9 \cdot z^{-2} \cdot \mathcal{Z}\left[\left(\frac{1}{3}\right)^n \cdot u(n)\right] + 1.3717\ \mathcal{Z}\left[(0.9)^n \cdot u(n)\right] =$$

$$= \frac{9 \cdot z^{-2}}{1-\frac{1}{3}z^{-1}} + \frac{1.3717}{1-0.9z^{-1}} = \frac{\frac{9}{z^2}}{1-\frac{1}{3z}} + \frac{1.3717}{1-\frac{0.9}{z}} =$$

$$= \frac{9}{z^2-\frac{1}{3}z} + \frac{1.3717 \cdot z}{z-0.9} = \frac{9z - 8.1 + 1.3717z^3 - 0.4572 \cdot z^2}{\left(z^2-\frac{1}{3}z\right)(z-0.9)} =$$

$$= \frac{1.3717 \cdot z^3 - 0.4572 z^2 + 9z - 8.1}{z^3 - 1.2333 z^2 + 0.3z}$$

MAPLE/MATLAB CHECKED !!!

ⓒ $x(n) = n \cdot \sin\left(\frac{\pi n}{3}\right) u(n) + (0.9)^n u(n-2)$

$$X(z) = \mathcal{Z}\left[n \cdot \underbrace{\sin\left(\frac{\pi n}{3}\right) u(n)}_{x(n)}\right] + (0.9)^{+2} z^{-2}\ \mathcal{Z}\left[(0.9)^n u(n)\right] =$$

$$\mathcal{Z}[n \cdot x(n)] = -z \frac{d}{dz} X(z)$$

$$\mathcal{Z}[\sin(\omega_0 n) u(n)] = \frac{\sin(\omega_0) z^{-1}}{1 - 2\cos\omega_0 z^{-1} + z^{-2}}$$

$$= -z \frac{d}{dz}\left[\frac{\overset{\sqrt{3}/2}{\cancel{\sin\frac{\pi}{3}}}\, z^{-1}}{1 - \cancel{2}\frac{1}{\cancel{2}} z^{-1} + z^{-2}}\right] + \frac{(0.9)^{+2} z^{-2}}{1-0.9z^{-1}}$$

$$\left(\frac{\frac{\sqrt{3}}{2} z^{-1}}{1 - z^{-1} + z^{-2}}\right)' = \left(\frac{\sqrt{3}}{2z - 2 + 2z^{-1}}\right)' = \left(\frac{\sqrt{3}\, z}{2z^2 - 2z + 2}\right)' =$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{\sqrt{3}}{2}\left(\frac{z}{z^2-z+1}\right) = \frac{z^2-z+1 - z(2z-1)}{(z^2-z+1)^2} = \frac{\sqrt{3}}{2}\frac{z^2 \cancel{-z} + 1 - 2z^2 \cancel{+z}}{(z^2 - z + 1)^2} = \frac{\sqrt{3}}{2}\frac{-z^2+1}{(z^2-z+1)^2}$$

MAPLE CHECKED

$$X(z) = \frac{\sqrt{3}}{2}\frac{z^3-z}{(z^2-z+1)^2} + \frac{z^{-2} \cdot (0.9)^{+2}}{1-0.9z^{-1}} = \frac{\sqrt{3}}{2} \cdot \frac{z^{-1}-z^{-3}}{\frac{(z^2-z+1)^2}{z^4}} + \frac{(0.9)^{+2} z^{-2}}{1-0.9z^{-1}} =$$

$$= \frac{\sqrt{3}}{2} \cdot \frac{z^{-1}-z^{-3}}{(1-z^{-1}+z^{-2})^2} + \frac{0.81 z^{-2}}{1-0.9z^{-1}} = \frac{z^6}{z^6}\ \frac{8.66z^{-1} + 0.305z^{-2} - 24.86z^{-3} + 32.09z^{-4} - 16.2z^{-5} + 8.1z^{-6}}{+10 - 29z^{-1} + 48z^{-2} - 47z^{-3} + 28z^{-4} - 9z^{-5}}$$

$$\boxed{X(z) = \frac{\sqrt{3}}{2}\frac{z^3-z}{(z^2-z+1)^2} + \frac{0.81}{z^2-0.9z} = \frac{8.66z^5 + 0.305z^4 - 24.86z^3 + 32.09z^2 - 16.2z + 8.1}{10z^6 - 29z^5 + 48z^4 - 47z^3 + 28z^2 - 9z}}$$

$$\to X(z) = \frac{+8.66z^5 + 0.305z^4 - 24.86z^3 + 32.09z^2 - 16.2z^1 + 8.1}{+10z^6 - 29z^5 + 48z^4 - 47z^3 + 28z^2 - 9z}$$

RESULT CHECKED WITH ZTRANS(f, n, z) IN MAPLE !!!

PROVERKA VO MATLAB: $x'(n) = \text{filter}(b, a, \text{impseq}(0,0,7))$
$x(n) = n \cdot \sin(\pi n/3) + (0.9)^n$

ⓓ $x(n) = \left(\frac{1}{2}\right)^n \cdot \cos\left(\frac{\pi n}{4} - \frac{\pi}{4}\right) \cdot \underline{u(n-1)}$

$$\mathcal{Z}\left[a^n \cdot \cos(w_0 n)\right] = \frac{1 - a\cos(w_0) z^{-1}}{1 - (2a\cos w_0) z^{-1} + a^2 z^{-2}}$$

$$\mathcal{Z}\left[x(n-n_0)\right] = z^{-n_0}\,\mathcal{Z}\left[x(n)\right]$$

$$x(n) = \frac{1}{2}\cdot\left(\frac{1}{2}\right)^{n-1} \cdot \cos\left(\left[\frac{\pi}{4}(n-1)\right] \cdot u(n-1)\right);$$

$$\mathcal{Z}\left[x(n)\right] = \frac{1}{2} z^{-1}\,\mathcal{Z}\left[\left(\frac{1}{2}\right)^n \cdot \cos\left(\frac{\pi}{4} \cdot n\right)\right] = \frac{1}{2} z^{-1} \frac{1 - \frac{1}{2}\overset{\sqrt{2}/2}{\cancel{\cos\frac{\pi}{4}}} z^{-1}}{1 - \cancel{2}\cdot\cancel{\tfrac{1}{2}}\cdot\underset{\sqrt{2}/2}{\cancel{\cos\left(\frac{\pi}{4}\right)}} z^{-1} + \left(\frac{1}{2}\right)^2 \cdot z^{-2}}$$

$$= \frac{\frac{1}{2} z^{-1} - \frac{1}{2} z^{-2} \cdot \frac{\sqrt{2}}{2} \cdot \frac{1}{2}}{1 - \frac{\sqrt{2}}{2} z^{-1} + \frac{1}{4} z^{-2}} = ⊛\left|\frac{\frac{1}{2} z^{-1} - \frac{\sqrt{2}}{8} z^{-2}}{1 - \frac{\sqrt{2}}{2} z^{-1} + \frac{1}{4} z^{-2}}\right| \cdot \frac{4 \cdot z^2}{4 \cdot z^2} =$$

$$= \boxed{\frac{2 \cdot z - \sqrt{2}/2}{4z^2 - 2\sqrt{2}z + 1}} \;⊛⊛$$

$b = [0, 2, -\sqrt{2}/2]$

$a = [4, -2\sqrt{2}, 1]$

$$⊛ \cdot \frac{4}{4} = \frac{2z^{-1} - \frac{\sqrt{2}}{2} z^{-2}}{4 - 2\sqrt{2} z^{-1} + z^{-2}}; \Rightarrow$$

$b = [0, 2, -\sqrt{2}/2]$

$a = [4, -2\sqrt{2}, 1]$

ⓔ $x(n) = (n-3)\left(\frac{4}{4}\right)^{n-2} \cos\left\{\frac{\pi}{2}(n-1)\right\} u(n)$

$$\cos\left(\alpha - \frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2} - \alpha\right)$$

$$\cos(\alpha - \beta) = \cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta$$

$$\cos\left(\alpha - \frac{\pi}{2}\right) = \cos\alpha \cdot \underset{0}{\cancel{\cos\frac{\pi}{2}}} + \sin\alpha \cdot \underset{1}{\cancel{\sin\frac{\pi}{2}}} = \sin\alpha$$

$$x(n) = (n-3) \cdot 4^2 \cdot \left(\frac{1}{4}\right)^n \cdot \sin\left(\frac{n\pi}{2}\right) \cdot u(n) = \overbrace{16\, n\left(\frac{1}{4}\right)^n \cdot \sin\left(\frac{n\pi}{2}\right)}^{x_1(n)} - \overbrace{48\left(\frac{1}{4}\right)^n \cdot \sin\left(\frac{n\pi}{2}\right)}^{x_2(n)}$$

$$X_2(z) = \mathcal{Z}\left[48\left(\frac{1}{4}\right)^n \cdot \sin\left(\frac{n\pi}{2}\right)\right] = \left|\mathcal{Z}\left[a^n \sin(w_0 n)\right] = \frac{a\sin(w_0) z^{-1}}{1 - 2a\cos(w_0) z^{-1} + a^2 z^{-2}}\right| =$$

$$= 48\,\frac{\frac{1}{4}\overset{1}{\cancel{\sin\frac{\pi}{2}}} z^{-1}}{1 - \cancel{2}\cdot\underset{2}{\cancel{\tfrac{1}{4}}}\,\underset{0}{\cancel{\cos\frac{\pi}{2}}} z^{-1} + \left(\frac{1}{4}\right)^2 z^{-2}} = \frac{12 z^{-1}}{1 + \frac{1}{16} z^{-2}} \cdot \frac{16 z^2}{16 z^2} = \frac{192 \cdot z^{+1}}{16z^2 + 1}$$

$$X_1(z) = 16 \cdot \mathcal{Z}\left[n \underbrace{\left(\frac{1}{4}\right)^n \sin\left(\frac{n\pi}{2}\right)}_{x_3(n)}\right] = \left|\mathcal{Z}\left[n x(n)\right] = -z\frac{d}{dz}\left[X(z)\right]\right|$$

$$\mathcal{Z}\left[\left(\frac{1}{4}\right)^n \sin\left(\frac{n\pi}{2}\right)\right] = \frac{1 - \frac{1}{4} z^{-1}}{1 - \frac{1}{2} z^{-1} + \frac{1}{16} z^{-2}} \cdot \frac{16z^2}{16z^2} = \frac{16z^2 - 4z}{16z^2 - 8z + 1}$$

$$\frac{d}{dz} X_3(z) = -\frac{4}{16z^2 - 8z + 1}; \quad \boxed{X_1(z) = \frac{1024 z}{16z^2 - 8z + 1}}$$

$$X_2(z) = \frac{768 z^2 - 192 z}{16z^2 - 8z + 1}; \quad \boxed{X(z) = \frac{-768 z^2 + 1216 z}{16z^2 - 8z + 1}}$$

$$\mathcal{X}_1(z) = 16\,\mathcal{Z}\left[n\underbrace{\left(\frac{1}{4}\right)^n\sin\left(\frac{n\pi}{2}\right)}_{x_3(n)}\right] = -16z\frac{d}{dz}X_3(z) = -16z\frac{d}{dz}\left[\frac{4z}{16z^2+1}\right]$$

$$\mathcal{X}_3(z) = \mathcal{Z}\left[\left(\frac{1}{4}\right)^n\cdot\sin\left(\frac{n\pi}{2}\right)\right] = \frac{\frac{1}{4}\cdot z^{-1}}{1+\frac{1}{16}z^{-2}}\cdot\frac{16z^2}{16z^2} = \frac{4z}{16z^2+1}$$

$$\mathcal{X}_1(z) = \frac{64z(16z^2-1)}{(16z^2+1)^2}\;; \qquad \mathcal{X}(z) = \mathcal{X}_1(z)-\mathcal{X}_2(z) = -\frac{256z(8z^2+1)}{(16z^2+1)^2}$$

$$\boxed{\mathcal{X}(z) = \frac{-2048z^3-256z}{256z^4+32z^2+1}}\;\frac{z^{-4}}{z^{-4}}$$

$b=[0,-2048,0,-256,0]$

$a=[256,0,32,0,1]$

$$\mathcal{X}(z) = \frac{-2048z^{-1}-256z^{-3}}{256+32z^{-2}+z^{-4}}$$

$a=[256,0,32,0,1]$

$b=[0,-2048,0,256,0]$

$\boxed{P.4.3}$ $\quad \mathcal{X}(z) = 1+2z^{-1} \qquad (z| \neq 0$

$\mathcal{Z}[x(n-n_0)] = z^{-n_0}\mathcal{Z}[x(n)]$ ROC: ROC_x

$\mathcal{Z}[x(-n)] = \mathcal{X}\left(\frac{1}{z}\right)$ ROC: inverted ROC_x

$\boxed{\delta(n)=\text{charfun}_0(n)}$

ⓐ $x_1(n) = \underbrace{x(3-n)}_{x_{11}(n)} + \underset{x_{12}(n)}{x(n-3)}$

$$\mathcal{Z}[x_{12}(n)] = \mathcal{Z}[x(n-3)] = z^{-3}\mathcal{Z}[x(n)] = z^{-3}+2z^{-4} = \mathcal{X}_{12}(z)$$

$$\mathcal{Z}[x_{11}(n)] = \mathcal{Z}[x(3-n)] = \mathcal{Z}[x(-(n-3))] = \mathcal{X}_{12}\left(\frac{1}{z}\right) = \left(\frac{1}{z}\right)^{-3}+2\left(\frac{1}{z}\right)^{-4}$$

$$\mathcal{X}_{11}(z) = z^3+2z^4\;; \quad \mathcal{X}_1(z) = \mathcal{X}_{11}(z)+\mathcal{X}_{12}(z) = z^3+2z^4+z^{-3}+2z^{-4}$$

$$= z^3+2z^4+\frac{1}{z^3}+\frac{2}{z^4} = \frac{z^7+2z^8+z+2}{z^4} \,.\; \frac{2z^8+z^7+z+2}{z^4}$$

$$\boxed{x(n) = \mathcal{Z}^{-1}[1+2z^{-1}] = \delta(n)+2\delta(n-1)}$$

$(x^{-1})' = -x^{-1-1}$

$= -\frac{1}{x^2}$

$$x_1(n) = \delta(n+3)+2\delta(n+4)+\delta(n-3)+2\delta(n-4)$$

$$x(-n) = \delta(-n)+2\delta(-n+1) = \delta(n)+2\delta(n+1)$$

$$u(-1-n)\;; \quad \overset{n=0}{u(-1)=0}\;; \quad \overset{n=-1}{u(-1+1)=u(0)=1}\;; \quad \overset{n-2}{u(-1+2)=u(1)=1}$$

$$x(-n+3) = \delta(n+3)+2\delta(n+4)$$

ⓑ $x_2(n) = (1+n+n^2)\,x(n)$

PROVERKA MAPLE | VO

$(x^2)' = 2\cdot x^{2-1}$

$$\mathcal{X}_2(z) = \mathcal{Z}\Big[\underbrace{x(n)}_{X_{21}(z)} + \underbrace{\overbrace{n\cdot x(n)}^{x_{22}(n)}}_{\mathcal{X}_{22}(z)} + \underbrace{n\cdot n\cdot x(n)}_{X_{23}(z)}\Big] = 1+\underline{2z^{-1}}+\underline{2z^{-1}}+\underline{2}z^{-1} = 1+6z^{-1}$$

$$\mathcal{Z}[n\,x(n)] = \frac{d}{dz}\mathcal{X}(z) = -z\frac{d}{dz}(1+2z^{-1}) = -z(-2)z^{-2} = 2z^{-1}$$

$$\mathcal{Z}[n\cdot x_{22}(n)] = -z\frac{d}{dz}\mathcal{X}_{22}(z) = -z\frac{d}{dz}(2z^{-1}) = -z(-2)z^{-1}$$

$$= +2z^{-1}$$

ⓒ $x_3(n) = \left(\frac{1}{2}\right)^n x(n-2)$; $X(z) = 1 + 2z^{-1}$; $x(n) = \delta(n) + 2\delta(n-1)$

$$\mathcal{Z}\left[\left(\frac{1}{2}\right)^n \underbrace{x(n-2)}_{x_{31}(n)}\right] = \mathcal{Z}\left[\left(\frac{1}{2}\right)^n x_{31}(n)\right] = X_{31}(2z)$$

$$X_{31}(z) = \mathcal{Z}[x(n-2)] = z^{-2}\,\mathcal{Z}[x(z)] = z^{-2}(1+2z^{-1}) = z^{-2} + 2z^{-3}$$

$$X_3(z) = X_{31}(2z) = \frac{1}{4}z^{-2} + 2\cdot\frac{1}{8}z^{-3} = \frac{1}{4}z^{-2} + \frac{1}{4}z^{-3}$$

MAPLE CHECKED

ⓓ $x_4(n) = x(n+2) * x(n-2)$

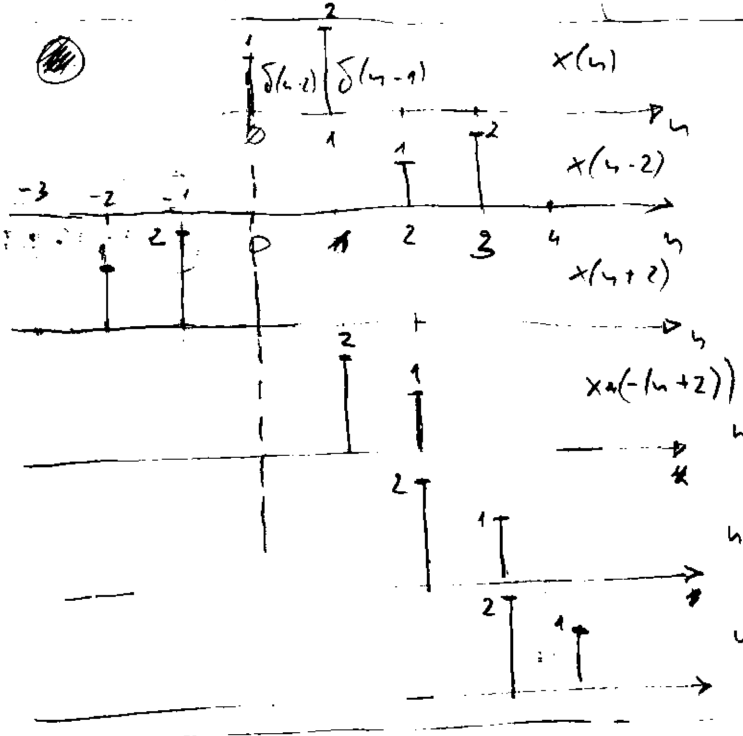
$$\mathcal{Z}[x(n+2) * x(n-2)] = \mathcal{Z}[x(n+2)]\,\mathcal{Z}[x(n-2)] = \cancel{z^{2}}(1+2z^{-1})\cdot\cancel{z^{-2}}(1+2z^{-1})$$

$$= 1 + 2z^{-1} + 2z^{-1} + 4z^{-2} = \underline{1 + 4z^{-1} + 4z^{-2}}$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k) \;; \quad \boxed{y(x) = \int_0^x h(t)\cdot x(x-t)\,dt}$$

$\boxed{\text{MATLAB CHECKED}}$

ⓓ



$e^{jx} = \cos x + j\sin x$

$e^{-jx} = \cos x - j\sin x$

$\cos x = \frac{1}{2}\left[e^{jx} + e^{-jx}\right]$

$n=0 \quad x_4(0) = 1\cdot 1 = 1$

$n=1 \quad x_4(1) = 2\cdot 1 + 1\cdot 2 = 4$

$n=2 \quad x_4(2) = 2\cdot 2 = 4$

$x_4(3) = 0$

$$\boxed{x_4(n) = x(n+2) * x(n-2) = [1, 4, 4, 0, \dots 0]}$$

ⓔ $x_5(n) = \cos(n\pi/2)\cdot x^*(n) = \left|\cos x = \frac{1}{2}\left[e^{jx} + e^{-jx}\right]\right| =$

$$= \underbrace{\frac{1}{2}e^{j\frac{n\pi}{2}}x^*(n)}_{x_{51}(n)} + \underbrace{\frac{1}{2}e^{-j\frac{n\pi}{2}}x^*(n)}_{x_{52}(n)} \;; \quad \mathcal{Z}[a^n x(n)] = X\left(\frac{z}{a}\right) \;; \quad a = e^{j\frac{\pi}{2}}$$

$\boxed{a = j}$

$\boxed{b = \frac{1}{a} = -j}$

$$\mathcal{Z}\left[\frac{1}{2}a^n\cdot x^*(n)\right] = \frac{1}{2}X^*\left(\left(\frac{z}{a}\right)^*\right) = \frac{1}{2}\left[1 + 2\left(\left(\frac{z}{a}\right)^*\right)^{-1}\right] =$$

$$= \frac{1}{2}\left[1 + 2\left(\left(\frac{z}{j}\right)^*\right)^{-1}\right] = \frac{1}{2}\left[1 + 2\left(\frac{z^*}{-j}\right)^{-1}\right] = \frac{1}{2}\left[1 + 2\frac{-j}{z^*}\right] =$$

$$= \frac{1}{2} - j\,\bar{z}^{-1} = X_{51}(z)$$

$$X_{s2}(z) = \mathcal{Z}\left[\frac{1}{2}\left(\frac{1}{a}\right)^n \cdot x^*(n)\right] = \frac{1}{2}\mathcal{Z}\left[b^n \cdot x^*(n)\right] = \frac{1}{2} X^*\left(\left(\frac{z}{b}\right)^*\right) =$$

$$= \frac{1}{2}\left[1 + 2\left(\left(\frac{z}{b}\right)^*\right)^{-1}\right] = \frac{1}{2}\left[1 + 2\left(\left(\frac{z}{-j}\right)^*\right)^{-1}\right] = \frac{1}{2}\left[1 + 2\left(\frac{z}{j}\right)^{-1}\right] =$$

$$= \frac{1}{2} + j z^{-1}$$

$$\boxed{X_5(z) = X_{s1}(z) + X_{s2}(z) = \frac{1}{2} - \cancel{j} z^{-1} + \frac{1}{2} + \cancel{j} z^{-1} = 1}\ !$$

CHECKED IN MATLAB

P.4.4 $X(z) = \dfrac{1 + z^{-1}}{1 + \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}}$, $|z| > \frac{1}{2}$

$$\boxed{\mathcal{Z}[a^n u(n)] = \frac{1}{1 - a z^{-1}}}$$

ⓐ $x_1(n) = \underbrace{x(3-n)}_{x_{11}(n)} + \underbrace{x(n-3)}_{x_{12}(n)}$

$$X(z) = \frac{-3}{1 + \frac{1}{2} z^{-1}} + \frac{4}{1 + \frac{1}{3} z^{-1}};$$

$$\boxed{x(n) = \left[-3\left(-\frac{1}{2}\right)^n + 4 \cdot \left(-\frac{1}{3}\right)^n\right] u(n)}$$

$$\boxed{\frac{z}{(z - a)} = \frac{1}{1 - a z^{-1}}}$$

$$\mathcal{Z}[x(-n)] = X\left(\frac{1}{z}\right)$$

$$X_{12}(z) = \mathcal{Z}[x(n-3)] = z^{-3} X(z) = \frac{z^{-3} + z^{-4}}{1 + \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}}$$

$$X_{11}(z) = \mathcal{Z}[x(-(n-3))] = z^{-3} X\left(\frac{1}{z}\right) = z^{-3} \frac{1 + z}{1 + \frac{5}{6} z + \frac{1}{6} z^2} =$$

$$= \frac{z^{-3} + z^{-2}}{z^{-2} + \frac{5}{6} z^{-1} + \frac{1}{6}} \quad \frac{z^{-2}}{1} = \frac{z^{-5} + z^{-4}}{z^{-2} + \frac{5}{6} z^{-1} + \frac{1}{6}} = \cancel{\frac{1}{z^2\left(1 + \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}\right)}}$$

~~$X(z) = z^{-3} + 2z^{-4} + z^{-5}$~~

$$\cancel{\frac{z^2(z^{-3} + z^{-2}) \cdot z^{-2}}{1 + \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}}} = \frac{z^{-3} + z^{-2}}{1 + \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}}$$

$$\boxed{X(z) = \frac{z^{-2} + 2z^{-3} + z^{-4}}{1 + \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}}}$$

$b = [0, 0, 1, 2, 1]$

$a = [1, 5/6, 1/6, 0, 0]$

$$X_{11}(z) = \mathcal{Z}[x(-(n-3))] = X_{12}\left(\frac{1}{z}\right) = \frac{z^3 + z^4}{1 + \frac{5}{6} z^1 + \frac{1}{6} z^2}$$

$$X(z) = X_{11}(z) + X_{12}(z) = \frac{36 z^8 + 66 z^7 + 36 z^6 + 6 z^5 + 6 z^3 + 36 z^2 + 66 z + 36}{6 z^6 + 35 z^5 + 62 z^4 + 35 z^3 + 6 z^2}$$

$$X(z)=\frac{36z^8+66z^7+36z^6+6z^5+6z^3+36z^2+66z+36}{6z^6+35z^5+62z^4+35z^3+6z^2}=$$

NE USPEAV DA GO ISCRTAM NITU VO MATLAB NITU VO MAPLE!!

$$=\frac{36z^2+66z+36+6z^{-1}+6z^{-3}+36z^{-4}+66z^{-5}+36z^{-6}}{6+35z^{-1}+62z^{-2}+35z^{-3}+6z^{-4}}$$

invztrans NE RABOTI VO MAPLE ZOŠTO IMAŠ NEGATIVNI SEQUENCE-I

$$X(z)=\underline{6z^2}-\underline{24z}+84+\underline{\underline{6z^{-2}}}+\frac{108}{3z+1}-\underline{\underline{24z^{-1}}}-\frac{324}{z+3}+\frac{48}{z+2}-\frac{24}{2z+1}$$

$$X(z)=6z^2-24z+84-24z^{-1}+6z^{-2}+\frac{36z^{-1}}{1+\frac{1}{3}z^{-1}}-\frac{324z^{-1}}{1+3z^{-1}}+\frac{48z^{-1}}{1+2z}-\frac{12z^{-1}}{1+\frac{1}{2}z^{-1}}$$

$$x(n)=6\delta(n+2)-24\delta(n+1)+84\delta(n)-24\delta(n-1)+6\delta(n-2)+36\left(-\frac{1}{3}\right)^{n-1}u(n-1)-$$
$$-324(-3)^{n-1}\cdot u(n-1)+48(-2)^{n-1}\cdot u(n-1)-12\left(-\frac{1}{2}\right)^{n-1}\cdot u(n-1)$$

(b.) $X(z)=\dfrac{1+z^{-1}}{1+\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}$; $x_2(n)=(1+n+n^2)x(n)$;

$$X_2(z)=Z\Big[\underbrace{x(n)}_{x_{21}}+\underbrace{nx(n)}_{x_{22}}+\underbrace{n^2x(n)}_{x_{23}}\Big]=X(z)\underbrace{-z\frac{d}{dz}X(z)}_{X_{22}(z)}-z\frac{d}{dz}X_{22}(z)=$$

$$\boxed{X(z)=\frac{z^2+z}{z^2+\frac{5}{6}z+\frac{1}{6}}}$$

MATLAB/MAPLE CHECKED !!!

$$X_2(z)=\frac{z^2+z}{z^2+\frac{5}{6}z+\frac{1}{6}}+\frac{6z(z^2-2z-1)}{(6z^2+5z+1)^2}+\frac{6z(6z^4-29z^3-21z^2-z+1)}{(6z^2+5z+1)^3}$$

$$=\frac{216z^6+648z^5+366z^4+66z^3+18z^2+6z}{216z^6+540z^5+558z^4+305z^3+93z^2+15z+1}$$

MATLAB CHECKED MAPLE GREŠI VO ZTRANS !?

(c.) $x_3(n)=\left(\frac{1}{2}\right)^n x(n-2)$; $Z[a^nx(n)]=X\left(\frac{z}{a}\right)$

$$X_3(z)=Z\Big[\left(\frac{1}{2}\right)^n\underbrace{x(n-2)}_{x_{31}}\Big]=X_{31}(2z)$$

$$X_{31}(z)=\frac{z^{-2}+z^{-3}}{1+\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}\ ;\quad X_3(z)=\frac{\frac{1}{4}z^{-2}+\frac{1}{8}z^{-3}}{1+\frac{5}{6}\cdot\frac{1}{2}z^{-1}+\frac{1}{6}\cdot\frac{1}{4}z^{-2}}$$

$$X_3(z)=\frac{\frac{1}{4}z^{-2}+\frac{1}{8}z^{-3}}{1+\frac{5}{12}z^{-1}+\frac{1}{24}z^{-2}}=\frac{6z^{-2}+3z^{-3}}{24+10z^{-1}+z^{-2}}=\frac{6+3z^{-1}}{24z^2+10z+1}$$

$$=\frac{6z+3}{24z^3+10z^2+z}$$

ⓓ $x_4(n) = x(n+2) * x(n-2)$

$$(-iz)(-iz) = +i^2 \cdot z^2 = -z^2$$

$$X_4(z) = \cancel{z^2} \cdot X(z) \cdot \cancel{z^{-2}} X(z) = X^2(z) = \frac{(z^2+z)^2}{\left(z^2+\frac{5}{6}z+\frac{1}{6}\right)^2} =$$

$$= \frac{36z^4 + 72z^3 + 36z^2}{36z^4+60z^3+37z^2+10z+1} \qquad (z+1)^2 = z^2+2z+1$$

ⓔ $x_5(n) = \cos(\pi n/2)\, x^*(n)$; $X(z) = \frac{1+z^{-1}}{1+\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}$ $|z| > \frac{1}{2}$

$$\cos(n\pi/2) = \frac{1}{2}e^{j\frac{n\pi}{2}} + \frac{1}{2}e^{-j\frac{n\pi}{2}} ; \quad x_5(n) = \underbrace{\frac{1}{2}\left(e^{j\frac{\pi}{2}}\right)^n \cdot x^*(n)}_{x_{51}(n)} + \underbrace{\frac{1}{2}\left(e^{-j\frac{\pi}{2}}\right)^n x^*(n)}_{x_{52}(n)}$$

$$x_5(n) = x_{51}(n) + x_{52}(n) = \frac{1}{2}a^n \cdot x^*(n) + \frac{1}{2}b^n x^*(n) \qquad a = e^{j\frac{\pi}{2}} \quad b = e^{-j\frac{\pi}{2}}\,!$$

$$\mathcal{Z}[a^n x(n)] = X\left(\frac{z}{a}\right), \quad \mathcal{Z}[x^*(n)] = X^*(z^*) ; \boxed{\mathcal{Z}[a^n x^*(n)] = X^*\left(\left(\frac{z}{a}\right)^*\right)}$$

$$X^*(z) = \left(\frac{z^2+z}{z^2+\frac{5}{6}z+\frac{1}{6}}\right)^* = \left(\frac{6z^2+6z}{6z^2+5z+1}\right)^* \qquad \boxed{a = i \qquad b = -i}$$

$$\left(\frac{1+2i}{2+3i}\right)^* = \left(\frac{(1+2i)(2-3i)}{4+9}\right)^* = \left(\frac{2-3i+4i+6}{4+9}\right)^* =$$

$$= \left(\frac{8+i}{13}\right)^* = \frac{8}{13} - \frac{i}{13}$$

$$\frac{(1+2i)^*}{2+3i} = \frac{(1-2i)(2-3i)}{4+9} = \frac{2-3i-4i-6}{13} = \frac{-4-7i}{13} \quad \#$$

$$X^*\left(\left(\frac{z}{a}\right)^*\right) = \frac{6\left(\frac{z}{a}\right)^2 + 6\left(\frac{z}{a}\right)}{6\left(\frac{z}{a}\right)^2 + 5\left(\frac{z}{a}\right)+1} = \frac{6\bar{z}^2 \cdot \left(e^{-j\pi}\right)^{*} + 6\bar{z}\cdot\left(e^{-j\frac{\pi}{2}}\right)^*}{6\bar{z}^2\cdot\left(e^{-j\pi}\right)^* + 5\bar{z}\cdot\left(e^{-j\frac{\pi}{2}}\right)^* + 1} =$$

-1 $\quad (-i)^* = +i \quad$ -1 $\quad (-i)^* = i$

$$= \frac{-6z^2 + i6\bar{z}}{-6z^2 + i5z + 1}$$

$$X^*\left(\left(\frac{z}{b}\right)^*\right) = \frac{6\left(\left(\frac{z}{b}\right)^*\right)^2 + 6\left(\frac{z}{b}\right)^*}{6\left(\left(\frac{z}{b}\right)^*\right)^2 + 5\left(\frac{z}{a}\right)^* + 1} = \frac{6\left((iz)^*\right)^2 + 6(+iz)^*}{6\left((iz)^*\right)^2 + 5(iz)^* + 1} =$$

$$= \frac{6\cdot(-i\bar{z})^2 - 6i\bar{z}}{6(-i\bar{z})^2 - 5i\bar{z}+1} = \frac{-6\bar{z}^2 - 6i\bar{z}}{-6\bar{z}^2 - 5i\bar{z}+1} = 2X_{52}(z)$$

$$x^{*}\left(\left(\frac{z}{a}\right)^{*}\right) = \frac{6\left(\left(\frac{z}{a}\right)^{*}\right)^{2} + 6\left(\frac{z}{a}\right)^{*}}{+6\left(\left(\frac{z}{a}\right)^{*}\right)^{2} + 5\left(\frac{z}{a}\right) + 1} = \Big| a = i \Big| = \frac{6\left((-iz)^{*}\right)^{2} + 6(-iz)^{*}}{6\left((-iz)^{*}\right)^{2} + 5(-iz)^{*} + 1} =$$

$$= \frac{-6\bar{z}^{2} + 6iz}{-6\bar{z}^{2} + 5iz + 1} = 2\cdot x_{s1}(z)$$

$$X_s(z) = \frac{1}{2}\,\frac{-6\bar{z}^{2} + i6z}{-6\bar{z}^{2} + 5iz + 1} + \frac{1}{2}\,\frac{-6z^{2} - 6i\bar{z}}{-6\bar{z}^{2} - 5iz + 1} = \boxed{\frac{36\bar{z}^{4} + 24\bar{z}^{2}}{36\bar{z}^{4} + 13\bar{z}^{2} + 1}}$$

$$x(n) = -3\left(-\frac{1}{2}\right)^{n} + 4\left(-\frac{1}{3}\right)^{n};\quad \mathcal{Z}\left[\cos\left(\frac{\pi n}{2}\right)\cdot x^{*}(n)\right] = \frac{36\bar{z}^{4} + 24\bar{z}^{2}}{36\bar{z}^{4} + 13\bar{z}^{2} + 1}$$

MATLAB CHECKED

$$X_s(e^{j\omega}) = \frac{36e^{-j4\omega} + 24e^{-j2\omega}}{36e^{-j4\omega} + 13e^{-j2\omega} + 1}$$

$\boxed{\text{P.4.5}}$ $\mathcal{Z}^{-1}[X(z)] = \left(\frac{1}{2}\right)^{n}u(n) = x(n)$

ⓐ $X_1(z) = \frac{z-1}{z}X(z) = \left[1 - \frac{1}{z}\right]X(z) = \left[1 - z^{-1}\right]X(z) = X(z) - z^{-1}X(z)$

$$\mathcal{Z}^{-1}[X_1(z)] = \mathcal{Z}^{-1}\left[X(z) - z^{-1}X(z)\right] = x(n) - x(n-1) = \underline{\left(\frac{1}{2}\right)^{n}u(n) - \left(\frac{1}{2}\right)^{n-1}u(n-1)}$$

$$X(z) = \mathcal{Z}\left[\left(\frac{1}{2}\right)^{n}\cdot u(n)\right] = \frac{1}{1 - \frac{1}{2}z^{-1}}$$

$$X_1(z) = \frac{z-1}{z}\,\frac{1}{1 - \frac{1}{2}z^{-1}} = \frac{z-1}{z}\,\frac{2z}{2z-1} = \frac{2z^{2} - 2z}{2z^{2} - z}$$

$$X_1(z) = \frac{2z-2}{2z-1} = \frac{1 - z^{-1}}{1 - \frac{1}{2}z^{-1}} = \frac{1}{1 - \frac{1}{2}z^{-1}} - z^{-1}\frac{1}{1 - \frac{1}{2}z^{-1}} =$$

$$= \underline{\left(\frac{1}{2}\right)^{n}u(n) - \left(\frac{1}{2}\right)^{n-1}\cdot u(n-1)};\quad \mathcal{Z}\left[\left(\frac{1}{2}\right)^{n-1}u(n-1)\right] = z^{-1}\mathcal{Z}\left[\left(\frac{1}{2}\right)^{n}\cdot u(n)\right] \underline{=} z^{-1}\frac{1}{1 - \frac{1}{2}z^{-1}}$$

ⓑ $X_2(z) = zX\left(\frac{1}{z}\right)$

$$\mathcal{Z}[x(1-n)] = \mathcal{Z}[x(-(n-1))] = z^{-1}X(z)\Big|_{z=\frac{1}{z}} = z\cdot X\left(\frac{1}{z}\right)$$

$$x_2(n) = x(1-n) = \left(\frac{1}{2}\right)^{1-n}\cdot u(1-n)$$

ⓒ $X_3(z) = \underbrace{2X(3z)}_{X_{31}(z)} + \underbrace{3X\left(\frac{z}{3}\right)}_{X_{32}(z)};$ $\boxed{x(n) = \left(\frac{1}{2}\right)^{n}u(n)}$

$\mathcal{Z}[a^{n}x(n)] = X\left(\frac{z}{a}\right)$

$$X_{31}(z) = 2\cdot X\left(\frac{z}{\left(\frac{1}{3}\right)}\right);\quad \mathcal{Z}^{-1}\left[2\cdot X\left(\frac{z}{a}\right)\right] = 2\cdot a^{n}\cdot x(n) = 2\cdot\left(\frac{1}{3}\right)^{n}\cdot x(n)$$

$X_{32}(z) = 3 \cdot X\left(\frac{z}{3}\right)$; $\quad Z^{-1}\left[3 \cdot X\left(\frac{z}{3}\right)\right] = \underline{3 \cdot 3^n \cdot x(n)}$

$$x_3(n) = \left[2 \cdot \left(\frac{1}{3}\right)^n + 3 \cdot 3^n\right] \cdot x(n) = \left[2 \cdot \left(\frac{1}{3}\right)^n + 3 \cdot 3^n\right] \cdot \left(\frac{1}{2}\right)^n \cdot u(n) =$$

$$= \left[2 \cdot \left(\frac{1}{6}\right)^n + 3 \cdot \left(\frac{3}{2}\right)^n\right] u(n)$$

$$X_3(z) = 2 \cdot \left.\frac{2z}{2z-1}\right|_{z=3z} + 3 \cdot \left.\frac{2z}{2z-1}\right|_{z=\frac{z}{3}} = 2\, \frac{6z}{6z-1} + 3 \cdot \frac{\frac{2}{3}z}{\frac{2}{3}z - 1} =$$

$$= 2 \cdot \frac{1}{1 - \frac{1}{6}z^{-1}} + 3 \cdot \frac{1}{1 - \frac{3}{2}z^{-1}} \Big/ Z^{-1} \Rightarrow x_3(n) = \left[2 \cdot \left(\frac{1}{6}\right)^n + 3 \cdot \left(\frac{3}{2}\right)^n\right] u(n)$$

(d) $X_4(z) = X(z) \cdot X(z^{-1})$

$$Z^{-1}[X_4(z)] = Z^{-1}\left[X(z) \cdot X(z^{-1})\right] = \underline{\underline{x(n) * x(-n)}}$$

$$X_4(z) = \frac{2z}{2z-1} \cdot \frac{2/z}{\frac{2}{z} - 1}\,\frac{z}{z} = \frac{2z}{2z-1} \cdot \frac{2}{2-z} = \frac{4z}{-2z^2+5z-2}$$

$$= -\frac{8}{3}\,\frac{1}{z-2} + \frac{4}{3}\,\frac{1}{2z-1} = -\frac{8}{3}\,\frac{z^{-1}}{1-2z^{-1}} + \frac{4}{3}\,\frac{\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

$$x_4(n) = -\frac{8}{3}\, 2^{n-1} u(n-1) + \frac{4}{3} \cdot \frac{1}{2} \cdot \left(\frac{1}{2}\right)^{n-1} \cdot u(n-1) =$$

$$= \left[-\frac{4}{3}\, 2^n + \frac{4}{3}\left(\frac{1}{2}\right)^n\right] u(n-1)$$

$x(n) = \left(\frac{1}{2}\right)^n u(n) \qquad x(-n) = \left(\frac{1}{2}\right)^{-n} u(-n) = 2^n$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k)$$

$$x_4(n) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k \cdot 2^{(n-k)} = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{-n} \left(\frac{1}{2}\right)^k =$$

$$= \left(\frac{1}{2}\right)^{-n} \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^{2k} = \left(\frac{1}{2}\right)^{-n} \sum_{k=-\infty}^{\infty} \left(\frac{1}{4}\right)^k = \left(\frac{1}{2}\right)^{-n} \left[\sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k\right.$$

$$\left. + \sum_{k=-\infty}^{-1} \left(\frac{1}{4}\right)^k\right] = \underbrace{\left(\frac{1}{2}\right)^{-n} \frac{1}{1-\frac{1}{4}}}_{(*)} + \underbrace{\left(\frac{1}{2}\right)^{-n} \sum_{k=-\infty}^{-1} \left(\frac{1}{4}\right)^k}_{(**)}$$

$$(*) = \frac{4}{3}\left(\frac{1}{2}\right)^{-n} = \frac{4}{3} \cdot (2)^n; \qquad (**) = \left(\frac{1}{2}\right)^{-n}\left[-1 + \sum_{k=-\infty}^{0} \left(\frac{1}{4}\right)^k\right] =$$

$$= \left(\frac{1}{2}\right)^{-n}\left[-1 + \sum_{k=0}^{\infty} 4^k\right]$$

$x(n) = \left(\frac{1}{2}\right)^n u(n)$ $\qquad x(-n) = \left(\frac{1}{2}\right)^{-n} u(-n) = 2^n \cdot u(-n)$

$$x_{41}(n) = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \cdot 2^{n-k} \cdot u(-n+k) = \frac{4}{3}(2^n)$$

$1 \quad (-n+k \geq 0, \; k \geq -n)$

$$x_{42}(n) = \sum_{k=-\infty}^{-1} \left(\frac{1}{2}\right)^{-k} \cdot 2^{-n+k} = \frac{4}{3}\left(\frac{1}{2}\right)^n \; ; \; \boxed{x_4 = x_{41} + x_{42}}$$

ⓔ $x_5(z) = z^2 \frac{d}{dz} X(z)$; $\qquad \mathcal{Z}[n\,x(n)] = -z \frac{d}{dz} X(z)$

$x(n) = \left(\frac{1}{2}\right)^n u(n) \qquad X(z) = \frac{2z}{2z-1}$

$\frac{d}{dz} X(z) = -\frac{2}{(2z-1)^2}$; $\boxed{X_5(z) = \frac{-2z^2}{(2z-1)^2} = \frac{-2z^2}{4z^2-4z+1}}$

$\mathcal{Z}[n^2 x(n)] = \mathcal{Z}[n\,x_{51}(n)] = -z \frac{d}{dz} X_{51}(z)$

$\mathcal{Z}[n\,x_{51}(n)] = -\cancel{z} \frac{d}{dz} X(z) \nRightarrow \qquad \frac{d}{dz}\left(-z \frac{d}{dz} X(z)\right) = -\frac{d}{dz} X(z) - z \frac{d^2}{dz^2} X(z)$

$\mathcal{Z}[-(n+1)\,x(n+1)] = -z\,\mathcal{Z}[n\,x(n)] = -z \cdot (-z) \frac{d}{dz} X(z) = z^2 \frac{d}{dz} X(z)$

$\int x\,dx = \frac{x^2}{2} + C \qquad \left(\frac{x^2}{2} + C\right)' = \frac{\cancel{2}x}{\cancel{2}} + \cancel{0} = x$

$x_5(n) = -(n+1) \cdot \left(\frac{1}{2}\right)^{n+1} u(n+1)$

$x_5(-1) = -(\overset{0}{-1+1}) \cancel{\left(\frac{1}{2}\right)^{-1+1}}^{1} \cancel{u(-1+1)}^{1} = 0$

$x_5(n) = -(n+1) \left(\frac{1}{2}\right)^{n+1} \cdot u(n) = -n\left(\frac{1}{2}\right)^{n+1} u(n) - \left(\frac{1}{2}\right)^{n+1} u(n)$

$\boxed{x_5(n) = -(n+1)\left(\frac{1}{2}\right)^{n+1} u(n)}$

~~$\mathcal{Z}[-(n-1)\,x(n-1)] = |m = n-1| = \mathcal{Z}[m\,x(m)] = z \frac{d}{dz} X(z)$~~

~~$\mathcal{Z}[(1-n)\,x(1-n)] = \mathcal{Z}[-(n-1)\,x(-(n-1))]$~~

~~$\mathcal{Z}[x(-(n-1))] = \frac{1}{z} X\left(\frac{1}{z}\right)$~~

$\boxed{P.4.6}$ $x_3(n) = x_1(n) * x_2(n)$

$$\sum_{n=-\infty}^{\infty} x_3(n) = \left(\sum_{n=\infty}^{\infty} x_1(n)\right)\left(\sum_{n=-\infty}^{\infty} x_2(n)\right)$$

$$\sum_{n=-\infty}^{\infty} x_3(n) = \sum_{n=-\infty}^{\infty} x_1(n)*x_2(n) = \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x_1(k)\cdot x_2(n-k)$$

$$\mathcal{Z}\left[x_1(n)*x_2(n)\right] = \sum_{n=-\infty}^{\infty} x_1(n)*x_2(n)\cdot z^{-n} = X_1(z)\,X_2(z)$$

$$\sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} x_1(k)\cdot x_2(n-k)\right]\cdot z^{-n}$$

$$-z\frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right) = \left|\begin{array}{l} u = 1-az^{-1}\\ du = -a(-1)\cdot z^{-2}dz\\ du = a\,z^{-2}dz\\ dz = \dfrac{du}{az^{-2}}\end{array}\right| \begin{array}{l} = -z(a\cdot z^{-2})\cdot\dfrac{d}{du}\left(\dfrac{1}{u}\right) = -az^{-1}(-1)\,u^{-2}\\ = +az^{-1}\dfrac{1}{(1-az^{-1})^2} = \dfrac{az^{-1}}{(1-az^{-1})^2}\end{array}$$

$$\sum_{k=-\infty}^{\infty} x_1(k)\left[\sum_{n=-\infty}^{\infty} x_2(n-k)\,z^{-n}\cdot z^{k}\right]\cdot z^{-k} = \underbrace{\sum_{k=-\infty}^{\infty} x_1(k)\cdot z^{-k}}_{X_1(z)}\;\underbrace{\sum_{n=-\infty}^{\infty} x_2(n-k)\cdot z^{-(n-k)}}_{X_2(z)}$$

$$\sum_{k=-\infty}^{\infty} x_1(k)\left[\underbrace{\sum_{n=-\infty}^{\infty} x_2(n-k)\,z^{-n}}_{t\text{-shift}}\right] = \underbrace{\sum_{k=-\infty}^{\infty} x_1(k)\cdot z^{-k}}_{X_1(z)}\cdot X_2(z) = X_1(z)\,X_2(z)$$

$$= \sum_{n=-\infty}^{\infty} x_1(n)\,z^{-n}\cdot\sum_{n=-\infty}^{\infty} x_2(n)\cdot z^{-n}$$

$$\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x_1(k)\cdot x_2(n-k) = \sum_{k=-\infty}^{\infty} x_1(k)\cdot\sum_{n=-\infty}^{\infty} x_2(n-k)\cdot z^{-k}\cdot z^{k} =$$

$$= \left|\begin{array}{l} n-k=m\\ k=n-m\end{array}\right| = \sum_{k=-\infty}^{\infty} x_1(k)\cdot z^{-k}\cdot\sum_{m=-\infty}^{\infty} x_2(m)\cdot z^{n}\cdot z^{-m} =$$

$$= z^{n}\underbrace{\sum_{k=-\infty}^{\infty} x_1(k)\,z^{-k}}_{X_1(z)}\cdot\underbrace{\sum_{m=-\infty}^{\infty} x_2(m)\cdot z^{-m}}_{X_2(z)} = z^{n}X_1(z)\cdot X_2(z)$$

$\boxed{P4.7}$ ⓐ $x_1(z) = (1 - 2z^{-1} + 3z^{-2} - 4z^{-3})(4 + 3z^{-1} - 2z^{-2} + z^{-3})$

$$= 4 - 5z^{-1} + 4z^{-2} - 2z^{-3} - 20z^{-4} + 11z^{-5} - 4z^{-6}$$

ⓑ $x_2(z) = (z^2 - 2z + 3 + 2z^{-1} - z^{-2})(z^3 - z^{-3}) =$

$$= \frac{(z^4 - 2z^3 + 3z^2 + 2z - 1)}{z^2} \cdot \frac{(z^6 - 1)}{z^3} =$$

$$= \frac{z^{10} - 2z^9 + 3z^8 + 2z^7 - z^6 - z^4 + 2z^3 - 3z^2 - 2z + 1}{z^5} =$$

$$= z^5 - 2z^4 + 3z^3 + 2z^2\ z - z^{-1} + 2z^{-2} - 3z^{-3} - 2z^{-4} + z^{-5}$$

ⓒ $x_3(z) = (1 - z^{-1} + z^{-2})^3 = 1 - 3z^{-1} + 6z^{-2} - 7z^{-3} + 6z^{-4} - 3z^{-5} + z^{-6}$

ⓓ $x_4(z) = X_1(z)X_2(z) + X_3(z)$ MATLAB + MAPLE

ⓔ $x_5(z) = (z^{-1} - 3z^{-3} + 2z^{-5} + 5z^{-7} - z^{-9})(z + 3z^2 + 2z^3 + 4z^4)$

$$x_5(z) = \frac{(z^8 - 3z^6 + 2z^4 + 5z^2 - 1)(z + 3z^2 + 2z^3 + 4z^4)}{z^9}$$

$$x_5(z) = (-z^{-9} + 5z^{-7} + 2z^{-5} - 3z^{-3} + z^{-1})(z + 3z^2 + 2z^3 + 4z^4)$$

$x_{s1} = [-1, 0, 5, 0, 2, 0, -3, 0, -1]$; $x_{s2} = [1, 3, 2, 4]$

$n_{s1} = [-9, -8, -7, -6, -5, -4, -3, -2, -1]$; $n_{s2} = [1, 2, 3, 4]$

$x_5 = \text{conv-m}(x_{s1}, n_{s1}, x_{s2}, n_{s2})$

$\boxed{P4.8}$ deconv-m

function [p, np, r, nr] = deconv-m(b, nb, a, na)

p - polynomial part $np1 \leq n \leq np2$

np = [np1, np2]

r - remainder part $nr1 \leq n \leq nr2$

nr = [nr1, nr2]

b - numerator

nb = [nb1, nb2]

a - denominator

na = [na1, na2]

$$X_5(z) = 4z^3 + 2z^2 - 9z - 5 - z^{-1} + z^{-2} + 26z^{-3} + 12z^{-4} + 11z^{-5} + 3z^{-6} - 3z^{-7} - z^{-8}$$

$$X_5(z) = (4z^4 + 2z^3 + 3z^2 + z)(z^{-1} - 3z^{-3} + 2z^{-5} + 5z^{-7} - z^{-9})$$

$$\frac{z^2 + z + 1 + z^{-1} + z^{-2} + z^{-3}}{z + 2 + z^{-1}} = (z - 1 + 2z^{-1} - 2z^{-2}) + \frac{3z^{-2} + 3z^{-3}}{z + 2 + z^{-1}}$$

TEST: $$\frac{4z^3+2z^2-9z-5-z^{-1}+z^{-2}+26z^{-3}+12z^{-4}+11z^{-5}+3z^{6}-3z^{-7}-z^{-8}+5z^{-9}}{z^{-1}-3z^{-3}+2z^{-5}+5z^{-7}-z^{-9}}=$$

$$=4\cdot z^4+2z^3+3z^2+z+\frac{5z^{-9}}{z^{-1}-3z^{-3}+2z^{-5}+5z^{-7}-z^{-9}}$$

MATLAB deconv.m

MAPLE: quo, rem devide

$\boxed{P4.9}$ $\mathcal{Z}^{-1}=?$ USING PARCIAL FRACTION EXPANSION METHOD

ⓐ $$X_1(z)=\frac{1-z^{-1}-4z^{-2}+4z^{-3}}{1-\frac{11}{4}z^{-1}+\frac{13}{8}z^{-2}-\frac{1}{4}z^{-3}}=$$

ROC: $\frac{1}{2}<|z|<1$

$z_1=\frac{1}{4}$

$z_2=\frac{1}{2}$

$$=-\frac{10}{1-\frac{1}{2}z^{-1}}+\frac{27}{1-\frac{1}{4}z^{-1}}-16.$$

MATLAB residuez

$$x_1(n)=-\left(\frac{1}{2}\right)^n\cdot 10\,u(n)+\left(\frac{1}{4}\right)^n\cdot 27\cdot u(n)-16\delta(n)$$

MAPLE CHECKED

$\boxed{z_1,z_2\le R_{x-}=\frac{1}{2}}$ ⇒ RIGHT SIDED SEQ

sequence is abs. summable

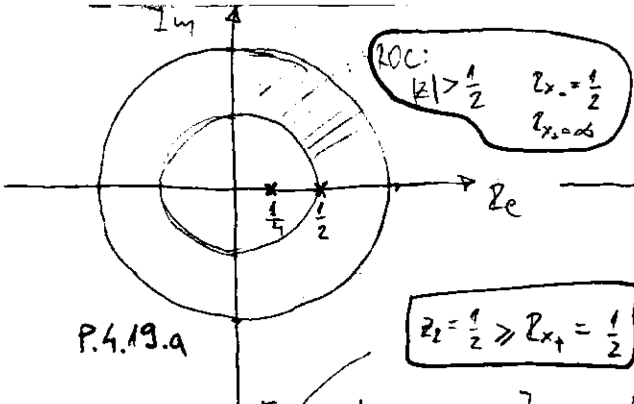
ⓑ $$X_2(z)=\frac{1-z^{-1}-4z^{-2}+4z^{-3}}{1-\frac{11}{4}z^{-1}+\frac{13}{8}z^{-2}-\frac{1}{4}z^{-3}}$$

ⓒ $$X_3(z)=\frac{z^3-3z^2+4z+1}{z^3-4z^2+z-0.16}$$ left sided sequence

$$\boxed{\mathcal{Z}^{-1}\left[\frac{z}{z-p_k}\right]=\begin{cases}p_k^n\,u(n) & |z_k|\le R_{x-}\\ -p_k^n\,u(-n-1) & |z_k|\ge R_{x+}\end{cases}}$$

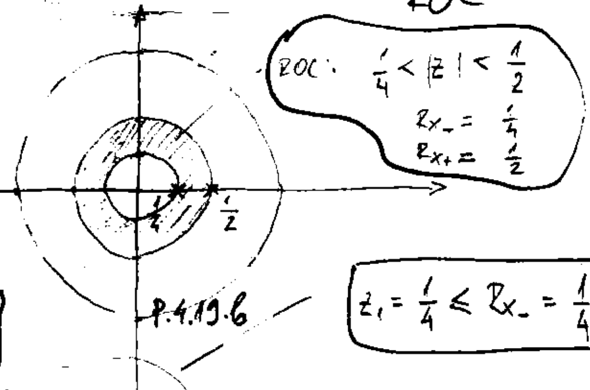
POLES ARE ON INTERIOR SIDE OF ROC

POLES ARE ON EXTERIOR SIDE OF ROC



P.4.19.a

$\boxed{z_2=\frac{1}{2}\ge R_{x+}=\frac{1}{2}}$



P.4.19.b

$\boxed{z_1=\frac{1}{4}\le R_{x-}=\frac{1}{4}}$

ⓑ $$x(n)=-10\left[\ominus\left(\frac{1}{2}\right)^n u(-1-n)\right]+27\left(\frac{1}{4}\right)^n\cdot u(n)+16\delta(n)=$$

$$=10\cdot\left(\frac{1}{2}\right)^n u(-1-n)+27\left(\frac{1}{4}\right)^n\cdot u(n)-16\delta(n)$$

ⓒ $R=[0.54;\ 3.4+j5.8;\ 3.4-j5.8];$ $\quad C=-6.25;$

$p=[3.7,\ 0.13+j0.16;\ 0.13-j0.16];$

$$X_3(z)=\frac{0.54}{1-0.54z^{-1}}+\frac{3.4+j5.8}{1-(0.13+j0.16)z^{-1}}+\frac{3.4-j5.8}{1-(0.13-j0.16)z^{-1}}-6.25$$

$R\doteq[0.54;\ 6.7\cdot e^{+j\frac{\pi}{3}};\ 6.7\cdot e^{-j\frac{\pi}{3}}]$ $\quad p=[3.7;\ 0.2\cdot e^{j0.9};\ 0.2\cdot e$

$$X_3(z)=\frac{0.54}{1-3.7\cdot z^{-1}}+\frac{6.7\cdot e^{j\frac{\pi}{3}}}{1-0.2e^{j0.9}z^{-1}}+\frac{6.7\cdot e^{-j\frac{\pi}{3}}}{1-0.2\cdot e^{-j0.9}z^{-1}}-6.25$$

Intellicom

$$x_3(n) = -0.54\cdot(3.7)^n\cdot u(-1-n) - 6.7\,e^{j\frac{\pi}{3}}\left(0.2\cdot e^{j0.9}\right)^n u(-1-n) - 6.7\cdot e^{-j\frac{\pi}{3}}\left(0.2\cdot e^{-j0.9}\right)^n u(-1-n) - 6.25\,\delta(n)$$

$$x_3(n) = \Big[-0{,}54(3{,}7)^n \underbrace{- 6.7\,e^{j\frac{\pi}{3}}\,0.2^n\,e^{j0{,}9n} - 6.7\cdot e^{-j\frac{\pi}{3}}\cdot 0.2^n\,e^{-j0{,}9n}}_{(*)}\Big]u(-1-n) - 6.25\cdot\delta(n)$$

$$(*) = -6.7\cdot(0{,}2)^n\Big[\underbrace{e^{j(\frac{\pi}{3}+0.9n)} + e^{-j(\frac{\pi}{3}+0.9n)}}_{2\cdot\cos\left(\frac{\pi}{3}+0.9n\right)}\Big] = -6.7\cdot(0.2)^n\cdot 2\cdot\cos\left(\frac{\pi}{3}+0.9n\right)$$

$\frac{406}{389} = 1{,}05 \quad \frac{2131}{2357}$

$$\boxed{x_3(n) = -\Big[0.54(3.7)^n + 13.4\cdot(0.2)^n\cdot\cos\left(\frac{\pi}{3}+0.9n\right)\Big]u(-1-n) - 6.25\cdot\delta(n)}$$

MAPLE CHECKED : ztrans. DAVA NAVISTINA POCINOM MNOGU PRIBLIZEN NA DADENIOT !!!

ⓓ $X_4(z) = \dfrac{z}{z^3+2z^2+1{,}25z+0{,}25}$ $\quad \underline{|z|>1}$

b = [0, 0, 1, 0]
a = [1, 2, 1.25, 0.25]

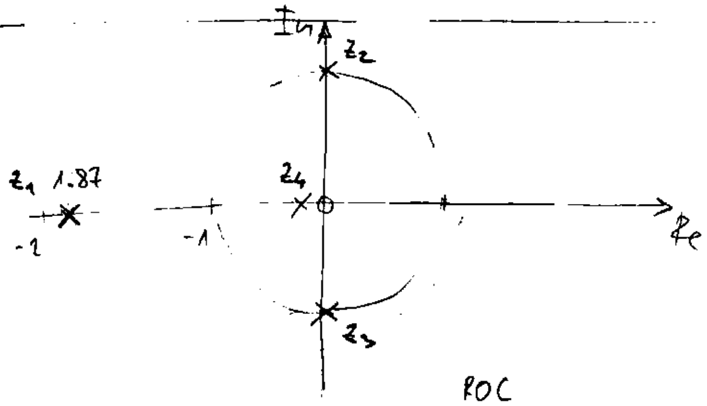
$X_4(z) = \dfrac{z^{-2}}{1+2z^{-1}+1.25z^{-2}+0.25z^{-3}}, \quad |z|>1$

b = [0, 0, 1, 0]
a = [1, 2, 1.25, 0.25]

$z = [0.24,\ 0.23\cdot e^{-j1.9},\ 0.23\,e^{j1.9},\ -0.07]$

$p = [-1{,}87,\ e^{j\frac{\pi}{2}},\ e^{-j\frac{\pi}{2}},\ -0{,}13]$

$\quad z_1 \quad z_2 \quad z_3 \quad z_4$



ROC

$\boxed{1<|z|<1.8}$

$$X(z) = \underbrace{\frac{0.24}{1+1{,}87\cdot z^{-1}}}_{z_1} + \underbrace{\frac{0.23\,e^{-j1.9}}{1-e^{j\frac{\pi}{2}}z^{-1}}}_{z_2} + \underbrace{\frac{0.23\,e^{j1.9}}{1-e^{-j\pi/2}z^{-1}}}_{z_3} - \underbrace{\frac{0.07}{1+0{,}13\,z^{-1}}}_{z_4}$$

$$x(n) = -0.24(-1.87)^n\cdot u(-1-n) + \Big[\underbrace{0.23e^{-j1.9}e^{j\frac{4\pi}{2}} + 0.23e^{j1.9}e^{-j\frac{4\pi}{2}}}_{*} - 0.07(-0.13)^n\Big]u(n)$$

$$(*) = 0.23\left[e^{j\left(\frac{4\pi}{2}-1.9\right)} + e^{-j\left(\frac{4\pi}{2}-1.9\right)}\right] = 0.23\cdot 2\,\cos\left(\frac{4\pi}{2}-1.9\right) = 0.46\cos\left(\frac{4\pi}{2}-1.9\right)$$

$$x(n) = -0.24(-1.87)^n u(-1-n) + \left[0.46\cos\left(\frac{4\pi}{2}-1.9\right) - 0.07(-0.13)^n\right]u(n)$$

ⓔ $X_5(z) = \dfrac{z}{(z^2-0.25)^2}$ $\quad \boxed{|z|<0.5}$

$$X_5(z) = \frac{z}{\left((z-0.5)(z+0.5)\right)^2} = \frac{z}{(z-0.5)^2(z+0.5)^2}$$

$$X_5(z) = \frac{z}{z^4-0{,}5z^2+0.0625} = \frac{z^{-3}}{1-0.5z^{-2}+0.0625z^{-4}}$$

b = [0, 0.0, 1, 0]
a = [1, 0, -0.5, 0, 0.0625]

$$x_5(z) = -\frac{4}{1-0.5z^{-1}} + \frac{2}{(1-0.5z^{-1})^2} + \frac{4}{1+0.5z^{-1}} - \frac{2}{(1+0.5z^{-1})^2}$$

$$x_5(z) = \frac{z}{(z-0{,}\text{[illegible]}5)(z-0{,}\text{[illegible]}5)(z+0{,}\text{[illegible]}5)(z+0{,}\text{[illegible]}5)} = \frac{z}{z^4-0.5z^2+0.0625} = \frac{z}{(z^2-0.25)^2}$$

Ex. 4.9

$$X(z) = \frac{1}{(1-0.9z^{-1})^2(1+0.9z^{-1})} = \frac{1}{1-0.9z^{-1}-0.81z^{-2}+0.729z^{-3}}$$

$[R, p, c] = \text{residuez}(b, a)$

$b = [1] = [1, 0, 0, 0]$

$a = [1, -0.9, -0.81, 0.729]$

$R = [0.25, 0.25, 0.5]$

$p = [-0.9, 0.9, 0.9]$

$$X(z) = \frac{0.25}{1+0.9z^{-1}} + \frac{0.25}{1-0.9z^{-1}} + \frac{0.5}{(1-0.9z^{-1})^2}$$

$\boxed{\text{MAPLE - MATLAB CHECKED}}$!!!

$b = [0, 0, 0, 1, 0]$

$a = [1, 0, -0.5, 0, 0.0625]$

$[R, p, c] = \text{residuez}(b, a)$

r - MULTIPLICITY OF POLE

$R = [-4, 2, 4, -2]$

$p = [0.5, 0.5, -0.5, -0.5]$

$$\sum_{l=1}^{r} \frac{R_{k,l}z^{-(l-1)}}{(1-p_kz^{-1})^l} = \left| r=2 \right| = \sum_{l=1}^{2} \frac{R_{k,l}z^{-(l-1)}}{(1-p_kz^{-1})^l} =$$

$$= \frac{R_{k,1}z^{-(1-1)}}{(1-p_kz^{-1})} + \frac{R_{k,2}z^{-(2-1)}}{(1-p_kz^{-1})^2}$$

$$\boxed{\sum_{l=1}^{r} \frac{R_{k,l}}{(1-p_kz^{-1})^l} = \frac{R_{k,1}}{(1-p_kz^{-1})} + \frac{R_{k,2}}{(1-p_kz^{-1})^2} + \dots \frac{R_{k,r}}{(1-p_kz^{-1})^r}}$$

FORMULA ZA r-TI POL

$$x_5(z) = -\frac{4}{(1-0.5z^{-1})} + 2\cdot\frac{z}{0.5}\frac{0.5z^{-1}}{(1-0.5z^{-1})^2} + \frac{4}{(1+0.5z^{-1})} - \frac{2z}{(-0.5)}\frac{-0.5z^{-1}}{(1+0.5z^{-1})^2}$$

$$\mathcal{Z}[n\cdot a^n u(n)] = \frac{az^{-1}}{(1-az^{-1})^2} \qquad \mathcal{Z}[-n b^n u(-n-1)] = \frac{bz^{-1}}{(1-bz^{-1})^2}$$

$$X_5(z) = -\frac{4}{(1-0.5z^{-1})} + z\frac{0.5z^{-1}}{(1-0.5z^{-1})^2} + \frac{4}{(1+0.5z^{-1})} + z\frac{-0.5z^{-1}}{(1+0.5z^{-1})^2} \qquad |z| < 0.5$$

$$x(n) = 4\cdot(0.5)^n u(-1-n) - (n+1)(0.5)^{n+1}u(-1-n-1) - 4(-0.5)^n \overset{u(-1-n)}{\vee} - (n+1)(-0.5)^{n+1}\cdot u(-1-n-1)$$

$$= 4[(0.5)^n - (-0.5)^n]u(-1-n) - (n+1)[(0.5)^{n+1} + (-0.5)^{n+1}]u(-2-n)$$

$\boxed{P.4.10}$

$$X(z) = \frac{2+3z^{-1}}{1-z^{-1}+0.81z^{-2}} \qquad |z| > 0.9$$

$b = [2, 3]$

$a = [1, -1, 0.81]$

ⓐ $x(n) = ?$ NO complex numbers

ⓑ Check in MATLAB

$$R = [1-i2.67, 1+i2.67] = [2.9\cdot e^{-j1.2}, 2.9\cdot e^{j1.2}]$$

$$p = [0.5+i0.75, 0.5-i0.75] = [0.9\cdot e^{j0.98}, 0.9\cdot e^{-j0.98}]$$

$$X(z) = \frac{2.9\cdot e^{-j1.2}}{1-0.9e^{j0.98}z^{-1}} + \frac{2.9\cdot e^{j1.2}}{1-0.9\cdot e^{-j0.98}z^{-1}}$$

$$X(z) = \frac{2.9 \cdot e^{-j1.2}}{1 - 0.9 \cdot e^{j0.98} z^{-1}} + \frac{2.9 \cdot e^{j1.2}}{1 - 0.9 \cdot e^{-j0.98} z^{-1}} ; \quad \mathcal{Z}\left[a^n u(n)\right] = \frac{1}{1 - a z^{-1}}$$

$\boxed{ROC: \ |z| > 0.9}$ right sided sequence

$$x(n) = \left(2.9 \cdot e^{-j1.2} \cdot (0.9)^n e^{j0.98n} + 2.9 \cdot e^{j1.2} (0.9)^n \cdot e^{-j0.98n}\right) u(n)$$

$$x(n) = 2.9 (0.9)^n \Big(\underbrace{e^{j(0.98n - 1.2)} + e^{-j(0.98n - 1.2)}}_{2\cos(0.98n - 1.2)} \Big) u(n)$$

$$\boxed{x(n) = 2 \cdot \underbrace{(2.8536)}_{5.7072} \cdot (0.9)^n \cos(0.9818\, n - 1.2128)\, u(n)}$$

MATLAB CHECKED
error = $5 \cdot 10^{-4}$

$\boxed{4.11}$: LTI system

(i) system function representation
(ii) difference equation - " -
(iii) pole-zero plot
(iv) $y(n)$ if input $x(n) = \left(\frac{1}{4}\right)^n u(n)$

$$\boxed{X(z) = \frac{1}{1 - \left(\frac{1}{4}\right) z}}$$

ⓐ $h(n) = 2 \cdot \left(\frac{1}{2}\right)^n u(n)$ (i) $\boxed{H(z) = \frac{2}{1 - \frac{1}{2} z^{-1}}}$; $H(z) = \frac{Y(z)}{X(z)}$

$$H(z) - \frac{1}{2} z^{-1} H(z) = 2 \quad ; \quad \frac{Y(z)}{X(z)} - \frac{1}{2} z^{-1} \frac{Y(z)}{X(z)} = 2$$

$$Y(z) - \frac{1}{2} z^{-1} Y(z) = 2X(z)$$

(ii) $\boxed{y(n) - \frac{1}{2} y(n-1) = 2x(n)}$

(iii) ROC: $|z| > \frac{1}{2}$

(iv) $y(n) - \frac{1}{2} y(n-1) = 2 \cdot \left(\frac{1}{4}\right)^n u(n) \ / \mathcal{Z}$

$$Y(z) - \frac{1}{2} z^{-1} Y(z) = \frac{2}{1 - \frac{1}{4} z^{-1}} \quad ; \quad Y(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} \cdot \frac{2}{1 - \frac{1}{4} z^{-1}}$$

$$Y(z) = \frac{2}{\left(1 - \frac{1}{2} z^{-1}\right)\left(1 - \frac{1}{4} z^{-1}\right)} = \frac{2}{1 - \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2}} = \frac{4}{1 - \frac{1}{2} z^{-1}} - \frac{2}{1 - \frac{1}{4} z^{-1}}$$

" poly([1/2 1/4]) "
conv([1 -1/2], [1 -1/4])

ROC_y $\boxed{|z| > \frac{1}{2}}$

$$\boxed{y(n) = \left[4 \cdot \left(\frac{1}{2}\right)^n - 2 \cdot \left(\frac{1}{4}\right)^n\right] u(n)}$$

MATLAB CHECKED

ROC_h: $|z| > \frac{1}{2}$ ROC_x: $|z| > \frac{1}{4}$

$ROC_y = ROC_h \cap ROC_x \Rightarrow$
ROC_y: $|z| > \frac{1}{2}$

$$y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{l=0}^{M} b_l \cdot x(n-l) \; ; \; / \mathcal{Z}$$

$$Y(z) + \sum_{k=1}^{N} a_k \cdot z^{-k} Y(z) = \sum_{l=0}^{M} b_l z^{-l} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{l=0}^{M} b_l z^{-l}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{B(z)}{A(z)} = \frac{(b_M z^{-M} + \cdots b_1 z^{-1} + b_0)}{(a_N z^{-N} + \cdots a_1 z^{-1} + 1)} =$$

$$= \frac{b_0 z^{-M} \left(\frac{b_M}{b_0} + \cdots + \frac{b_1}{b_0} z^{M-1} + \cdots z^{M} \right)}{z^{-N} \left(a_N + \cdots a_1 z^{N-1} + \cdots z^{N} \right)} = b_0 z^{N-M} \frac{z^{M} + \cdots + \frac{b_{M-1}}{b_0} z^{1} + \frac{b_M}{b_0}}{z^{N} + \cdots + a_{N-1} z + a_N}$$

$$\boxed{H(z) = b_0 z^{N-M} \frac{\prod_{l=1}^{M} (z - z_l)}{\prod_{k=1}^{N} (z - p_k)}}$$

ZEROS (SYSTEM z)

SYSTEM POLES

FREQUENCY - RESPONSE FCN: $H(e^{j\omega}) = b_0 e^{j(N-M)\omega} \frac{\prod_{l=1}^{M} (e^{j\omega} - z_l)}{\prod_{k=1}^{N} (e^{j\omega} - p_k)}$

$$H(e^{j\omega}) = |b_0| \frac{|e^{j\omega} - z_1| \cdots |e^{j\omega} - z_M|}{|e^{j\omega} - p_1| \cdots |e^{j\omega} - p_N|}$$

$$\angle H(e^{j\omega}) = [0 \text{ or } \pi] + \underbrace{[(N-M)\omega]}_{\text{linear}} + \underbrace{\sum_{1}^{M} \angle(e^{j\omega} - z_l) - \sum_{1}^{N} \angle(e^{j\omega} - p_k)}_{\text{nonlinear}}$$

$$\boxed{\begin{array}{l} [H, w] = \text{freqz}(b, a, N) \; ; \; [H, w] = \text{freqz}(b, a, N, \text{'whole'}) \\ H = \text{freqz}(b, a, w) \end{array}}$$

ⓑ $h(n) = n \left(\frac{1}{3}\right)^n u(n) + \left(-\frac{1}{4}\right)^n u(n) = h_{11}(n) + h_{12}(n)$

$$\mathcal{Z}[n x(n)] = -z \frac{dX(z)}{dz} \qquad -z \frac{d}{dz} \left(\frac{1}{1 - \frac{1}{3} z^{-1}} \right) = \frac{\frac{1}{3} z^{-1}}{\left(1 - \frac{1}{3} z^{-1}\right)^2}$$

(i)

$$H(z) = \frac{\frac{1}{3} z^{-1}}{\left(1 - \frac{1}{3} z^{-1}\right)^2} + \frac{1}{1 + \frac{1}{4} z^{-1}} = \frac{36 - 12 z^{-1} + 7 z^{-2}}{36 - 15 z^{-1} - 2 z^{-2} + z^{-3}}$$

$bb = [36, -12, 7, 0]$
$ab = [36, -15, -2, +1]$

(ii) $36 Y(z) - 15 z^{-1} Y(z) - 2 z^{-2} Y(z) + z^{-3} Y(z) = (36 - 12 z^{-1} + 7 z^{-2}) X(z)$

$$\boxed{36 y(n) - 15 y(n-1) - 2 y(n-2) + y(n-3) = 36 x(n) - 12 x(n-1) + 7 x(n-2)}$$

(iv) $Y(z) = \frac{36 - 12 z^{-1} + 7 z^{-2}}{36 - 15 z^{-1} - 2 z^{-2} + z^{-3}} \cdot \frac{1}{1 - \frac{1}{4} z^{-1}}$

$$Y(z) = \frac{-16}{1 - \frac{1}{3} z^{-1}} + \frac{4}{\left(1 - \frac{1}{3} z^{-1}\right)^2} + \frac{0.5}{1 + \frac{1}{4} z^{-1}} + \frac{12.5}{1 - \frac{1}{4} z^{-1}}$$

DUPLI POL

$$Y(z)=\frac{-16}{\left(1-\frac{1}{3}z^{-1}\right)}+\frac{\frac{1}{3}z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)^2}\frac{4z}{\frac{1}{3}}+\frac{0.5}{\left(1+\frac{1}{4}z^{-1}\right)}+\frac{12.5}{\left(1-\frac{1}{4}z^{-1}\right)}\Rightarrow \quad |z|>\frac{1}{3}$$

$$y(n)=-16\left(\frac{1}{3}\right)^n\overset{u(n)}{}+\left(12n\left(\frac{1}{3}\right)^n u(n)\right)\Bigg|_{n=n+1}+0.5\left(-\frac{1}{4}\right)^n u(n)+12.5\left(\frac{1}{4}\right)^n u(n)$$

$$y(n)=\left[-16\left(\frac{1}{3}\right)^n+0.5\left(-\frac{1}{4}\right)^n+12.5\left(\frac{1}{4}\right)^n\right]u(n)+12(n+1)\left(\frac{1}{3}\right)^{n+1}u(n+1)$$

$$y(-1)=0+0=0$$

$$y(n)=\left[-16\left(\frac{1}{3}\right)^n+0.5\left(-\frac{1}{4}\right)^n+12.5\left(\frac{1}{4}\right)^n+4n\cdot\left(\frac{1}{3}\right)^n+4\left(\frac{1}{3}\right)^n\right]u(n)$$

$$\boxed{y(n)=\Big[\underbrace{-12\left(\frac{1}{3}\right)^n+0.5\left(-\frac{1}{4}\right)^n+12.5\left(\frac{1}{4}\right)^n}+\bigcirc\!\!\!\!\!\!\;4n\left(\frac{1}{3}\right)^n\Big]u(n)}$$

MATLAB CHECKED error = 10^{-16}

ČLEN AKO NE GO TRETIRAV DUPLIOT POL „KAKO DUPLI"

ČLEN POSLEDICA NA DUPLIOT POL !!!

VO TOJ SLUČAJ SE JAVUVA GOLEMA GREŠKA (error = 1.333) VO PROVERKATA VO MATLAB

ⓒ $h(n)=3\cdot(0.9)^n\cdot\cos(\pi n/4+\pi/3)\,u(n+1)$

$$\mathcal{Z}^{+}\left[a^n\cos(\omega_0 n)\right]=\frac{1-a\cos(\omega_0)z^{-1}}{1-2a\cos(\omega_0)z^{-1}+a^2z^{-1}}\;;\;\mathcal{Z}\left[a^n\cos(\omega_0 n)\right]=\frac{(a\cdot\sin(\omega_0))z^{-1}}{1-2a\cos(\omega_0)z^{-1}+a^2z^{-2}}$$

$$h(n)=3\cdot\frac{1}{0.9}(0.9)^{n+1}\cdot\cos\left(\frac{(n+1)\pi}{4}-\frac{\pi}{4}+\frac{\pi}{3}\right)u(n+1)\Rightarrow$$

$$h(n)=\frac{10}{3}\cdot 0.9^{n+1}\underbrace{\cos\left((n+1)\pi/4-\pi/12\right)}_{*}u(n+1)$$

$$\cos(\alpha-\beta)=\cos\alpha\cdot\cos\beta+\sin\alpha\cdot\sin\beta$$

$$h(n)=\frac{10}{3}(0.9)^{n+1}\,\underline{0.97}\cdot\cos\left((n+1)\pi/4\right)\cdot u(n+1)+\frac{10}{3}(0.9)^{n+1}\,\underline{\underline{0.26}}\sin\left((n+1)\pi/4\right)u(n+1)$$

$$H(z)=\frac{10}{3}\cdot z\cdot\mathcal{Z}\left[(0.9)^n\cdot 0.97\cdot\cos(n\pi/4)\,u(n)+(0.9)^n\cdot 0.26\,\sin(n\pi/4)\,u(n)\right]=$$

$$=\frac{10}{3}z\cdot\left[0.97\cdot\frac{1-0.9\cdot\cos\left(\overset{\sqrt{2}/2}{\pi/4}\right)z^{-1}}{1-2\cdot0.9\cdot\cos(\pi/4)\cdot z^{-1}+(0.9)^2\cdot z^{-2}}+0.26\,\frac{0.9\sin\overset{\sqrt{2}/2}{\pi/4}\;z^{-1}}{1-2\cdot0.9\cdot\cos\left(\frac{\pi}{4}\right)z^{-1}+(0.9)^2z^{-2}}\right]$$

$$=\frac{10}{3}z\;\frac{0.97-0.97\cdot0.9\cdot0.71\cdot z^{-1}+0.26\cdot0.9\cdot0.71\cdot z^{-1}}{1-1.2728z^{-1}+0.81z^{-2}}=$$

$$\frac{10}{3}z\;\frac{\overset{\cos(\pi/12)}{0.97}-0.45\cdot z^{-1}}{1-1.2728z^{-1}+0.81z^{-2}}=\frac{3.2198z-1.5}{1-1.2728z^{-1}+0.81z^{-2}}$$

$$H(z) = \frac{(3.2198 - 1.5z^{-1})z}{1 - 1.2728z^{-1} + 0.81z^{-2}} = \frac{3.2198 - 1.5z^{-1}}{z^{-1} - 1.2728z^{-2} + 0.81z^{-3}} = \frac{3.2198z - 1.5}{1 - 1.2728z^{-1} + 0.81z^{-2}}$$

$b = [3,2198; -1,5]$

$a = [0, +1, -1.2728, 0.81]$ ⇒ can't use in zplane (zero leading coeff in denominator)

(ii) $Y(z) - 1.2728\, z^{-1}\, Y(z) + 0.81\, z^{-2}\, Y(z) = 3.2198 \cdot z \cdot X(z) - 1.5\, X(z)$

(iv) $Y(z) = H(z) \cdot X(z) = \dfrac{3.2198z - 1.5}{1 - 1.2728z^{-1} + 0.81z^{-2}} \cdot \dfrac{1}{1 - \frac{1}{4}z^{-1}}$

$$Y(z) = \frac{3.2198z - 1.5}{1 - 1.5228z^{-1} + 1.1282z^{-2} - 0.2025z^{-3}}$$

$$\boxed{Y(z) = z \cdot \frac{3.2198 - 1.5z^{-1}}{1 - 1.5228z^{-1} + 1.1282z^{-2} - 0.2025z^{-3}} = z \cdot \underline{Y_1(z)}}$$

$r = [2.0148\, e^{-j0.5015},\ 2.0148\, e^{j0.5015},\ -0.3135]$

$p = [0.9\, e^{+j\pi/4},\ 0.9 \cdot e^{-j\pi/4},\ 0.25]$

$$Y_1(z) = \frac{2.0148\, e^{-j0.5015}}{1 - 0.9e^{j\pi/4}z^{-1}} + \frac{2.0148\, e^{j0.5015}}{1 - 0.9\, e^{j\pi/4}z^{-1}} - \frac{0.3135}{1 - \frac{1}{4}z^{-1}}$$

$$y_1(n) = \left[2.0148 \cdot (0.9)^n \cdot e^{jn\pi/4} e^{-j0.5015} + 2.0148 \cdot (0.9)^n \cdot e^{-jn\pi/4} \cdot e^{j0.5015} - \left(\frac{1}{4}\right)^n \cdot 0.3135\right] u(n)$$

~~$y_1(n) = \left[2 \cdot 2,0148 \cdot (0.9)^n \cdot \cos(n\pi/4) - 0.3135 \left(\frac{1}{4}\right)^n\right] u(n)$~~

~~$y(n) = \left[4,0296 \cdot (0.9)^{n+1} \cos((n+1)\pi/4) - 0.3135 \left(\frac{1}{4}\right)^{n+1}\right] u(n+1)$~~

~~$n = -1: \quad y(n) = 4,0296 - 0.3135$~~

$$y_1(n) = \left[4.0296 \cdot (0.9)^n \cdot \cos(n\pi/4 - 0.5015) - 0.3135 \left(\frac{1}{4}\right)^n\right] u(n)$$

MATLAB CHECKED

$$y(n) = \left[4.0296\,(0.9)^{n+1} \cos((n+1)\pi/4 - 0.5015) - 0.3135 \left(\frac{1}{4}\right)^{n+1}\right] u(n+1)$$

ⓓ $h(n) = n\left[u(n) - u(n-10)\right] = n\,u(n) - n\,u(n-10)$

$$\boxed{\mathcal{Z}\left[n a^n u(n)\right] = \frac{a z^{-1}}{(1 - a z^{-1})^2} \quad ; \quad \mathcal{Z}\left[n\, u(n)\right] = \frac{z^{-1}}{(1 - z^{-1})^2}}$$

$$H(z) = \frac{z^{-1}}{(1 - z^{-1})^2} - \mathcal{Z}\left[(n-10)\,u(n-10) + 10\,u(n-10)\right] =$$

$$= \frac{z^{-1}}{(1 - z^{-1})^2} - z^{-10} \frac{z^{-1}}{(1 - z^{-1})^2} + \frac{10}{1 - z^{-1}} = \frac{z^{-1} - z^{-11} + 10(1 - z^{-1})}{(1 - z^{-1})^2}$$

$$= \frac{z^{-1} - z^{-11} + 10 - 10z^{-1}}{(1 - z^{-1})^2} = \frac{10 - 9z^{-1} - z^{-11}}{1 - 2z^{-1} + z^{-2}} \quad \boxed{(i)}$$

$H(z) = \dfrac{10 - 9z^{-1} - z^{-11}}{1 - 2z^{-1} + z^{-2}}$; $\quad Y(z) - 2z^{-1}Y(z) + z^{-2}Y(z) = 10X(z) - 9z^{-1}X(z) - z^{-11}X(z)$

$y(n) - 2y(n-1) + y(n-2) = 10x(n) - 9x(n-1) - x(n-11)$ $\boxed{(ii)}$

$$\boxed{Y(z) = \frac{10 - 9z^{-1} - z^{-11}}{1 - 2z^{-1} + z^{-2}} \quad \frac{1}{1 - \frac{1}{4}z^{-1}}}$$

$R = [3.6, 2.7, 4.7]$

$p = [1, 1, 0.25]$

$C = [-466020, -116496, -29116, -7272, -1812, -448, -108, -24, -4]$

$$\boxed{X(z) = \sum_{k=1}^{N} \frac{R_k}{1 - p_k z^{-1}} + \sum_{k=0}^{M-N} C_k \cdot z^{-k}}$$

*

$$Y(z) = \frac{27}{1 - z^{-1}} + z \underset{n=-1}{\overset{\approx 0}{\cancel{\frac{2.7z^{-1}}{(1-z^{-1})^2}}}} = 0 + \frac{466037}{1 - \frac{1}{4}z^{-1}} + 466020 + 116496z^{-1} + 29116z^{-2} + 7272z^{-3} + \cdot + 1812z^{-4}, + 448z^{-5}, +108z^{-6}, +24z^{-7}, +4z^{-8}$$

~~$y(n) = 3.6\,u(n) + 2.7(n+1)^{x} \cdot u(n+1) + 4.7\left(\frac{1}{4}\right)^{n} u(n) - 466020\,\delta(n) - 116496\,\delta(n-1) - 29116\,\delta(n-2) - 7272\,\delta(n-3) - 1812\,\delta(n-4) - 448\,\delta(n-5) - 108\,\delta(n-6) - 24\,\delta(n-7) - 4\,\delta(n-8)$~~

$$Y(z) = \frac{27}{1 - z^{-1}} - \frac{466037}{1 - \frac{1}{4}z^{-1}} + ⊛$$

$$\boxed{Y(t) = 27\,u(n) - 466037\left(\frac{1}{4}\right)^{n} u(n) + 466020\,\delta(n) + 116496\,\delta(n-1) + \ldots + 4\delta(n-8)}$$

ⓔ $h(n) = [2 - \sin(n\pi)]\,u(n) = 2 \cdot u(n) - \sin(n\pi)\,u(n)$

$$H(z) = \frac{2}{1 - z^{-1}} - \frac{\overset{\approx 0}{\sin(\pi)} \cdot z^{-1}}{1 - 2\cos(\pi)z^{-1} + z^{-2}} = \frac{2}{1 - z^{-1}} \quad \boxed{(i)}$$

$$Y(z) = \frac{2}{1 - z^{-1}} \cdot \frac{1}{1 - \frac{1}{4}z^{-1}}$$

$Y(z) - z^{-1} \cdot Y(z) = 2X(z)$

$y(n) - y(n-1) = 2x(n)$ $\boxed{(ii)}$

$$Y(z) = \frac{8/3}{1 - z^{-1}} - \frac{2/3}{1 - \frac{1}{4}z^{-1}} \quad \boxed{(iv)}$$

MATLAB CHECKED

$$y(n) = \frac{8}{3}u(n) - \frac{2}{3}\left(\frac{1}{4}\right)^{n} u(n)$$

$\boxed{P.4.12}$ LTI system determine

(i) impulse response representation

(ii) difference equation – " –

(iii) pole zero plot

(iv) $y(n)$ if $\boxed{x(n) = 3\cos(n\pi/3)\,u(n)}$

$$\boxed{x(n) = 3\cos(\pi n/3)\, u(n)} \qquad \mathcal{Z}\left[a^n \cos(\omega_0 n)\right] = \frac{1 - a\cdot\cos(\omega_0) z^{-1}}{1 - 2a\cos(\omega_0) + z^{-2}}$$

$$\boxed{X(z)} = 3\,\frac{1 - \cos(\pi/3)\, z^{-1}}{1 - 2\cos(\pi/3) z^{-1} + z^{-2}} = \frac{3 - \frac{3}{2} z^{-1}}{1 - z^{-1} + z^{-2}} = \boxed{\frac{3 - 1.5 z^{-1}}{1 - z^{-1} + z^{-2}}}$$

ⓐ causal

$$H(z) = \frac{z+1}{z-0.5} = \frac{1 + z^{-1}}{1 - 0.5 z^{-1}}$$

$b_h = [1, 1]$
$a_h = [1, -0.5]$

$b_x = [3, -1.5]$
$a_x = [1, -1, 1]$

$$Y(z) = \frac{1 + z^{-1}}{1 - 0.5 z^{-1}} \cdot \frac{3 - 1.5 z^{-1}}{1 - z^{-1} + z^{-2}}$$

$$Y(z) = \frac{3 + 1.5 z^{-1} - 1.5 z^{-2}}{1 - 1.5 z^{-1} + 1.5 z^{-2} - 0.5 z^{-3}} \qquad (i)$$

$$\frac{Y(z)}{X(z)} = \frac{1 + z^{-1}}{1 - 0.5 z^{-1}}$$

$$y(n) - 0.5 y(n-1) = x(n) + x(n-1) \qquad (iii)$$

(iv)

$$Y(z) = \frac{3\cdot e^{-j\pi/3}}{1 - e^{j\pi/3} z^{-1}} + \frac{3\cdot e^{j\pi/3}}{1 - e^{-j\pi/3} z^{-1}}$$

$$y(n) = \left(3\cdot e^{-j\pi/3}\cdot e^{jn\pi/3} + 3\cdot e^{j\pi/3}\cdot e^{-jn\pi/3}\right) u(n) = 6\cdot\cos\left(n\tfrac{\pi}{3} - \tfrac{\pi}{3}\right) u(n)$$

$$\boxed{y(n) = 6\cdot\cos\left((n-1)\pi/3\right)\cdot u(n)}$$

ⓑ stable

$$H(z) = \frac{1 + z^{-1} + z^{-2}}{1 + 0.5 z^{-1} - 0.25 z^{-2}}$$

$$Y(z) = \frac{1 + z^{-1} + z^{-2}}{1 + 0.5 z^{-1} - 0.25 z^{-2}} \cdot \frac{3 - 1.5 z^{-1}}{1 - z^{-1} + z^{-2}} = \frac{3 + 1.5 z^{-1} + 1.5 z^{-2} - 1.5 z^{-3}}{1 - 0.5 z^{-1} + 0.25 z^{-2} + 0.75 z^{-3} - 0.25 z^{-4}}$$

$$Y(z) = \frac{2.16\cdot e^{-j0.9}}{1 - e^{j\pi/3} z^{-1}} + \frac{2.16\cdot e^{j0.9}}{1 - e^{-j\pi/3} z^{-1}} + \frac{1.21}{1 + 0.81 z^{-1}} - \frac{0.9}{1 - 0.31 z^{-1}}$$

MATLAB CHECKED

$$y(n) = \left(2.16\cdot 2\cos\left(n\pi/3 - 0.9\right) + 1.21\cdot(-0.81)^n - 0.9\cdot(0.31)^n\right) u(n)$$

ⓒ anticausal

$$H(z) = \frac{z^2 - 1}{(z-3)^2} = \frac{z^2 - 1}{z^{3} - 6z^2 + 9}$$

$b_h = [1, \cancel{0}, -1]$
$a_h = [1, -6, 9]$

$$X(z) = \frac{3z^2 - 1.5z}{z^2 - z + 1}$$

!!?

$b_x = [3, -1.5]$
$a_x = [1, -1, 1]$

?

GRESKA

$$Y(z) = \frac{3z^4 - 4.5\cdot z^3 + 1.5 z^2}{z^4 - 7z^3 + 16z^2 - 15z + 9} = \frac{1.2}{1 - 3z^{-1}} + \frac{3\cdot z^{-1}}{(1 - 3z^{-1})^2}\,\frac{2.1\cdot 2}{3} + \frac{0.2\cdot e^{-j2.5}}{1 - e^{j\pi/3} z^{-1}} + \frac{0.2\, e^{j2.5}}{1 - e^{-j\pi/3} z^{-1}}$$

$$y(n) = -\underset{u(-1-n)}{3^n} - \left[n\, 3^n\, u(-n-1)\cdot\frac{2.1}{3}\right]\Bigg|_{n=n+1} - \left[0.2\cdot e^{-j2.5}\cdot e^{j\frac{n\pi}{3}} + 0.2\cdot e^{j2.5}\cdot e^{-j\frac{\pi n}{3}}\right] u(-1-n)$$

ANTICAUSAL

$$y(n) = -\overset{1.2\cdot u(-1-n)}{\cancel{3}^n} - 0.7(n+1)\, 3^{n+1} u(-2-n) - 0.4\cos\left(n\pi/3 - 2.5\right)\cdot u(-1-n)$$

$$y(n) = -\left[1.2\cdot 3^n + 0.4\cos\left(n\pi/3 - 2.5\right)\right]\cdot u(-1-n) - 2.1\cdot 3^n (n+1)\, u(-2-n)$$

IF CAUSAL THEN

$$y(n) = 1.2(3)^n u(n) + \overset{2.1}{\cancel{0.7}}\,(n+1)\, 3^{(n+1)} u(n+1) + 0.4\cdot\cos\left(\frac{\pi n}{3} - 2.5\right) u(n)$$

$$\boxed{y(n) = \left[1.1939\; 3^n + 0.4286\cos(n\pi/3 - 2.4746)\right] u(n) + 2.1429\,(n+1)\cdot 3^n\cdot u(n+1)}$$ CAUSAL

$$y(n) = -\left[1.1939\; 3^n + 0.4286\cos(n\pi/3 - 2.4746)\right] u(-1-n) - 2.1429\,(n+1)\, 3^n\, u(-2-n)$$

ANTICAUSAL

$H(z) = \dfrac{z^2-1}{(z-3)^2}$; $\boxed{x(z) = \dfrac{3z^2 - 1.5z}{z^2 - z + 1}}$ $b_x = [3, 1.5, 0]$ $a_x = [1, -1, 1]$

$\boxed{ROC_x: \quad |z| > 1}$

$H(z) = \dfrac{z^2 - 1}{z^2 - 6z + 9}$; $b_h = [1, 0, -1]$ $a_h = [1, -6, 9]$

$$Y(z) = H(z)X(z) = \frac{3z^4 - 1.5z^3 - 3z^2 - 1.5z}{z^4 - 7z3 + 16z^2 - 15z + 9} =$$

ROC_y: $\boxed{1 < |z| < 3}$

$$= \frac{0.8776}{1-3z^{-1}} + \frac{2.8571 \cdot 3z^{-1}}{(1-3z^{-1})^2} \cdot \frac{z}{3} + \frac{0.3712 \cdot e^{-j3}}{1 - e^{j\pi/3} z^{-1}} + \frac{0.3712 \cdot e^{j3}}{1 - e^{-j\pi/3} z^{-1}}$$

$$y(n) = -0.8776 \cdot (3)^n \cdot u(-1-n) - \frac{2.8571}{\cancel{3}}(+3)^{n+1}(n+1)u(-2-n) + 0.3712 \cdot 2 \cdot \cos(n\pi/3 - 2.9982) \cdot u(n)$$

~~$y(n) = [0.8776 (3)^n + 2.8571 (+3)^n (n+1)] u(-1-n) + 0.7424 \cos(n\pi/3 - 2.9982) u(n)$~~

$$\boxed{y(n) = -0.8776\,(3)^n u(-1-n) - 2.8571\,(3)^n (n+1)\, u(-2-n) + 0.7424 \cos(n\pi/3 - 2.9982)\, u(n)}$$

$$H(z) = \frac{0.2222}{1-3z^{-1}} + \frac{0.8889 \cdot 3z^{-1}}{(1-3z^{-1})^2} \cdot \frac{z}{3} - 0.1111$$

$\boxed{ROC_h: \ |z| < 3}$

$$h(n) = -0.2222\ 3^n u(-1-n) + \frac{0.8889}{3}\ 3^{n+1}(n+1)\, u(-2-n) - 0.1111\,\delta(n)$$

$$h(n) = -0.2222\ 3^n u(-1-n) - 0.8889\ 3^n (n+1)\, u(-2-n) - 0.1111\,\delta(n)$$

$$x(n) = 3 \cdot \cos(n\pi/3)\, u(n)$$

MATLAB CHECKED WITH conv-m (h, n, x, n)

ⓓ stable

$$H(z) = \frac{z}{z - 0.25} + \frac{1 - 0.5z^{-1}}{1 + 2z^{-1}} = \frac{1}{1 - 0.25z^{-1}} + \frac{1 - 0.5z^{-1}}{1 + 2z^{-1}} =$$

$$= \frac{1 + 2z^{-1} + (1 - 0.25z^{-1})^2}{(1 - 0.25z^{-1})(1 + 2z^{-1})} = \frac{1 + 2z^{-1} + 1 - 0.5z^{-1} + 0.0625z^{-2}}{1 + 1.75z^{-1} - 0.5z^{-2}} = \frac{2 + 1.5z^{-1} + 0.0625z^{-2}}{1 + 1.75z^{-1} - 0.5z^{-2}}$$

$\boxed{ROC_h: 0.25 < |z| < 2}$

~~$Y(z) = H(z) \cdot X(z) = \dfrac{2.6786}{1 + 2z^{-1}} + \dfrac{1.7087\, e^{-j2.4}}{1 - e^{+j\pi/3} z^{-1}} + \dfrac{1.7087\, e^{j2.4}}{1 - e^{-j\pi/3} z^{-1}}$~~

~~$\boxed{0.5 < |z| < 1}$~~

~~$y(n) = [2.6786 (-2)^n + \underbrace{2 \cdot 1.7087}_{3.4174} \cdot \cos(n\pi/3 - 2.375)]\, u(n)$~~

~~$\boxed{y(n) = +2.6786 (-2)^n u(n) - 3.4174 \cos(n\pi/3 - 2.375)\, u(-1-n)}$~~

~~$ROC_x: |z| < 1$, $ROC_h: 0.5 < |z| < 2$; $ROC_y = ROC_x \cap ROC_h : 0.5 < |z| < 1$~~

$$Y(z) = H(z)X(z) = \frac{2.4107}{1 + 2z^{-1}} + \frac{1.9 \cdot e^{+j0.01}}{1 - e^{+j\pi/3} z^{-1}} + \frac{1.9 \cdot e^{-j0.01}}{1 - e^{-j\pi/3} z^{-1}} - \frac{0.23}{1 - 0.25z^{-1}}$$

$\boxed{ROC_x: \ |z| > 1 \qquad ROC_h: 0.25 < |z| < 2}$ $ROC_y: ROC_x \cap ROC_h \Rightarrow$ ROC_y: $1 < |z| < 2$

$$\boxed{y(n) = -2.41 \cdot (-2)^n u(-1-n) + 3.8 \cos(n\pi/3 + 0.01)\, u(n) - 0.23 (0.25)^n u(n)}$$

$$H(z) = \frac{2 + 1.5z^{-1} + 0.0625z^{-2}}{1 + 1.75z^{-1} - 0.5z^{-2}} = \frac{1.1250}{1 + 2 \cdot z^{-1}} + \frac{1}{1 - 0.25z^{-1}} - 0.1250$$

$\boxed{0.25 < |z| < 2}$

$$\boxed{h(n) = -1.1250(-2)^n \, u(-1-n) + (0.25)^n \cdot u(n) - 0.1250\,\delta(n)}$$

USED FOR MATLAB CHECK !!!!

$$X(z) = \frac{3 - 1.5z^{-1}}{1 - z^{-1} + z^{-2}} = \frac{1.5}{1 - e^{j\pi/3}z^{-1}} + \frac{1.5}{1 - e^{-j\pi/3}z^{-1}} \Rightarrow \Big|\ |z| > 1\ \Big| \boxed{x(n) = 3\cos(n\pi/3)\,u(n)}$$

$$\boxed{\text{IF } |z| < 1 : \ x(n) = -3\cos(n\pi/3)\,u(-1-n)}$$

ⓔ $$H(z) = (1 + z^{-1} + z^{-2})^2 = 1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4} = \frac{z^4 + 2z^3 + 3z^2 + 2z + 1}{z^4}$$

$$\Big| h(n) = \delta(n) + 2\delta(n-1) + 3\delta(n-2) + 2\delta(n-3) + \delta(n-4) \Big|$$

$\boxed{ROC_Y : |z| > 0}$

$\boxed{\text{STABLE + CAUSAL}}$

$\boxed{ROC_X : |z| > 1}$

$\boxed{ROC_Y : |z| > 1}$

$$Y(z) = H(z) \cdot X(z) = \frac{(1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4})(3 - 1.5z^{-1})}{1 - z^{-1} + z^{-2}}$$

$$Y(z) = \frac{3 + 4.5z^{-1} + 6z^{-2} + 1.5z^{-3} + 0 - 1.5z^{-4}}{1 - z^{-1} + z^{-2}}$$

$$Y(z) = \frac{6 \cdot e^{-j2.1}}{1 - e^{j\pi/3}z^{-1}} + \frac{6 \cdot e^{j2.1}}{1 - e^{-j\pi/3}z^{-1}} + 9 + 1.5z^{-1} - 1.5z^{-2} - 1.5z^{-3}$$

$$\boxed{y(n) = 12 \cdot \cos(n\pi/3 - 2.0944) + 9\delta(n) + 1.5\delta(n-1) - 1.5\delta(n-2) - 1.5\delta(n-3)}$$

$\boxed{P.4.13}$ (i) $Y(z)$; (ii) $H(z)$; (iii) pole zero plot; (iv) $y(n) = ?$ if

$x(n) = 2 \cdot (0.9)^n u(n)$ $\quad \boxed{X(z) = \frac{2}{1 - 0.9 \cdot z^{-1}}}$

ⓐ $y(n) = \frac{1}{4}x(n) + \frac{1}{2}x(n-1) + \frac{1}{4}x(n-3) \quad / \mathcal{Z}$

$$Y(z) = \frac{1}{4}\frac{2}{1 - 0.9z^{-1}} + \frac{1}{2}z^{-1}\frac{2}{1 - 0.9z^{-1}} + \frac{1}{4}z^{-3}\frac{2}{1 - 0.9z^{-1}}$$

(ii)

$$Y(z) = \left(\frac{1}{4} + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-3}\right)X(z), \quad \boxed{H(z) = \frac{Y(z)}{X(z)} = \left(\frac{1}{4} + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-3}\right)}$$

(i)

$$Y(z) = \frac{2 \cdot \left(\frac{1}{4} + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-3}\right)}{1 - 0.9z^{-1}} = \frac{0.5 + z^{-1} + 0.5z^{-3}}{1 - 0.9z^{-1}}$$

b = [0.5, 1, 0, 0.5]
a = [1, -0.9]

$$Y(z) = \frac{2.2970}{1 - 0.9z^{-1}} - 1.7970 - 0.6173z^{-1} - 0.5556z^{-2}$$

$$\Big| y(n) = 2.2970 \cdot (0.9)^n \cdot u(n) - 1.7970\,\delta(n) - 0.6173\,\delta(n-1) - 0.5556\,\delta(n-2) \Big|$$

ⓑ $y(n) = x(n) + 0.5x(n-1) - 0.5y(n-1) + 0.25y(n-2)$

$y(n) + 0.5y(n-1) - 0.25y(n-2) = x(n) + 0.5x(n-1) \quad / \mathcal{Z}$

$Y(z)\,[1 + 0.5z^{-1} - 0.25z^{-2}] = [1 + 0.5z^{-1}]\,X(z)$

$$H(z) = \frac{1 + 0.5z^{-1}}{1 + 0.5z^{-1} - 0.25z^{-2}} = \frac{0.2764}{1 + 0.809z^{-1}} + \frac{0.7236}{1 - 0.309z^{-1}}$$

$$Y(z) = \frac{1 + 0.5z^{-1}}{1 + 0.5z^{-1} - 0.25z^{-2}} \cdot \frac{2}{1 - 0.9z^{-1}} = \frac{2 + z^{-1}}{1 - 0.4z^{-1} - 0.7z^{-2} + 0.2250z^{-3}}$$

$$Y(z) = \frac{2.4950}{1-0.9z^{-1}} + \frac{0.2617}{1+0.809z^{-1}} - \frac{0.7567}{1-0.309z^{-1}}$$

$$H(z) = \frac{0.2764}{1+0.809z^{-1}} + \frac{0.7236}{1-0.309z^{-1}} \; ; \quad \underline{h(n) = 0.2764(-0.809)^n \cdot u(n) + 0.7236(0.309)^n u(n)}$$

$$y(n) = 2.4950(0.9)^n u(n) + 0.2617(-0.809)^n u(n) - 0.7567(0.309)^n \cdot u(n)$$

ⓒ $y(n) = 2x(n) + 0.9y(n-1)$; $\quad y(n) - 0.9y(n-1) = 2x(n) \; / \mathcal{Z}$

$$Y(z) = \frac{2}{1-0.9z^{-1}} X(z) \; ; \qquad H(z) = \frac{2}{1-0.9z^{-1}} \; ;$$

$$Y(z) = \frac{4}{(1-0.9z^{-1})^2} = \frac{4}{1-1.8z^{-1}+0.81z^{-2}} = 4 \cdot \frac{0.9z^{-1}}{(1-0.9z^{-1})^2} \cdot \frac{z}{0.9} \Rightarrow$$

$$y(n) = \frac{4}{\cancel{0.9}} (n+1) \, 0.9^{n\cancel{+1}} \cdot u(n+1) \; ; \qquad n=-1 \Rightarrow y(-1) = 0 \; ;$$

$$\boxed{y(n) = 4 \cdot (n+1) \cdot 0.9^n \cdot u(n)}$$

ⓓ $y(n) = -0.45x(n) - 0.4x(n-1) + x(n-2) + 0.4y(n-1) + 0.45y(n-2)$

$$y(n) - 0.4y(n-1) - 0.45y(n-2) = -0.45x(n) - 0.4x(n-1) + x(n-2)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{-0.45-0.4z^{-1}+z^{-2}}{1-0.4z^{-1}-0.45z^{-2}} = \frac{0.2187}{1-0.9z^{-1}} + \frac{1.5536}{1+0.5z^{-1}} - 2.2222$$

$$Y(z) = \frac{-0.45-0.4z^{-1}+z^{-2}}{1-0.4z^{-1}-0.45z^{-2}} \cdot \frac{2}{1-0.9z^{-1}} = \frac{-0.9-0.8z^{-1}+2z^{-2}}{1-1.3z^{-1}-0.09z^{-2}+0.405z^{-3}}$$

$$Y(z) = \frac{-2.4470}{1-0.9z^{-1}} + 0.4373 \frac{0.9 \cdot z^{-1}}{(1-0.9z^{-1})^2} \cdot \frac{z}{0.9} + \frac{1.1097}{1+0.5z^{-1}}$$

$$y(n) = -2.4(0.9)^n \cdot u(n) + \frac{0.4373}{\cancel{0.9}} (0.9)^{n\cancel{+1}}(n+1)\,u(n+1) + 1.1 \cdot (-0.5)^n \cdot u(n)$$

$$\boxed{n=-1} \qquad \boxed{y(-1) = 0} \Rightarrow y(n) = \left[-2.4(0.9)^n + 0.4373 \cdot (0.9)^n (n+1) + 1.1(-0.5)^n\right] u(n)$$

ⓔ $y(n) = \sum\limits_{m=0}^{4} (0.8)^m x(n-m) - \sum\limits_{l=1}^{4} (0.9)^l y(n-l) \; / \mathcal{Z}$

$$Y(z) + \sum_{l=1}^{4} (0.9)^l z^{-l} Y(z) = \sum_{m=0}^{4} (0.8)^m z^{-m} X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{\sum\limits_{m=0}^{4} (0.8)^m z^{-m}}{1 + \sum\limits_{l=1}^{4} (0.9)^l z^{-l}} = \frac{1+0.8 \cdot z^{-1} + 0.8^2 z^{-2} + 0.8^3 z^{-3} + 0.8^4 z^{-4}}{1+0.9z^{-1}+0.9^2 z^{-2} + 0.9^3 z^{-3} + 0.9^4 z^{-4}}$$

$$\boxed{Y(z) = \frac{2 + 1.6z^{-1} + 1.28 \cdot z^{-2} + 1.024 \cdot z^{-3} + 0.8192 z^{-4}}{1 - 0.5905 z^{-5}}}$$

$$Y(z) = \frac{0.1598 \cdot e^{-j0.8612}}{1-0.9 \cdot e^{j1.25} z^{-1}} + \frac{0.16 e^{j0.86}}{1-0.9e^{-j1.25} z^{-1}} + \frac{1.6}{1-0.9z^{-1}} + \frac{0.1 \cdot e^{-j0.3}}{1-0.9e^{j2.5} z^{-1}} + \frac{0.1 \cdot e^{+j0.3}}{1-0.9 \cdot e^{-j2.5} z^{-1}}$$

$$y(n) = 2 \cdot 0.16 \; 0.9^n \cos(1.25n - 0.86) + 1.6(0.9)^n + 2 \cdot 0.1 \; 0.9^n \cdot \cos(2.5n - 0.3)$$

$\boxed{P.4.14}$ Separate $y(n)$ into (i) homogenous part (ii) particular part, (iii) trancient response (iv) steady state response

---

- One-sided z-Transform

$$\mathcal{Z}^+[x(n)] = \mathcal{Z}[x(n)u(n)] = \mathcal{X}^+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$\mathcal{Z}^+[x(n-k)] = \mathcal{Z}[x(n-k)u(n)] = \sum_{n=0}^{\infty} x(n-k) z^{-n} = \left| \begin{matrix} m = n-k \\ n=0 \; m=-k \\ n = n+k \end{matrix} \right| =$$

$$= \sum_{m=-k}^{\infty} x(m) z^{-(m+k)} = \sum_{m=-k}^{-1} x(m) z^{-(m+k)} + z^{-k} \sum_{m=0}^{\infty} x(m) \cdot z^{-m}$$

$$\boxed{\mathcal{Z}^+[x(n-k)] = x(-1) z^{1-k} + x(-2) z^{2-k} + \dots + x(-k) + z^{-k} \mathcal{X}^+(z)}$$

Use for solving the difference equation

$$1 + \sum_{k=1}^{N} a_k y(n-k) = \sum_{m=1}^{M} b_m x(n-m)$$

s.t. initial condition: $\{y(i), i=-1, \dots, -N\}, \{x(i), i=-1, \dots -M\}$

USE FUNCTIONS: $\boxed{y = \text{filter}(b, a, x, xic) \quad xic = \text{filteric}(b, a, x, y)}$

ⓐ $y(n) = \frac{1}{4}x(n) + \frac{1}{2}x(n-1) + \frac{1}{4}x(n-3)$; $x(n) = 2 \cdot (0.9)^n u(n)$

$$\mathcal{X}(z) = \frac{2}{1-0.9z^{-1}} \; ; \quad Y(z) = \underbrace{\left[\frac{1}{4} + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-3}\right]}_{H(z)} \cdot \mathcal{X}(z)$$

$$y(n) = \underbrace{\overbrace{2.2970 (0.9)^n u(n)}^{(**)} \underbrace{- 1.7970 - 0.6173\,\delta(n-1) - 0.5556\,\delta(n-2)}_{(*)}}_{(11)}$$

(*) homogenous
(**) particular
(1) steady state
(11) transient

ⓑ $y(n) = x(n) + 0.5x(n-1) - 0.5y(n-1) + 0.25y(n-2)$

$$Y(z)\left[1 + 0.5z^{-1} - 0.25z^{-2}\right] = \left[1 + 0.5z^{-1}\right]\mathcal{X}(z)$$

$$Y(z) = \frac{1+0.5z^{-1}}{1+0.5z^{-1}-0.25z^{-2}} \cdot \frac{2}{1-0.9z^{-1}} \; ; \quad H(z) = \frac{0.28}{1+0.8z^{-1}} + \frac{0.7}{1-0.3z^{-1}}$$

$$y(n) = \underbrace{\underbrace{2.5 \cdot (0.9)^n u(n)}_{(**)} + \underbrace{0.26(-0.809)^n u(n) - 0.76(0.309)^n u(n)}_{(*)}}_{(11)}$$

ⓒ $y(n) = 2x(n) + 0.9\,y(n-1)$; $Y(z) = \frac{2}{1-0.9z^{-1}}\,X(z)$; $X(z) = \frac{2}{1-0.9z^{-1}}$

$$y(n) = 4(n+1)\,0.9^n u(n) = \underbrace{\underbrace{4\cdot n\,0.9^n u(n)}_{\text{particular } (**)} + \underbrace{4\cdot 0.9^n u(n)}_{\text{homogenous } (*)}}_{\text{transient } (1)}$$

ⓓ $H(z) = \frac{0.2187}{1-0.9z^{-1}} - \frac{1.5536}{1+0.5z^{-1}} - 0.2222$

$$y(n) = \Big[\underbrace{\underbrace{-2.4(0.9)^n + 0.44\,n(0.9)^n}_{(**)} + \underbrace{0.44\cdot(0.9)^n + 1.1(-0.5)^n}_{(*)}}_{(11)}\Big]u(n)$$

ⓔ $H(z) = \frac{0.0942\cdot e^{+j0.0191}}{1-(0.9)e^{+j2.5}z^{-1}} + \frac{0.0942\cdot e^{-j0.0191}}{1-(0.9)e^{j2.5}z^{-1}} + \frac{0.0939\,e^{-j0.08}}{1-(0.9)e^{+j1.3}} + \frac{0.0939\,e^{j0.08}}{1-0.9\,e^{j1.3}}$

$$y(n) = \Big[\underbrace{\underbrace{2{,}016\;0.9^n\cos(1.25n-0.86) + 2{,}01\cdot 0.9^n\cos(2.5n-0.3)}_{(*)} + \underbrace{1.6\;0.9^n}_{(**)}}_{(11)}\Big]u(n)$$

$\boxed{\text{P4 15}}$ Stable system: $z_1 = j$; $z_2 = -j$; $p_1 = -\frac{1}{2} + j\frac{1}{2}$; $p_2 = -\frac{1}{2} - j\frac{1}{2}$

$H(e^{j0}) = 0.8$

ⓐ $H(z) = ?$ ROC = ? ; ⓑ diference equation represent ; ⓒ $y_{ss}(n) = ?$

ⓓ $y_{tr}(n) = ?$; $\boxed{x(n) = \frac{1}{\sqrt{2}}\sin\left(\frac{\pi n}{2}\right)u(n)}$

ⓐ $H(z) = \frac{\left(1-e^{+j\frac{\pi}{2}}z^{-1}\right)\left(1-e^{-j\frac{\pi}{2}}z^{-1}\right)}{\left(1-\frac{\sqrt{2}}{2}e^{+j\frac{3\pi}{4}}z^{-1}\right)\left(1-\frac{\sqrt{2}}{2}e^{-j\frac{3\pi}{4}}z^{-1}\right)}\cdot 1.3656$ $\quad H(e^{j0}) = 0.8$

ROC: $|z| > \sqrt{2}/2 = 0.7071$

$\boxed{H(z) = \frac{1+z^{-2}}{1+z^{-1}+0.5z^{-2}} \qquad H(1) = 0.8}$ DIREKTNO VO MAPLE

$p\_mag = \sqrt{\frac{1}{4}+\frac{1}{4}} = \frac{1}{\sqrt{2}}$

$\frac{\sqrt{2}}{2}\cdot e^{+j\frac{3\pi}{4}} = \frac{\sqrt{2}}{2}\left[\cos\left(\frac{3\pi}{4}\right) + j\sin\left(\frac{3\pi}{4}\right)\right] = \frac{\sqrt{2}}{2}\left[-\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}\right] = -\frac{1}{2} + j\frac{1}{2}$

$$H(z) = \alpha\cdot\frac{(1-jz^{-1})(1+jz^{-1})}{\left(1-(-0.5+j0.5)z^{-1}\right)\left(1-(-0.5-j0.5)z^{-1}\right)} = \alpha\cdot\frac{1+z^{-2}}{\underbrace{\left(1+(0.5-j0.5)z^{-1}\right)\left(1+(0.5+j0.5)z^{-1}\right)}_{*}}$$

$$(*) = 1 + (0.5+\cancel{j0.5})z^{-1} + (0.5-\cancel{j0.5})z^{-1} + (0.5-j0.5)(0.5+j0.5)z^{-2} =$$

$$= 1 + 2\cdot 0.5\,z^{-1} + (0.25+0.25)z^{-2} = 1 + z^{-1} + 0.5z^{-2}$$

MATLAB CMETKO

$H(z) = \alpha\cdot\frac{1+z^{-2}}{1+z^{-1}+0.5z^{-2}}$ $\qquad H(e^{j\omega}) = \alpha\cdot\frac{1+e^{-j2\omega}}{1+e^{-j\omega}+0.5e^{-j2\omega}}$

$H(e^{j0}) = \alpha\cdot\frac{1+1}{1+1+0.5} = \alpha\cdot\frac{2}{2.5}$ $\qquad \alpha\cdot\frac{2}{2.5} = 0.8$; $\alpha = \frac{\overset{0.4}{\cancel{0.8}}\cdot 2.5}{\cancel{2}}$ $\boxed{\alpha = 1}$

88

$$\frac{Y(z)}{X(z)} = \frac{1+z^{-2}}{1+z^{-1}+0.5z^{-2}} = \frac{1.5811\cdot e^{+j1.89}}{1-0.71\cdot e^{+j3\pi/4}z^{-1}} + \frac{1.5811\cdot e^{-j1.89}}{1-0.71\cdot e^{-j3\pi/4}z^{-1}} + 2$$

ⓑ $Y(z) + z^{-1}Y(z) + 0.5z^{-2}Y(z) = X(z) + z^{-2}X(z)$

$y(n) + y(n-1) + 0.5\,y(n-2) = x(n) + x(n-2)$

ⓒ $x(n) = \frac{1}{\sqrt{2}}\sin\left(\frac{\pi n}{2}\right)u(n)$

$$X(z) = \frac{\sqrt{2}}{2}\,\frac{\overset{\to 1}{\cancel{\sin}}\left(\frac{\pi}{2}\right)z^{-1}}{1-2\cdot\underset{\to 0}{\cancel{\cos}}\left(\frac{\pi}{2}\right)z^{-1}+z^{-2}} = \frac{\frac{\sqrt{2}}{2}z^{-1}}{1+z^{-2}}$$

$$Y(z) = \frac{\cancel{1+z^{-2}}}{1+z^{-1}+0.5z^{-2}}\cdot\frac{z^{-1}}{\cancel{1+z^{-2}}} = \frac{\frac{\sqrt{2}}{2}z^{-1}}{1+z^{-1}+0.5z^{-2}} = \frac{\frac{\sqrt{2}}{2}e^{-j\frac{\pi}{2}}}{1-0.71\cdot e^{j\frac{3\pi}{4}}z^{-1}} + \frac{\frac{\sqrt{2}}{2}e^{j\frac{\pi}{2}}}{1-0.71e^{-j\frac{3\pi}{4}}z^{-1}}$$

SE PONISTUVA NULATA NA H(z) SO POLOT NA X(z)

$$y(n) = \cancel{2}\frac{\sqrt{2}}{\cancel{2}}\left(\frac{\sqrt{2}}{2}\right)^n\cdot\cos\left(\frac{3n\pi}{4}-\frac{\pi}{2}\right) = \sqrt{2}\cdot\left(\frac{\sqrt{2}}{2}\right)^n\sin\left(\frac{3n\pi}{4}\right)$$

$$h(n) = 2\cdot 1.5811\cdot\left(\frac{\sqrt{2}}{2}\right)^n\cdot\cos\left(\frac{3\pi}{4}n+1.89\right) + 2\delta(n)$$

$$\boxed{\begin{array}{l} y_{ss} = 0 \xrightarrow{} ⓓ \\ y_{tr} = y(n) \xrightarrow{} ⓔ \end{array}}$$

$$y_{ss} = |H(e^{j\omega})|\,\cos(\omega n + \angle H(e^{j\omega_0}))$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\cdot e^{-j\omega n}$$

$x(n) = \frac{1}{\sqrt{2}}\sin\left(\frac{\pi n}{2}\right)$ $\quad \omega_0 = \frac{\pi}{2}$

$$H(e^{j\omega_0}) = \left.\frac{1+z^{-2}}{1+z^{-1}+0.5z^{-2}}\right|_{z=e^{j\omega_0}} = \frac{1+e^{-j2\cdot\frac{\pi}{2}}}{1+e^{-j\frac{\pi}{2}}+0.5e^{-j\pi}} = \frac{1+(-1)}{1-j+0.5(-1)} =$$

$$= \frac{1-1}{1-j-0.5} = \frac{0}{0.5-j} = 0 \Rightarrow \boxed{y_{ss}=0}$$

$$x(n) = \frac{\sqrt{2}}{2}\sin\left(\frac{\pi n}{2}\right) = \frac{\sqrt{2}}{2}\cos\left(\frac{\pi}{2}-\frac{\pi n}{2}\right) = \frac{\sqrt{2}}{2}\cos\left(\frac{\pi n}{2}-\frac{\pi}{2}\right)$$

$\omega_0 = \frac{\pi}{2} \quad \theta = -\frac{\pi}{2}$

$$\frac{\sqrt{3}}{2}+\frac{1}{2} = \frac{1+\sqrt{3}}{2}$$

$$A\cdot\cos(\omega_0 n+\theta_0) \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow A\cdot|H(e^{j\omega})|\cdot\cos(\omega_0 n+\theta+\angle H(e^{j\omega}))$$

$\boxed{P4.16}$ $y(n) = x(n) + x(n-1) + 0.9\,y(n-1) - 0.81\,y(n-2)$

ⓐ $|H(e^{j\omega})| = ?$ $\angle H(e^{j\omega}) = ?$ use freqz ; note: $\omega = \frac{\pi}{3}$ & $\omega = \pi$

ⓑ $x(n) = \sin(\pi n/3) + 5\cos(\pi n)$; Compare y_{ss} PORTION !!

$$y(n) - 0.9\,y(n-1) + 0.81\,y(n-2) = x(n) + x(n-1)$$

$$Y(z) = \frac{1+z^{-1}}{1-0.9z^{-1}+0.81z^{-2}}\cdot X(z)$$

$bh = [1, 1]$

$ah = [1, -0.9, 0.81]$

$$\underline{\underline{H(z)}} = \frac{1+z^{-1}}{1-0.9z^{-1}+0.81z^{-2}} = \frac{1.056\cdot e^{-j1.08}}{1-0.9\cdot e^{j\pi/3}\cdot z^{-1}} + \frac{1.056\cdot e^{+j1.08}}{1-0.9\cdot e^{-j\pi/3}z^{-1}}$$

$$\underline{X(z)} = \frac{\sin\left(\frac{\pi}{3}\right)^{\to\sqrt{3}/2}z^{-1}}{1-2\cos\left(\frac{\pi}{3}\right)z^{-1}+z^{-2}} + \frac{5(1-\cos(\pi)^{\to -1}z^{-1})}{1-2\underset{\to -1}{\cos}(\pi)z^{-1}+z^{-2}} =$$

$$\frac{\sqrt{3}}{2}\cdot\frac{5}{2}$$

$$\cancel{= \frac{0.8660z^{-1}}{1-z^{-1}+z^{-2}} + \frac{5-2.5z^{-1}}{1-z^{-1}+z^{-2}} = \frac{5-1.634z^{-1}}{1-z^{-1}+z^{-2}}}$$

$$\cancel{= \frac{3.3514\,e^{-j0.7288}}{1-e^{-j\pi/3}\cdot z^{-1}} + \frac{3.3514\cdot e^{j0.7288}}{1-e^{j\pi/3}z^{-1}}}$$

$$Y(z) = \frac{1+z^{-1}}{1-0.9z^{-1}+0.81z^{-2}} \cdot \frac{5-1.634z^{-1}}{1-z^{-1}+z^{-2}} = \frac{26.8\cdot e^{-j1.2}}{1-e^{+j\pi/3}z^{-1}} + \frac{26.8\cdot e^{j1.2}}{1-e^{-j\pi/3}z^{-1}} +$$

$$\frac{22.8\cdot e^{j1.87}}{1-0.9\,e^{+j\pi/3}z^{-1}} + \frac{22.8\, e^{-j1.87}}{1-0.9\cdot e^{j\pi/3}z^{-1}}$$

$$(1+z^{-1})^2$$

$$1+2z^{-1}+z^{-2}$$

$$Y(n) = \underbrace{53.6\cdot\cos\left(\frac{n\pi}{3}-1.75\right)}_{\text{particulal + steady}} + \underbrace{61.38\,(0.9)^n\cos\left(n\pi/3+1.29\right)}_{\text{homogenous + transient}}$$

$$X(z) = \underbrace{\frac{\frac{\sqrt{3}}{2}z^{-1}}{1-z^{-1}+z^{-2}}}_{Y_1} + \underbrace{\frac{5+5z^{-1}}{1+2z^{-1}+z^{-2}}}_{Y_2} = \cancel{\frac{10+(\sqrt{3}/2-5)z^{-1}+(\sqrt{3}/2+5)z^{-2}}{1+z^{-3}}}$$

$$\cancel{X(z) = \frac{5+\frac{\sqrt{3}}{2}z^{-1}+\sqrt{3}z^{-2}+\left(\frac{\sqrt{3}}{2}+5\right)z^{-3}}{1+z^{-1}+z^{-3}+z^{-4}}}$$

$$X(z) = \frac{\frac{\sqrt{3}}{2}z^{-1}}{1-z^{-1}+z^{-2}} + \frac{5\cancel{(1+z^{-1})}}{(1+z^{-1})^{2}} = \frac{\frac{\sqrt{3}}{2}z^{-1}+\frac{\sqrt{3}}{2}z^{-2}+5-5z^{-1}+5z^{-2}}{1+z^{-3}}$$

MATLE CHECKED

$$\boxed{X(z) = \frac{5+\left(\frac{\sqrt{3}}{2}-5\right)z^{-1}+\left(\frac{\sqrt{3}}{2}+5\right)z^{-2}}{1+z^{-3}} = \frac{0.5e^{-j\pi/2}}{1-e^{+j\pi/3}z^{-1}} + \frac{0.5e^{-j\pi/2}}{1-e^{-j\pi/3}z^{-1}} + \frac{5}{1+z^{-1}}}$$

$$Y(z) = \frac{1+z^{-1}}{1-0.9z^{-1}+0.81z^{-2}} \quad \frac{5-4.13z^{-1}+5.87z^{-2}}{1+z^{-3}} = \frac{5.26\,e^{-j2.6}}{1-e^{j\pi/3}z^{-1}} + \frac{5.26\cdot e^{j2.6}}{1-e^{-j\pi/3}z^{-1}} +$$

$$+ \frac{7.01\cdot e^{+j0.11}}{1-0.9\cdot e^{+j\pi/3}z^{-1}} + \frac{7.01\cdot e^{-j0.11}}{1-0.9e^{-j\pi/3}z^{-1}}$$

$$Y(n) = \underbrace{10.521\cdot\cos\left(n\pi/3-2.6\right)}_{\text{particular + steady}} + \underbrace{14.02\cdot(0.9)^n\cdot\cos\left(n\pi/3+0.11\right)}_{\text{homogenous + transient}}$$

$$\boxed{X(n) = \cos\left(\frac{n\pi}{3}-\frac{\pi}{2}\right) + 5\cdot\overset{\cos(n\pi)}{(-1)^n} = \cos\left(\frac{\pi}{2}-\frac{n\pi}{3}\right) + 5\cdot\cos(n\pi) = \sin\frac{n\pi}{3} + 5\cos(n\pi)}$$

$$H(e^{j\omega_0})\Big|_{\omega_0=\frac{\pi}{3}} = \frac{1+e^{-j\omega_0}}{1-0.9\,e^{-j\omega_0}+0.81\cdot e^{-j2\omega_0}}\Big|_{\omega_0=\frac{\pi}{3}} = \frac{1+e^{-j\pi/3}}{1-0.9e^{-j\pi/3}+0.81\cdot e^{-j\frac{2\pi}{3}}} = 10.52\cdot e^{-j1.01}$$

$$H(e^{j\omega_0})\Big|_{\omega_0=\pi} = \frac{1+\cancel{e^{-j\pi}}^{\,-1}}{1-0.9\,e^{-j\pi}+0.81e^{-j2\pi}} = 0$$

$$\boxed{Y_{ss} = 10.52\cdot\cos\left(\frac{n\pi}{3}-1.01\right)}$$

$\boxed{P.4.17}$ $y(n) = 0.5\,y(n-1) + 0.25\,y(n-2) + x(n), \quad n \geqslant 0$ $\boxed{\begin{matrix} y(-1) = 1 \\ y(-2) = 2 \end{matrix}}$

$x(n) = (0.8)^n u(n)$

$$\mathcal{Z}\left[x(n-k)u(k)\right] = \sum_{n=0}^{\infty} x(n-k) \cdot z^{-n} = \left| \begin{matrix} m = n-k \\ n = m+k \\ n=0;\ m=-k \end{matrix} \right| = \sum_{m=-k}^{\infty} x(m) \cdot z^{-(m+k)} =$$

$$= \sum_{m=-k}^{-1} x(m) \cdot z^{-(m+k)} + \left[\sum_{m=0}^{\infty} x(m) \cdot z^{-m}\right] z^{-k}$$

$$\boxed{\mathcal{Z}\left[x(n-k)u(k)\right] = x(-1)\,z^{1-k} + x(-2)\,z^{2-k} + \dots x(-k) + z^{-k}\mathcal{X}^+(z) = \mathcal{Z}^+[x(n-k)]}$$

$$y(n) - 0.5y(n-1) - 0.25y(n-2) = x(n) \quad / \mathcal{Z}^+$$

$$Y^+(z) - 0.5\left[\cancelto{1}{y(-1)} \cdot z^0 + z^{-1} Y^+(z)\right] - 0.25\left[\cancelto{1}{y(-1)} z^{-1} + \cancelto{2}{y(-2)} + z^{-2} Y^+(z)\right] = \mathcal{X}^+(z)$$

$$Y^+(z) - 0.5\left[1 + z^{-1} Y^+(z)\right] - 0.25\left[z^{-1} + 2 + z^{-2} Y^+(z)\right] = \mathcal{X}^+(z)$$

$$\underline{Y^+(z)} - \underline{\underline{0.5}} - \underline{0.5 z^{-1} Y^+(z)} - \underline{\underline{0.25 z^{-1}}} - \underline{0.5} - \underline{0.25 z^{-2} Y^+(z)} = \mathcal{X}^+(z)$$

$$Y^+(z)\left[1 - 0.5z^{-1} - 0.25z^{-2}\right] = \mathcal{X}^+(z) + 1 + 0.25z^{-1}$$

$$Y^+(z) = \frac{\mathcal{X}^+(z)}{1 - 0.5z^{-1} - 0.25z^{-2}} + \frac{1 + 0.25z^{-1}}{1 - 0.5z^{-1} - 0.25z^{-2}} \qquad xic = filtic(b, a, Y, X)$$

$$Y^+(z) = \frac{1}{1 - 0.5z^{-1} - 0.25z^{-2}} \cdot \frac{1}{1 - 0.8z^{-1}} + \frac{1 + 0.25z^{-1}}{1 - 0.5z^{-1} - 0.25z^{-2}} = \frac{2 - 0.55z^{-1} - 0.2z^{-2}}{1 - 1.3z^{-1} + 0.15z^{-2} + 0.2z^{-3}}$$

$$= \frac{65.8702}{1 - 0.809z^{-1}} - \frac{64.00}{1 - 0.800 \cdot z^{-1}} + \frac{0.1298}{1 + 0.309z^{-1}}$$

$$\boxed{y(n) = \left[65.8702 \cdot (0.809)^n - 64 \cdot (0.8)^n + 0.1298(-0.309)^n\right] u(n)}$$

$\boxed{P.4.18}$ $y(n) - 0.4\,y(n-1) - 0.45\,y(n-2) = 0.45\,x(n) + 0.4\,x(n-1) - x(n-2) \quad / \mathcal{Z}^+$

$x(n) = 2 + \left(\frac{1}{2}\right)^n u(n) \qquad y(-1) = 0 \quad y(-2) = 3; \quad x(-1) = x(-2) = 2$

$$Y^+(z) - 0.4\left[\cancelto{0}{y(-1)} + z^{-1} Y^+(z)\right] - 0.45\left[\cancelto{0}{y(-1)} z^{-1} + \cancelto{3}{y(-2)} + z^{-2} Y^+(z)\right] =$$

$$0.45\mathcal{X}^+(z) + 0.4\left[\cancelto{2}{x(-1)} + z^{-1}\mathcal{X}^+(z)\right] - \left[\cancelto{2}{x(-1)} z^{-1} + \cancelto{2}{x(-2)} + z^{-2}\mathcal{X}^+(z)\right]$$

$$\underline{Y^+(z)} - \underline{0.4z^{-1} Y^+(z)} - 0.45 \cdot 3 - \underline{0.45 \cdot z^{-2} Y^+(z)} = \underline{0.45\mathcal{X}^+(z)} + 0.8 + \underline{0.4z^{-1}\mathcal{X}^+(z)} - 2z^{-1} - 2 - \underline{z^{-2}\mathcal{X}^+(z)}$$

$$Y^+(z)\left[1 - 0.4z^{-1} - 0.45z^{-2}\right] = \left[0.45 + 0.4z^{-1} - 2z^{-2}\right]\mathcal{X}^+(z) + 0.15 - 2z^{-1}$$

$$Y^+(z) = \frac{0.45 + 0.4z^{-1} - z^{-2}}{1 - 0.4z^{-1} - 0.45z^{-2}}\,\mathcal{X}^+(z) + \frac{0.15 - 2z^{-1}}{1 - 0.4z^{-1} - 0.45z^{-2}}$$

$$\boxed{H(z) = \frac{0.45 + 0.4z^{-1} - z^{-2}}{(1 - 0.9z^{-1})(1 + 0.5z^{-1})}}$$

ERROR!!

$$\mathcal{X}^+(z) = 2; + \frac{1}{1 - 0.5z^{-1}} = \frac{2 - 1z^{-1} + 1}{1 - 0.5z^{-1}} = \frac{3 - z^{-1}}{1 - 0.5z^{-1}}$$

$$Y^+(z) = \underbrace{\frac{0.45 + 0.4z^{-1} - z^{-2}}{1 - 0.4z^{-1} - 0.45z^{-2}} \cdot \frac{3 - z^{-1}}{1 - 0.5z^{-1}}}_{Y_1} + \underbrace{\frac{0.15 - 2z^{-1}}{1 - 0.4z^{-1} - 0.45z^{-2}}}_{Y_2} \cdot \frac{1 - 0.5z^{-1}}{1 - 0.5z^{-1}}$$

$$Y_1(z) = -\frac{0.9293}{1 - 0.9z^{-1}} + \frac{1.7188}{1 - 0.5z^{-1}} - \frac{3.8839}{1 + 0.5} + 4.4444$$

$$Y_2(z) = -\frac{1.3321}{1 - 0.9z^{-1}} + \frac{1.4821}{1 + 0.5z^{-1}}$$

ERROR!

$$y(n) = \underbrace{\underbrace{-0.9293(0.9)^n + 1.7188(0.5)^n - 3.8839(-0.5)^n + 4.4444\delta(n)}_{y_1(n)\ \text{zero state response}} - \underbrace{1.3321(0.9)^n + 1.4821(-0.5)^n}_{y_2(n)\ \text{zero input response}}}_{\text{transient response}}$$

$$\boxed{y(n) = -2.2614(0.9)^n + 1.7188(0.5)^n - 2.4018(-0.5)^n + 4.4444\delta(n)}$$ MATLAB CHECKED

$$X^+(z) = \mathcal{Z}\left[\left(2 + \left(\frac{1}{2}\right)^n\right)u(n)\right] = \frac{2}{1 - z^{-1}} + \frac{1}{1 - 0.5z^{-1}} = \frac{2 - z^{-1} + 1 - z^{-1}}{(1 - z^{-1})(1 - 0.5z^{-1})}$$

$$\boxed{X^+(z) = \frac{3 - 2z^{-1}}{1 - 1.5z^{-1} + 0.5z^{-2}}}$$

$$Y^+(z) = \underbrace{\frac{0.45 + 0.4z^{-1} - z^{-2}}{1 - 0.4z^{-1} - 0.45z^{-2}} \cdot \frac{3 - 2z^{-1}}{1 - 1.5z^{-1} + 0.5z^{-2}}}_{y_1} + \underbrace{\frac{0.15 - 2z^{-1}}{1 - 0.4z^{-1} - 0.45z^{-2}}}_{y_2}$$

$$y(n) = \left[\underbrace{-2 + 3.4438(0.9)^n - 1.8125(-0.5)^n + 1.7187(0.5)^n}_{\text{zero state response}} - \underbrace{1.3321(0.9)^n + 1.4821(-0.5)^n}_{\text{zero input response}}\right]u(n)$$

$$\boxed{y(n) = [\underbrace{-2}_{\text{steady state response}} + \underbrace{2.116(0.9)^n - 0.3304(-0.5)^n + 1.7187(0.5)^n}_{\text{transient response}}]u(n)}$$

$\boxed{\text{P4.19}}$ $y(n) = y(n-1) + y(n-2) + x(n-1)$

ⓐ $H(z) = ?$ ⓑ pole-zero plot ⓒ $h(n) = ?$ ⓓ is the system stable = ?

ⓐ $y(n) - y(n-1) - y(n-2) = x(n-1) \quad / \mathcal{Z}$

$Y(z) - z^{-1}Y(z) - z^{-2}Y(z) = z^{-1}X(z);$ $\quad H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - z^{-1} - z^{-2}}$

$H(z) = +\frac{0.4472}{1 - 1.618z^{-1}} - \frac{0.4472}{1 + 0.618z^{-1}}$, $\quad \boxed{\text{ROC: } |z| > 1.618}$

ⓒ CAUSAL ⇒ $h(n) = +0.4472 \cdot (1.618)^n u(n) - 0.4472(-0.618)^n u(n)$

ⓓ system is not stable

It could be stable if: $\boxed{\text{ROC: } 0.618 < |z| < 1.618}$

$$\boxed{h(n) = +0.4472 \cdot (1.618)^n u(-1-n) - 0.4472(-0.618)^n u(n)}$$

$$Y^+(z) - [y(-1) + z^{-1}Y^+(z)] - [y(-1)z^{-1} + y(-2) + z^{-2}Y^+(z)] = x(-1) + z^{-1}X^+(z)$$

$$Y^+(z)[1 - z^{-1} - z^{-2}] = z^{-1}X^+(z) + x(-1) + y(-1) + y(-2) + y(-1)z^{-1}$$

$$Y^+(z) = \frac{z^{-1}X^+(z)}{1 - z^{-1} - z^{-2}} + \frac{[x(-1) + y(-1) + y(-2)] + y(-1)z^{-1}}{1 - z^{-1} - z^{-2}}$$

$X^+(z) = 1 \quad (x(n) = \delta(n)) \qquad Y^+(z) = H(z)$

$$H(z)=\frac{z^{-1}+x(-1)+y(-1)+y(-2)+y(-1)z^{-1}}{1-z^{-1}-z^{-2}}=\frac{x(-1)+y(-1)+y(-2)+(1+y(-1))z^{-1}}{(1-0.618z^{-1})(1+1.618z^{-1})}$$

$x(-1)+y(-1)+y(-2)=1$

$1+y(-1)=-1.618$ ⇒ ~~$y(-1)=0 \Rightarrow \left\{ x(-1)+y(-2)=1 \text{ if } x(-1)=0 \Rightarrow \boxed{y(-2)=1} \right.$~~

$\boxed{y(-1)=-2.618}$

$\boxed{x(-1)=0}$ ⇒ $y(-1)+y(-2)=1$ $\quad \boxed{y(-2)=1+2.618=3.618}$

THE SISTEM COULD BE STABLE AND CAUSAL IF:

$$y(n)=y(n-1)+y(n-2)+x(n-1)$$

$Y=[-2.618,\ 3.618]$

$\boxed{y(-1)=-2.618 \quad y(-2)=3.618}$

$x(-1)=0 \quad x(-2)=0$

$y(n)-y(n-1)-y(n-2)=x(n-1)\ \ /\mathcal{Z}^{+}$

$$\underline{Y^{+}(z)}-[y(-1)+\underline{z^{-1}Y^{+}(z)}]-[y(-1)\cdot z^{-1}+y(-2)+\underline{z^{-2}Y^{+}(z)}]=z^{-1}X^{+}(z)$$

$$Y^{+}(z)[1-z^{-1}-z^{-2}]=z^{-1}X^{+}(z)+y(-1)+y(-1)z^{-1}+y(-2)$$

$x(n)=\delta(n) \Rightarrow Y^{+}(z)=H(z)$ $\quad -2.618 \quad 3.618 \quad -2.618$

$$H(z)=\frac{z^{-1}}{1-z^{-1}-z^{-2}}+\frac{y(-1)+y(-2)+y(-1)z^{-1}}{1-z^{-1}-z^{-2}}=\frac{\cancel{1+1.618z^{-1}}}{(1+0.618z^{-1})\cancel{(1+1.618z^{-1})}}$$

$H(z)=\dfrac{1}{1+0.618z^{-1}}$; $\quad \boxed{h(n)=(-0.618)^n u(n)}$

$H(z)=\dfrac{z^{-1}}{1-z^{-1}-z^{-2}}+\dfrac{1-2.618z^{-1}}{1-z^{-1}-z^{-2}}$ — xic

MATLAB CHECKED

$\boxed{h\_check=filter(bh, ah, \delta(n), xic)}$

$\boxed{P.4.20}$ Determine zero-state response?

$y(n)-0.25y(n-1)=x(n)+3x(n-1)$; $\quad y(-1)=2$, $\quad x(n)=e^{j\frac{n\pi}{4}}u(n)$

$$Y(z)-0.25[\underset{2}{y(-1)}+z^{-1}Y^{+}(z)]=X^{+}(z)+3[\underset{0}{x(-1)}+z^{-1}X^{+}(z)]$$

$Y(z)-0.5-0.25z^{-1}Y^{+}(z)=X^{+}(z)[1+3z^{-1}]$

$Y^{+}(z)=\dfrac{1+3z^{-1}}{1-0.25z^{-1}}\cdot X^{+}(z)+\dfrac{0.5}{1-0.25z^{-1}}$; $\quad X^{+}(z)=\dfrac{1}{1-e^{j\frac{\pi}{4}}z^{-1}}$

$$Y^{+}(z)=\underbrace{\frac{1+3z^{-1}}{1-0.25z^{-1}}\cdot\frac{1}{1-e^{j\frac{\pi}{4}}z^{-1}}}_{Y_1}+\underbrace{\frac{0.5}{1-0.25z^{-1}}}_{Y_2}$$

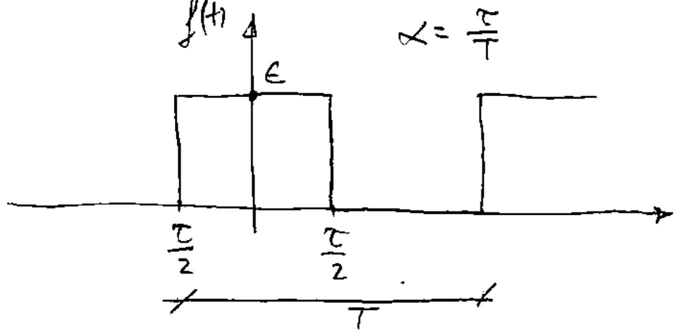
$Y_1(z)=\dfrac{4.4822\cdot e^{-j0.8084}}{1-e^{+j0.7854}z^{-1}}+\dfrac{3.8599\cdot e^{j2.1447}}{1-0.25z^{-1}}$; $\quad y_1(n)=\left[4.48\cdot e^{-j0.81}\cdot e^{+j0.79n}+3.8599\cdot e^{j2.1}\cdot(0.25)^n\right]u(n)$

$$y=y_1+y_2=\underbrace{4.48\cdot e^{-j0.81}\cdot e^{j0.79n}+3.8599e^{j2.1}(0.25)^n}_{\text{zero state response}}+\underbrace{0.5(0.25)^n}_{\text{zero input response}}$$

$$Y^{+}(z)=\frac{4.48\cdot e^{-j0.8084}}{1-e^{j0.7854}z^{-1}}+\frac{3.61\cdot e^{j2.0282}}{1-0.25z^{-1}}$$

$$y(n)=\Big[\underbrace{4.48\cdot e^{-j0.81}\cdot e^{j0.79n}}_{\text{steady state}}+\underbrace{3.61\cdot e^{j2.0282}\cdot(0.25)^n}_{\text{transient}}\Big]u(n)$$

THE DISCRETE FOURIER TRANSFORM



$\alpha = \frac{\tau}{T}$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t)$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega_0 t)\,dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega_0 t)\,dt$$

$f(t) = f(-t)$ – ПАРНА

$f(-t) = -f(t)$ – НЕПАРНА

$\boxed{b_n = 0}$

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2}\epsilon\cdot dt = \epsilon\left[\frac{\tau}{2}+\frac{\tau}{2}\right] = 2\cdot\frac{\epsilon\cdot\tau}{T} = 2\epsilon\cdot\alpha$$

$$a_n = \frac{2}{T}\int_{-\tau/2}^{\tau/2}\epsilon\cdot\cos(n\omega_0 t)\,dt = \left|\begin{matrix} n\omega_0 t = u \\ dt = \frac{du}{n\omega_0} \\ t = \tau/2 \Rightarrow \\ u = \frac{n\omega_0\tau}{2}\end{matrix}\right| = \frac{\epsilon}{n\omega_0}\int_{-\frac{n\omega_0\tau}{2}}^{\frac{n\omega_0\tau}{2}}\cos(u)\,du$$

$$a_n = \frac{2}{T}\frac{\epsilon}{n\omega_0}\left[\sin\frac{n\omega_0\tau}{2} + \sin\frac{n\omega_0\tau}{2}\right] = \frac{\epsilon}{\frac{n\omega_0}{2}}\cdot\sin\frac{n\omega_0\tau}{2} = \frac{2}{T}\epsilon\overset{\alpha}{\tau}\frac{\sin\frac{n\omega_0\tau}{2}}{\frac{n\omega_0\tau}{2}}$$

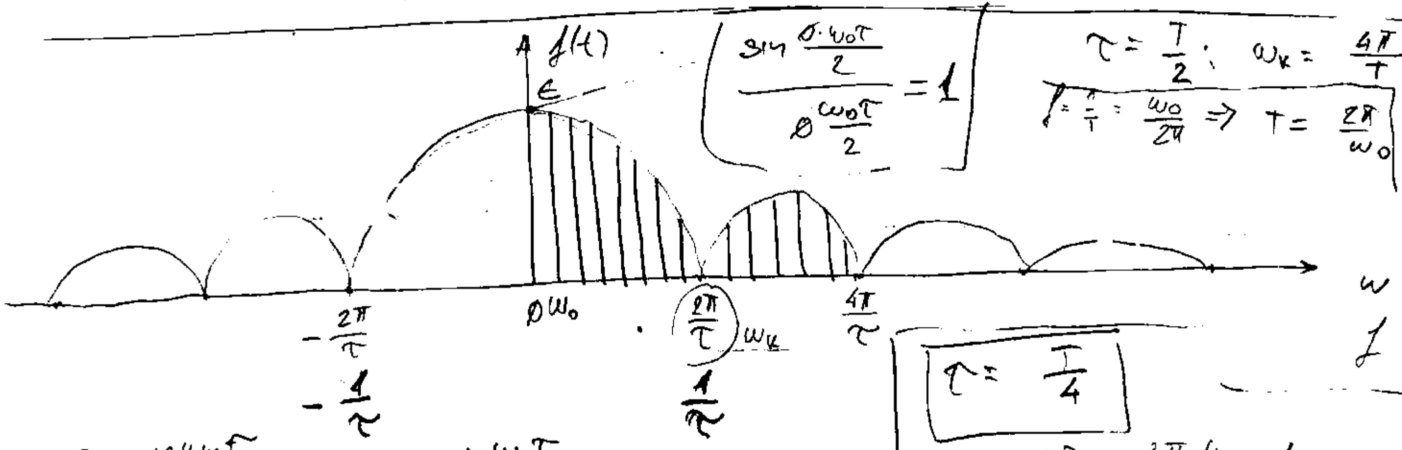
$$\boxed{f(t) = \frac{\cancel{2}\epsilon\alpha}{\cancel{2}} + 2\epsilon\alpha\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\omega_0\tau}{2}\right)}{\frac{n\omega_0\tau}{2}}\cdot\cos(n\omega_0 t)} = \left|\omega_0 = \frac{2\pi}{T}\right| \Longrightarrow \frac{\omega_0\tau}{2} = \frac{2\pi}{T}\cdot\frac{\overset{\alpha}{\tau}}{\cancel{2}} = \pi\alpha$$

$$f(t) = \epsilon\alpha + 2\epsilon\cdot\alpha\sum_{n=1}^{\infty}\frac{\sin n\pi\alpha}{n\pi\alpha}\cos(n\omega_0 t)$$

IF: $\tau = \frac{T}{2} \Rightarrow$ $$\boxed{f(t) = \frac{\epsilon}{2} + \epsilon\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{\frac{n\pi}{2}}\cdot\cos(n\omega_0 t)}$$

$n\pi\alpha = 2\pi \Rightarrow n = \frac{2\pi}{\alpha\pi} = \frac{2}{\alpha} = \frac{2T}{\tau} = 2\cdot\frac{2\pi}{\tau} =$

$n\pi\alpha = \pi$ $\quad n\cancel{\pi}\cdot\frac{\tau}{T} = \cancel{\pi}$; $T = \frac{2\pi}{\omega_0}$ $\quad n\tau = \frac{2\pi}{\omega_0}$; $n\omega_0 = \frac{2\pi}{\tau}$; $nf_0 = \frac{1}{\tau}$



$$\frac{\sin\frac{0\cdot\omega_0\tau}{2}}{0\frac{\omega_0\tau}{2}} = 1$$

$\tau = \frac{T}{2}$; $\omega_k = \frac{4\pi}{T} = \frac{\cancel{4\pi}\omega_0}{\cancel{2\pi}}$

$f = \frac{1}{T} = \frac{\omega_0}{2\pi} \Rightarrow T = \frac{2\pi}{\omega_0}$ $\quad \omega_k = 2\omega_0$

$\boxed{\tau = \frac{T}{4}}$

PR: $\frac{2\pi}{\tau} = \frac{2\pi\cdot4}{T} = 4\omega_0$

$\sin\frac{n\omega_0\tau}{2} = 0$ $\qquad \frac{n\omega_0\tau}{2} = K\cdot\pi$ $\qquad n\omega_0 = \frac{2K\pi}{\tau}$ $\quad K = 1,2,3$

$$\boxed{f(t) = \frac{E\tau}{T} + \frac{2E\tau}{T} \sum_{n=1}^{\infty} \frac{\sin \frac{n\omega_0\tau}{2}}{\frac{n\omega_0\tau}{2}} \cdot \cos(n\omega_0 t)}$$

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} \frac{E\tau}{T} \cdot \frac{\sin \frac{n\omega_0\tau}{2}}{\frac{n\omega_0\tau}{2}} \cdot e^{+jn\omega_0 t}}$$

Fourier Series for Discrete-Time Periodic Signals

- $x(n)$ - PERIODIC SEQUENCE

$x(n) = x(n+N)$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n}$$

harmonics: $e^{j2\pi kn/N}$ $k = 0, 1, \dots N-1$

$$\underbrace{e^{j2\pi k \frac{n \cdot \Delta t}{N \cdot \Delta t}} = e^{j2\pi k \frac{t}{T}} = \left| \frac{2\pi}{T} = \omega \right| = e^{j\omega k t}}_{\text{RELATION TO CONTINUOUS. TIME FOURIER SERIES}}$$

$$\boxed{X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt}$$

CT FOURIER TRANSFORM

$$x(n) = \sum_{k=0}^{N-1} c_k \cdot e^{j2\pi k n/N}$$

$\boxed{DT}$

$$\sum_{n=0}^{N-1} e^{j2\pi kn/N} = \begin{cases} N, & k = 0, \pm N, \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$$

$\boxed{CT}$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \cdot e^{j2\pi k F_0 t} = \sum_{k=-\infty}^{\infty} c_k \cdot e^{jk\Omega_0 t} \Big/ \cdot e^{-j2\pi F_0 l t} \qquad \boxed{T_p = \frac{1}{F_0}}$$

$$\{ e^{j2\pi l F_0 t}, \; k = 0, \pm 1, \pm 2, \dots \}$$

$$\int_{t_0}^{t_0+T_p} x(t) e^{-j2\pi F_0 l t} dt = \underbrace{\int_{t_0}^{t_0+T_p} e^{-j2\pi l F_0 t} \left(\sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t} \right) dt}_{⊛}$$

$$⊛ = \sum_{k=-\infty}^{\infty} c_k \int_{t_0}^{t_0+T_p} e^{j2\pi F_0(k-l)t} dt = \sum_{k=-\infty}^{\infty} c_k \left. \frac{e^{j2\pi F_0(k-l)t}}{j2\pi F_0(k-l)} \right|_{t_0}^{t_0+T_p}$$

$\boxed{4.1.3}$

$\boxed{k \neq l}$

$$e^{j2\pi F_0(k-l)t_0} \left[\cancel{e^{j2\pi F_0(k-l) \cdot T_p}} \xrightarrow{} 1 \; - 1 \right] = \left| T_p = \frac{1}{F_0}; \; e^{j2\pi \cancel{F_0}(k-l)\frac{1}{\cancel{F_0}}} = e^{j2\pi \cdot n} \right| = 0$$

$$\cancel{\cos(2\pi n)} \to 1 + j \cancel{\sin(2\pi n)} \to 0$$

$\boxed{k = l}$

$$\int_{t_0}^{t_0+T_p} dt = t \Big|_{t_0}^{t_0+T_p} = T_p$$

$$c_l \cdot T_p = \int_{t_0}^{t_0+T_p} x(t)\, e^{-j2\pi F_0 l t} dt \qquad c_l = \frac{1}{T_p} \int_{t_0}^{t_0+T_p} x(t)\, e^{-j2\pi F_0 l t} dt$$

$$\boxed{c_l = \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} x(t)\, e^{-j2\pi l F_0 t} dt}$$

Dirichlet conditions

$$\sum_{k=-\infty}^{\infty} c_k \cdot e^{jk2\pi F_0} \to x(t)$$

$$\text{if} \int_{-T_p/2}^{T_p/2} |x(t)| dt < \infty$$

$$\sum_{n=0}^{N} a^n = \begin{cases} N & a=1 \\ \frac{1-a^N}{1-a} & a\neq 1 \end{cases}$$

$$\sum_{n=0}^{N-1} e^{j2\pi kn/N} = \frac{1-\overset{\neq 1}{e^{j2k\pi n}}}{1-e^{j2\pi kn/N}} = 0$$

$$\boxed{x(n) = \sum_{k=0}^{N-1} c_k \cdot e^{j2\pi kn/N} \Big/ \cdot e^{-j2\pi ln/N}}$$

$$\sum_{n=0}^{N-1} x(n) \cdot e^{-j2\pi ln/N} = \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} c_k e^{j2\pi(k-l)n/N} = \sum_{k=0}^{N-1}\sum_{n=0}^{N-1} c_k e^{j2\pi(k-l)n/N}$$

$$\sum_{n=0}^{N-1} e^{j2\pi(k-l)n/N} = \begin{cases} N, & k-l=0, N, 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$$

$$\boxed{k=l} \quad N\cdot c_l = \sum_{n=0}^{N-1} x(n)\cdot e^{-j2\pi ln/N} \qquad \boxed{c_l = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi ln/N}}$$

$$l = 0, 1, 2, \ldots \quad N-1$$

FREQUENCY ANALYSIS OF DISCRETE-TIME PERIOD. SIGN.

SYNTHESIS : $x(n) = \sum_{k=0}^{N-1} c_k \cdot e^{+j2\pi kn/N}$ → DTFS (series)

ANALYSIS : $c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}$ → Fourier Coeficients

$$s_k(n) = e^{j2\pi kn/N} = e^{j\omega_k n} \qquad \boxed{\omega_k = 2\pi k/N}$$ } FREQUENCY COMPONENT

$$s_k(t) = e^{jk\Omega_0 t} = e^{jk2\pi F_0 t}$$

$\frac{2\pi}{k2\pi F_0}$ = FUNDAMENTAL PERIOD = $\frac{1}{kF_0} = \frac{T_P}{k}$

$$c_{k+N} = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi(k+N)n/N} = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N} \cdot \cancel{e^{-j2\pi n}}^{\,1} = c_k$$

- DSP using MATLAB

$$\tilde{x}(n) = \tilde{x}(n+kN) \quad \forall n, k$$

FINITE NUMBER OF HARMONICS:

$$\left\{\frac{2\pi}{N}\cdot k,\ k=0,1,\ldots,N-1\right\}$$

FUNDAMENTAL FRQ $= \frac{2\pi}{N} = \omega_0$ (FUNDAMEN. PERIOD: N)

$$\tilde{x}(n) = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}(k)\, e^{j\frac{2\pi}{N}kn}$$

$$\boxed{\tilde{x}(k) = \sum_{n=0}^{N-1} \tilde{x}(n)\, e^{-j\frac{2\pi}{N}\cdot k\cdot n} \qquad k = 0, \pm 1, \pm 2, \ldots}$$ DTFS coefficents

$$\boxed{\tilde{x}(k+N) = \tilde{x}(k)}$$

- $\boxed{W_N \triangleq e^{-j\frac{2\pi}{N}}}$ $\Rrightarrow$

$$\boxed{\begin{aligned} &\tilde{X}(k) \triangleq \mathrm{DFS}[\tilde{x}(n)] = \sum_{n=0}^{N-1} x(n)\cdot W_N^{nk} && \text{ANALYSIS} \\ &\tilde{x}(n) \triangleq \mathrm{IDFS}[\tilde{X}(k)] = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-nk} && \text{SYNESIS} \end{aligned}}$$

Proakis & Manolakis

Ex. 4.2.1 Determine the spectra of the signals

ⓐ $x(n) = \cos\sqrt{2}\pi n$

$\omega_0 = \frac{2\pi}{N} = \sqrt{2}\pi \qquad N = \frac{2\pi}{\sqrt{2}\pi} = \sqrt{2} \qquad f_0 = \frac{\sqrt{2}\pi}{2\pi} = \frac{1}{\sqrt{2}}$

$\omega_0 = 2\pi f_0$ SIGNAL IS NOT PERIODIC

ⓑ $x(n) = \cos(\pi n/3) \qquad \omega_0 = \frac{\pi}{3} \qquad f_0 = \frac{\omega_0}{2\pi} = \frac{\pi}{3\cdot 2\pi} = \frac{1}{6}$

$\omega_0 = \frac{2\pi}{N} \Rightarrow N = \frac{2\pi}{\omega_0} = \frac{2\pi}{\frac{\pi}{3}} = 6 \Rightarrow$ FUNDAMENTAL PERIOD

$c_k = \frac{1}{N}\sum_{n=0}^{5} x(n)\cdot e^{j2\pi kn/N}$

$x(n) = \underbrace{\frac{1}{2} e^{-jn\pi/3}}_{k=-1} + \underbrace{\frac{1}{2} e^{jn\pi/3}}_{k=1}$ $= e^{j5n\pi/3}$

$\boxed{k=1}$

$x(0) = \cos(0) = 1$ $x(1) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ $x(2) = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$	$x(3) = \cos\pi = -1$ $x(4) = \cos\frac{4\pi}{3} = -\frac{1}{2}$ $x(5) = \cos\frac{5\pi}{3} = \frac{1}{2}$
$W_N^{nk}(0) = e^{j0} = 1$ $W_N^{nk}(1) = e^{j2\pi/6}$ $W_N^{nk}(2) = e^{j4\pi/6}$	$W_N^{nk}(3) = e^{j3\pi/6}$ $W_N^{nk}(4) = e^{j4\pi/6}$ $W_N^{nk}(5) = e^{j5\pi/6}$

$c_1 = \frac{1}{2}$ 1

$C_{-1} = \frac{1}{2}; \quad e^{-jn\pi/3}\cdot \cancel{e^{j2\pi n}} =$

$= e^{j(2\pi - n\pi/3)} = e^{j\frac{6\pi-\pi}{3}n}$

$= e^{j\frac{5\pi}{3}n} \Rightarrow c_5 = \frac{1}{3}$

$c_0 = 0,\ c_1 = \frac{1}{2},\ c_2 = 0,\ c_3 = 0,\ c_4 = 0,\ c_5 = \frac{1}{2}$ $\boxed{\text{MATLAB CHECKED}}$

ⓒ $x(n)$ is PERIODIC WITH PERIOD $N = 4$

$x(n) = \{1, 1, 0, 0\}$

$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N} = \frac{1}{4}\sum_{n=0}^{3} x(n) e^{-j2\pi kn/N}$

$c_1 = \frac{1}{4}\left[1\cdot e^{-j0} + e^{-j2\pi/4}\right]$ $\quad\boxed{c_k = \frac{1}{4}\left(1 + e^{-j2\pi k/4}\right)}$ $\quad c_0 = \frac{1}{4}(1+1) = \frac{1}{2}$

$c_1 = \frac{1}{4}\left(1 + e^{-j\frac{\pi}{2}}\right) = \frac{1}{4}(1 - j) \qquad c_2 = \frac{1}{4}\left(1 + e^{-j\pi}\right) = \frac{1}{4}(1-1)$

$c_3 = \frac{1}{4}\left(1 + e^{-3\pi/2}\right) = \frac{1}{4}(1 + j)$

$\lvert c_0\rvert = \frac{1}{2}$	$\lvert c_1\rvert = \sqrt{2}/4$	$\lvert c_2\rvert = 0$	$c_3 = \sqrt{2}/4$
$\measuredangle c_0 = 0$	$\measuredangle c_1 = -\pi/4$	$\measuredangle c_2 = 0$	$\measuredangle c_3 = \pi/4$

$\boxed{\text{EXAMPLE 5.1}}$ DFS = ? $\tilde{x}(n) = \{\ldots, \underbrace{0,1,2,3,}\overset{\uparrow}{0}, \underbrace{1,2,3,}_{N=4} 0,1,2,3,\ldots\}$

$$\tilde{X}(k) = \text{DFS}[x(n)] = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} = \sum_{n=0}^{3} x(n) e^{-j2\pi kn/4}$$

$$\tilde{X}(k) = 0 + e^{-j2\pi k/4} + 2 \cdot e^{-j4\pi k/4} + 3 \cdot e^{-j6\pi k/4}$$

$$X(0) = 1 + 2 + 3 = 6$$

$$\tilde{X}(k) = \sum_{n=0}^{3} x(n) W_4^{nk} = \left| \begin{array}{l} W_4 = e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}} \\ \quad = -j \end{array} \right| = \sum_{n=0}^{3} x(n)(-j)^{n \cdot k}$$

$$\tilde{X}(2) = \sum_{n=0}^{3} x(n)(-j)^{2n} = \sum_{n=0}^{3} x(n)(-1)^n = 0 - 1 + 2 - 3 = -4 + 2 = -2$$

$$X = x * W_N^{nk}$$

$$\boxed{x = X * W_N^{*nk}} , \qquad x = \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} = \sum_{k=0}^{N-1} X(k) W_N^{*nk}$$

$\boxed{\text{EXAMPLE 5.2}}$

$$\tilde{x}(n) = \begin{cases} 1, & mN \leq n \leq mN + L - 1 \\ 0 & mN + L \leq (m+1)N - 1 \end{cases} ; \quad m = 0, \pm 1, \pm 2, \ldots$$

N . fundamental period

L/N - duty circle

ⓐ Determine $|\tilde{X}(k)|$ in terms of L & N

ⓑ Plot magnitube $|\tilde{X}(k)|$ for $L=5, N=20$; $L=5, N=40$; $L=5, N=60$; And $L=7, N=60$

ⓐ $$\tilde{X}(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j2\pi kn/N} = \sum_{n=0}^{L-1} e^{-j2\pi kn/N} = \left| a = e^{-j2\pi k/N} \right| =$$

$$= \sum_{n=0}^{L-1} a^n = \frac{1 - a^L}{1 - a} = \frac{1 - e^{-j2\pi kL/N}}{1 - e^{-j2\pi k/N}} \qquad \boxed{k < N}$$

$$\begin{array}{l} S = 1 + a + a^2 + \ldots a^{L-1} \\ \underline{a \cdot S = a + a^2 + a^3 + \ldots a^L} \quad - \\ S \cdot aS = 1 - a^L ; \quad S = \dfrac{1 - a^L}{1 - a} \end{array}$$

$\boxed{k = N}$

$$\underline{\tilde{X}(k)} = \sum_{n=0}^{N-1} \left(e^{-j2\pi}\right)^n = \sum_{n=0}^{L-1} (1)^n = L$$

- RELATION TO THE z-TRANSFORM

$$x(n) = \begin{cases} \text{NONZERO}, & 0 \leq n \leq N-1 \\ 0, & \text{elsewere} \end{cases}$$

$$X(z) = \sum_{n=0}^{N-1} x(n) z^{-n} , \qquad x(n) = \begin{cases} \tilde{x}(n) & 0 \leq n \leq N-1 \\ 0, & \text{elsewere} \end{cases}$$

$$X(k) = \sum_{n=0}^{N-1} \tilde{x}(n) e^{-j\frac{2\pi}{N}nk} = \sum_{n=0}^{N-1} x(n) \left[e^{j\frac{2\pi}{N}k}\right]^{-n} ; \quad \tilde{X}(k) = X(z)\Big|_{z = e^{j\frac{2\pi}{N}k}}$$

DTFT $X(e^{j\omega}) = \sum_{n=0}^{N-1} x(n) e^{-j\omega n} = \sum_{n=0}^{N-1} \tilde{x}(n) e^{-j\omega n}$

$\tilde{X}(k) = X(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N}k}$

Let: $\omega_1 \triangleq \frac{2\pi}{N}$ $\quad \omega_k = k\cdot\frac{2\pi}{N} = k\cdot\omega_1$; DFS: $X(k) = X(e^{j\omega_k}) = X(e^{jk\omega_1})$

$\boxed{\text{ex. 5.3}}$ $\quad x(n) = \{\underset{\uparrow}{0},1,2,3\}$

ⓐ DTFT = ? $\quad X(e^{j\omega}) = ?$

ⓑ sample $X(e^{j\omega})$ at $k\omega_1 = \frac{2\pi}{4}k$ $\quad k=0,1,2,3$
show it is equal to $X(k)$ from example 5.1.

$X(e^{j\omega}) = \sum_{n=0}^{\infty} x(n)\cdot e^{-j\omega n} = e^{-j\omega} + 2e^{-j2\omega} + 3e^{-j3\omega}$

SAMPLING & RECONSTRUCTION

$X(z) = \sum_{m=-\infty}^{\infty} x(m) z^{-m}, \quad \tilde{X}(k) = X(z)\Big|_{z=e^{j\frac{2\pi}{N}k}} \quad k=0,\pm1,\pm2,\dots$

$\tilde{X}(k) = \sum_{m=-\infty}^{\infty} x(m)\cdot e^{-j\frac{2\pi}{N}km} = \sum_{m=-\infty}^{\infty} x(m) W_N^{km}$; $\tilde{x}(n) = \text{IDFS}[\tilde{X}(k)]$

$\tilde{x}(n) = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{X}(k) W_N^{-kn} = \frac{1}{N}\sum_{n=0}^{N-1}\left\{\sum_{m=-\infty}^{\infty} x(m) W_N^{km}\right\} W_N^{-kn} =$

$= \cancel{N} \sum_{m=-\infty}^{\infty} x(m)\cdot \underbrace{\frac{1}{N}\sum_{n=0}^{N-1} W_N^{-k(n-m)}}_{=\begin{cases}1, & n-m=r\cdot N\\ 0, & \text{otherwise}\end{cases}} = \sum_{m=-\infty}^{\infty} x(m) \sum_{r=-\infty}^{\infty} \delta(n-m-rN)$

$\boxed{W_N = e^{-j\frac{2\pi}{N}}}$

$= \sum_{r=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x(m)\,\delta(n-m-rN)$

$= \sum_{r=-\infty}^{\infty} x(n-rN) = \dots + x(n+N) + x(n) + x(n-N) + x(n-2N) \dots$

$x(n) = \tilde{x}(n)$ for $0 \le n \le N-1$

$x(n) = \tilde{x}(n)\cdot R_N(n) = \tilde{x}(n)\begin{cases}1, & 0\le n\le N-1\\ 0, & \text{else}\end{cases}$

Theorem: if $x(n)$ is finate sequence $[0, N-1]]$, then N samples of $X(z)$ on the unit circle determines $X(z)$ for all z.

$\boxed{\text{ex-5.4}}$ $\quad x_1(n) = \{6,5,4,3,2,1\}$ $\quad X_1(e^{j\omega})$ sampled $\omega_k = \frac{2\pi k}{4}$
$k = 0,\pm1,\pm2,\pm3,\dots$

$x_1(n) = 6\delta(n) + 5\delta(n-1) + 4\delta(n-2) + 3\delta(n-3) + 2\delta(n-4) + \delta(n-5)$

$X_1 = 6 + 5z^{-1} + 4z^{-2} + 3z^{-3} + 2z^{-4} + z^{-5}$

$x_2(n) = \sum_{r=-\infty}^{\infty} x(\underset{\substack{0 \cdots 4\\5}}{n} - rN) = \boxed{N=4} \sum_{r=-\infty}^{\infty} x(n - r\cdot 4)$

```
    2 1 6 5 4 3 2 1
6 5 4 3 2 1 6 5 4 3 2 1
    8 6 10 8 6 4 8 6
```

-4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10 11 12 13

6 5 4 3 2 1 $\underbrace{6\ 5\ 4\ 3\ 2\ 1}_{x(n-8)}$

$\underbrace{6\ 5\ 4\ 3\ 2\ 1}_{x(n+4)}$ $\underbrace{6\ 5\ 4\ 3\ 2\ 1}_{x(n-4)}$

8 6 4 3 8 6 4 3 8 6 - - - -

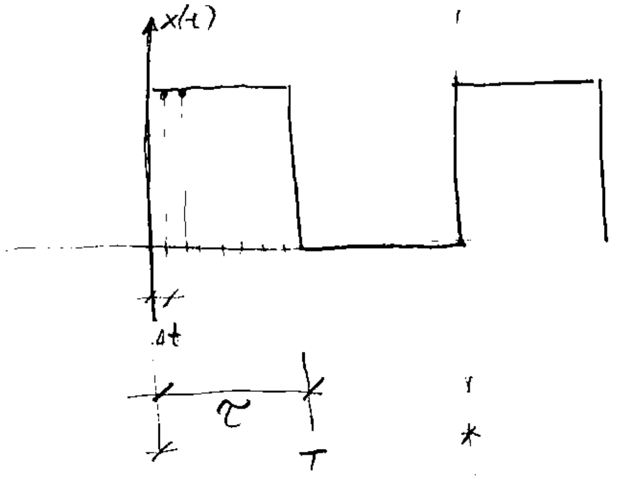
$\boxed{EX. 5.5}$ $X(z) = \frac{1}{1-0.7z^{-1}} = \frac{z}{z-0.7}$ $|z| > 0.7$

$x(n) = (0.7)^n u(n)$ ↗ SAMPLE THE $\mathcal{Z}$ TRANSFORM WITH $N = 5, 10$ 20, 50 and study the efect in time domain

$$\tilde{X}(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} x(n)\, W_N^{nk}$$

$$x(\ ) = \frac{1}{N} \sum_{n=0}^{N-1} X(k)\, W_N^{-nk}$$

- RECONSTRUCTION FORMULA

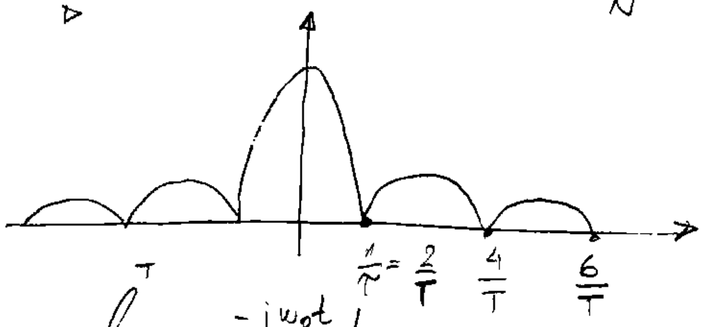


$f = \frac{1}{T}$

$T = 10^{-3} = 1ms$

$f = 10^3 = 1KHz$

$\Delta t = \frac{T}{N}$



$$x(t) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{+j\omega_0 t n} \qquad c_k = \frac{1}{T}\int_0^T x(t)\, e^{-j\omega_0 t}\, dt$$

$$\omega_0 = \frac{\omega}{N} = \frac{2\pi f}{N} = \frac{2\pi}{N \cdot T} = \frac{2\pi}{\Delta t}$$

$N-1-(N/2 - 1) = N-1-N/2-1+1$
$= N/2 - 2 + 1$
$= N/2 - 1$

$N/2 - \cancel{1} + \cancel{1} = N/2$

$\tau = 10 \cdot 10^{-5} = 10^{-4}$ $\frac{1}{\tau} = \frac{1}{10^{-4}} = 10^4 = 10\ 10^3 = 10KHz$

$\omega = n\omega_0$

$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{\Delta t}$

$$X(k) = \sum_{k=0}^{N-1} x(n) \cdot e^{-j2\pi k n/N}$$

$\omega = \frac{2\pi k}{N} \Rightarrow \Omega_0 = \frac{2\pi}{N \cdot \Delta t}$ $\Omega = \overbrace{[0:N]}^{k}\, \omega_0 = [0:N]\, \frac{2\pi}{N \cdot \Delta t}$

$x(n)\cos(\omega_0 n) \Rightarrow \quad \omega = \frac{2\pi}{\cancel{1}2}[0:100] \quad \boxed{\omega_0 = \pi},$

$$\boxed{\begin{aligned}\sin x &= -\frac{j}{2}\left[e^{+jx} - e^{-jx}\right]\\ e^{+jx} &= \cos x + j\sin x\\ e^{-jx} &= \cos x - j\sin x\end{aligned}}$$

$$x(t) = \cos(\Omega_0 t) = \left| t = n\Delta t \right| = \cos\left(\frac{2\pi}{2\Delta t}\cdot n\Delta t\right) \qquad \Omega_0 = \frac{\pi}{\Delta t}$$

$$\Omega_0 = \frac{\pi}{10^{-5}} \qquad F_0 = \frac{\Omega_0}{2\pi} = \frac{\pi}{2\pi\cdot 10^{-5}} = 0.5\cdot 10^5 = 5\cdot 10^4 = 50\cdot 10^3$$
$$= 50\cdot \text{kHz}$$

Reconstruction Formula

samples of $X(z)$

$x(n) \quad [0, N-1]$

$$X(z) = \mathcal{Z}[x(n)] = \mathcal{Z}[\tilde{x}(n)\cdot R_N(n)] = \mathcal{Z}[\text{IDFS}\{\overbrace{\tilde{X}(k)}\}R_N(n)]$$

$$X(z) = \sum_{n=0}^{N-1}\left\{\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}(k)\cdot W_N^{-kn}\right\}z^{-n} = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}(k)\sum_{n=0}^{N-1}W_N^{-kn}z^{-n} =$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}(k)\sum_{n=0}^{N-1}\left(W_N^{-k}z^{-1}\right)^n = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}(k)\,\frac{1 - W_N^{-kN}z^{-N}}{1 - W_N^{-k}z^{-1}}$$

$$\left| W_N^{-kN} = e^{+j\frac{2\pi}{N}\cdot kN} = e^{j2\pi k} = 1 \right| \Rightarrow X(z) = \frac{1 - z^{-N}}{N}\sum_{k=0}^{N-1}\frac{\tilde{X}(k)}{1 - W_N^{-k}z^{-1}}$$

$$\boxed{X(e^{j\omega})} = \frac{1 - e^{-j\omega N}}{N}\sum_{k=0}^{N-1}\tilde{X}(k)\,\frac{1}{1 - e^{j2\pi k/N}\cdot e^{-j\omega} \to 1}$$

$$\frac{1 - e^{-j\omega N}}{N\left(1 - e^{-j\left(\omega - \frac{2\pi k}{N}\right)}\right)} = \frac{1 - e^{-j\omega N}\cancel{e^{+j\frac{2\pi k}{N}N}}}{N\left(1 - e^{-j\left(\omega - \frac{2\pi k}{N}\right)}\right)} = \frac{1 - e^{-j\left(\omega - \frac{2\pi k}{N}\right)\cdot N}}{N\left(1 - e^{-j\left(\omega - \frac{2\pi k}{N}\right)}\right)} =$$

$$= \frac{e^{-j\left(\omega - \frac{2\pi k}{N}\right)\frac{N}{2}}}{e^{-j\left(\omega - \frac{2\pi k}{N}\right)\cdot\frac{1}{2}}}\cdot\frac{e^{+j\left(\omega - \frac{2\pi k}{N}\right)\frac{N}{2}} - e^{-j\left(\omega - \frac{2\pi k}{N}\right)\frac{N}{2}}}{N\cdot\left(e^{+j\left(\omega - \frac{2\pi k}{N}\right)\frac{1}{2}} - e^{-j\left(\omega - \frac{2\pi k}{N}\right)\cdot\frac{1}{2}}\right)} =$$

$$= e^{-j\left(\omega - \frac{2\pi k}{N}\right)\left(\frac{N}{2} - \frac{1}{2}\right)}\cdot\frac{\sin\left(\omega - \frac{2\pi k}{N}\right)\frac{N}{2}}{N\cdot\sin\left(\omega - \frac{2\pi k}{N}\right)\frac{1}{2}} = e^{-j\omega\frac{N-1}{2}}\cdot\frac{\sin\left(\omega - \frac{2\pi k}{N}\right)\frac{N}{2}}{N\cdot\sin\left(\omega - \frac{2\pi k}{N}\right)\frac{1}{2}}$$

$$\boxed{X(e^{j\omega}) = \sum_{k=0}^{N-1}\tilde{X}(k)\cdot\underbrace{\left(e^{-j\omega\frac{N-1}{2}}\frac{\sin\left(\omega - \frac{2\pi k}{N}\right)\frac{N}{2}}{N\cdot\sin\left(\omega - \frac{2\pi k}{N}\right)\frac{1}{2}}\right)}_{\phi\left(\omega - \frac{2\pi k}{N}\right)} = \sum_{k=0}^{N-1}\tilde{X}(k)\,\phi\left(\omega - \frac{2\pi k}{N}\right)}$$

$\phi\left(\omega - \frac{2\pi k}{N}\right)$ – INTERPOLATION POLYNOMIAL $\Rightarrow$ DIGITAL SYNC FUNCTION

- time domain interpolation:

$$x_a(t) = \sum_{n=-\infty}^{\infty}x(n)\cdot\text{sinc}\left[F_s\cdot(t - nT_s)\right]$$

Intellicom

THE DISCRETE FOURIER TRANSFORM

$$\tilde{x}(n) = \sum_{r=-\infty}^{\infty} x(n - rN) \quad ; \quad \tilde{x}(n) = x(n \bmod N)$$

$$x((n))_N \triangleq x(n \bmod N) \quad ; \quad \tilde{x}(n) = x((n))_N \ ; \quad x(n) = x((n))_N \; R_N(n)$$

$$X(k) \triangleq \mathrm{DFT}[x(n)] = \begin{cases} \tilde{X}(k) & , \ 0 \le k \le N-1 \\ 0 & ; \ \text{elsewhere} \end{cases} = \tilde{X}(k)\, R_N(k)$$

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \qquad k = 0 : N-1}$$

$$\tilde{X}(k) = X((k))_N = X(k \bmod N)$$

$$x(n) \triangleq \mathrm{IDFT}[X(k)] = \tilde{x}(n)\, R_N(n)$$

$$\boxed{x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-kn} \qquad 0 \le n \le N-1} \qquad \mathrm{IDFT}$$

- MATLAB IMPLEMENTATION.

$$X = W_N \cdot x \ ; \qquad x = \frac{1}{N} W_N^{*} X \qquad W_N - \text{DFT MATRIX}$$

Ex. 5.6

$$x(n) = \begin{cases} 1, & 0 \le n \le 3 \\ 0; & \text{otherwise} \end{cases}$$

ⓐ DTFT = ? magnitude + phase

ⓑ DFT = ?

ⓐ
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n} = \sum_{n=0}^{3} e^{-j\omega n} = \underbrace{1 + e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega}}_{\boxed{\text{roots}([1,1,1,1]) = [-1, j, -j]}}$$

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} \tilde{X}(k)\, e^{-j\omega \frac{N-1}{2}} \frac{\sin(\omega - \frac{2\pi k}{N})\frac{N}{2}}{\sin(\omega - \frac{2\pi k}{N})\frac{1}{2}}$$

$$= |N = 4| : \sum_{n=0}^{3} \tilde{X}(k)\, e^{-j\frac{3\omega}{2}} \frac{\sin(\omega - \frac{2\pi k}{4}) \cdot 2}{\sin(\omega - \frac{2\pi k}{4})\frac{1}{2}}$$

$$X(e^{j\omega}) = (e^{-j\omega} + 1)(e^{-j\omega} - j)(e^{-j\omega} + j) \cdot \frac{(e^{-j\omega} - 1)}{(e^{-j\omega} - 1)} =$$

$$= \frac{(e^{-j2\omega} - 1)(e^{-j2\omega} + 1)}{(e^{-j\omega} - 1)} = \frac{(e^{-j4\omega} - 1)}{(e^{-j\omega} - 1)} = \underline{\underline{\frac{1 - e^{-j4\omega}}{1 - e^{-j\omega}}}}$$

$$X(e^{j\omega}) = \frac{1-e^{-j4\omega}}{1-e^{-j\omega}} = \frac{e^{-j2\omega}\left(e^{j2\omega}-e^{-j2\omega}\right)}{e^{-j\omega/2}\left(e^{j\omega/2}-e^{-j\omega/2}\right)} = e^{-j\frac{3\omega}{2}}\,\frac{\sin(2\omega)}{\sin(\omega/2)}$$

$$|X(e^{j\omega})| = \left|\frac{\sin 2\omega}{\sin(\omega/2)}\right|$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\,e^{j\omega n}$$

$$\angle X(e^{j\omega}) = \begin{cases} -\frac{3\omega}{2} & \frac{\sin(2\omega)}{\sin(\omega/2)} > 0 \\ -\frac{3\omega}{2} \pm \pi & \frac{\sin(2\omega)}{\sin(\omega/2)} < 0 \end{cases}$$

$$X(e^{j\omega}) = -\left|\frac{\sin(2\omega)}{\sin\left(\frac{\omega}{2}\right)}\right| \cdot e^{-j\frac{3\omega}{2}} = \left|\frac{\sin(2\omega)}{\sin\left(\frac{\omega}{2}\right)}\right| e^{\pm j\pi} \cdot e^{-j\frac{3\omega}{2}}$$

$$\omega = 0.2\pi \qquad \angle X(e^{j\omega}) = -\frac{3 \cdot 0.2\pi}{2} = -0.3\pi$$

$$\omega = 0.5\pi \qquad \angle X(e^{j\omega}) = -\frac{3 \cdot 0.5\pi}{2} = \frac{1.5}{2}\pi = -0.75\pi$$

ex. 5.7 $x(n) = \{\underset{\uparrow}{1}, 1, 1, 1, 0, 0, 0, 0\}$

$$X_8(k) = \sum_{n=0}^{7} x(n)\, W_8^{nk} \;; \qquad W_8 = e^{-j\frac{2\pi}{N}} = e^{-j\frac{2\pi}{8}} = e^{-j\frac{\pi}{4}}$$

$$\omega_1 = \frac{2\pi}{N} = \frac{2\pi}{8} = \frac{\pi}{4} \qquad \omega = \frac{2\pi}{N}[0 \ldots N] \qquad \boxed{\omega_{max} = 2\pi}$$

ex. 5.8 $x(n) = \cos(0.48\pi n) + \cos(0.52\pi n)$

ⓐ DTFT = ? $0 \le n \le 10$

ⓑ DTFT = ? $0 \le n \le 100$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\cdot e^{-j\omega n} = \frac{1}{2}\sum_{n=0}^{10}\left\{\left[e^{j\alpha n}+e^{-j\alpha n}\right] + \left[e^{j\beta n}+e^{-j\beta n}\right] e^{-j\omega n}\right.$$

$$= \frac{1}{2}\sum_{n=0}^{10} e^{-j(\omega n-\alpha n)} + e^{-j(\omega n+\alpha n)} + e^{-j(\omega n-\beta n)} + e^{-j(\omega n+\beta n)} =$$

$$= \frac{1}{2}\left[\frac{1-e^{-j(\omega-\alpha)N}}{1-e^{-j(\omega-\alpha)}} + \frac{1-e^{-j(\omega+\alpha)N}}{1-e^{-j(\omega+\alpha)}} + \frac{1-e^{-j(\omega-\beta)N}}{1-e^{-j(\omega-\beta)}} + \frac{1-e^{-j(\omega+\beta)N}}{1-e^{-j(\omega+\beta)}}\right]$$

$$x(t) = \sum_{n=-\infty}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=-\infty}^{\infty} b_n \sin(n\omega_0 t)$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} \cos(\omega_0 t)\cdot\cos(n\omega_0 t)\,dt$$

$$x(t) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{+j\omega_0 n t}$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\cdot e^{-j\omega_0 n t}\,dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} \frac{1}{2}\cos(n\omega_0-\omega_0)t + \frac{1}{2}\cos(n\omega_0+\omega_0)t\;dt$$

$$= \left[\frac{1}{T}\cdot\circledast \sin(n\omega_0-\omega_0)t + \frac{1}{T}\cdot\circledast\!\circledast \sin(n\omega_0+\omega_0)t\right]\Bigg|_{-T/2}^{T/2}$$

?

$$\circledast = \frac{1}{n\omega_0-\omega_0}$$

$$\circledast\!\circledast = \frac{1}{n\omega_0+\omega_0}$$

(10)

$$a_n = \left[\frac{1}{T\circledast} \sin(n\omega_0 - \omega_0)t + \frac{1}{T\circledast}\sin(n\omega_0+\omega_0)t\right]\Bigg|_{-T/2}^{T/2}$$

$\sin(-\alpha) = -\sin\alpha$

$\boxed{n=1}$

$$a_1 = \frac{1}{T\circledast}\left[\sin(\omega_0+\omega_0)t\right]\Bigg|_{-T/2}^{T/2} = \frac{1}{T}\left[\sin\left(2\cdot\omega_0 \frac{T}{2}\right) \overset{\times}{-} \sin\left(-2\omega_0 \frac{T}{2}\right)\right]$$

$$= \frac{1}{T}\left[\sin(\omega_0 T) + \sin(\omega_0 T)\right] = \frac{1}{T}\cdot 2 \sin \cancel{\omega_0}\frac{2\pi}{\cancel{\omega_0}} = \frac{2}{T}\cdot\sin 2\pi = 0$$

$$T = \frac{1}{f_0} = \frac{1}{\frac{\omega_0}{2\pi}} = \frac{2\pi}{\omega_0}$$

$\boxed{n=1}$

$$a_1 = \frac{2}{T}\int_{-T/2}^{T/2} \cos(\omega_0 t)\cdot\cos(\omega_0 t)\,dt = \frac{2}{T}\int_{-T/2}^{T/2}\cos^2(\omega_0 t)\,dt = \left|\begin{array}{l} u = \omega_0 t \\ du = \omega_0 dt \\ dt = \frac{du}{\omega_0} \\ u_1 = -\omega_0\cdot T/2 = -\pi \\ u_2 = +\frac{\cancel{2\pi}}{\cancel{T}}\cdot\frac{\cancel{T}}{\cancel{2}} = \pi \end{array}\right|$$

$$a_1 = \frac{2}{T}\cdot\frac{1}{\omega_0}\int_{-\pi}^{\pi}\cos^2 u\,du = \frac{\cancel{2}}{\cancel{T}}\,\frac{1}{\frac{2\pi}{\cancel{T}}}\cdot\left[\frac{1}{2}\cos(x)\sin(x) + \frac{1}{2}x\right|_{-\pi}^{\pi} =$$

$\neq 0$

$$= \frac{1}{\pi}\left[\frac{1}{2}\cancel{\cos(x)\sin x}\Big|_{-\pi}^{\pi} + \frac{1}{2}\pi + \frac{1}{2}\pi\right] = \frac{1}{\cancel{\pi}}\cdot\cancel{\pi} = \boxed{1}$$

$$\boxed{a_0 = 0\ ;\quad a_n = 0 \quad \text{ZA} \quad n > 1}$$

PROPERTIES OF DFT

- $X(k)$ N-point DFT of sequence $x(n)$

1.) Linearity

$$DFT\left[ax_1(n) + bx_2(n)\right] = a\,DFT\left[x_1(n)\right] + b\,DFT\left[x_2(n)\right]$$

If $x_1(n)$ N_1 length seq $x_2(n)$ N_2 length seq $\Rightarrow$

$DFT\left(ax_1(n) + bx_2(n), N_3\right)$

2.) Circular folding

$$x((-n))_N = \begin{cases} x(0) & n = 0 \\ x(N-n) & 1 \leq n \leq N-1 \end{cases}$$

$$DFT\left[x((-n))_N\right] = X((-k))_N = \begin{cases} X(0) & k = 0 \\ X(N-k) & 1\leq k \leq N-1 \end{cases}$$

Ex 5.9 $x(n) = 10(0.8)^n \qquad 0 \le n \le 10$

ⓐ Determine & Plot $x((-n))_{N=11}$

ⓑ Verify circular folding property

$$x((-n))_N = \begin{cases} x(0) & n=0 \\ x(N-n) & 1 \le n \le 10 \end{cases}$$

$$X((k))_N = \begin{cases} X(k) & k=0 \\ X(N-k) & 1 \le k \le 10 \end{cases}$$

- PROPERTY Nº 3 conjugation property
- $\text{DFT}[x^*(n)] = X^*((-k))_N$
- PROPERTY Nº 4 SYMETRY PROPERTY OF REAL SEQUENCES

$$X(k) = X^*((-k))_N \leftarrow$$

If $x(n)$ is real valued sequence $x(n) = x^*(n)$

$\text{Re}[X(k)] = \text{Re}[X((-k))]$ Circular-even seq

$\text{Im}[X(k)] = -\text{Im}[X((N-k))]$ Circular-odd seq

$|X(k)| = |X((-k))_N|$ Circular-even seq

$\angle X(k) = -\angle X((-k))_N$ Circular-odd

$x(0) = x^*(0) = x(-0) \qquad \omega = k\omega_1 = 0 \cdot \omega_1 = 0 \Rightarrow$ DC component

$x\left(\frac{N}{2}\right) = x^*\left(\left(-\frac{N}{2}\right)\right)_N = x^*\left(\frac{N}{2}\right)$

$$x_{ec}(n) \triangleq \frac{1}{2}\left[x(n) + x((-n))_N\right] = \begin{cases} x(0) & n=0 \\ \frac{1}{2}[x(n) + x(N-n)] & 1 \le n \le N-1 \end{cases}$$

$$x_{oc}(n) \triangleq \frac{1}{2}\left[x(n) - x((-n))_N\right] = \begin{cases} x(0) & n=0 \\ \frac{1}{2}[x(n) - x(N-n)] & 1 \le n \le N-1 \end{cases}$$

$$\boxed{\begin{aligned} &\text{DFT}[x_e(n)] = \text{Re}[X(k)] = \text{Re}[X((-k))_N] \\ &\text{DFT}[x_o(n) = +j\,\text{Im}[X(k)] = -j\,\text{Im}[X((-k))_N] \end{aligned}}$$

PROPERTY 5.34

Ex. 5.10 $x(n) = 10(0.8)^n \qquad 0 \le n \le 10$

ⓐ $x_{ec}(n), x_{oc}(n) = ?$

ⓑ verify 5.34

$x(n) = x_{ec}(n) + x_{oc}(n)$

$x((-n))_N = x_{ec}(n) + x_{oc}(n)$

$+$

$x_{ec}(n) = \frac{1}{2}\left[x(n) + x((-n))_N\right]$

$x_{oc}(n) = \frac{1}{2}\left[x(n) - x((-n))_N\right]$

• PERIODIC SHIFT VS. CIRCULAR SHIFT

$$\tilde{x}(n-m) = x((n-m))_N \Rightarrow \boxed{\text{PERIODIC SHIFT}}$$

$$\tilde{x}(n-m)R_N(n) = x((n-m))_N R_N(n) \Rightarrow \boxed{\text{CIRCULAR SHIFT}}$$

$$\text{DFT}\left[x((n-m))_N R_N(n)\right] = W_N^{km} X(k)$$

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j2\pi kn/N} \; ; \quad \text{DFT}\left[x((n-m))_N\right] = \sum_{n=0}^{N-1} x((n-m))_N \cdot e^{-j2\pi kn/N} =$$

$$= \left| \begin{matrix} n-m=u \; ; \; n=u+m \\ n=0 \; ; \; u=-m \\ n=N-1 \; , \; u=N-1-m \end{matrix} \right| = \sum_{u=0}^{u=N-1} x(u) \cdot e^{-j2\pi k(u+m)} =$$

$$= e^{-j\frac{2\pi km}{N}} \sum_{u=0}^{N-1} x(u)\, e^{-j2\pi ku} = \underline{W_N^{km} \cdot X(k)} \; , \quad W_N = e^{-j\frac{2\pi}{N}}$$

EX. 5.11 $x(n) = 10(0.8)^n, \quad 0 \le n \le 10$

ⓐ Sketch $x((n+4))_{11} R_{11}(n)$

ⓑ Sketch $x((n-3))_{15} R_{15}(n)$

EX. 5.12 $x(n) = 10(0.8)^n \quad 0 \le n \le 10$

Determine and plot .

$x((n-6))_{15}$

PROPERTY 6 Circular Shift in frequency domain

$$\text{DFT}\left[W_N^{-ln} x(n)\right] = X((k-l))_N R_N(k)$$

PROPERTY 7 Circular Convolution

$$\boxed{x_1(n) \textcircled{N} x_2(n) = \sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N \; , \quad 0 \le n \le N-1}$$

$$\text{DFT}\left[x_1(n) \textcircled{N} x_2(n)\right] = X_1(k) \cdot X_2(k)$$

EX. 5.13 $x_1(n) = \{1, 2, 2\} \quad x_2(n) = \{1, 2, 3, 4\}$

$x_1(n) \textcircled{4} x_2(n) = ?$

$x_1 = \{\overset{0}{1}, \overset{1}{2}, \overset{2}{2}\}$; $x_2 = \{\overset{0}{1}, \overset{1}{2}, \overset{2}{3}, \overset{3}{4}\}$ $y = x_1 \textcircled{4} x_2$

$n_y = 0:5$

	-4	-3	-2	-1	0	1	2	3	4	④	~~y(n)~~	y(n)	
x_1					1	2	2	0					
$x_2(-k)$		4	3	2	1	4	3	2	1	0	15	1	$x_2(-k+4)$
			4	3	2	1	4	3	2	1	12	4	
				4	3	2	1	4	3	2	9	9	
					4	3	2	1	4	3	14	14	
		4	3	2	1	4	3	2	1	4	15	14	
			4	3	2	1	4	3	2	5	12	8	

$x_1 \textcircled{4} x_2 = \{15, 12, 9, 14\}$

$(-j)^{-1} = -\frac{1}{j} \cdot \frac{j}{j} = j$

$$X_1(k) = \sum_{n=0}^{3} x(n) \cdot W_N^{+kn} = \left| \begin{matrix} W_N = e^{-j2\pi/N} = \\ = e^{-j\frac{\pi}{2}} = -j \end{matrix} \right| = \sum_{n=0}^{3} x(n)(-j)^{+kn} = \sum_{n=0}^{3} x(n) \cdot j^{-kn}$$

$$X_1(1) = \sum_{n=0}^{3} \{1,2,2\} \cdot j^{n} = 1 \cdot (j)^{0} + 2 \cdot (j)^{-1} + 2 (j)^{-2} = 1 - 2j - 2 = -1 + 2j$$

$$X_1(0) = \sum_{n=0}^{3} \{1,2,2\} = 1 + 2 + 2 = 5$$

$$X_1(2) = \sum_{n=0}^{3} \{1,2,2\}\, j^{2n} = 1 \cdot j^{0} + 2 \cdot (j)^{2} + 2 \cdot j^{4} = 1 - 2 + 2(j)^2(j)^2 = 1 - 2 + 2 = 1$$

$$X_1(3) = \sum_{n=0}^{3} \{1,2,2\}\, j^{-3n} = 1 \cdot j^{0} + 2 (j)^{-3} + 2 \cdot j^{-6} = 1 - 2 + 2j = -1 - 2j$$

$+j$; -1

EX 5.15 $x_1 = \{1,2,2\}$; $x_2 = \{1,2,3,4\}$

ⓐ $x_1(n) \textcircled{5} x_2(n) = ?$ $Y = \{9,4,9,14,14\}$

ⓑ $x_2(n) \textcircled{6} x_2(n) = ?$ $Y = \{1,4,9,14,14,8\}$

aliasing

FOR $N = 6$ (N_1+N_2-1)

CIRCULAR CONV ≡ LINEAR CONV

MMV

VAŽI ZA: $N \geq N_1 + N_2 - 1$

⑧ Multiplication

$$DFT[x_1(n) \cdot x_2(n)] = \frac{1}{N} X_1(k) \textcircled{N} X_2(k)$$

⑨ Parseval's relation

$$E_x = \sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

$\frac{|X(k)|^2}{N}$ → energy spectrum of finite duration sequence

LINEAR CONVOLUTION USING DFT

$x_1(n)$ N_1 POINT SEQUENCE

$x_2(n)$ N_2 POINT SEQUENCE

$$x_3(n) = x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k) = \sum_{k=0}^{N_1-1} x_1(k) \cdot x_2(n-k)$$

⇒ x_3 $N_1 + N_2 - 1$ POINT SEQUENCE

$N = \max(N_1, N_2)$ $x_1 * x_2 \neq x_1 \textcircled{N} x_2$

$N = N_1 + N_2 - 1$ x_1, x_2 N poin $(N = \max(N_1, N_2))$

$$x_4(n) = x_1(n) \circledcirc{N}\, x_2(n) = \left[\sum_{k=0}^{N-1} x_1(k)\, x_2((n-k))_N\right] R_N(n) =$$

$$= \left[\sum_{k=0}^{N-1} x_1(k) \sum_{r=-\infty}^{\infty} x_2(n-k-rN)\right] R_N(n) = \left[\sum_{r=-\infty}^{\infty} \underbrace{\sum_{k=0}^{N-1} x_1(k)\, x_2(n-k-rN)}_{x_3(n-rN)}\right] R_N(n)$$

$$= \left[\sum_{r=-\infty}^{\infty} x_3(n-rN)\right] R_N(n)$$

$\boxed{N = N_1 + N_2 - 1}$

$x_4(n) = x_3(n)$ $0 \le n \le N-1$

$\boxed{\text{Ex. 5.16}}$ $x_1(n) = \{1, 2, 2, 1\}$

$x_2(n) = \{1, -1, -1, 1\}$

a.) Linear conv = ? $x_3 = x_1 * x_2$

b.) $x_4(n) = ?$ $x_4 = x_3$

$\boxed{\text{ERROR ANALYSIS}}$ $\boxed{N \geq \max(N_1, N_2)}$

$\max(N_1, N_2) \le N \le N_1 + N_2 - 1$

$$x_4(n) = \left[\sum_{r=-\infty}^{\infty} x_3(n-rN)\right] R_N(n)$$

$$e(n) = x_4(n) - x_3(n) = \left[\sum_{r \neq 0} x_3(n-rN)\right] R_N(n) \quad . \quad \boxed{r = \pm 1}$$

$$e(n) = [x_3(n-N) + x_3(n+N)]\, R_N(n)$$

$x_1(n), x_2(n) = 0$ $n < 0$ causal $\Rightarrow$ $x_3(n)$ causal

$x_3(n-N) = 0$ $0 \le n \le N-1$

$x_3(n+N) \neq 0$

$e(n) = x_3(n+N)$ $0 \le n \le N-1$

$\boxed{\text{Ex. 5.17}}$ $x_1 = \{1, 2, 2, 1\}$; $x_2 = \{1, -1, -1, 1\}$ $N = 6, 5, 4$

$\boxed{e = ?}$ $x_3(n) = \{1, 1, -1, -2, -1, 1, 1\}$

$N = 6$ $x_4(n) = \{2, 1, -1, -2, -1, 1\}$

$e(n) = x_4(n) - x_3(n) = \{1, 1, -1, -2, -1, 1\} - \{2, 1, -1, -2, -1, 1\} =$

$0 \le n \le N-1$

$= \{1, 0, 0, 0, 0, 0\} = x_3(n+6)$

$x_3(0+6) = e(0) = x_3(6) = 1$

$x_3(1+6) = e(1) = 0$

$$x_3(n) = \{ \overset{0}{1}, \overset{1}{1}, \overset{2}{-1}, \overset{3}{-2}, \overset{4}{-1}, \overset{5}{1}, \overset{6}{1} \}$$

$\boxed{N=5}$ $x_4(n) = \{2, 2, -1, -2, -1\}$

$e(n) = x_4 - x_3(n) = \| \; n = 0 \ldots 5 \; | = -\{1, 1, -1, -2, -1\} + \{2, 2, -1, -2, -1\}$

$e(n) = \{1, 1, 0, 0, 0\} = x_3(n+N) = x_3(n+5)$

$x_3(0+5) = x_3(5) = 1$

$x_3(1+5) = x_3(6) = 1$

$x_3(2+5) = x_3(7) = 0$

$\boxed{N=4}$ $x_4(n) = \{0, 2, 0, -2\}$

$e(n) = x_4(n) - x_3(n) = |n = 0 \ldots 4| = \{-1, 1, 1, 0\} = x_3(n+4)$

$x_3(0+4) = x_3(4) = -1$

$x_3(1+4) = x_3(5) = 1$

$x_3(2+4) = x_3(6) = 1$

OBSERVATION: If $N = \max(N_1, N_2)$ THEN FIRS $M-1$ samples are in error; $M = \min(N_1, N_2)$

ex.: $N = 4 \Rightarrow x_4(n) = \{0, 2, 0, 2\}$

$N_1 = \text{lenght}(x_1) = 4$ \
$N_2 = \text{lenght}(x_2) = 4$ $\Big\}$ $\min(N_1, N_2) = 4 = M$ \
errors $= M - 1 = 4 - 1 = 3$

BLOCK CONVOLUTIONS

$x(n)$ sectioned in N-point sequences

$h(n)$ M-point sequence

$\boxed{M < N}$

OVERLAP AND SAVE METHOD

$\boxed{\text{ex. } 5.18}$ $x(n) = (n+1)$ $\quad 0 \leq n \leq 9 \quad$ $h(n) = \{\underset{\uparrow}{1}, 0, -1\}$

implement overlapping save method using $N = 6$

$y(n) = x(n) * h(n)$

$N_1 + N_2 - 1 = 10 + 3 - 1 = \underline{13} - 1 = 12$

$N = 6$ \
$M = 3$

$\boxed{N - M + 1 = 6 - 3 + 1 = 3 + 1 = 4}$

↳ SAVE THE LAST

MATLAB IMPLEMENTATION

$\hat{x}(n) \triangleq \{\underbrace{0, 0, \ldots, 0}_{M-1 \text{ zeros}}, \underbrace{x(n)}\} \quad n \geq 0$

$\boxed{L = N - M + 1}$ → DVE GLAVNI FORMULI !!!

- kth block : $x_k(n)$ $\quad 0 \leq n \leq N-1$

$x_k(n) = \hat{x}(n)$ $\quad \boxed{kL \leq n \leq k \cdot L + N - 1}$ $\quad k \geq 0$

$0 \leq n \leq N-1$

$$\hat{x}(n) \triangleq \{\underbrace{0,0,0,\dots,0}_{M-1}, \underbrace{x(n)}_{N_x}\}$$

$$x_k(n) = \hat{x}(n) \qquad kL \le n \le kL+N-1 \qquad \begin{matrix} k \ge 0 \\ 0 \le n \le N-1 \end{matrix}$$

$$\hat{N}_x = N_x + L - 1$$

$$K = \frac{N_x + M - 1 + K(M-1)}{N} \, ?\,; \qquad \boxed{K = \left\lfloor \frac{N_x+M-2}{L} \right\rfloor + 1}$$

$$x(n) = \{\overset{0}{1}, \overset{1}{2}, \overset{2}{3}, \overset{3}{4}, \overset{4}{5}, \overset{5}{6}, \overset{6}{7}, \overset{7}{8}, \overset{8}{9}, \overset{9}{10}\}$$

$$\hat{x}(n) = \{\overset{0}{0}, \overset{1}{0}, \overset{2}{1}, \overset{3}{2}, \overset{4}{3}, \overset{5}{4}, \overset{6}{5}, \overset{7}{6}, \overset{8}{7}, \overset{9}{8}, \overset{10}{9}, \overset{11}{10}\} \overset{12}{0} \; \overset{13}{0}$$

$x_0 \quad x_1 \quad x_2$

$M = 3, \; N = 6$

$\underline{L = N - M + 1 = 4}$

$$\left.\begin{matrix} x_0 = [0,0,1,2,3,4] \\ x_1 = [3,4,5,6,7,8] \\ x_2 = [7,8,9,10,0,0] \end{matrix}\right\} \quad K = 3$$

$\boxed{k=1}$

$$x_1(n) = \hat{x}(n) \qquad kL \le n \le kL + N - 1$$

$$k=1 \Rightarrow \qquad L \le n \le \overset{1}{k}L + N - 1$$

$$\underline{4} \le n \le 4 + 6 - 1 = \underline{9}$$

$\boxed{k=0}$ $x_0(n) = \hat{x}(n) \qquad 0 \le n \le 5$

$k = 2 \quad x_2(n) = \hat{x}(n) \qquad 8 \le n \le 13$

$$N_x = 12 \qquad K = \frac{10+3-2}{4} + 1 = \left\lfloor \frac{11}{4} \right\rfloor + 1 = 2 + 1 = 3$$

$$\boxed{y_k(n) = x_k(n) \; \textcircled{N} \; h(n)}$$

$$K \ast L + N = 3 \cdot 4 + 6 = 12 + 6 = 16$$

$$K \ast L + N - N_x = 16 - 10 = 6$$

THE FAST FOURIER TRANSFORM

$$X(k) = \sum_{n=0}^{N-1} x(n) \, W_N^{nk} \qquad \Big/ \; W_N = e^{-j\frac{2\pi}{N}} \Big/ \qquad k = 0, 1, 2 \dots N-1$$

$$C_N = O(N^2) \qquad \dots$$

$$W_N^{kn} = W_N^{k(n+N)} = W_N^{(k+N)n}$$

$$W_N^{kn + \frac{N}{2}} = -W_N^{kn}$$

$$e^{-j\frac{2\pi}{N}\left(kn+\frac{N}{2}\right)} = e^{-j\frac{2\pi}{N}kn - j\frac{2\pi}{N}\frac{N}{2}} = e^{-j\frac{2\pi}{N}kn}\cdot \overset{-1}{e^{-j\pi}} = -e^{-j\frac{2\pi}{N}kn}$$

$$= -W_N^{kn}$$

$\boxed{ex\ 5.20}$ $X(k) = \sum_{n=0}^{3} x(n)\cdot W_4^{nk}$, $0 \le k \le 3$, $W_4 = e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}} = -j$

$$\begin{bmatrix} X(0)\\ X(1)\\ X(2)\\ X(3)\end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0\\ W_4^0 & W_4^1 & W_4^2 & W_4^3\\ W_4^0 & W_4^2 & W_4^4 & W_4^6\\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} x(0)\\ x(1)\\ x(2)\\ x(3)\end{bmatrix}$$

$$\boxed{W_N^{n(k+N)} = W_N^{nk}}$$ efficient algorithm using the periodicity

$$e^{-j\frac{2\pi}{4}\cdot 2} = e^{-j\frac{\pi}{2}} = e^{j\pi}\ 1$$

$W_4^0 = W_4^4 = 1$ $\qquad W_4^1 = W_4^9 = -j$

$W_4^2 = W_4^6 = -1$ $\qquad W_4^3 = \cancel{W_4^7} = j$

$$e^{-j\frac{2\pi}{4}\cdot 6} = e^{-j3\pi}$$

$$e^{-j\frac{2\pi}{4}\cdot 3} = e^{-j\frac{\pi}{2}\cdot 3} = j$$

$$\begin{bmatrix} X(0)\\ X(1)\\ X(2)\\ X(3)\end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} x(0)\\ x(1)\\ x(2)\\ x(3)\end{bmatrix}$$

$$X(0) = x(0)+x(1)+x(2)+x(3) = \overbrace{x(0)+x(2)}^{g_1} + \overbrace{x(1)+x(3)}^{g_2}$$

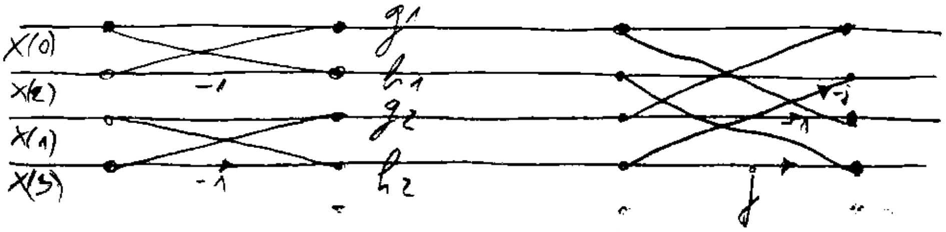
$$X(1) = x(0)-jx(1)-x(2)+jx(3) = \overbrace{x(0)-x(2)}^{h_1} - j\overbrace{[x(1)-x(3)]}^{h_2}$$

$$X(2) = x(0)-x(1)+x(2)-x(3) = \overbrace{x(0)+x(2)}^{g_1} - \overbrace{[x(1)+x(3)]}^{g_2}$$

$$X(3) = x(0)+jx(1)-x(2)-jx(3) = \underbrace{x(0)-x(2)}_{h_1} + j\underbrace{[x(1)-x(3)]}_{h_2}$$

Step 1	Step 2
$g_1 = x(0)+x(2)$	$X(0) = g_1+g_2$
$g_2 = x(1)+x(3)$	$X(1) = h_1-jh_2$
$h_1 = x(0)-x(2)$	$X(2) = g_1-g_2$
$h_2 = x(1)-x(3)$	$X(3) = h_1+jh_2$

ONLY 2 COMPLEX MULTIPLICATIONS



$X(0) = g_1+g_2$

$X(1) = h_1-jh_2$

$X(2) = g_1-g_2$

$X(3) = h_1+jh_2$

$$\left[\begin{bmatrix} x(0)\\ x(2)\end{bmatrix}, \begin{bmatrix} x(1)\\ x(3)\end{bmatrix}\right] = \begin{bmatrix} x(0) & x(1)\\ x(2) & x(3)\end{bmatrix}$$

$$W_2 = \begin{bmatrix} W_2^0 & W_2^0\\ W_2^0 & W_2^1\end{bmatrix} \qquad W = e^{-j\frac{2\pi}{N}} = e^{-j\frac{2\pi}{2}} = e^{-j\pi}$$

$$W_2^0 = e^{-j0} = 1 \qquad W_2^1 = e^{-j\pi} = -1$$

$$W_2 = \begin{bmatrix} 1 & 1\\ 1 & -1\end{bmatrix}; \quad \begin{bmatrix} 1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix} x(0) & x(1)\\ x(2) & x(3)\end{bmatrix} = \begin{bmatrix} x(0)+x(2), & x(1)+x(3)\\ x(0)-x(2), & x(1)-x(3)\end{bmatrix}$$

$$= \begin{bmatrix} g_1 & g_2\\ h_1 & h_2\end{bmatrix}$$

$\boxed{W_4^{pq}}$ p - row index
q - column index

$$W_4 = e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}}$$

$$W_4^{pq} = \begin{bmatrix} W_4^0 , & W_4^0 \\ W_4^0 , & W_4^1 \end{bmatrix} = \begin{bmatrix} 1 , & 1 \\ 1 , & e^{-j\frac{\pi}{2}} \end{bmatrix} = \begin{bmatrix} 1 , & 1 \\ 1 , & -j \end{bmatrix}$$

$$W_4^{pq} * \begin{bmatrix} g_1 & g_2 \\ h_1 & h_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -j \end{bmatrix} * \begin{bmatrix} g_1 & g_2 \\ h_1 & h_2 \end{bmatrix} = \begin{bmatrix} g_1 & g_2 \\ h_1 & -jh_2 \end{bmatrix}$$

$$\begin{bmatrix} g_1 & g_2 \\ h_1 & -jh_2 \end{bmatrix} W_2 = \begin{bmatrix} g_1 & g_2 \\ h_1 & -jh_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} g_1+g_2 , & g_1-g_2 \\ h_1-jh_2 , & h_1+jh_2 \end{bmatrix} = \begin{bmatrix} X(0) & X(2) \\ X(1) & X(3) \end{bmatrix}$$

DIVIDE AND COMBINE METHOD

$\boxed{N = L \cdot M}$ $L^2 + M^2 \ll N^2$ for large N

DEVIDE SEQUENCE IN M SMALLER SEQ WITH LENGHT L

M SMALLER L-point DFT ; $\boxed{N = M \cdot L}$

$n = Ml + m$ $0 \leq l \leq L-1$; $0 \leq m \leq M-1$

$k = p + Lq$ $0 \leq p \leq L-1$; $0 \leq q \leq M-1$

$\max(n) = M(L-1) + M-1 = M \cdot L - 1$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk} \qquad nk = \boxed{Mlp} + MLlq + \boxed{mp} + \underline{\underline{mLq}}$$

$$X(p \cdot q) = \sum_{m=0}^{M-1} \sum_{l=0}^{L-1} x(l,m) W_N^{(Ml+m)(p+Lq)} =$$

$$= \left| \begin{array}{l} n_{max} = M(L-1) + M - 1 = ML - M + M - 1 = ML - 1 \\ k_{max} = L - 1 + L(M-1) = \cancel{L} - 1 + LM - \cancel{L} = \underline{\underline{LM - 1}} \end{array} \right| =$$

$$= \sum_{m=0}^{M-1} \left\{ W_N^{\boxed{mp}} \left[\sum_{l=0}^{L-1} x(l,m) W_N^{\boxed{Mlp}} \right] \right\} W_N^{Lmq}$$

$N = M \cdot L$

$$= \left| W_N^{MLlq} = W_N^{N \cdot lq} = e^{-j\frac{2\pi}{N} N lq} = 1 \qquad W_N^{Mlp} = e^{-j\frac{2\pi}{N} \cdot Mlp} = e^{-j\frac{2\pi}{L} \cdot lp} \right| =$$

$$= \underbrace{\sum_{m=0}^{M-1} \left\{ W_N^{mp} \underbrace{\left[\sum_{l=0}^{L-1} x(l,m) W_L^{lp} \right]}_{L \text{ point DFT}} \right\} W_M^{mq}}_{M \text{ point DFT}}$$

① $F(p,m) = \sum_{l=0}^{L-1} x(l,m) W_L^{lp}$; $0 \le p \le L-1$

$\boxed{0 \le m \le M-1}$

twidle factor

② $G(p,m) = W_N^{pm} F(p,m)$ $0 \le p \le L-1$, $0 \le m \le M-1$

③ $X(p,q) = \sum_{m=0}^{M-1} G(p,m) W_M^{mq}$ $0 \le q \le M-1$

$\boxed{0 \le p \le M-1}$

$C_N = ML^2 + N + M^2 \cdot L \le O(N^2)$

RADIX - R COMPOSITE FFT ALGORITHMS ; $N = R_1^{v_1} \cdot R_2^{v_2} \ldots$

→ $\boxed{N = R^v}$ RADIX - 2 FFT ALGORITHMS   ~~MIXED RADIX FFT~~

DIT-FFT

$N = 2^v$; $M = 2 \quad L = N/2$ DIVIDE $x(n)$ ACCORDING 5.48

$n = M \cdot l + m$ $0 \le l \le L-1$, $0 \le m \le M-1$

$k = p + Lq$ $0 \le p \le L-1$, $0 \le q \le M$

$g_1(n) = x(2n)$
$g_2(n) = x(2n+1)$ $0 \le n \le \frac{N}{2} - 1$

ex..

⓪ 1 2	3 4 5	⑥ 7 8	9 10 11	⑫ 13 14	15 16 17
1, 2, 3	4, 5, 6	7, 8, 9	10, 11, 12	13, 14, 15	16, 17, 18

$L = 3$
$M = 6$

$g_1(n) = 1, 7, 13$; $g_2(n) = 2, 8, 14$; $g_3(n) = 3, 10, 16$

$0 \le l \le 2$
$0 \le m \le 5$

$m = 0$; $l = 0..2$ $n = 0 \quad l = 0$
$n = 6 \cdot 1 + 0 = 6 \quad l = 1$
$n = 6 \cdot 2 + 0 = 12 \quad l = 2$

...

$G_1(k) = DFT[g_1(n)]$ $G_2(k) = DFT[g_2(n)]$

$$X(p,q) = \sum_{m=0}^{1} \left\{ W_N^{mp} \left[\sum_{l=0}^{N/2-1} x(l,m) W_{N/2}^{lp} \right] \right\} W_2^{mq} =$$

$$= W_N^{\cancel{mp}\,0} \sum_{l=0}^{N/2-1} \underline{x(l,0)} W_{N/2}^{lp} + W_N^{p} \cdot W_{N}^{\frac{2 \cdot L}{2 \cdot L}} \cdot \sum_{l=0}^{N/2-1} \underline{x(l,1)} W_{N/2}^{lp}$$

$$\boxed{X(k) = G_1(k) + W_N^k G_2(k)}$$ $0 \le k \le N-1$

$C_N = 2\left(\frac{N^2}{4}\right) + (N) = \frac{N^2}{2} + N = O\left(\frac{N^2}{2}\right)$

calculation of $G_1(k)$ OR $G_2(k)$

→ calculation of $W_N^k \cdot G_2(k)$

DECIMATION IN TIME : $C_N = N_V = N \log_2 N$

$$\boxed{N = 8 \qquad M = 2 \qquad L = 4}$$

$$\boxed{X(k) = G_1(k) + W_N^k G_2(k)}$$

$$x(n) = \begin{array}{cccc|cccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1, & 2, & 3, & 4 & 5, & 6, & 7, & 8 \end{array}$$

$m = 0$ $\qquad n = 2\,0 + 0 = 0 \qquad l = 0$

$\qquad n = 2\,1 + 0 = 2 \qquad l = 1$

$\qquad n = 2\,2 + 0 = 4 \qquad l = 2$

$\qquad n = 2 \cdot 3 + 0 = 6 \qquad l = 3$

$m = 1 \qquad n = 2 \cdot 0 + 1 = 1 \qquad l = 0$

$g_1(n) = x(2n)$

$g_2(n) = x(2n+1)$

$n = M \cdot l + m \qquad 0 \leq l \leq L-1$

$\qquad 0 \leq m \leq M-1$

$l = 0 \dots 3$

$m = 0 \dots 1$

$g_1(0,2,4,6) = \underline{1,3,5,7}$

$g_2(1,3,5,7) = 2,4,6,8$

$$G_1(k) = W_4^{nk} \cdot g_1(n) = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} g_1(0) \\ g_1(1) \\ g_1(2) \\ g_1(3) \end{bmatrix} \begin{array}{c} x(0) \\ x(2) \\ x(4) \\ x(6) \end{array}$$

$$G_2(k) = W_4^{nk}\, g_2(n) = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} g_2(0) \\ g_2(1) \\ g_2(2) \\ g_2(3) \end{bmatrix} \begin{array}{c} x(1) \\ x(3) \\ x(5) \\ x(7) \end{array}$$

$$G_1(0) = W_4^0 x(0) + W_4^0 x(2) + W_4^0 x(4) + W_4^0 x(6) = W_4^0 [x(0) + x(2) + x(4) + x(6)]$$

$$G_2(0) = W_4^0 [x(1) + x(3) + x(5) + x(7)]$$

$$X_1(0) = G_1(0) + \cancel{W_N^{\overset{1}{\cancel{0}}}}\, G_2(0)$$

$$X_1(1) = G_1(1) + W_8^1 G_2(0) = \left| \begin{array}{l} e^{-j\frac{2\pi}{8}} = e^{-j\frac{\pi}{4}} \\ = \frac{\sqrt{2}}{2}(1+j) \end{array} \right|$$

$$X(0) = W_N^0 x(0) + W_N^0 x(4) + W_N^0 x(2) + W_N^0 x(6) + W_N^0 [x(1) + x(5)] + W_N^0 [x(3) + x(7)]$$

$$\boxed{\begin{array}{ll} F(p,m) = \sum_{l=0}^{L-1} x(l,m)\, W_L^{lp} & 0 \leq p \leq L-1 \\ 0 \leq m \leq M-1 & \end{array}}$$

$$\boxed{\begin{array}{l} \text{DIF-FFT} \\ L = 2 \\ M = N/2 \end{array}}$$

$F(0,m) = x(0,m) + x(1,m)\, W_2^0 \qquad 0 \leq m < \frac{N}{2} - 1$

$p = 0$

$n = M \cdot l + m$

$F(0,m) = x(n) + x(n + N/2) \qquad 0 \leq n \leq \frac{N}{2}$

$$x(n) = \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8 \end{array}$$

$n = M \cdot l + m \qquad M = 4 \qquad l = 0 \dots 1$

$m = 0 \qquad n = 0 \qquad l = 0 \qquad m = 0 \dots 3$

$\qquad n = 4 \qquad l = 1$

$g_1 = [1,5]; \; g_2 = [2,6] \dots$

$$F(0,m) = x(0,m) + x(1,m)\,W_2^0 = x(n) + x\left(n+\frac{N}{2}\right)$$

$$n = M\cdot l + m \Rightarrow \begin{array}{l} x(0,m) = x(M\cdot 0 + m) = x(m) = \left| m = 0 \dots \frac{N}{2} \right| = x(n) \\ x(1,m) = x(M\cdot 1 + m) = x\left(m + \underset{M}{\boxed{\frac{N}{2}}}\right) = \left| n = 0 \dots \frac{N}{2} \right| x\left(n+\frac{N}{2}\right) \end{array}$$

$$F(1,m) = x(0,m) + x(1,m)\cdot W_2^1 \quad ; \quad \left| W_2^1 = e^{-j\frac{2\pi}{2}} = e^{-j\pi} = -1 \right|$$

$$F(1,n) = x(n) - x\left(n+\frac{N}{2}\right) \qquad 0 \leq n \leq \frac{N}{2}$$

$$G(p,m) = W_N^{pm}\cdot F(p,m)$$

$$\left.\begin{array}{l} G(0,m) = F(0,n)\cdot W_N^0 = x(n) + x\left(n+\frac{N}{2}\right) \\ G(1,m) = F(1,m)\,W_N^m = \left[x(n) - x\left(n+\frac{N}{2}\right)\right]\cdot W_N^n \end{array}\right\} \quad 0 \leq n \leq \frac{N}{2}$$

$$X(p,q) = \sum_{m=0}^{M-1} G(p,m)\cdot W_{M}^{mq} \qquad 0 \leq q \leq M-1$$

$$X(\underline{0,q}) = \sum_{n=0}^{\frac{N}{2}-1} G(0,n)\,W_{N/2}^{nq} = \qquad \left| \begin{array}{l} G(0,n) = d_1(n) \\ G(1,n) = d_2(n) \end{array} \right|$$

$$= \left| \begin{array}{l} n = 2m+1 \\ m = 0 \\ \left(\frac{N}{2}-1\right)2 = N-2 \end{array} \right| = \sum_{n=0}^{\frac{N}{2}-1} d_1(n)\cdot W_N^{2nq} = \sum_{n=0}^{N-2} d_1(n)\cdot W_N^{nq} = D_1(q) = \underline{X(2q)} \qquad \boxed{\begin{array}{l} k = \overset{0}{p} + L\cdot q \\ \quad = 2\cdot q \end{array}}$$

$$W_{N/2}^{nq} = e^{-j\frac{2\pi}{N}\cdot 2\cdot nq} = W_N^{2nq}$$

$$X(1,q) = \sum_{n=0}^{\frac{N}{2}-1} G(1,n)\,W_{N/2}^{nq} = D_2(q) = \underline{X(2q+1)}$$

$$\rightarrow \boxed{k = p + L\cdot q = 1 + 2\cdot q = 2q+1}$$

EXAMPLE 5.21 study execution time of fft for $1 \leq N \leq 2048$

FAST CONVOLUTIONS

$$N = 2^{\lceil \log_2(N_1+N_2-1) \rceil} \qquad \begin{array}{l} N_1 = \text{length}(x_1) = 4 \\ N_2 = \text{length}(x_2) = 6 \end{array} \qquad N_1+N_2-1 = 9$$

$$\lceil \log_2 9 \rceil = 4 \; ;$$

$$\boxed{N = 2^4 = 16}$$

$$x_1(n) * x_2(n) = \text{IFFT}\left[\text{FFT}\left[x_1(n)\right]\cdot \text{FFT}\left[x_2(n)\right]\right]$$

EX. 5.21 demonstrate effectiveness of high speed convolution

$x_1(n) = \text{rand}(1,L)$

$x_2(n) = \text{randn}(1,L)$

average execution time for

$1 \leq L \leq 150$

$\boxed{\text{HIGH SPEED BLOCK CONVOLUTIONS}}$

- High speed overlap-save function implementation

$\boxed{\text{PROBLEMS}}$

$\boxed{P.5.1}$ DFS = ? USING DFS DEFINITION

ⓐ $\tilde{x}_1(n) = \{2, 0, 2, 0\}$ $N = 4$

$$X_1(k) = \sum_{n=0}^{N-1} \tilde{x}_1(n)\, e^{-j\frac{2\pi}{N}kn} \qquad k = 0, \pm 1, \pm 2 \ldots \pm\infty$$

$$k=0 \quad X_1(0) = \sum_{n=0}^{3} \tilde{x}_1(n)\, \overset{\to 1}{\cancel{e^{-j\frac{2\pi}{4}\cdot 0}}} = x_1(0) + x_1(1) + x_1(2) + x_1(3) = 4$$

$$k=1 \quad X_1(1) = \sum_{n=0}^{3} \tilde{x}_1(n)\, \overset{\to -j}{\cancel{e^{-j\frac{\pi}{2}\cdot n}}} = 2\cdot\overset{\to 1}{(\cancel{-j})^0} + 0(-j)^1 + 2\cdot(-j)^2 + 0 = +\cancel{j}2 + 2(-1)$$
$$= +2 - 2\cancel{j} = 0$$

$$k=2 \quad X_1(2) = \sum_{n=0}^{3} \tilde{x}_1(n)\, e^{j\,\cancel{2}\frac{\pi}{\cancel{2}}\cdot n} = 2\cdot(-1)^0 + 0\cdot(-1)^1 + 2\cdot(-1)^2 + 0 = 2 + 2 = 4$$

$\boxed{P.5.2}$ x = ? Use IDFS DEFINITION

$$\tilde{x}(n) = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}(k)\cdot e^{+j\frac{2\pi}{N}nk} = \frac{1}{N}\sum_{n=0}^{N-1} X(k)\, W_N^{-nk} = \text{IDFS}[\tilde{X}(k)]$$

- ⓐ $\tilde{X}_1(k) = \{5, -2j, 3, 2j\}$ $N = 4$

$\boxed{n=0}$

$$\tilde{x}_1(n) = \frac{1}{4}\sum_{k=0}^{3} \tilde{X}_1(k)\, \overset{\to 1}{\cancel{e^{j\frac{\pi}{2}\cdot 0}}} = \frac{1}{4}(5 - 2j + 3 - 2j) = \frac{1}{4}(8 - 4j) =$$
$$= 2 - j$$

$$x_1(1) = \frac{1}{4}\sum_{k=0}^{3} \tilde{X}_1(k)\, e^{+j\frac{\pi}{2}k} = \frac{1}{4}\left[5\cdot 1 - 2j\cdot(+j) + 3\cdot(-1) + 2\cdot j(-j)\right] =$$

$$= \frac{1}{4}[5 + 2 - 3 + 2] = \frac{6}{4} = \frac{3}{2} = 1.5$$

$\boxed{P.5.3}$

$$x_1(n) = \begin{cases} n\cdot e^{-0.3n}, & 0 \le n \le 25 \\ 0, & 26 \le n \le 49 \end{cases} \qquad \boxed{N = 50}$$

$$x_2(n) = \begin{cases} n\, e^{-0.3n}, & 0 \le n \le 25 \\ 0, & 26 \le n \le 99 \end{cases} \qquad \boxed{N = 100}$$

$\boxed{P.5.4}$ $x_1(n)$ from problem P5.3

$$\tilde{x}_3(n) = [\tilde{x}_1(n), \tilde{x}_1(n)]_{\text{PERIODIC}} \qquad x_M(n) = [\underbrace{\tilde{x}_1(n), \ldots \tilde{x}_1(n)}_{M \text{ TIMES}}]_{\text{PERIODIC}}$$

$$\tilde{X}_M(Mk) = M\,\tilde{X}_1(k) \qquad k = 0, 1, 2 \ldots N-1$$

$$\tilde{X}_M(k) = 0 \qquad k \ne 0, M, \ldots, MN$$

$$\tilde{x}_1(k) = \sum_{n=0}^{N_1-1} \tilde{x}_1(n)\, e^{-j\frac{2\pi}{N_1}\cdot nk} \qquad k = 0, \pm 1, \pm 2 \ldots$$

$$\tilde{x}_2(k) = \sum_{n=0}^{N_2-1} \tilde{x}_2(n)\, e^{-j\frac{2\pi}{N_2} nk} = \Big/ \; N_2 = 2\cdot N_1 \; \Big/$$

$$\tilde{x}_2(k) = \sum_{n=0}^{2N_1-1} [x_1(n), x_1(n)] \cdot e^{-j\frac{2\pi}{2N_1}\cdot nk} \qquad e^{-j\frac{2\pi}{2N_1} 2nk + j\frac{2\pi}{2N_1}\cdot nk}$$

$n = [n_1, n_2]$ $\qquad n_1 = 0 \ldots N_1 - 1$

$\qquad\qquad\qquad\quad n_2 = N_1 \ldots 2N_1 - 1$

$$x_2(k) = \sum_{n=0}^{2N_1-1} [x_1(n), x_1(n)] \cdot e^{-j\frac{2\pi}{2N_1}[n_1, n_2]}$$

$$\tilde{x}_3(n) = \tilde{x}_1(n) + \tilde{x}_1(n+N)$$

$$x_3(k) = \sum_{n=0}^{2N-1} [x_1(n) + x_1(n+N)] \cdot e^{-j\frac{2\pi}{2N}\cdot nk} =$$

$$= \sum_{n=0}^{2N-1} x_1(n) \cdot e^{-j\frac{2\pi}{2N}\cdot nk} + \sum_{n=0}^{2N-1} x_1(n+N) e^{-j\frac{2\pi}{2N}\cdot n\cdot k}$$

$$= \underbrace{\sum_{n=0}^{N-1} x_1(n) \cdot e^{-j\frac{2\pi}{N}\cdot n\cdot \frac{k}{2}}}_{(*)} + \underbrace{\sum_{n=N}^{2N-1} x_1(n+N) \cdot e^{-j\frac{2\pi}{2N}\cdot n\cdot k}}_{(**)}$$

$k = 0, 1, 2, \ldots 2N-1$

$k = 1 \qquad e^{-j\frac{2\pi}{2N}\cdot n} = e^{-j\frac{\pi n}{N}}$

(*) for $k = 0, 2, \ldots 2N-2$ $\qquad$ (*) $= x_1(2k)$ $\qquad k = 0, 1, 2, 3 \ldots N-1$

$\overset{+1}{\underset{-1}{}}$

$$X_3(2k) = 2 \cdot X_1(k)$$

$$(**) = \left| \begin{array}{l} m = n - N \\ n = N \quad m = 0 \\ n = 2N-1 \quad m = N-1 \\ n+N = m+\cancel{N}+\cancel{N} = m \end{array} \right| = \sum_{m=0}^{N-1} x_1(m) \cdot e^{-j\frac{2\pi}{2N}\cdot(m+N)\cdot k} \qquad \cancel{e^{-j\pi\cdot k}}$$

$k = 2k_1$

$k = 0, 2, 4, 6$

$$e^{-j\frac{2\pi}{2N}\cdot m\cdot k} \cdot \cancel{e^{-j\frac{2\pi}{2N}\cdot N\cdot k}}$$

$\boxed{P.5.5}$ $x(n) = \{2, 5, 3, -4, -2, 6, 0, -3, -3, 2\}$

ⓐ $y_1(n) = \overset{3\ \text{points}}{IDFS}\left[X(e^{j0}), X(e^{j\frac{2\pi}{3}}), X(e^{j\frac{4\pi}{3}})\right]$

$\boxed{y_1(n) = ?}$ DTFT $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n}$

$$\boxed{y_1(n) = \sum_{r=-\infty}^{\infty} x_1(n - rN)}$$

	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12
x(n)				2	5	3	-4	-2	6	0	-3	-3	2			
x(n-3)							2	5	3	-4	-2	6	0	-3	-3	2
	2	5	3	-4	-2	6	0	-3	-3	2						
	-4	-2	6	0	-3	-3	2									
	0	-3	-3	2												
				0	0	6										

$X(0) = \sum_{n=0}^{9} x(n) = 2 + 5 + 3 - 4 - 2 + 6 + 0, -3 - 3 + 2 = 6$

$X(1) = 2 \cdot e^{-j\frac{2\pi}{3} \cdot 0} + 5 \cdot e^{-j\frac{2\pi}{3} \cdot 1} + 3 \cdot e^{-j\frac{2\pi}{3} \cdot 2} + \ldots = -3 + 5.2i$

$X(2) = 2 \cdot e^{-j\frac{4\pi}{3} 0} + 5\, e^{-j\frac{4\pi}{3} \cdot 1} + \ldots = -3 - 5.2i$

$y = [0, 0, 6]$ CHECKED WITH MATLAB USIND IDFS

$\boxed{P.5.6}$ $x(n) = \{1, 2, 3, 4, 5, 6, 6, 5, 4, 3, 2, 1\}$

$(f_s \geq 2 f_{max})$

$\frac{1}{T_s} \geq \frac{2}{T}$ $\boxed{T_s < T/2}$

ⓐ DFT = ?

$$W_N^{-kN} = \left(e^{-j\frac{2\pi}{N}}\right)^{-kN} = e^{+j2\pi \cdot k} = 1$$

$$X(z) = \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{\tilde{X}(k)}{1 - W_N^{-k} z^{-1}}$$

$$X(e^{j\omega}) = \frac{1 - e^{-j\omega N}}{N} \sum_{k=0}^{N-1} \frac{\tilde{X}(k)}{1 - W_N^{-k} e^{-j\omega}}$$

$$X(e^{j\omega}) = \sum_{k=0}^{N-1} X(k)\, \phi\left(\omega - \frac{2\pi k}{N}\right) \qquad \phi(\omega) = \frac{\sin\left(\frac{\omega N}{2}\right) \cdot e^{-j\omega\left(\frac{N-1}{2}\right)}}{N \cdot \sin\left(\frac{\omega}{2}\right)}$$

$$\phi\left(\omega - \frac{2\pi k}{N}\right) = \frac{\sin\left[\left(\omega - \frac{2\pi k}{N}\right)\frac{N}{2}\right]}{N \cdot \sin\left[\left(\omega - \frac{2\pi}{N}k\right)\frac{1}{2}\right]} \cdot e^{-j\left(\omega - \frac{2\pi k}{N}\right)\frac{N-1}{2}}$$

$N \cdot T_s = T$; $f_s \geq 2 f_{max}$ $\frac{1}{T_s} \geq \frac{2}{T}$ $T \geq 2T_s$

$N T_s \geq 2 T_s$ $\boxed{N \geq 2}$?? $\omega = \frac{2\pi}{N} \cdot k$ $\frac{\omega}{\pi} = \frac{2\pi k}{N} \cdot \frac{1}{\pi}$

$\boxed{P.5.7}$

ⓐ $x_1(n) = 2\cos(0.2\pi n)\left[u(n) - u(n-10)\right]$

ⓑ $x_2(n) = \sin(0.45\pi n) \cdot \sin(0.55\pi n)$ $0 \leq n \leq 50$

ⓒ $x_3(n) = 3 \cdot 2^n$ $-10 \leq n \leq 10$

ⓓ $x_4(n) = (-0.5)^n$, $-10 \leq n \leq 10$

ⓔ $x_5(n) = 5 \cdot (0.9 e^{j\pi/4})^n \, u(n)$

ⓑ $\cos(\alpha - \beta) = \cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta$

$\cos(\alpha + \beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta$ $-$

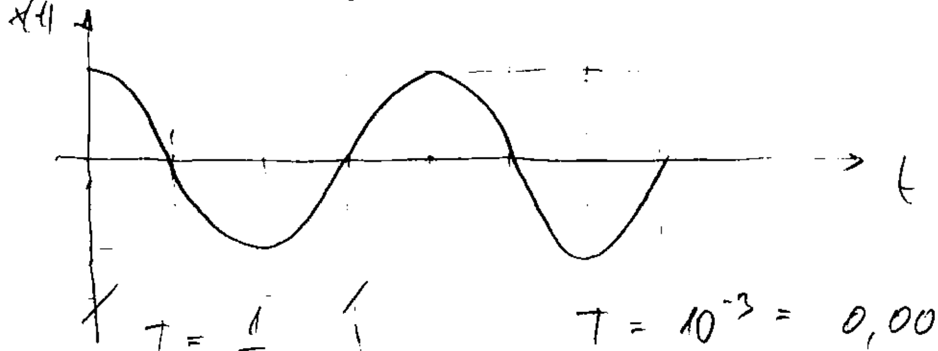
$2\sin\alpha \cdot \sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$

$\sin\alpha \cdot \sin\beta = \frac{1}{2}\left[\cos(\alpha - \beta) - \cos(\alpha + \beta)\right]$

$\boxed{(\alpha + \beta) = 0.45 + 0.55 = 1}$

$0.55 - 0.45 = 0,1$

$\omega = 2\pi f$ $f = 1000\,H$ $\omega = 2\pi \cdot 1000 \cdot t$



$T = \frac{1}{f}$ $T = 10^{-3} = 0,001 \, ms$

$t = \Delta t \cdot n$ $n = 0 \div 100$ $\Delta t = 0,001 / 100 = 10^{-3} \cdot 10^{-2} = 10^{-5}$

$2000 \cdot 10^{-5} = 2 \cdot 10^{3} \cdot 10^{-5} = 2 \cdot 10^{-2} = 0,02$

$\omega = \frac{2\pi}{N} k$ $\Omega = \omega \cdot t = \underbrace{\left(\frac{2\pi}{N} k\right)}_{\omega} \Delta t$

$k = 1$ $\frac{2\pi}{N} \cdot 1 \cdot$

$\cos(2\pi \cdot 1000 t)$ $\omega = 2\pi \cdot 1000$

$\boxed{0.55 - 045 = 0,1}$ $\omega = \frac{2\pi}{N} \cdot k$ $\boxed{\begin{array}{r} 0.55 \\ 0.45 \\ \hline 1.00 \end{array}}$

$\boxed{\frac{\omega}{\pi} = \frac{2k}{N}}$

$k = -\frac{N}{2}+1 : \frac{N}{2} = \Big| N = 100 \Big| = -49 : 50$

$50 - (-49) + 1 = 50 + 49 + 1 = 100$

ⓓ $x = (-0.5)^n \qquad -10 \le n \le 10$

ⓔ $x = 5\cdot(0.9 \cdot e^{+j\pi/4})^n u(n)$

$x = 5\cdot(0.9)^n \cdot e^{+j n\pi/4} = 5\cdot(0.9)^n \left(\cos\left(\frac{\pi}{4}n\right) + j\sin\left(\frac{\pi}{4}n\right)\right)$

$\sigma^2 = \left(\xi - \bar{\xi}\right)^2$

$x = \{127, 78, 120, 130, 95\}$

$x_i = \{132, 76, 122, 129, 91\}$

$\sigma^2 = \frac{1}{5}\left[(127-132)^2 + (78-76)^2 + (120-122)^2 + (130-129)^2 + (95-91)^2\right]$

$= \frac{1}{5}\left[25 + 4 + 4 + 1 + 16\right] = \frac{1}{5}50 = 10$

$\sigma^2 = 10 \qquad \boxed{\sigma = \sqrt{10} = 3{,}16}$

$> \sigma^2 = \frac{1}{5}\left[(132-110)^2 + (76-110)^2 + (122-110)^2 + (129-110)^2 + (91-110)^2\right]$

$\frac{1}{5}\left[22^2 + 34^2 + 12^2 + 19^2 + 19^2\right] = 501{,}2$

$\boxed{P5.8.}$ ⓐ $\mathrm{Re}\left\{H(e^{j\omega})\right\} = \sum_{k=0}^{5}(0.5)^k\cos(k\omega) =$

$= 1 + 0.5\cos\omega + 0.25\cdot\cos(2\omega) + 0.125\cdot\cos(3\omega) + 0.0625\cdot\cos(4\omega) + 0.03125\cdot\cos(5\omega)$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n)\cdot e^{-j\omega n}$$

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})\cdot e^{j\omega n}\,d\omega$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} 0.5\cos(\omega)\cdot e^{j\omega n}\,d\omega = \frac{1}{\pi(n^2-1)}\cdot\left(0.25\, I\cdot n\left(-1 + \overset{1}{\cancel{\cos(2\pi n)}}\right) + I\cdot\overset{0}{\cancel{\sin(2\pi n)}}\right)$$

$$\left(\underset{-1}{\cancel{\cos(\pi n)}} - I\underset{0}{\cancel{\sin(\pi n)}}\right) =$$

$$\frac{1}{2\pi}\int_{0}^{\pi}(0.5)^k\cdot\cos(k\omega)\,e^{j\omega n}\,d\omega = \frac{(0.5)^{k+1}\cdot i n\left(1 + e^{j\pi n}\right)}{\pi\left(n^2 - k^2\right)}$$

$k = 1, 2 .. 5$

$$\frac{1}{2\pi}\int_0^{\pi}(0.5)^k\cdot\cos(k\omega)\cdot e^{j\omega n}d\omega = \frac{(0.5)^{k+1}\cdot j\cdot n\left(1+e^{j\pi n}\right)}{\pi\left(n^2-k^2\right)} = ⊛$$

$$e^{j\pi} = \cos\pi + j\cancel{\sin\pi}^{\,0} = \boxed{-1} \quad ; \qquad \left(1+e^{j\pi n}\right) = e^{+j\frac{\pi n}{2}}\left(e^{-j\frac{\pi n}{2}} + e^{j\frac{\pi n}{2}}\right)$$

$$= \cancel{e^{j\frac{\pi n}{2}}}\,{}^{v}(j)^n \cdot 2\cdot\cos\left(\frac{\pi n}{2}\right) = \underline{\underline{2\cdot(j)^n\cos\left(\frac{\pi n}{2}\right)}}$$

$$e^{jt} = \cos t + j\sin t$$
$$e^{-jt} = \cos t - j\sin t$$

$$⊛ = \frac{(0.5)^{k+1}\cdot j\cdot n\cdot 2(j)^n\cos\left(\frac{\pi n}{2}\right)}{\pi\left(n^2-k^2\right)} = \left|\, \cancel{-\frac{(0.5)^{k+1}\,2n\cdot\cos\left(\frac{\pi n}{2}\right)}{\pi(n+k)(n-k)}} = \cancel{\frac{(0.5)^{k+1}\,2n\cos\frac{\pi n}{2}}{\left(k^2-n^2\right)}}\right.$$

$$= \frac{(0.5)^{k+1}\,2n\cdot(j)^{n+1}}{\pi\left(n^2-k^2\right)}\cdot\cos\left(\frac{\pi n}{2}\right) \qquad\qquad e^{\cdot} =$$

$$\operatorname{Re}\left\{H\left(e^{j\omega}\right)\right\} = \sum_{k=0}^{5}(0.5)^k\cdot\cos(k\omega)$$

$$H\left(e^{j\omega}\right) = \sum_{k=0}^{5}(0.5)^k\cdot e^{-j\omega\cdot k} = \sum_{k=0}^{5}(0.5)^k\cdot\left(\underbrace{\cos(\omega k)}_{\operatorname{Re}} + j\sin(\omega k)\right)$$

$$h_n = 0.5^n \qquad n = 0\ldots5$$

$$\boxed{h_n = [1, 0.5, 0.25, 0.125, 0.0625, 0.03125]}$$

TK-59M 15.12.04

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\mu = 0; \qquad \sigma = 1 \qquad f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}$$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{x^2}{2}}dx$$

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$$

$$\frac{1}{\sqrt{2\pi}}\int_{-2}^{2} e^{-\frac{x^2}{2}}dx = \frac{1}{\sqrt{2\pi}}\left[\int_0^2 e^{-\frac{x^2}{2}}dx - \underbrace{\int_{-2}^{0} e^{-\frac{x^2}{2}}dx}_{*}\right] = \frac{2}{\sqrt{2\pi}}\int_0^2 e^{-\frac{x^2}{2}}dx = \frac{1}{\sqrt{2}}\operatorname{erf}(x)$$

$$* = \left|\begin{array}{l} y = -x \\ dy = -dx \\ x = -2 \rightarrow y = 2 \\ x = 0 \quad y = 0 \end{array}\right| = -\int_2^0 e^{-\frac{y^2}{2}}dy = -\int_0^2 e^{-\frac{x^2}{2}}dx$$

BINOMIAL DISTRIBUTION

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

n - NUMBER OF TRIALS
p - PROBABILITY OF SUCCESS IN EACH TRIAL

POISSON

$$\boxed{f(x) = e^{-\lambda} \frac{\lambda^x}{x!}} \qquad P(k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

GAUSSIAN DISTRIBUTION:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$SEP = \sqrt{\frac{p(1-p)}{N}}$$

p - AVERAGE VALUE
N - SIZE OF POPULATION

$\boxed{P.5.8}$ ⓑ $\quad Im\left\{H(e^{j\omega})\right\} = \sum_{l=0}^{5} 2l \sin(l\omega) \quad ; \quad \int_{-\pi}^{\pi} H(e^{j\omega})\, d\omega = 0$

$$H(e^{j\omega}) = \sum_{l=-\infty}^{\infty} -2l \cdot e^{-j\omega l} = \sum_{l=0}^{5} -2l \left(\cos(\omega l) - j\sin(\omega l)\right) =$$

$$= -\sum_{l=0}^{5} 2l\cos(\omega l) + j \underbrace{\sum_{l=0}^{5} 2l \sin(\omega l)}_{Im\left\{H(e^{j\omega})\right\}}$$

$$H(e^{j\omega}) = \sum_{l=-2}^{2} l\, e^{-j\omega l} = \sum_{l=-2}^{2} l \cdot (\cos \omega l - j \sin \omega l) =$$

$l = -2$ — ZA PROVERKA

$$= -2 \cdot \cos(-2\omega l) + 2j\sin(-2\omega l) - \cos(-\omega l) + j\sin(-\omega l) +$$
$$2 \cdot \cos(2\omega l) - 2j\sin(2\omega l) + \cos(\omega l) - j\sin(\omega l) =$$

$$= -2\cancel{\cos}(2\omega l) - 2j\sin(2\omega l) - \cancel{\cos}(\omega l) - j\sin(\omega l)$$
$$+ 2\cancel{\cos}(2\omega l) - 2j\sin(2\omega l) + \cancel{\cos}(\omega l) - j\sin(\omega l) = -\left[2j\sin(l\omega) + 4j\sin(l\omega)\right]$$

$$\boxed{H(e^{j\omega}) = -2\sum_{l=0}^{2} l \cdot j\sin(l\omega)}$$

$$H(e^{j\omega}) = \sum_{l=-5}^{5} (-l)\, e^{-j\omega l} = 2\sum_{l=0}^{5} l \cdot j \cdot \sin(l\omega)$$

$$\underline{Re\left\{H(e^{j\omega})\right\} = \sum_{l=0}^{5} 2 \cdot l \sin(l\omega)}$$

$\boxed{P.5.9}$ $$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}nk} \quad k = 0 \ldots N-1$$ $x(n)$ – N POINT SEQUENCE

ⓐ If DFT of $X(k)$ computed to optain another N POINT SEQUENCE, show that

$$\boxed{x_1(n) = N\, x((-n))_N} \qquad W_N = e^{-j\frac{2\pi}{N}}$$

$$\sum_{m=0}^{N-1} X(m)\, e^{-j\frac{2\pi}{N}mn} = \sum_{m=0}^{N-1} X(m) \cdot W_N^{mn} = \sum_{m=0}^{N-1} \left(\sum_{n=0}^{N-1} x(n) \cdot W_N^{nk} \right) W_N^{mn} = ⊛$$

$x(n)$ finite duration sequence $n = 0, 1, \ldots N-1$

$\tilde{x}(n)$ PERIODIC SIGNAL OF PERIOD „N"

$$\tilde{x}(n) = \sum_{r=-\infty}^{\infty} x(n - rN) \qquad \tilde{x}(n) = x((n))_N = x(n \bmod N)$$

$\tilde{x}(n) = x((n))_N$ PERIODIC extension

$x(n) = \tilde{x}(n)\, R_N(n)$ WINDOW OPERATION

$$⊛ = N \cdot \sum_{n=0}^{N-1} x(n) \underbrace{\frac{1}{N} \sum_{m=0}^{N-1} W_N^{n(k+m)}}_{①} = \Bigg| \begin{matrix} W_N = e^{-j\frac{2\pi}{N}} \\ e^{-j\frac{2\pi}{N} \cdot n(k+m)} \end{matrix} = \begin{cases} 1, & (k+m) = r \cdot N \\ 0, & \text{otherwise} \end{cases}$$

$$① : \frac{1 - W_N^{N}}{1 - W_N^{\cancel{N}}} = \frac{1 - e^{-j\frac{2\pi}{\cancel{N}}\cdot\cancel{N}}}{1 - e^{-j\frac{2\pi}{N}}} = \frac{1 - 1}{1 - e^{-j\frac{2\pi}{N}}} = 0$$

$$⊛ = N \cdot \sum_{n=0}^{N-1} x(n) \sum_{r=0}^{N-1} \delta(k + m - rN) = \cancel{N} \sum_{n=0}^{N-1} \sum_{r=0}^{N-1} x(n) \cdot \delta(k + m - rN)$$

$$= N \cdot \sum_{n=0}^{N-1} x(n - rN) = N\Big(x(n) + x(n-N) + x(n-2N) \ldots x(n - N \cdot N) \Big)$$

$$x_1 = N \cdot \tilde{x}(n) = N \cdot x((n))_N$$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk} \quad k = 0, 1, \ldots N-1 \qquad \tilde{X}(k) = X((k))_N \qquad X(k) = \tilde{X}(k)\, R_N(n)$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{nk} \quad n = 0, 1, \ldots N-1$$

Intellicom

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-nk} \qquad n = 0, 1, \ldots N-1$$

046 262 040

$\hookrightarrow$ VITE

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk}$$

$$x_1(n) = \text{DFT}\left[X(k)\right] = \sum_{k=0}^{N-1} X(k) \cdot W_N^{nk} \qquad n = 0, 1, 2 \ldots N-1$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-nk} \qquad n = 0, 1, 2, \ldots N-1 \qquad x(-n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{nk} \qquad n = 0, -1, -2, \ldots -N+1$$

$$x((-n))_N = x(\text{mod}(-n, N)) = x(-n \bmod N) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{nk} \qquad n = 0, N-1, \ldots 3, 2, 1$$

$$\text{mod}(-n, N) = 0, 1, 2 \ldots N-1$$

$$x((-n))_N = \frac{1}{N}\, x_1(n) \Rightarrow \boxed{x_1(n) = N \cdot x((-n))_N}$$

CHECKED IN MATLAB js_pr_5_09_fold.m

MMV

$$\boxed{\text{DFT}\left[x((-n))_N\right] = X((-k))_N = \begin{cases} X(0), & k = 0 \\ X(N-k), & k = 1, 2, \ldots N-1 \end{cases}}$$

- $\text{DFT}\left[x^*(n)\right] = X^*((-k))_N$ conjuction property
- $x(n)$ - real valued N-point sequence

$$X(k) = X^*((-k))$$

$$X\left(\frac{N}{2}\right) = X^*\left(\left(-\frac{N}{2}\right)\right)_N = X^*\left(\frac{N}{2}\right) \quad ; \quad k = \frac{N}{2} \qquad \omega = \frac{2\pi}{\cancel{N}} \cdot \frac{\cancel{N}}{2} = \pi$$

DIGITAL NYQUIST FREQUENCY

$$x_{ec}(n) \triangleq \frac{1}{2}\left[x(n) + x((-n))_N\right] = \begin{cases} x(0) & n = 0 \\ \left[x(n) + x(N-n)\right] \cdot \frac{1}{2} & n = 1, 2 \ldots N-1 \end{cases}$$

$$x_{oc}(n) \triangleq \frac{1}{2}\left[x(n) - x((-n))_N\right] = \begin{cases} x(0) & n = 0 \\ \left[x(n) - x(N-n)\right]\frac{1}{2} & n = 1, 2, \ldots N-1 \end{cases}$$

$$\text{DFT}\left[x_{ec}(n)\right] = \text{Re}\left[X(k)\right] = \text{Re}\left[X((-k))_N\right]$$

$$\text{DFT}\left[x_{oc}(n)\right] = \text{Im}\left[X(k)\right] = -\text{Im}\left[X((-k))_N\right]$$

$$x(n) = x_e(n) + x_o(n)$$

$$x_o(n) = -x_o(-n) \quad ; \quad x_e(n) = x_e(-n)$$

$$x_e(n) = \frac{1}{2}\left[x(n) + x(-n)\right] \qquad x_o(n) = \frac{1}{2}\left[x(n) - x(-n)\right]$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} = \left/ \begin{matrix} x(n) = x_I + j x_R \\ e^{-j\omega n} = \cos(\omega n) - j \sin(\omega n) \end{matrix} \right/ =$$

$$= \sum_{n=-\infty}^{\infty} (x_R + j x_I)(\cos(\omega n) - j \sin(\omega n))$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_R \cdot \cos(\omega n) - j x_R \sin(\omega n) + j x_I \cos(\omega n) + x_I \sin(\omega n) =$$

$$= \sum_{n=-\infty}^{\infty} \underbrace{\left[x_R \cdot \cos(\omega n) + x_I \sin(\omega n)\right]}_{X_R(\omega)} - j \underbrace{\left[x_R \cdot \sin(\omega n) - x_I \cos(\omega n)\right]}_{X_I(\omega)}$$

$$x(n) = x_e(n) + x_o(n)$$
$$x(-n) = x_e(-n) + x_o(-n)$$

$$X_R(\omega) = \sum_{n=-\infty}^{\infty} x_R(n) \cdot \cos(\omega n) + x_I \sin(\omega n)$$
$$X_I(\omega) = -\sum_{n=-\infty}^{\infty} x_R \sin(\omega n) - x_I \cos(\omega n)$$

$x(n)$ - REAL SEQUENCE $\quad x_R(n) = x(n) \quad x_I(n) = 0$

$$\boxed{X_R(\omega) = \sum_{n=-\infty}^{\infty} x_R(n) \cdot \cos(\omega n) \qquad X_I(\omega) = -\sum_{n=-\infty}^{\infty} x_R(n) \cdot \sin(\omega n)}$$

$$X_R(-\omega) = X_R(\omega) \qquad \cos(-\omega n) = \cos(\omega n) \quad \text{even}$$
$$X_I(-\omega) = -X_I(\omega) \qquad \sin(-\omega n) = -\sin(\omega n) \quad \text{odd}$$

$$\boxed{X^*(\omega) = X(-\omega)}$$

1° $x(n)$ Real & even $\Rightarrow x(-n) = x(n)$

$$\boxed{X_R(\omega) = x(0) + 2\sum_{n=1}^{\infty} x(n) \cdot \cos(\omega n)}$$
$$X_I(\omega) = 0$$

$$x(-n) = x(n)$$
$$\cos(-\omega n) = \cos(\omega n)$$
$$x(n) = \frac{1}{\pi}\int_0^{\pi} X_R(\omega)\cos(\omega n)\,d\omega$$

2° $x(n)$ Real & Odd $\Rightarrow x(-n) = -x(n)$

$$X_R(\omega) = \underbrace{\sum_{n=-\infty}^{-1} x_R(n) \cdot \cos(\omega n)}_{*} + \cancel{x(0)} \rightarrow 0 + \sum_{n=1}^{\infty} x_R(n)\cos(\omega n)$$

$$* = \left| \begin{matrix} m = -n \\ n = -\infty \quad m = \infty \\ n = -1 \quad m = 1 \end{matrix} \right| = \sum_{m=1}^{\infty} \underset{-x_R(m)}{\underset{\parallel}{x_R(-m)}} \, \underset{+\cos(\omega m)}{\underset{\parallel}{\cos(-\omega m)}} = -\sum_{n=1}^{\infty} x_R(n) \cos(\omega n)$$

$$X_R(\omega) = -\sum_{n=1}^{\infty} x_R(n) \cdot \cos(\omega n) + \sum_{n=1}^{\infty} x_R(n) \cdot \cos(\omega n) = 0$$

$$\boxed{X_I(\omega) = -2\sum_{n=1}^{\infty} x(n) \cdot \sin(\omega n)}$$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Big[\underbrace{X_R(\omega)\cos(\omega n)}_{\text{even}} - \underbrace{X_I(\omega) \cdot \sin(\omega n)}_{\text{even}}\Big] d\omega =$$

$$x(n) = \frac{1}{\pi}\int_0^{\pi} \left[X_R(\omega)\cos(\omega n) - X_I(\omega)\sin(\omega n)\right] d\omega$$

$$\boxed{x(n) = -\frac{1}{\pi}\int_0^{\pi} X_I(\omega)\sin(\omega n)\,d\omega}$$

- circulary even

$$x(N-n) = x(n) \qquad 1 \leq n \leq N-1$$

- circulary odd

$$x(N-n) = -x(n) \qquad 1 \leq n \leq N-1$$

- time reversal

$$x((-n))_N = x(N-n) \quad , \quad 0 \leq n \leq N-1$$

- Definition of even & odd sequences for the associated periodic sequence $x_p(n)$ is given:

even: $x_p(n) = x_p(-n) = x_p(N-n)$

odd: $x_p(n) = -x_p(-n) = -x_p(N-n)$

- If periodic sequence is complex valued we have:

conjugate even: $x_p(n) = x_p^*(N-n)$

conjugate odd: $x_p(n) = -x_p^*(N-n)$

$$x_p(n) = x_{pe}(n) + x_{po}(n)$$

$$x_{pe}(n) = \frac{1}{2}\left[x_p(n) + x_p^*(N-n)\right]$$

$$x_{po}(n) = \frac{1}{2}\left[x_p(n) - x_p^*(N-n)\right]$$

- $$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}nk} \quad k = 0,1,\dots N-1 \qquad x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi}{N}nk}$$

$$x(n) = x_R(n) + j\,x_I(n) \qquad X(k) = X_R(k) + j\,X_I(k)$$

$$X(k) = \sum_{n=0}^{N-1}\left[x_R(n) + j\,x_I(n)\right]\left[\cos\left(\frac{2\pi kn}{N}\right) - j\sin\left(\frac{2\pi kn}{N}\right)\right] =$$

$$= \sum_{n=0}^{N-1} x_R(n)\cos\left(\frac{2\pi kn}{N}\right) - \underline{j\,x_R(n)\cdot\sin\left(\frac{2\pi kn}{N}\right)} + j\,x_I(n)\cos\left(\frac{2\pi kn}{N}\right) + x_I(n)\cdot\sin\left(\frac{2\pi kn}{N}\right)$$

$$\boxed{\begin{aligned} X_R(k) &= \sum_{n=0}^{N-1} x_R(n)\cos\left(\frac{2\pi kn}{N}\right) + x_I(n)\cdot\sin\left(\frac{2\pi kn}{N}\right) \\ X_I(k) &= -\sum_{n=0}^{N-1} x_R(n)\sin\left(\frac{2\pi kn}{N}\right) - x_I(n)\cdot\cos\left(\frac{2\pi kn}{N}\right) \end{aligned}}$$

① 5.2.20

② 5.2.21

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left[X_R(k) + j\,X_I(k)\right]\left[\cos\left(\frac{2\pi kn}{N}\right) + j\sin\left(\frac{2\pi kn}{N}\right)\right]$$

$$\boxed{\begin{aligned} x_R(n) &= \frac{1}{N}\sum_{k=0}^{N-1} X_R(k)\cos\left(\frac{2\pi kn}{N}\right) - X_I(k)\cdot\sin\left(\frac{2\pi kn}{N}\right) \\ x_I(n) &= \frac{1}{N}\sum_{k=0}^{N-1} X_R(k)\cdot\sin\left(\frac{2\pi kn}{N}\right) + X_I(k)\cdot\cos\left(\frac{2\pi kn}{N}\right) \end{aligned}}$$

③ 5.2.22

④ 5.2.23

• REAL VALUED SEQUENCE:

$$X(k)=\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi kn}{N}} \quad k=0,1,2,\dots N-1 \quad = \underbrace{\sum_{n=0}^{N-1} x(n)\cdot\cos\frac{2\pi kn}{N}}_{X_R} \underbrace{- j\, x(n)\cdot\sin\frac{2\pi kn}{N}}_{X_I}$$

$$X(-k)=\sum_{n=0}^{N-1} x(n)\cos\frac{2\pi kn}{N} + j\, x(n)\cdot\sin\frac{2\pi kn}{N} = X^*(k)$$

$$X(N-k)=\sum_{n=0}^{N-1} x(n)\, \overset{1}{\cancel{e^{-j\frac{2\pi Nn}{N}}}}\, e^{j\frac{2\pi kn}{N}} = \sum_{n=0}^{N-1} x(n)\, e^{j\frac{2\pi kn}{N}} = X^*(k)$$

$k=0,1,2,\dots N-1$

$$\boxed{\boxed{X(N-k)=X(-k)=X^*(k)} \qquad \boxed{X(N-k)=X((-k))_N}}$$

$$|X(N-k)|=|X(k)| \qquad \measuredangle X(N-k)=-\measuredangle X(k) \qquad x_I=0$$

$$① \Rightarrow x(n)=x_R(n)=\frac{1}{N}\sum_{k=0}^{N-1} X_R(k)\cos\frac{2\pi kn}{N} - \frac{1}{N}\sum_{k=0}^{N-1} X_I(k)\cos\frac{2\pi kn}{N}$$

• REAL & EVEN: $x(n)=x(N-n)$ $\quad 0\le n\le N-1$

$$\boxed{X(k)=X_R(k)=\sum_{n=0}^{N-1} x(n)\cdot\cos\left(\frac{2\pi kn}{N}\right) \quad k=0,1,\dots N-1}$$

$X_I(k)=0$

even

$$\boxed{x(n)=x_R(n)=\frac{1}{N}\sum_{k=0}^{N-1} X(k)\cdot\cos\left(\frac{2\pi kn}{N}\right) \quad n=0,1,2\dots N-1}$$

ИМАМЕ even + odd И ПРИМЕРОЦИТЕ ДО $N/2$ СЕ ТОА СЛУЧАЈ СО ОНИЕ ОД $N/2 \div N$!!!

MMV

$$\boxed{\begin{aligned} X_R(k) &= \sum_{n=0}^{N-1} x_R(n)\cos\frac{2\pi kn}{N} \quad \text{even}\cdot\text{even} \\ X_I(k) &= -\sum_{n=0}^{N-1} x_R(n)\sin\frac{2\pi kn}{N} \quad \text{odd}\cdot\text{odd} \end{aligned}}$$

odd

• REAL & ODD $\quad x(n)=-x(N-n) \Rightarrow$ [illegible]$(k)=0$

$$\boxed{X(k)=-j\sum_{n=0}^{N-1} x(n)\sin\left(\frac{2\pi kn}{N}\right); \quad k=0,1,\dots N-1 \qquad x(n)=j\frac{1}{N}\sum_{k=0}^{N-1} X(k)\sin\left(\frac{2\pi kn}{N}\right) \quad n=0,1,\dots N-1}$$

• purely imaginary sequence, $\quad x(n)=j\,x_I(n) \quad x_R(n)=0$

$$\boxed{\begin{aligned} X_R(k) &= \sum_{n=0}^{N-1} x_I(n)\cdot\sin\left(\frac{2\pi kn}{N}\right) & x_I(n) \text{ odd} \quad X_I(k)=0 \\ X_I(k) &= \sum_{n=0}^{N-1} x_I(n)\cdot\cos\left(\frac{2\pi kn}{N}\right) & x_I(n) \text{ even} \quad X_R(k)=0 \end{aligned}}$$

$$\boxed{\begin{aligned} x(n) &= x_R^e(n)+x_R^o(n)+j\,x_I^e(n)+j\,x_I^o(n) \\ X(k) &= X_R^e(k)+X_R^o(k)+j\,X_I^e(k)+j\,X_I^o(k) \end{aligned}}$$

$\boxed{P.5.10}$ $x(n)$ complex valued N-point sequence

$$x_{ec}(n) \triangleq \frac{1}{2}\left[x(n) + x^*((-n))_N\right]$$

$$x_{oc}(n) \triangleq \frac{1}{2}\left[x(n) - x^*((-n))_N\right]$$

$$\text{DFT}\left[x_{ec}(n)\right] = \text{Re}\left[X(k)\right] = \text{Re}\left[X((-k))_N\right]$$

$$\text{DFT}\left[x_{oc}(n)\right] = j\,\text{Im}\left[X(k)\right] = -j\,\text{Im}\left[X((-k))_N\right]$$

$$\boxed{\begin{array}{l}\text{DFT}[x((-n))_N] \\ = X((-k))_N\end{array}}$$

$$\text{DFT}\left[x^*(n)\right] = X^*((-k))_N$$

IF $x(n)$ – REAL THEN

$$x^*(n) = x(n) \Rightarrow$$

$$\text{DFT}\left[x(n)\right] = X^*((-k))_N$$

$$X(k) = X^*((-k))_N$$

x(n) REAL

$$\boxed{\begin{array}{ll} X(k) = X^*((-k))_N = X^*(N-k) & |X(k)| = |X((-k))_N| \\ \text{Re}[X(n)] = \text{Re}[X((-k))_N] & \measuredangle X(k) = -\measuredangle X((-k))_N \\ \text{Im}[X(k)] = -\text{Im}[X((-k))_N] & \end{array}}$$

$\boxed{\text{MMV}}$

SYMETRY PROPERTIES

$$\text{DFT}\left[x_{ec}(n)\right] = \frac{1}{2}\left[\underbrace{\sum_{n=0}^{N-1} x(n)\cdot e^{-j\frac{2\pi k n}{N}}}_{X(k)} + \underbrace{\sum_{n=0}^{N-1} x^*((-n))\, e^{-j\frac{2\pi k n}{N}}}_{\circledast =}\right] =$$

$$= \left|\begin{array}{l} \underbrace{\text{DFT}\left[x^*((+n))\right] = \underline{\underline{X^*((-k))_N}}}_{\text{PROPERTY 3 (conjugation)}} = X(k) \\ \text{DFT}\left[x((-n))\right] = X((-k))_N \\ X^*(k) = X((-k))_k \end{array}\right|$$

$$\circledast = \left(\sum_{n=0}^{N-1} x((-n))\, e^{j\frac{2\pi k n}{N}}\right)^* = \left(\underbrace{\sum_{n=1}^{N-1} x(N-n)\, e^{j\frac{2\pi k n}{N}}}_{(*1)}\right)^* + \left(x(0)\, e^{j\frac{2\pi k \cdot 0}{0}}\right)^*$$

$$\underbrace{\overline{(a+jb)}\cdot(a+jb)}_{(**)} = \underbrace{\overline{(a+jb)\,\overline{(c+jb)}}}_{(***)}$$

$$(**) = ac + jab - jbc + b^2; \quad (***) = \overline{ac - jab + jbc + b^2} = ac + jab - jbc + b^2$$

$$(*1) = \sum_{n=1}^{N-1} x(N-n)\, e^{j\frac{2\pi k n}{N}} = \left|\begin{array}{l} m = N-n \\ n=1 \quad m = N-1 \\ n = N-1 \quad m = 0 \end{array}\right| = \sum_{m=N-1}^{1} x(m)\cdot \underbrace{\cancel{e^{j\frac{2\pi k N}{N}}}}_{1}\cdot e^{-j\frac{2\pi k m}{N}}$$

$$= \sum_{n=1}^{N-1} x(n)\cdot e^{-j\frac{2\pi k n}{N}}$$

$$\circledast = \left(\sum_{n=1}^{N-1} x(n)\, e^{-j\frac{2\pi k n}{N}}\right)^* + x^*(0) = \left(\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi k n}{N}}\right)^* = X^*(k)$$

$$\text{DFT}\left[x_{ec}(n)\right] = \frac{1}{2}\left[X(k) + X^*(k)\right] = \frac{1}{2}\left[X_R + \cancel{jX_I} + X_R - \cancel{jX_I}\right] = \underline{\underline{X_R(k)}}$$

$$DFT[x_{oc}(n)] = \frac{1}{2}\left[\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}} - \underbrace{\sum_{n=0}^{N-1} x^*((-n))e^{-j\frac{2\pi kn}{N}}}_{X^*(k)}\right] = \frac{1}{2}[X(k) - X^*(k)] = jX_I(k)$$

$$\boxed{DFT[x^*((-n))_N] = X^*(k)}$$

NEW PROPERTY

$$\boxed{DFT[x^*(n)] = X^*((-n))_N}$$

OLD PROPERTY

$\boxed{P\boxed{5.11}}$ 8-point DFT $[x(n)]$: $\{0.25, 0.125 - j0.3, 0, 0.125 - j0.6, 0.5\}$

DFT = ?

ⓐ $x_1(n) = x((2-n))_8$

ⓑ $x_2(n) = x((n+5))_{10}$

ⓒ $x_3(n) = x^2(n)$

ⓓ $x_4(n) = x(n) \textcircled{8} x((-n))_8$

ⓔ $x_5(n) = x(n)e^{j\pi n/4}$

$x(n)$ – N POINT SEQUENCE

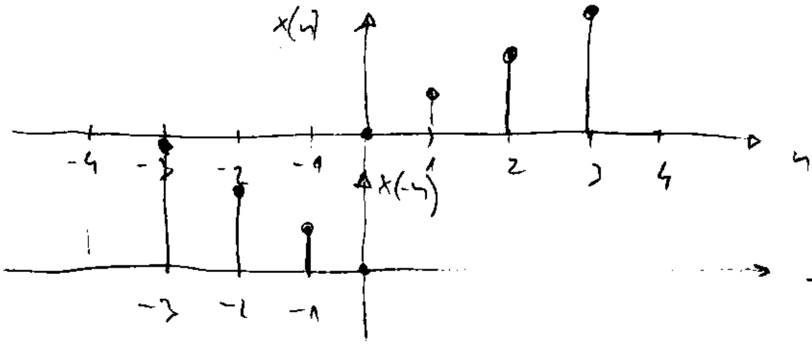
- convert to periodic $\tilde{x}(n)$ sequence

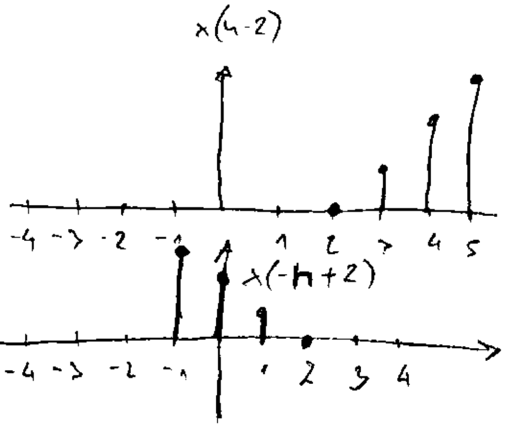
$\tilde{x}(n-m) = x((n-m))_N$

$\tilde{x}(n-m) \cdot R_N(n) = x((n-m))_N \, R_N(n) \Rightarrow$ circshif of $x(n)$

$DFT[x((n-m))_N R_N(n)] = W_N^{kn} X(k)$

$$\boxed{DFT[x((-n))_N] = X((-k))_N}$$





$$x_1(n) \textcircled{N} x_2(n) = \sum_{m=0}^{N-1} x_1(m) \cdot x_2((n-m))_N \qquad n = 0, 1, \dots N-1$$

$$\boxed{DFT[x_1(n) \textcircled{N} x_2(n)] = X_1(k) . X_2(k)}$$

$$DFT[x_1(n) . x_2(n)] = \frac{1}{N} X_1(k) \textcircled{N} X_2(k)$$

ⓓ $DFT[x_4(n)] = DFT[x(n) \textcircled{8} x((-n))_N] = X(k) \cdot X((-k))_N$

ⓔ $DFT[W_N^{-ln} x(n)] = X((k-l))_N R_N(k)$

$W_N^{kn} = e^{-j\frac{2\pi kn}{N}}$

$W_N^{ln} = e^{-j\frac{2\pi ln}{N}}$

$e^{j\frac{2\pi n}{8}} \Rightarrow \boxed{k = 1}$

$\boxed{P.5.12}$ $X(k)$ DFT of N-point complex valued sequence

$$x(n) = x_R(n) + j\,x_I(n) \qquad\qquad x(n) = x_{ec}(n) + x_{oc}(n)$$

$$X_R(k) \triangleq \text{DFT}[x_R(n)] = X_{ec}(k)$$

$$j\,X_I(k) \triangleq \text{DFT}[x_I(n)] = X_{oc}(k)$$

ⓐ $$X(k) = \sum_{n=0}^{N-1} [x_R(n) + j x_I(n)]\, e^{-j\frac{2\pi k}{N}\cdot n} = \sum_{n=0}^{N-1} (x_R(n) + j x_I(n))\left(\cos\frac{2\pi nk}{N} - j\sin\frac{2\pi nk}{N}\right)$$

$$= \sum_{n=0}^{N-1} x_R(n)\cos\left(\frac{2\pi nk}{N}\right) + x_I \sin\left(\frac{2\pi nk}{N}\right) + j\left[x_I(n)\cos\left(\frac{2\pi nk}{N}\right) - x_R \sin\left(\frac{2\pi nk}{N}\right)\right]$$

$$\text{DFT}[x(n)] = \text{DFT}[x_R(n)] + j\,\text{DFT}[x_I(n)]$$

$$X_R(k) = \sum_{n=0}^{N-1} x_R(n)\cdot\cos\left(\frac{2\pi nk}{N}\right) + x_I\cdot\sin\left(\frac{2\pi nk}{N}\right)$$

$$X_I(k) = -\sum_{n=0}^{N-1} x_R(n)\cdot\sin\left(\frac{2\pi nk}{N}\right) - x_I\cdot\cos\left(\frac{2\pi nk}{N}\right)$$

$$X(k) = \sum_{n=0}^{N-1} (x_{ec} + x_{oc})\left(\cos\frac{2\pi nk}{N} - j\sin\left(\frac{2\pi nk}{N}\right)\right)\Bigg] = \qquad \nabla 0$$

$$= \sum_{n=0}^{N-1} x_{ec}\cdot\cos\frac{2\pi nk}{N} + \overset{\nabla 0}{\cancel{x_{oc}\cos\left(\frac{2\pi nk}{N}\right)}} - j\left[x_{ec}\sin\left(\frac{2\pi nk}{N}\right) - x_{oc}\sin\left(\frac{2\pi nk}{N}\right)\right]$$

$$= \underbrace{\sum_{n=0}^{N-1} x_{ec}\cos\left(\frac{2\pi nk}{N}\right)}_{X_R} + \underbrace{j\,x_{oc}\sin\left(\frac{2\pi nk}{N}\right)}_{X_I}$$

$$x(n) = \frac{1}{N}\sum_{n=0}^{N-1} (X_R(k) + j X_I(k))\left(\cos\frac{2\pi nk}{N} + j\sin\frac{2\pi nk}{N}\right)$$

$$x(n) = \frac{1}{N}\sum_{n=0}^{N-1} \left[X_R(k)\cdot\cos\left(\frac{2\pi nk}{N}\right) - X_I(k)\sin\left(\frac{2\pi nk}{N}\right)\right] + j\left[X_R(k)\sin\left(\frac{2\pi nk}{N}\right) + X_I(k)\cos\frac{2\pi nk}{N}\right]$$

$$x_R(n) = \frac{1}{N}\sum_{n=0}^{N-1} X_R(k)\cdot\cos\left(\frac{2\pi nk}{N}\right) - X_I(k)\sin\left(\frac{2\pi nk}{N}\right)$$

$$x_I(n) = \frac{1}{N}\sum_{n=0}^{N-1} X_R(k)\sin\left(\frac{2\pi nk}{N}\right) + X_I(k)\cos\left(\frac{2\pi nk}{N}\right)$$

$x_{ec}(n) = \frac{1}{2}\left[x(n) + x^*((-n))_N\right]$

$x_{oc}(n) = \frac{1}{2}\left[x(n) - x^*((-n))_N\right]$

$\cancel{X((-k))_N = X^*(k)}$

$\cancel{X^*((-k))_N = X(k)}$

$\circ\ \mathrm{DFT}[x_{ec}(n)] = \frac{1}{2}\left\{\mathrm{DFT}[x(n)] + \mathrm{DFT}[x^*((-n))_N]\right\} = \cancel{\frac{1}{2}\left[X(k) + X((-k))_N\right]}$

$= \frac{1}{2}\left[X(k) + X^*(k)\right] = X_R(k) \Rightarrow \underline{X_R(k) = \mathrm{DFT}[x_R(n)] = X_{ec}(k)}$

$\circ\ \mathrm{DFT}[x_{oc}(n)] = \frac{1}{2}\left\{\mathrm{DFT}[x(n)] - \mathrm{DFT}[x^*((-n))_N]\right\} = \frac{1}{2}\left[X(k) - X^*(k)\right]$

$= jX_I(k) \Rightarrow \underline{jX_I(k) = j\,\mathrm{DFT}[x_I(n)] = X_{oc}(k)}$

ⓒ $x_1(n) = \cos(0.25\pi n)$; $x_2(n) = \sin(0.75\pi n)$; $0 \le n \le 63$

$x = x_1 + jx_2$

gpedit.msc

$\boxed{\mathrm{DFT}[x^*(n)] = X^*((-k))_N}$; $\mathrm{DFT}[x^*((-n))_N] = X^*(k)$

$X(k) = X^*((-k))_N$; $X^*(k) = X((-k))_N$;

DOKAZANI SE EKSPERIME-NTALNO VO MATLAB,

js-pr-5-12-symetry-conjugation

P.5.13 circular shift using frequency domain approach

$x((n-m))_N$; given N_1-point sequence $x(n)$ $N_1 \le N$

$\boxed{\mathrm{DFT}[x((n-m))_N R_N(n)] = W_N^{mk} X(k)}$ $\mathrm{DFT}[x(-n-m) R_N(n)] = W_N^{mk} X((-k))_N$

$\boxed{x_1(n) = 11 - n \quad 0 \le n \le 10}$

$|a \cdot e^{j\alpha} + b e^{j\beta}|^2 =$

$|a\cdot\cos\alpha + ja\sin\alpha + b\cdot\cos\beta + jb\sin\beta|^2$

$(a\cos\alpha + b\cos\beta)^2 + (a\sin\alpha + b\sin\beta)^2 =$

$a^2\cos^2\alpha + 2ab\cos\alpha\cos\beta + b^2\cos^2\beta +$

$a^2\sin^2\alpha + 2ab\sin\alpha\sin\beta + b^2\sin^2\beta = ⊛$

P.5.14 $\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{+j\frac{2\pi nk}{N}}$$

$$\sum_{n=0}^{N-1}\left|\frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi nk}{N}}\right|^2 = \frac{1}{N^2}\sum_{n=0}^{N-1}\left|\sum_{k=0}^{N-1} X(k)\left(\cos\left(\frac{2\pi nk}{N}\right) + j\cos\left(\frac{2\pi nk}{N}\right)\right)\right|^2$$

$|a + jb + c + jd|^2 = (a+b)^2 + (b+d)^2$

$|a + jb|^2 + |c + jd|^2 = a^2 + b^2 + c^2 + d^2$

$$\circledast = \left| a e^{j\alpha} + b \cdot e^{j\beta} \right|^2 = a^2 + b^2 + 2ab(\cos\alpha \cos\beta + \sin\alpha \sin\beta) =$$

$$= a^2 + b^2 + 2ab \cdot \cos(\alpha - \beta)$$

$$① = \frac{1}{N^2} \sum_{n=0}^{N-1} \left| \sum_{k=0}^{N-1} x(k) \left[\cos\left(\frac{2\pi nk}{N}\right) + j \sin\left(\frac{2\pi nk}{N}\right) \right] \right|^2 =$$

$$= \frac{1}{N^2} \sum_{n=0}^{N-1} \left| \sum_{k=0}^{N-1} x(k) \cos\left(\frac{2\pi nk}{N}\right) + j \sum_{k=0}^{N-1} x(k) \sin\left(\frac{2\pi nk}{N}\right) \right|^2 =$$

$$= \frac{1}{N^2} \sum_{n=0}^{N-1} \left[\left[\sum_{k=0}^{N-1} x(k) \cos\left(\frac{2\pi nk}{N}\right) \right]^2 + \left[\sum_{k=0}^{N-1} x(k) \cdot \sin\left(\frac{2\pi nk}{N}\right) \right]^2 \right]$$

$$\sum_{n=0}^{N} q^n = \frac{1 - q^{N+1}}{1 - q}$$

Circular Correlation:

$$x(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X(k)$$

$$y(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} Y(k)$$

$$r_{xy}(l) = \sum_{n=0}^{N-1} x(n) \cdot y^*((n-l))_N$$

$$r_{xy}(l) = x(l) \, Ⓝ \, y(-l) \Rightarrow \left| \; DFT\left[x^*((-n))_N\right] = X^*(k) \; \right| \Rightarrow \underline{\underline{R_{xy}(k) = X(k) \cdot Y^*(k)}}$$

$$\text{IF: } \quad y(n) = x(n) \quad \Rightarrow \quad r_{xx}(l) \overset{DFT}{\underset{N}{\longleftrightarrow}} R_{xx}(k) = |X(k)|^2$$

$$\boxed{\text{Parseval's theorem}}$$

$$\sum_{n=0}^{N-1} x(n) \cdot y^*(n) = \frac{1}{N} \sum_{n=0}^{N-1} X(k) \cdot Y^*(k)$$

$$\sum_{n=0}^{N-1} x(n) y^*(n) = \underline{r_{xy}(0)} \; ; \quad \underline{\underline{r_{xy}(l)}} = \frac{1}{N} \sum_{k=0}^{N-1} R_{xy}(k) e^{j\frac{2\pi kl}{N}} =$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) Y^*(k) e^{j\frac{2\pi kl}{N}} \; ;$$

$$\Swarrow$$

$$r_{xy}(0) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) Y^*(k) = \sum_{n=0}^{N-1} x(n) y^*(n)$$

If: $y(n) = x(n) \Rightarrow \sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} [x_1\ x_2\ x_3]$

$\boxed{P.5.15}$ $\quad DFT[x(n) Ⓝ y(n)] = X(k) \cdot Y(k)$

$\boxed{P.5.18}$ Compute N-point circular convolution:

ⓐ $x_1(n) = \{1,1,1,1\}$, $x_2(n) = \cos(n\pi/4) R_N(n)$; $N=8$
ⓑ $x_1(n) = \cos(2\pi n/N) R_N(n)$, $x_2(n) = \sin(2\pi n/N) R_N(n)$; $N=32$

ⓒ $x_1(n) = (0.8)^n R_N(n)$, $x_2(n) = (-0.8)^n R_N(n)$; $N=20$
ⓓ $x_1(n) = n R_N(n)$, $x_2(n) = (N-n) R_N(n)$; $N=10$
ⓔ $x_1(n) = \{1,-1,1,-1\}$, $x_2(n) = \{1,0,-1,0\}$; $N=4$

$\boxed{P.5.19}$ Compute: (i) N point circular convolution $x_3 = x_1 Ⓝ x_2$
(ii) linear convolution: $x_4 = x_1 * x_2$
(iii) error sequence: $e(n) = x_3(n) - x_4(n)$

verify that: $\boxed{e(n) = x_4(n+N)}$

BLOCK CONVOLUTIONS

$\hat{x}(n) = \{\underbrace{0, 0 \ldots, 0}_{M-1 \text{ zeros}}, x(n)\}$, $n \geq 0$, $L = N - M + 1$

$\boxed{x_k(n) = \hat{x}(m);\ kL \leq m \leq kL + N - 1}$ $\quad k \geq 0;\quad 0 \leq n \leq N-1$

$K = \left\lfloor \frac{N_x + M - 2}{L} \right\rfloor + 1 \qquad N_x = \text{length}(x(n))$

$$\tilde{x} = \{\underbrace{\overset{0}{0}, \overset{1}{0}, \overset{2}{1}, \overset{3}{2}, \overset{4}{3}, \overset{5}{4}}_{x_1}, \ldots\} \quad \tilde{x} = \{0,0,1,2,\underbrace{3,4,5,6,7,8}_{x_2}, \ldots\} \quad \tilde{x} = \{\ldots, \underbrace{7,8,9,10,0,0}_{x_3}\}$$

$\tilde{x} = \{0, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 0, 0\}$ (indices $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13$)

$h = \{1, 0, -1\}$

$N_x = 10$
$N = 6,$
$M = 3 \quad M1 = M-1 = 2$
$L = N - M + 1 = 4$

$x_k = \hat{x}(k \cdot L : k \cdot L + N - 1)$
$x_1 = \{0, 0, 1, 2, 3, 4\}$
$x_2 = \{4, 5, 6, 7, 8, 9\}$
$x_3 = \{7, 8, 9, 10, 0, 0\}$

— segments

$\boxed{P5.20}$ OVERLAP-ADD METOD

ⓐ $x(n)$ - long sequence of length $N_x = M \cdot L$ $M, L \gg 1$

$\{x_m(n),\ m = 1, \dots, M\}$ length$(x_m\} = L$

$$x_m(n) = \begin{cases} x(n), & mL \le n \le (m+1)L - 1 \\ 0, & \text{elsewhere} \end{cases} \qquad x(n) = \sum_{m=0}^{M-1} x_m(n)$$

$h(n)$ - L point impulse response

$$y(n) = x(n) * h(n) = \sum_{m=0}^{M-1} x_m(n) * h(n) = \sum_{m=0}^{M-1} y_m(n); \quad y_m(n) = x_m(n) * h_m(n)$$

$y_m(n) \Rightarrow 2L-1$ - point sequence $\boxed{N \ge 2L-1}$

$x_1 = \{1, 2, 3, 4\}$
$x_2 = \{1, 0, -1\}$

$$N_x + L - 1 - (2L - 1) = N_x + L \cancel{-1} - 2L \cancel{+1} = \underline{N_x - L}$$

ⓑ radix-2 FFT incorporate it in the impenentation!

ⓒ $x(n) = \cos(4\pi/500) R_{4000}(n)$, $h(n) = \{1, -1, 1, -1\}$

$\boxed{\text{DIVIDE AND COMBINE FFT ALGORITHM}}$

$n = M \cdot l + m$ $0 \le l \le L-1$ $0 \le m \le M-1$

$\underline{k = p + L \cdot q}$ $0 \le p \le L-1$ $0 \le q \le M-1$

$\max(n) = \underline{M \cdot (L-1)} + \underline{M} - 1 = M \cdot L - 1$

$\max(k) = \underline{L} - 1 + \underline{L \cdot (M-1)} = L \cdot M - 1$

$\to Mlp + \cancel{MLlq} + mp + Lmq$ 1 $\frac{2\pi}{N} \cdot ML = \frac{2\pi}{\cancel{N}} \cdot \cancel{N} = 2\pi$

$$X(k) = X(p, q) = \sum_{m=0}^{M-1} \sum_{l=0}^{L-1} x(m, l) \cdot W_N^{(Ml+m)(p+Lq)}$$

$$= \sum_{m=0}^{M-1} W_N^{pm} \left\{ \sum_{l=0}^{L-1} x(m, l) \cdot \overset{W_L^{lp}}{W_N^{Mlp}} \right\} \overset{W_M^{mq}}{W_N^{Lmq}} = \underbrace{\sum_{m=0}^{M-1} \left\{ W_N^{pm} \underbrace{\left[\sum_{l=0}^{L-1} x(m, l) W_L^{lp} \right]}_{L \text{ point DFT}} \right\} W_M^{mq}}_{M \text{ point DFT}}$$

1.) $F(p, m) = \sum_{l=0}^{L-1} x(l, m) W_L^{lp}$ $0 \le p \le L-1$, $0 \le m \le M-1$

2.) $G(p, m) = W_N^{pm} \cdot F(p, m)$ $0 \le p \le L-1$, $0 \le m \le M-1$

3.) $X(p, q) = \sum_{m=0}^{M-1} G(p, m) W_M^{mq}$ $0 \le p \le L-1$, $0 \le q \le M-1$

C_{NF} $M \cdot L^2 + L \cdot M + M^2 L$
$C_N < O(N^2)$

RADIX 2 FFT ALGORITHM

$N = M \cdot L \; ; \quad N = 2^v \; ; \quad M = 2 \; ; \quad N = 2L \rightarrow \boxed{L = \frac{N}{2}}$

$\boxed{m = 0:1 \; ; \qquad l = 0 : \frac{N}{2} - 1}$

$\boxed{\begin{array}{l} n = M \cdot l + m \\ k = p + Lq \end{array}}$

$$X(p,q) = \sum_{m=0}^{1} \left\{ W_N^{pm} \left[\sum_{l=0}^{\frac{N}{2}-1} x(m,l)\, W_L^{lp} \right] \right\} W_M^{mq} =$$

$$\begin{array}{l} x(0,l) = g_1(n) = x(2n) \\ x(1,l) = g_2(n) = x(2n+1) \\ m=0 \quad n = 2M \\ m=1 \quad n = 2M+1 \\ \mathrm{DFT}[g_1(n)] = G_1(k) \\ \mathrm{DFT}[g_2(n)] = G_2(k) \end{array} =$$

$$= \underbrace{\sum_{l=0}^{\frac{N}{2}-1} x(0,l)\, W_L^{lp}}_{G_1(p)} + W_N^{p} \cdot W_M^{q} \underbrace{\sum_{l=0}^{N/2-1} x(1,l)\, W_L^{q}}_{G_2(p)}$$

$$= G_1(p) + W_N^{p}\, W_N^{q \cdot L} \cdot G_2(p) = G_1(p) + W_N^{k} G_2(p) = \underline{G_1(k) + W_N^{k} G_2(k)}$$

$$\boxed{X(k) = G_1(k) + W_N^{k} G_2(k) \qquad 0 \leq k \leq N-1}$$

$$o(n) = \cancel{2} \cdot \left(\frac{N}{2}\right)^2 + N = \frac{N^2}{2} + N \sim o(N^2/2)$$

$\boxed{L = 2 \; ; \quad M = N/2}$

$l = 0 : 1$
$m = 0 : \frac{N}{2} - 1$

$\boxed{\begin{array}{l} n = M \cdot l + m \\ k = p + L \cdot q \end{array}}$

$$F(0,m) = x(0,m) + x(1,m)\, W_2^{0} = x(n) + x\left(n + \frac{N}{2}\right) \qquad \boxed{n = 0 : \frac{N}{2}}$$

$n = M \cdot l + m$

$x(0,m) = x(M \cdot l + m) = x(m)$

$x(1,m) = x(M \cdot 1 + m) = x(M+m) = x\left(m + \frac{N}{2}\right)$ ⓜ

$$F(1,m) = x(0,m) + x(1,m)\, W_2^{1} = x(0,m) - x(1,m) = x(n) - x\left(n + \frac{N}{2}\right) \qquad \boxed{n = 0 : N/2}$$

$G(0,n) = W_N^{0} F(0,m) = F(0,m) = x(n) + x\left(n + \frac{N}{2}\right) \qquad 0 \leq n \leq \frac{N}{2}$

$G(1,m) = W_N^{m} F(1,m) = \left[x(n) + x\left(n + \frac{N}{2}\right)\right] W_N^{n} \qquad 0 \leq n \leq \frac{N}{2}$

$G(0,n) = d_1(n) \qquad G(1,n) = d_2(n) \qquad 0 \leq n \leq \frac{N}{2} - 1$

$$X(0,q) = \sum_{m=0}^{\frac{N}{2}-1} G(0,m)\, W_M^{mq} = D_1(q) = X(2q) \qquad X(k) = X(p + 2 \cdot q)$$

$$X(1,q) = \sum_{m=0}^{\frac{N}{2}-1} G(1,m)\, W_M^{mq} = D_2(q) = X(2q+1)$$

FAST CONVOLUTION

$$N = 2^{\lceil \log_2 [N_1 + N_2 - 1] \rceil}$$

$x_1(n) \qquad n = 0:9 \; ; \quad N_1 = 10$
$x_2(n) \qquad n = 0:3 \; ; \quad N_2 = 4$

$\boxed{N_1 + N_2 - 1 = 13}$

$\log_2 13 = 4 \qquad \boxed{N = 2^4 = 16}$

$x = \cos(\pi n / 500)\, R_{4000}(n) \qquad h(n) = \{1, -1, 1, -1\}$

$\boxed{P.\ 5.21}$ $x_1(n) = \{2, 1, 1, 2\}$; $x_2(n) = \{1, -1, -1, 1\}$;
$N = 4, 7, 8$

$\longrightarrow$ circular even

$\boxed{P\ 5.22}$ $$x(n) = \begin{cases} A \cdot \cos(2\pi l n / N), & 0 \leq n \leq N-1 \\ \emptyset, & \text{elsewhere} \end{cases}$$

l - integer ; x contains exactly l periods of cosine waveform in N samples

ⓐ Show that DFT $X(k)$ is real sequence.

$$X(k) = \frac{AN}{2}\delta(k-l) + \frac{AN}{2}\delta(k-N+l); \quad 0 \leq k \leq N-1, \quad 0 < l < N$$

$l = 2$, $N = 10$ $\Rightarrow$ $x(n) = A \cdot \cos\left(\frac{2\pi\, 2 \cdot n}{N}\right) R_N(n)$

$$X(k) = \sum_{n=0}^{N-1} A \cdot \cos(2\pi \cdot 2n/N) \cdot e^{-j\frac{2\pi n k}{N}} = \sum_{n=0}^{N-1} \frac{A}{2}\cdot\left(e^{+j\frac{2\pi\cdot 2n}{N}} + e^{-j\frac{2\pi 2n}{N}}\right) e^{-j\frac{2\pi n k}{N}}$$

$$= \frac{A}{2}\sum_{n=0}^{N-1} e^{j\frac{2\pi n}{N}(2-k)} + e^{-j\frac{2\pi n}{N}(2+k)} = \frac{A}{2}\sum_{n=0}^{N-1} e^{-j\frac{2\pi n}{N}(k-2)} + e^{-j\frac{2\pi}{N}(k+2)}$$

$k = \emptyset$ $$X(0) = \frac{A}{2}\sum_{n=0}^{N-1} e^{j\frac{2\pi n}{N}\cdot 2} + e^{-j\frac{2\pi n}{N}\cdot 2} = \underline{A \cdot \sum_{n=0}^{N-1} \cos\left(2 \cdot \frac{2\pi n}{N}\right) = \emptyset}$$

$$\cos(\alpha + \alpha) = \cos\alpha \cdot \cos\alpha - \sin\alpha \cdot \sin\alpha = \cos^2\alpha - \sin^2\alpha$$

$k = 1$ $$X(1) = A \sum_{n=0}^{N-1} \cos\left(2 \cdot \frac{2\pi n}{N}\right) = \emptyset \quad \triangledown 1$$

$k = 2$ $$X(2) = \frac{A}{2}\sum_{n=0}^{N-1} e^{-j\frac{2\pi n}{N}\cdot(\underset{2}{\cancel{k}} - 2)} + e^{-j\frac{2\pi n}{N}(\overset{2}{\cancel{k}} + 2)} =$$

$$= \frac{A}{2}\left[N + \underbrace{\sum_{n=0}^{N-1} e^{-j\frac{2\pi n}{N}\cdot 4}}_{\emptyset}\right] = \boxed{\frac{A \cdot N}{2}}$$

$\boxed{k = N-2}$ $N = 10 \Rightarrow k = 8$

$$X(k) = \frac{A}{2}\underset{\blacktriangledown 1}{\sum_{n=0}^{N-1}} e^{-j\frac{2\pi n}{N}(N-2-2)} + \overset{1}{\cancel{e^{-j\frac{2\pi n}{N}\cdot N}}} =$$

$$= \frac{A}{2}\left[\underbrace{\sum_{n=0}^{N-1} \cancel{e^{-j\frac{2\pi n}{N}\cdot N}} \cdot e^{-j\frac{2\pi n}{N}\cdot(-4)}}_{\emptyset} + N\right] = \frac{A}{2}\cdot N$$

$$\boxed{X(k) = \frac{AN}{2}\delta(k-l) + \frac{AN}{2}\delta(k-N+l)}$$

ⓑ $\boxed{l=0}$ $x(n) = A \cdot \cos\left(\frac{2\pi \cdot l n}{N}\right) R_N(n) = A \cdot R_N(n)$

$$DFT[x(n)] = DFT[A \cdot R_N(n)] = \sum_{n=0}^{N-1} A \cdot e^{-j\frac{2\pi n k}{N}}$$

$\boxed{k=0}$ $x(k) = \sum_{n=0}^{N-1} A = \underline{\underline{N \cdot A}}$

$\boxed{k \neq 0}$ $x(k) = \sum_{n=0}^{N-1} A \cdot e^{-j\frac{2\pi n k}{N}} = 0$

$x(n) = A \cdot \cos\left(\frac{2\pi l n}{N}\right) R_N(n)$ $x(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi n k}{N}}$

$$x(k) = \sum_{n=0}^{N-1} A \cdot \cos\left(\frac{2\pi l n}{N}\right)\left[\cos\left(\frac{2\pi n k}{N}\right) + j\sin\left(\frac{2\pi n k}{N}\right)\right] =$$

$$= \sum_{n=0}^{N-1} A \cdot \cos\left(\frac{2\pi l n}{N}\right)\cos\left(\frac{2\pi n k}{N}\right) + \underbrace{jA \cdot \cos\frac{2\pi l n}{N} \cdot \sin\left(\frac{2\pi n k}{N}\right)}_{\text{even} * \text{odd} = 0}$$

primerocite se ponistuvaat

$\cos(\alpha+\beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta$

$\cos(\alpha-\beta) = \cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta$

$\cos\alpha \cdot \cos\beta = \frac{1}{2}\cos(\alpha+\beta) + \frac{1}{2}\cos(\alpha-\beta)$

$$\boxed{x(k) = \sum_{n=0} \frac{A}{2}\cos\frac{2\pi n}{N}(k+l) + \frac{A}{2}\cos\frac{2\pi n}{N}(k-l)}$$

$l=0$ $x(k) = \sum_{n=0}^{N-1} \frac{A}{2} \cdot 2\cos\frac{2\pi n k}{N} = A\sum_{n=0}^{N-1}\cos\left(\frac{2\pi n k}{N}\right)$

$k=0$ $\boxed{x(k) = A \cdot N}$

$k \neq 0$ $x(k) = 0$ SE PONISTUVAAT PRIMEROCITE T.E. KOLKU STO IMA NEGATIVNI VREDNOSTI TOLKU IMA I POZITIVNI (SE POMINUVAAT SITE ČETIRI KVADRANTI)

ⓒ $\boxed{l<0}$ $x(k) = \sum_{n=0}^{N-1} \frac{A}{2}\cos\left(\frac{2\pi n}{N}\right)(k-|l|) + \frac{A}{2}\cos\frac{2\pi n}{N}(k+|l|)$

$\boxed{l>N}$ $l = N + l_1$ (ISTOTO VAZI ZA: $l = m \cdot N + l_1$ $m=0,1,2\dots$)

$x(n) = A \cdot \cos\frac{2\pi \cdot n(N+l_1)}{N} = A \cdot \cos\left(\frac{2\pi n l_1}{N} + \frac{2\pi n N}{N}\right) =$

$= A \cdot \cos\left(\frac{2\pi n l_1}{N}\right) \cdot \cos(2\pi n)^{\to 1} - A\sin\left(\frac{2\pi n l_1}{N}\right) \cdot \cancel{\sin(2\pi n)}^{\to 0} = A\cos\left(\frac{2\pi n l_1}{N}\right)$

ⓓ (i) $x_1(n) = 3\cos(0.04\pi n) R_{200}(n)$

(ii) $x_2(n) = 5 R_{50}(n)$

(iii) $x_3(n) = [1 + 2\cos(0.5\pi n) + \cos(\pi n)] R_{100}(n)$

(iv) $x_4(n) = \cos(2\pi n/16) R_{64}(n)$

(v) $x_5(n) = [4\cos(0.1\pi n) - 3\cos(1.9\pi n)] R_N(n)$

(i) $x_1(n) = 3\cos(0.04\pi n) R_{200}(n)$

$$0.04\pi = \frac{2\pi}{N} \Rightarrow N = \frac{2}{0.04} = \frac{2}{4\cdot 10^{-2}} = 0.5\cdot 10^{2} = 50$$

$$x_1(n) = 3\cdot\cos\left(\frac{2\pi}{50}\cdot n\right) R_{200}(n) = 3\cdot\cos\left(\frac{2\pi\cdot \overset{\textcircled{l}}{\textcircled{4}}}{200} n\right) R_{200}(n) \Rightarrow$$

$$x(k) = \frac{A\cdot N}{2}\cdot\delta(k-l) + \frac{A\cdot N}{2}\cdot\delta(k-N+l) =$$

$$= 300\cdot\delta(k-4) + 300\,\delta(k-196)$$

(ii) $x_2(n) = 5\cdot R_{50}(n)$; $x(k) = A\cdot N\,\delta(k) = 5\cdot 50\,\delta(k) = 250\,\delta(k)$

(iii) $x_3(n) = x_{31}(n) + x_{32}(n) + x_{33}(n)$; $N = 100$;

$$\underline{x_{31}(k) = A\cdot N\cdot\delta(k) = 100\cdot\delta(k)};$$

$x_{32}(n) = 2\cdot\cos(0.5\pi n)$ $0.5\pi = \frac{2\pi}{N}$ $N = \frac{2}{0.5} = \frac{20}{5} = 4$

$$x_{32}(n) = 2\cdot\cos\left(\frac{2\pi\cdot n}{4\cdot 25}\cdot 25\right) = 2\cdot\cos\left(\frac{2\pi n}{100}\cdot 25\right)$$

$$x_{32}(k) = \overset{\textcircled{AN/2}}{\frac{2\cdot 100}{2}}\,\delta(k-25) + \frac{2\cdot 100}{2}\,\delta(k-N+2\pi)$$

$$x_{32}(k) = 100\cdot\delta(k-25) + 100\,\delta(k-75)$$

$$x_{33}(n) = \cos(\pi n) = \cos\left(\frac{2\pi n}{100}\cdot 50\right)$$

$$x_{33}(k) = \frac{100}{2}\cdot\delta(k-50) + \frac{100}{2}\delta(k-50) = 100\,\delta(k-50)$$

$$x_3(k) = x_{31}(k) + x_{32}(k) + x_{33}(k) = \underline{100\,\delta(k) + 100\,\delta(k-25) + 100\,\delta(k-75) + 100\,\delta(k-50)}$$

(iv) $x_4(n) = \cos\left(\frac{2\pi n}{16}\right) R_{64}(n) = \cos\left(\frac{2\pi n}{64}\cdot 4\right) R_{64}(n)$

$$x_4(k) = 32\cdot\delta(k-4) + 32\,\delta(k-60)$$

(v) $x_5(n) = x_{51}(n) - x_{52}(n)$

$x_{51}(n) = 4\cdot\cos(0.1\pi n)$ $0.1\pi = \frac{2\pi l_1}{N}$; $2l_1 = 0.1N$; $N = 20 l_1$

$x_{52}(n) = 3\cdot\cos(1.9\pi n)$ $1.9\pi = \frac{2\pi l_2}{N}$; $1.9N = 2\cdot l_2$

$l_1 = \frac{N}{20}$; $l_2 = \frac{1.9}{2}\cdot N = 0{,}95\cdot N$

if: $N = 20 \Rightarrow \boxed{l_1 = 1}$ $l_2 = 20\cdot 0{,}95 = 19$

if: $N = 40 \Rightarrow l_1 = 2$ $l_2 = 38$

$$\boxed{\begin{matrix} l_1 = 0{,}05\ N \\ l_2 = 0{,}95\ N \end{matrix}}$$

$$x_{51}(n) = 4 \cdot \cos\left(\frac{2\pi n}{N} \cdot l_1\right)$$

$$x_{52}(n) = 3 \cdot \cos\left(\frac{2\pi n}{N} \cdot l_2\right)$$

$$x_5(n) = 4 \cdot \cos\left(\frac{2\pi n}{N} \cdot 0.05N\right) - 3 \cdot \cos\left(\frac{2\pi n}{N} \cdot 0.95 \cdot N\right)$$

~~$X_5(k) = 2 \cdot N \cdot \delta(k - 0.05N) - \frac{3 \cdot N}{2} \cdot \delta(k - 0.95N)$~~

~~$N = 20 \Rightarrow X_5(k) = 40\,\delta(k-1)$~~

$$X_5(k) = 2N \cdot \delta(k - 0.05N) + 2N\,\delta(k - N + 0.05N) - \frac{3N}{2}\delta(k - 0.95N) - \frac{3N}{2}\delta(k - N + 0.95N)$$

$$X_5(k) = \underline{2N \cdot \delta(k - 0.05N)} + \underline{2N\,\delta(k - 0.95N)} - \underline{\frac{3N}{2}\,\delta(k - 0.95N)} - \underline{\frac{3N}{2}\delta(k - 0.05N)}$$

$$= \left(2N - \frac{3N}{2}\right)\delta(k - 0.05N) + \left(2N - \frac{3N}{2}\right)\delta(k - 0.95N)$$

$$X_5(k) = \frac{N}{2}\,\delta(k - 0.05N) + \frac{N}{2}\,\delta(k - 0.95N)$$

$\boxed{P.5.23}$ $x(n) = A\cos(w_0 n)\,R_N(n)$ w_0 - real number

$$X(k) = X_R(k) + j\,X_I(k)$$

$$X_R(k) = \frac{A}{2}\cos\frac{\pi(N-1)}{N}(k - f_0N)\,\frac{\sin[\pi(k - f_0N)]}{\sin[\pi(k - f_0N)/N]} +$$

$$+ \frac{A}{2}\cos\frac{\pi(N-1)}{N}(k + f_0N)\,\frac{\sin[\pi(k - N + f_0N)]}{\sin[\pi(k - N + f_0N)/N]}$$

$$X_I(k) = -\frac{A}{2}\sin\frac{\pi(N-1)}{N}(k - f_0N)\,\frac{\sin[\pi(k - f_0N)]}{\sin[\pi(k - f_0N)/N]} -$$

$$- \frac{A}{2}\sin\frac{\pi(N-1)}{N}(k + f_0N)\,\frac{\sin[\pi(k - N + f_0N)]}{\sin[\pi(k - N + f_0N)/N]}$$

$$X(k) = \sum_{n=0}^{N-1} A\underset{\textcircled{\beta}}{\cos(w_0 n)}\,\underset{\textcircled{\alpha}}{e^{-j\frac{2\pi nk}{N}}} = \sum_{n=0}^{N-1} A\cos(w_0 n)\left[\cos\left(\frac{2\pi nk}{N}\right) - j\sin\left(\frac{2\pi nk}{N}\right)\right]$$

$$= \sum_{n=0}^{N-1} A\cos(w_0 n)\cos\left(\frac{2\pi nk}{N}\right) - jA\sum_{n=0}^{N-1} \overset{\textcircled{\beta}}{\cos(w_0 n)}\,\overset{\textcircled{\alpha}}{\sin\left(\frac{2\pi nk}{N}\right)}$$

$$= \left|\begin{matrix} \cos\alpha \cdot \cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)] \\ \sin\alpha \cdot \cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)] \end{matrix}\right| =$$

$$= \frac{A}{2}\left[\sum_{n=0}^{N-1}\underbrace{\cos\left(\frac{2\pi k}{N} + w_0\right)n}_{\textcircled{1}} + \underbrace{\cos\left(\frac{2\pi k}{N} - w_0\right)}_{\textcircled{2}} - j\underbrace{\sin\left(\frac{2\pi nk}{N} + w_0\right)n}_{[illegible]} - j\underbrace{\sin\left(\frac{2\pi nk}{N} - w_0\right)}_{[illegible]}\right]$$

$$① = \sum_{n=0}^{N-1} \cos\left(\frac{2\pi n k}{N} + \omega_0 n\right) = \sum_{n=0}^{N-1} \cos\left(\frac{2\pi k}{N} + \omega_0\right) n$$

$$\sum_{n=0}^{N-1} \cos(\alpha n) = \frac{1}{2} \sum_{n=0}^{N-1} \left(e^{+j\alpha n} - e^{-j\alpha n}\right) = \frac{1}{2}\left[\underbrace{\sum_{n=0}^{N-1} e^{j\alpha n}}_{(*)} - \underbrace{\sum_{n=0}^{N-1} e^{-j\alpha n}}_{(**)}\right]$$

$$(*) = \left| \frac{1-q^{N}}{1-q} = \sum_{n=0}^{N-1} q^n \qquad \begin{array}{l} S_n = 1 + q + q^2 + \cdots q^{N-1} \\ q S_n = q + q^2 + q^2 + \cdots q^{N} \end{array} \right| =$$

$$= \frac{1 - e^{j\alpha N}}{1 - e^{\alpha}} = \frac{e^{j\frac{\alpha N}{2}}\left(e^{-j\frac{\alpha N}{2}} - e^{j\frac{\alpha N}{2}}\right)}{e^{j\alpha/2}\left(e^{-j\alpha/2} - e^{j\alpha/2}\right)} =$$

$$= \left| \sin(x) = -\frac{j}{2}\left(e^{jx} - e^{-jx}\right) \; ; \quad e^{jx} - e^{-jx} = 2j\sin(x) \right| =$$

$$= e^{j\frac{\alpha}{2}(N-1)} \frac{\cancel{2j}\sin\left(\frac{\alpha N}{2}\right)}{\cancel{2j}\sin\left(\frac{\alpha}{2}\right)} = \boxed{e^{j\frac{\alpha}{2}(N-1)}} \frac{\sin\left(\frac{\alpha N}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}$$

$$\sum_{n=0}^{N-1} \cos(\alpha n) = \frac{1}{2}\left[\frac{\cos(\alpha)\cdot\sin(\alpha N)}{\sin\alpha} + \frac{\sin(\alpha N)}{\sin(\alpha)} - \frac{1}{2}\cos(\alpha N) + \frac{1}{2}\right]$$

MAPLE

MATHEMATICA

$$\boxed{\sum_{n=0}^{N-1} \cos(\alpha \cdot n) = \cos\left[\frac{\alpha}{2}(1-N)\right] \Bigg| \frac{\sin\left(\frac{N\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}}$$

MATEMATICA
VEZI VARANO
NE ZNAM DA
GO DOKAŽAM!

$$① = \left[\cos\frac{\alpha}{2}(N-1) + j\sin\frac{\alpha}{2}(N-1)\right] \cdot \frac{\sin\left(\frac{\alpha N}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}$$

$$⑪ \quad \sin\frac{\alpha}{2}(N-1) \cdot \sin\left(\frac{\alpha N}{2}\right) = \frac{1}{2}\left[\cos(\alpha - \beta) - \cos(\alpha + \beta)\right] =$$

$$= \frac{1}{2}\left[\cos\left[\frac{\alpha}{2}(N-1) - \frac{\alpha N}{2}\right] - \cos\left[\frac{\alpha}{2}(N-1) + \frac{\alpha N}{2}\right]\right] =$$

$$= \frac{1}{2}\left[\cos\left(-\frac{\alpha}{2}\right) - \cos\left[\alpha N - \frac{\alpha}{2}\right]\right]$$

$$⑪ \quad \frac{1}{\sin\left(\frac{\alpha}{2}\right)} = \frac{1}{2}\left[\frac{\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)} - \frac{\cos\alpha N \cdot \cos\frac{\alpha}{2} + \sin\alpha N \cdot \sin\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}\right] -$$

$$(*) = e^{j\frac{\alpha}{2}(N-1)} \frac{\sin\left(\frac{\alpha N}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)} = e^{j\frac{\alpha N}{2}} \cdot e^{-j\frac{\alpha}{2}} \cdot \frac{\sin\left(\frac{\alpha N}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}$$

~~$⑪\ \frac{1}{\sin\left(\frac{\alpha}{2}\right)} = \frac{1}{2}\left[\operatorname{ctan}\left(\frac{\alpha}{2}\right) - \cos\alpha N \cdot \operatorname{ctan}\left(\frac{\alpha}{2}\right) - \sin(\alpha N)\right]$~~

$$\cos\frac{\alpha}{2}(N-1) \cdot \sin\left(\frac{N\alpha}{2}\right) = \sin\left(\frac{N\alpha}{2}\right) \cdot \cos\left(\frac{\alpha}{2}(N-1)\right) =$$

$$= \left| \sin\alpha \cdot \cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right] \right| =$$

$$= \frac{1}{2}\left[\sin\left(\frac{N\alpha}{2} + \frac{\alpha N}{2} - \frac{\alpha}{2}\right) + \sin\left(\cancel{\frac{N\alpha}{2}} - \cancel{\frac{\alpha N}{2}} + \frac{\alpha}{2}\right)\right] =$$

$$= \frac{1}{2}\left[\sin\left(\alpha N - \frac{\alpha}{2}\right) + \sin\left(\frac{\alpha}{2}\right)\right]$$

$$① = \frac{1}{2}\left[\sin\left(\alpha N - \frac{\alpha}{2}\right) + \sin\left(\frac{\alpha}{2}\right) + j\cos\left(\frac{\alpha}{2}\right) - j\cos\left(\alpha N - \frac{\alpha}{2}\right)\right] \cdot \frac{1}{\sin\left(\frac{\alpha}{2}\right)}$$

$$A_n = \sum_{n=0}^{N-1} \cos(\alpha n) = 1 + \cos\alpha + \cos 2\alpha + \dots \cos(N-1)\alpha$$

$$jB_n = j\sum_{n=0}^{N-1} \sin(\alpha n) = j\sin\alpha + j\sin 2\alpha + \dots j\sin(N-1)\alpha$$

$$A_n + jB_n = 1 + e^{j\alpha} + e^{j2\alpha} + \dots + e^{j(N-1)\alpha}$$

$$C_n = \sum_{n=0}^{N-1} e^{jn\alpha} = \frac{1 - e^{jN\alpha}}{1 - e^{j\alpha}} = e^{j\frac{\alpha}{2}(N-1)} \frac{\sin\frac{\alpha N}{2}}{\sin\frac{\alpha}{2}} =$$

$$= \left[\cos\frac{\alpha}{2}(N-1) + j\sin\frac{\alpha}{2}(N-1)\right] \frac{\sin\frac{\alpha N}{2}}{\sin\frac{\alpha}{2}} =$$

$$= \underbrace{\cos\frac{\alpha}{2}(N-1)\frac{\sin\frac{\alpha N}{2}}{\sin\frac{\alpha}{2}}}_{A_n} + j\underbrace{\sin\frac{\alpha}{2}(N-1)\frac{\sin\frac{\alpha N}{2}}{\sin\frac{\alpha}{2}}}_{B_n}$$

MNU

POEDNO-STAVEN PRISTAP!

$$(**) = \sum_{n=0}^{N-1} e^{-j\alpha n} = \frac{1 - e^{-j\alpha N}}{1 - e^{-j\alpha}} = \frac{e^{-j\frac{\alpha N}{2}}\left(e^{j\frac{\alpha N}{2}} - e^{-j\frac{\alpha N}{2}}\right)}{e^{-j\frac{\alpha}{2}}\left(e^{j\frac{\alpha}{2}} - e^{-j\frac{\alpha}{2}}\right)} =$$

$$= \boxed{e^{-j\frac{\alpha}{2}(N-1)}} \frac{\sin\frac{\alpha N}{2}}{\sin\frac{\alpha}{2}}$$

$$(*) + (**) = \frac{1}{2}\left(e^{j\frac{\alpha}{2}(N-1)} + e^{-j\frac{\alpha}{2}(N-1)}\right)\frac{\sin\frac{\alpha N}{2}}{\sin\frac{\alpha}{2}} =$$

$$(\square\square) = \frac{1}{2}\left(\frac{2\pi k}{N} + \pi f_0\right)N = \frac{\pi}{\cancel{N}}(k+Nf_0)\cancel{N} = \pi(k+Nf_0)$$

$$= \cos\frac{\alpha}{2}(N-1)\cdot\frac{\sin\left(\frac{\alpha N}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)} = \sum_{n=0}^{N-1}\cos(\alpha n)$$

$$① = \sum_{n=0}^{N-1}\cos\left(\frac{2\pi k}{N} + \omega_0\right)n = \cos\Big[\underbrace{\frac{1}{2}\left(\frac{2\pi k}{N} + \omega_0\right)(N-1)}_{(\square)}\Big]\cdot\frac{\overbrace{\sin\frac{1}{2}\left(\frac{2\pi k}{N}+\omega_0\right)N}^{(\square\square)}}{\sin\frac{1}{2}\left(\frac{2\pi k}{N}+\omega_0\right)}$$

$$(\square) = \frac{1}{2}\left(\frac{2\pi k}{\cancel{N}}\cdot\cancel{N} - \frac{2\pi k}{N} + \omega_0 N - \omega_0\right) = \frac{1}{2}\left(2\pi k - \frac{2\pi k}{N} + 2\pi f_0 - 2\pi f_0\right)$$

$$(\square) = \frac{1}{2}\left(\frac{2\pi k}{N} + 2\pi f_0\right)(N-1) = \frac{\cancel{2}\pi}{N}\cdot\frac{1}{\cancel{2}}\left(k + N\cdot f_0\right)(N-1) = \frac{\pi(N-1)}{N}(k+N f_0)$$

$$\boxed{① = \cos\frac{\pi(N-1)}{N}(k+Nf_0)\frac{\sin\pi(k+N\cdot f_0)}{\sin\pi(k+N\cdot f_0)/N}}$$

$$② = \sum_{n=0}^{N-1}\cos\left(\frac{2\pi k}{N} - \omega_0\right)n = \cos\left[\frac{1}{2}\left(\frac{2\pi k}{N} - \omega_0\right)(N-1)\right]\cdot\frac{\sin\frac{1}{2}\left(\frac{2\pi k}{N}-\omega_0\right)N}{\sin\frac{1}{2}\left(\frac{2\pi k}{N}-\omega_0\right)}$$

$$\boxed{② = \cos\left[\frac{\pi(N-1)}{N}(k-N\cdot f_0)\right]\cdot\frac{\sin\left[\pi(k-N\cdot f_0)\right]}{\sin\left[\pi(k-N\cdot f_0)/N\right]}}$$

$$\sin\pi(k+Nf_0) = \sin(\alpha) \qquad \sin(N\cdot\pi - \alpha)$$

$$\sin(\alpha-\pi) = -\sin(\pi-\alpha) = -\sin(\alpha)$$

$$\sin(\alpha-2\pi) = -\sin(2\pi-\alpha) = \sin(\alpha)$$

$$\frac{\sin\pi(k - N + N\cdot f_0)}{\sin\pi(k-N+N\cdot f_0)/N} = \frac{\sin\left[\pi(k+Nf_0) - N\pi\right]}{\sin\left[\frac{\pi(k+Nf_0)}{N} - \pi\right]} =$$

$$= \frac{\sin\pi(k+Nf_0)\cdot\overset{(-1)^N}{\cos(N\pi)} - \cos\left[\pi(k+Nf_0)\right]\cdot\cancel{\sin(N\pi)}\to 0}{-\sin\left[\pi(k+Nf_0)/N\right]}$$

$$= (-1)^{N+1}\sin\left[\pi(k+Nf_0)\right]$$

$$x(k) = \sum_{n=0}^{N-1} A\cos(\omega_0 n)\left[\cos\left(\frac{2\pi k n}{N}\right) - j\sin\left(\frac{2\pi k n}{N}\right)\right] =$$

$$= A\sum_{n=0}^{N-1} \underset{(\alpha)}{\cos\left(\frac{2\pi k n}{N}\right)}\cdot \underset{(\beta)}{\cos(\omega_0 n)} - j\,\underset{(\alpha)}{\sin\left(\frac{2\pi k n}{N}\right)}\cdot \underset{(\beta)}{\cos(\omega_0 n)}$$

$$= \Big|\ \sin\alpha\cdot\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

$$\sin(\alpha+\beta) = \sin\alpha\cdot\cos\beta - \sin\beta\cdot\cos\alpha$$
$$\sin(\alpha-\beta) = \sin\alpha\cdot\cos\beta + \sin\beta\cos\alpha$$

$$= \frac{A}{2}\sum_{n=0}^{N-1} \cos\left(\frac{2\pi k n}{N} + \omega_0\right)n + \cos\left(\frac{2\pi k n}{N} - \omega_0\right)n - j\underbrace{\sin\left(\frac{2\pi k}{N} + \omega_0\right)n}_{(*)} - j\underbrace{\sin\left(\frac{2\pi k}{N} - \omega_0\right)n}_{(**)}$$

$$\sum_{n=0}^{N-1} \sin(\alpha n) = \sin\frac{\alpha}{2}(N-1)\ \frac{\sin\frac{\alpha N}{2}}{\sin\frac{\alpha}{2}}$$

$$(*) = \sin\left[\frac{1}{2}(N-1)\left(\frac{2\pi k}{N} + \omega_0\right)\right]\ \frac{\sin\frac{N}{2}\left(\frac{2\pi k}{N}+\omega_0\right)}{\sin\frac{1}{2}\left(\frac{2\pi k}{N}+\omega_0\right)} = \sin\frac{\pi(N-1)}{N}(k + N\cdot f_0)\frac{\sin\pi(k+N\cdot f_0)}{\sin\pi(k+Nf_0)/N}$$

VO KNIGATA E BEZ OVIE CLENOVI !!!

$$x(k) = \frac{A}{2}\cos\frac{\pi(N-1)}{N}(k - Nf_0)\frac{\sin\pi(k-Nf_0)}{\sin\pi(k-Nf_0)/N} + \frac{A}{2}\cos\frac{\pi(N-1)}{N}(k\overset{-N}{\vee}+Nf_0)\ \frac{\sin\pi(k-N+Nf_0)}{\sin\pi(k-N+Nf_0)/N}$$

$$-j\frac{A}{2}\sin\frac{\pi(N-1)}{N}(k-Nf_0)\ \frac{\sin\pi(k-Nf_0)}{\sin\pi(k-Nf_0)/N} - j\frac{A}{2}\sin\frac{\pi(N-1)}{N}(k\overset{-N}{\vee}+Nf_0)\ \frac{\sin\pi(k-N+Nf_0)}{\sin\pi(k-N+Nf_0)/N}$$

MMV

$$\cos\left[\frac{\pi(N-1)}{N}(k - N + Nf_0)\right] = \cos\left[\frac{\pi(N-1)}{N}(k+Nf_0) - \pi(N-1)\right] =$$

$$= \cos\frac{\pi(N-1)}{N}(k+Nf_0)\cdot \underset{(-1)^{N-1}}{\cos\pi(N-1)} = (-1)^{N-1}\cos\frac{\pi(N-1)}{N}(k+Nf_0)$$

$$\boxed{1 = (-1)^{2N-2}}$$

$$\cos\frac{\pi(N-1)}{N}(k+Nf_0)\cdot\frac{\sin\pi(k+Nf_0)}{\sin\pi(k+Nf_0)} = \cancel{(-1)^{N-1}}\cdot(-1)^{N-1}\cos\frac{\pi(N-1)}{N}(k-N+Nf_0)\frac{\sin\pi(k-N+Nf_0)}{\sin\pi(k-N+Nf_0)/N}$$

MMV

$$\sin\frac{\pi(N-1)}{N}(k-N+Nf_0) = \sin\left[\frac{\pi(N-1)}{N}(k+Nf_0) - \pi(N-1)\right] =$$

$$= \sin\frac{\pi(N-1)}{N}(k+Nf_0)\cdot\cos\pi(N-1) = (-1)^{N-1}\sin\frac{\pi(N-1)}{N}(k+Nf_0)$$

$$\sin\frac{\pi(N-1)}{N}(k+Nf_0)\cdot\frac{\sin\pi(k+Nf_0)}{\sin\pi(k+Nf_0)} = (-1)^{N-1}(-1)^{N-1}\sin\frac{\pi(N-1)}{N}(k-N+Nf_0)\ \frac{\sin\pi(k-N+Nf_0)}{\sin\pi(k-N+Nf_0)}$$

$$\frac{5\pi}{99}\cdot l = \frac{2\pi}{N}\cdot l = \frac{2\pi}{200}$$

ⓒ VERIFY LEAKAGE PROPERTY :

$$x(n) = \cos(5\pi n/99)\ R_{200}(n)$$

$$\omega_0 = \frac{5\pi}{99} \qquad f_0 = \frac{\omega_0}{2\pi} = \frac{5\pi}{2\pi\cdot 99} = \frac{5}{188}$$

$$x(n) = \cos\left(\boxed{\frac{5\pi}{99}} n\right) R_{200}(n)$$

$$\cos(0.04\pi n)\, R_{200}(n) = \cos\left(\frac{2\pi}{200} \cdot n \cdot 4\right)$$

$$0.04\pi = \frac{2\pi}{200} \cdot l \qquad l = 0.04 \cdot 100 \qquad \boxed{l = 4}$$

$$\boxed{\frac{5\pi}{99} = \frac{2\pi}{200} \cdot l_0} ; \quad l_0 = \frac{500}{99} = 5.0505$$

$$\omega_0 = \frac{5\pi}{99} ; \quad 2\pi f_0 = \frac{5\pi}{99} \Rightarrow f_0 = \frac{5}{188}$$

5.24

$$x(n) = \begin{cases} A \sin(2\pi l n/N) & 0 \le n \le N-1 \\ 0 & \text{, elsewhere} \end{cases}$$

$$x(n) = A \cdot \sin(2\pi l n/N) \cdot R_N(n) = A \cdot \sin(\omega_0 n) R_N(n) \qquad \boxed{\omega_0 = \frac{2\pi l}{N}}$$

$$X(k) = \sum_{n=0}^{N-1} A \cdot \sin(\omega_0 n)\, e^{-j\frac{2\pi n k}{N}} = \sum_{n=0}^{N-1} A \cdot \sin(\overset{\beta}{\omega_0 n}) \cdot \left[\cos\left(\overset{\alpha}{\frac{2\pi n k}{N}}\right) - j \sin\left(\overset{\alpha}{\frac{2\pi n k}{N}}\right)\right] =$$

$$= \frac{A}{2} \sum_{n=0}^{N-1} \underbrace{\sin\left(\frac{2\pi n k}{N} + \omega_0 n\right)}_{①} - \underbrace{\sin\left(\frac{2\pi n k}{N} - \omega_0 n\right)}_{②} - j \underbrace{\cos\left(\frac{2\pi n k}{N} - \omega_0 n\right)}_{③} + j \underbrace{\cos\left(\frac{2\pi n k}{N} + \omega_0 n\right)}_{④}$$

$$\sin(\alpha + \beta) = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta$$
$$\sin(\alpha - \beta) = \sin\alpha \cdot \cos\beta - \cos\alpha \cdot \sin\beta$$
$$\cos\alpha \cdot \sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$$

$$\cos(\alpha + \beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta$$
$$\cos(\alpha - \beta) = \cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta$$
$$\sin\alpha \cdot \sin\beta = \frac{1}{2}\left[-\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$$

$$S_1 = \sum_{n=0}^{N-1} \cos(\alpha n) = 1 + \cos\alpha + \cos(2\alpha) + \dots \cos((N-1)\alpha)$$

$$jS_2 = j\sum_{n=0}^{N-1} \sin(\alpha n) = j\sin\alpha + j\sin(2\alpha) + j\sin((N-1)\alpha) \quad +$$

$$S_1 + jS_2 = 1 + e^{j\alpha} + e^{+j2\alpha} + \dots + e^{j(N-1)\alpha} = \sum_{n=0}^{N-1} e^{jn\alpha} = \frac{1 - e^{j\alpha N}}{1 - e^{j\alpha}}$$

$$S_1 + jS_2 = \frac{e^{j\frac{\alpha N}{2}}\left(e^{-j\frac{\alpha N}{2}} - e^{j\frac{\alpha N}{2}}\right)}{e^{j\frac{\alpha}{2}}\left(e^{-j\frac{\alpha}{2}} - e^{j\frac{\alpha}{2}}\right)} = e^{j\frac{\alpha}{2}(N-1)} \frac{\sin\frac{\alpha N}{2}}{\sin\frac{\alpha}{2}}$$

$$S_1 + jS_2 = \underbrace{\cos\frac{\alpha}{2}(N-1) \cdot \frac{\sin\frac{\alpha N}{2}}{\sin\frac{\alpha}{2}}}_{\sum \cos(\alpha n)} + j \underbrace{\sin\frac{\alpha}{2}(N-1) \frac{\sin\frac{\alpha N}{2}}{\sin\frac{\alpha}{2}}}_{\sum \sin(\alpha n)}$$

$$① = \sum_{n=0}^{N-1} \sin\left(\frac{2\pi k}{N} + \omega_0\right) n = \sin\frac{1}{2}\left(\frac{2\pi k}{N} + \omega_0\right)(N-1) \frac{\sin\frac{1}{2}\left(\frac{2\pi k}{N} + \omega_0\right) N}{\sin\frac{1}{2}\left(\frac{2\pi k}{N} + \omega_0\right)}$$

$$\frac{1}{2}\left(\frac{2\pi k}{N} + 2\pi f_0\right) N = \pi k + \pi f_0 N = \pi(k + f_0 N)$$

$$\frac{1}{2}\left(\frac{2\pi k}{N} + 2\pi f_0\right)(N-1) = \pi\left(\frac{k}{N} + f_0\right)(N-1) = \pi\left(k + f_0 N - \frac{k}{N} - f_0\right) = \pi(k + Nf_0)\frac{N-1}{N}$$

$$① = \sin\left(\pi(k+Nf_0)\frac{N-1}{N}\right) \frac{\sin(\pi(k+Nf_0))}{\sin(\pi(k+Nf_0)/N)}$$

$$X(k) = \frac{A}{2}\sin\frac{\pi(N-1)}{N}(k+Nf_0)\frac{\sin\pi(k+Nf_0)}{\sin\pi(k+Nf_0)/N} - \frac{A}{2}\sin\frac{\pi(N-1)}{N}(k-Nf_0)\frac{\sin\pi(k-Nf_0)}{\sin\pi(k+Nf_0)/N}$$
$$+j\frac{A}{2}\cos\frac{\pi(N-1)}{N}(k+Nf_0)\frac{\sin\pi(k+Nf_0)}{\sin\pi(k+Nf_0)/N} - j\frac{A}{2}\cos\frac{\pi(N-1)}{N}(k-Nf_0)\frac{\sin\pi(k-Nf_0)}{\sin\pi(k+Nf_0)/N}$$

$\omega_0 = 2\pi f_0$ $\qquad$ $\omega_0 = \frac{5\pi}{99} = 2\pi f_0$ $\qquad$ $\boxed{f_0 = \frac{5}{198}}$

$$\boxed{\begin{aligned} X(k) = &-\frac{A}{2}\sin\frac{\pi(N-1)}{N}(k-Nf_0)\frac{\sin\pi(k-Nf_0)}{\sin\pi(k-Nf_0)} + \frac{A}{2}\sin\frac{\pi(N-1)}{N}(k-N+Nf_0)\frac{\sin\pi(k-N+Nf_0)}{\sin\pi(k-N+Nf_0)/N} \\ &-j\frac{A}{2}\cos\frac{\pi(N-1)}{N}(k-Nf_0)\frac{\sin\pi(k-Nf_0)}{\sin\pi(k-Nf_0)} + j\frac{A}{2}\cos\frac{\pi(N-1)}{N}(k-N+Nf_0)\frac{\sin\pi(k-N+Nf_0)}{\sin\pi(k-N+Nf_0)/N}\end{aligned}}$$

$x(n) = A\cdot\sin(\omega_0 n)$; $\omega_0 = \frac{2\pi}{N}\cdot l$; $x(n) = \sin\left(\frac{2\pi}{N}\cdot l\cdot n\right)$

$$X(k) = \frac{AN}{2j}\delta(k-l) - \frac{AN}{2j}\delta(k-N+l)$$

$l = \frac{\omega_0\cdot N}{2\pi}$ $\qquad$ $\omega_0 = 2\pi f_0$ $\qquad$ $\boxed{l = N\cdot f_0}$

$$\circledast = \sin\left(\frac{\pi(N-1)}{N}\cdot(k-l)\right)\frac{\sin\pi(k-l)}{\sin\pi(k-l)/N} \qquad (k-l) = ⓡ$$

$$\circledast = \sin\left(\frac{\pi(N-1)}{N}\cdot r\right)\frac{\sin\pi r}{\sin\pi r/N}$$

$$\sin\frac{\pi r}{N}(N-1) = \sin\left(\frac{\pi r}{N}\cdot N - \frac{\pi r}{N}\right) = \sin\left(\pi r - \frac{\pi r}{N}\right) = \underline{\underline{(-1)^{r+1}\sin\left(\frac{\pi r}{N}\right)}}$$

$\sin(\pi-\alpha) = \sin(\alpha)$; $\sin(-\alpha) = -\sin\alpha$ $\quad$ $\sin(2\pi-\alpha) = -\alpha$

$\sin(r\pi-\alpha) = (-1)^{r+1}\sin(+\alpha)$

$$\circledast = (-1)^{r+1}\cancel{\sin\left(\frac{\pi r}{N}\right)}\cdot\frac{\sin(\pi r)}{\cancel{\sin\pi r/N}} = (-1)^{r+1}\cdot\cancelto{0}{\sin(\pi r)} \underline{\underline{= 0}}$$

$$(**) = \cos\frac{\pi(N-1)}{N}\cdot(k-l)\frac{\sin\pi(k-l)}{\sin\pi(k-l)/N} = \underbrace{\cos\frac{\pi(N-1)}{N}\cdot r}_{①}\frac{\sin\pi r}{\sin\pi r/N}$$

$$① = \cos\left(\frac{\pi N}{N} - \frac{\pi}{N}\right)r = \cos\left(\pi r - \frac{\pi r}{N}\right) =$$

$\cos\left(0-\frac{\pi r}{N}\right) = \cos\left(\frac{\pi r}{N}\right)$ $\quad$ $\cos\left(\pi - \frac{\pi r}{N}\right) = -\cos\frac{\pi r}{N}$; $\cos\left(2\pi - \frac{\pi r}{N}\right) = \cos\frac{\pi r}{N}$

$$① = (-1)^r\cos\left(\frac{\pi r}{N}\right)$$

$(**) = (-1)^r\cos\left(\frac{\pi r}{N}\right)\cdot\frac{\sin\pi r}{\sin\pi r/N}$; $\quad r \neq 0$ $\quad (**) = 0$ $\qquad$ $\boxed{\begin{matrix} r=0 \\ k=l\end{matrix}}$ $\quad (**) = N$

$$w_0 = \frac{2\pi}{N} \cdot l$$

$$\boxed{\begin{aligned} x(k) &= -j\frac{AN}{2}\delta(n-l) + j\frac{AN}{2}\delta(n-N+l) = \\ &= +\frac{AN}{2j}\delta(n-l) - \frac{AN}{2j}\delta(n-N+l) \end{aligned}}$$

(b) $\boxed{l=0}$ $\quad x(n) = A\sin\left(\frac{2\pi l n}{N}\right) R_N(n)$

$$x(k) = \sum_{n=0}^{N-1} A\sin\left(\frac{2\pi l n}{N}\right) e^{-j\frac{2\pi n k}{N}} = \left| \begin{matrix} l=0 \\ \sin 0 = 0 \end{matrix} \right| = 0$$

(d) (i) $x_1(n) = 3\sin(0.04\pi n) R_{200}(n)$

(ii) $x_2(n) = 5\sin(10\pi n) R_{50}(n)$

(iii) $x_3(n) = [2\sin(0.5\pi n) + \sin(\pi n)] R_{100}(n)$

(iv) $x_4(n) = \sin\left(\frac{25\pi n}{16}\right) R_{64}(n)$

(v) $x_5(n) = [4\sin(0.1\pi n) - 3\sin(1.9\pi n)] R_N(n)$

(i) $\frac{2\pi}{N} = \frac{2\pi \cdot l}{200} = 0.04\pi\ ;\ l = \frac{0.04 \cdot 200}{2} = 4$

(ii) $\frac{2\pi}{50} l = 10\pi \quad l = \frac{500}{2} = 250$

$$x_2(k) = \frac{5 \cdot 50}{2j}\delta(n-250) - \frac{5 \cdot 50}{2j}\delta(n-N+250) =$$

$$= \frac{125}{j}\delta(n-250) - \frac{125}{j}\delta(n+200)$$

$\underline{n = 0 \dots N-1 = 0 \dots 49}$ $\qquad x_2(n) = 0$

(iii) $\frac{2\pi l}{N} = \frac{2\pi l}{100} = 0.5\pi\ ;\ l = \frac{50}{2} = 25\ ;\quad \frac{2\pi l}{100} = \pi \quad \boxed{l = 50}$

$$x_3(k) = \left[\frac{2 \cdot 100}{2j} \cdot \delta(k-25) - \frac{2 \cdot 100}{2j}\delta(k-100+25)\right] +$$

$$\left[\cancel{\frac{100}{2j}\delta(n-50)} - \cancel{\frac{100}{2j}\delta(n-50)}\right] = -100j\,\delta(k-25) + 100j\,\delta(k-75)$$

(iv) $\frac{2\pi l}{64} = \frac{25\pi}{16}$ $\qquad \boxed{l = 32\,\frac{25}{16} = 50}$

$$x_4(k) = \frac{64}{2j}\delta(n-50) - \frac{64}{2j}\delta(n-14) = -32j\,\delta(n-50) + 32j\,\delta(n-14)$$

(v) $\frac{2\pi}{N} \cdot l = 0.1\pi \quad l = \frac{0.1N}{2} = 0.05N$

$\frac{2\pi}{N} \cdot l = 1.9\pi \quad l = \frac{1.9N}{2} = 0.95N$

$$x_5(k) = \frac{4N}{2j}\delta(n-0.05N) - \frac{4N}{2j}\delta(n-0.95N) - \frac{3N}{2j}\delta(n-0.95N) + \frac{3N}{2j}\delta(n-0.05N)$$

$$= \frac{7N}{2j}\delta(n-0.05N) - \frac{7N}{2j}\delta(n-0.95N) = -\frac{7N}{2}j\,\delta(n-0.05n) + \frac{7N}{2}j\,\delta(n-0.95n)$$

$\boxed{5.26}$ $x_a(t) = 2\cdot\sin(4\pi t) + 5\cos(8\pi t)$; sampled $\boxed{t = 0.01n}$

$n = 0, 1, \dots, N-1$

ⓐ Chose N that woul provide accurate estimate of spectrum of $x_a(t)$. Plot real and imaginary of DFT spectrum $|X(k)|$

(i) $N=40$; (ii) $N=50$ (iii) $N=60$

$t = 0:0.01:N-1 = 0\cdot 0.01:3900$ $\boxed{N=39}$

$4\pi t = 2\pi$ $t = T$; $N = 39$ $t = 0.01\ 39 = \underline{\underline{0.39}}$

$\sin(4\pi\cdot 0.39) = \sin(1.56\pi)$

$x(n) = 2\cdot\sin(\overset{T_1}{4\pi\cdot 0.01n}) + 5\cos(\overset{T_2}{8\pi\cdot 0.01n})$

$x(n) = 2\cdot\sin(0.04\pi n) + 5\cdot\cos(0.08\pi n)$

$t = 0:0.5$ sec $x_a(t) = 2\cdot\sin(2\pi) + 5\cos(4\pi)$ ..

$\Delta t = 0.001$ $n = 0:499$ $M = 500$

$t = (0:M-1)\Delta t$

$N = 50$ $n = 0:49$

$T_1 = 0.5$ sec $T_2 = 0.25$ sec $f_1 = \frac{1}{0.5} = 2$ Hz $f_2 = 4$ Hz

$\omega_1 = 0.04\pi$ $0.2/5 = \underline{\underline{0.04}}$

$\omega_2 = 0.08\pi$

$f_1 = \frac{\omega_1}{2\pi} = \frac{0.04\cancel{\pi}}{2\cancel{\pi}} = 0.02$ Hz

$f_2 = \frac{\omega_2}{2\pi} = \frac{0.08\cancel{\pi}}{2\cancel{\pi}} = 0.04$ Hz

---

$\omega = \frac{2\pi k}{N}$ $k = \frac{\omega\cdot N}{2\pi}$ $2\pi f = \frac{2\pi k}{N}$ $f = \frac{\cancel{2\pi}k}{\cancel{2\pi}N} =$

$\Omega = \frac{\omega}{T_s} = \frac{\omega}{\Delta t}$ $F = \frac{\Omega}{2\pi} = \frac{\omega}{2\pi\Delta t} = \frac{\cancel{2\pi}k}{\cancel{2\pi}N\Delta t} = \frac{k}{N\Delta t}$

$\Delta t = 0.01$ $M = 500$ $f_1 = 2$ Hz

$t = [0:M-1]\Delta t = \underline{0:5\ \text{sec}}$ $f_2 = 4$ Hz

$x(t) = \underbrace{2\sin(4\pi t)}_{T_1} + 5\underbrace{\cos(8\pi t)}_{T_2}$

$\frac{\cancel{20\pi}}{10} = 2\pi$ $T_1 = \frac{5}{10} = 0.5$ sec $T_2 = \frac{5}{20}$ $T_2 = 0.25$ sec

$2 \cdot f_{max} \leq f_s$ $\quad \frac{1}{T_s} \geq \frac{2}{T_{max}}$ $\quad T_s \geq \frac{T_{max}}{2} = \frac{0.25}{2} = 0.125$

$\boxed{\frac{0.5}{10} = 0.05}$; $\quad \frac{0.5}{x} = 0.125 \quad x = \frac{0.5}{0.125}$

(a) $x_a(t) = 2\sin(4\pi t) + 5\cos(8\pi t)$, $\quad \boxed{t = 0.01n}$

$n = 0 : N-1$ $\quad \boxed{T = 100 \cdot N}$

$t = 0 : T-1$

$N = 40$ $\quad T = 4000$ $\quad t = 0 : T-1 = 0 : 3999$

$\omega = \frac{2\pi k}{N} = 2\pi f \qquad f = \frac{k}{N}$

$$\left.\begin{aligned} \Omega &= \omega/\Delta t \\ 2\pi F &= \omega/\Delta t \end{aligned}\right\} \quad F = \frac{1}{\cancel{2\pi}} \cdot \frac{\cancel{2\pi} k}{N\Delta t} \qquad \boxed{F = \frac{k}{N\Delta t}}$$

$$\boxed{\sin(\Omega t) = \sin(\Omega \cdot n \cdot \overset{T_s}{\Delta t}) = \sin(\Omega\Delta t \cdot n) = \sin(\omega n)}$$

$x = 2\sin(4\pi \cdot m \cdot \Delta t) + 5\cos(8\pi \cdot m \cdot \Delta t)$

$\omega_1 = 4\pi\Delta t \cdot n$; $\quad F_1 = \frac{\omega_1}{2\pi\Delta t} = \frac{4\pi\cancel{\Delta t} n}{2\pi\cancel{\Delta t}} = \underline{\underline{2\ Hz}}$

$\omega_2 = 8\pi\Delta t\, n$; $\quad F_2 = \frac{\omega_2}{2\pi\Delta t} = \frac{4\,\cancel{8\pi\Delta t}\, n}{\cancel{2\pi\Delta t}} = \underline{\underline{4\ Hz}}$

$T_1 = \frac{1}{F_1} = 0.5$; $\qquad T_2 = \frac{1}{F_2} = 0.25$

$x_a(t) = \overbrace{2 \cdot \sin(4\pi\Delta t\, m)}^{x_{a1}(t)} + \overbrace{5\cos(8\pi\Delta t\, m)}^{x_{a2}(t)}$

$\left.T_1 = 0.5 \qquad M = \frac{T_1}{\Delta t} = \left|\Delta t = 0.0001\right| = 5000\right\}$ AKO SAKAŠ DA PRIKAZES SAMO EDNA PERIODA OD x_{a1}

$\Delta t = \frac{T_1}{M} = \frac{0.5}{6000} = 83{,}3 \cdot 10^{-6}\ \underline{\underline{sec}}$

$t = m \cdot \cancel{\Delta t} = 0.01\, n\, \cancel{\Delta t}$ $\qquad \boxed{n = 100\, m}$

$T_s = 0{,}01\, \underline{T_n} \qquad T_n = 100 \cdot \underline{\underline{T_s}} = 100\,\Delta t$

$\boxed{P.5.29}$ $\quad x(n) = \cos(\pi n/99) \qquad 0 \leq n \leq N-1$

$N = 4^v \qquad v = 5, 6, \ldots 10$

$\boxed{T_{exec} \sim N \log_4 N}$ $\qquad \log_2 x = \frac{\ln x}{\ln 2}$

NUMERICAL METHODS

$f(x) = \cos(x)$ $\quad f'(x) = -\sin(x)$ $\quad F(x) = \int \cos(x)\, dx = +\sin(x) + C$

POINT : $\left(\frac{\pi}{2}, 0\right)$ SLOPE : $f'(x) = -\sin\left(\frac{\pi}{2}\right) = -1$

$$y_{tan} = m\left(x - \frac{\pi}{2}\right) = f'\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right) = -x + \frac{\pi}{2}$$

POINT : $\left(\underset{x_1}{\frac{\pi}{4}}, \underset{y_1}{\frac{\sqrt{2}}{2}}\right)$

$$y_t - \frac{\sqrt{2}}{2} = f'\left(\frac{\pi}{4}\right)\left(x - \frac{\sqrt{2}}{2}\right)$$

- $0 \le x \le \frac{\pi}{2}$ $\quad$ AREA $= \int_0^{\pi/2} \cos(x)\, dx = +\sin(x)\Big|_0^{\pi/2} = \sin\frac{\pi}{2} - 0 = 1$

- **Limits & Continuity**

$f(x)$ defined on set S of real numbers.

L - limit at $x = x_0$

$$\lim_{x \to x_0} f(x) = L$$

$\varepsilon > 0$, there exists $\delta > 0$, whenever $x \in S$ $\quad 0 < |x - x_0| < \delta$

$|f(x) - L| < \varepsilon$; $\quad x = x_0 + h$ $\quad h = x - x_0$

$$\lim_{h \to 0} f(x_0 + h) = L$$

DEFINITION 1.2: $f(x)$ is defined on set S of real numbers $x_0 \in S$

f is continous in x_0 if:

$\lim_{x \to x_0} f(x) = f(x_0)$ $\quad C^n(S) \Rightarrow$ $\boxed{\text{f \& its fist n derivates are continous on S}}$

$C^n[a,b]$ continous in interval $[a,b]$

EXAMPLE: $f(x) = x^{\frac{4}{3}}$ $\quad f'(x) = \frac{4}{3} x^{\frac{4}{3}-1} = \frac{4}{3} x^{\frac{1}{3}}$

$$\boxed{f''(x) = \frac{4}{3} \cdot \frac{1}{3} x^{\frac{1}{3}-1} = \frac{4}{9} \cdot x^{-\frac{2}{3}}}$$

DEFINITION 1.3:

$\{x_n\}_{n=1}^{\infty}$ infinite sequence; $\boxed{\lim_{n \to \infty} x_n = L}$

$\varepsilon > 0$ $\quad N = N(\varepsilon)$ $\quad n > N \Rightarrow$ $\quad |x_n - L| < \varepsilon$

conbergent sequence

$\boxed{x_n \to L \text{ as } n \to \infty}$ $\qquad \boxed{\lim_{n \to \infty} (x_n - L) = 0}$

$\{\varepsilon_n\}_{n=1}^{\infty} = \{x_n - L\}_{n=1}^{\infty}$ error sequence

THEOREM 1.1 $\quad x_0 \in S \quad f(x)$

ⓐ f continuous at x_0

ⓑ If $\lim_{n \to \infty} x_n = x_0$ $\qquad \lim_{n\to\infty} f(x_n) = f(x_0)$

$\boxed{\text{Ex. 1.1}}$ $\quad f(x) = \cos(x-1)$ $\quad$ continuous $[0, 1]$

$L = 0.8 \in [\cos(0), \cos(1)]$ $\qquad \boxed{f(a) \le L \le f(b)}$

continous: $[1, 2.5]$ $\qquad f(x) = 0.8$ $\qquad [1, 2.5]$ $\qquad x = 1.643$

$\boxed{\text{THEOREM 1.2} \quad f \in C[a,b] \quad f(a) \le L \le f(b) \quad c\text{-exist, with } c \in (a,b) \text{ such that } f(c) = L}$

$\boxed{\text{INTERMEDIATE VALUE THEOREM}}$ $\boxed{\text{FOR CONTINOUS FUNCTION}}$

THEOREM 1.3 (extreme value theorem)

$f \in C[a,b]$ $\quad$ THERE EXIST LOWER BOUND M_1, AND UPPER BOUND M_2 AND TWO NUMBERS $x_1, x_2 \in [a,b]$ SUCH THAT

$$M_1 = f(x_1) \le f(x) \le f(x_2) = M_2 \qquad x \in [a,b]$$

$$M_1 = \min_{a \le x \le b}\{f(x)\}; \qquad M_2 = f(x_2) = \max_{a \le x \le b}\{f(x)\}$$

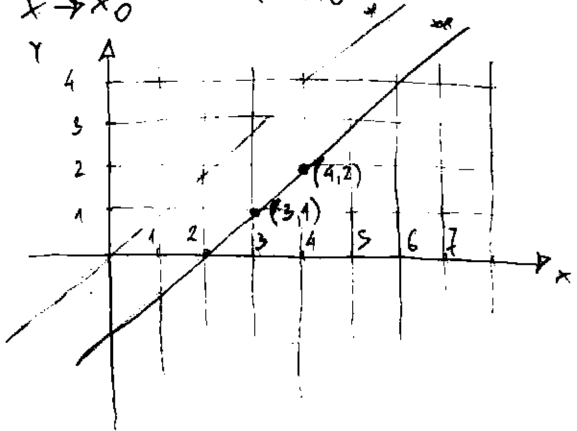
$$f(x) = 35 + 59.5x - 66.5x^2 + 15x^3$$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$\boxed{\text{Differentiable Function}}$

diferentiable at „x_0"

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \ \boxed{\text{exist}}; \qquad \lim \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0)$$



* $y = x$;

** $\Rightarrow y = x - 2$

$$y - 1 = \frac{2-1}{4-3}(x - 3)$$

$y - 1 = x - 3$; $\quad \boxed{y = x - 2}$

Theorem 1.6 (Mean Value Theorem)

$f \in C[a,b]$ $f'(x)$ exist for all $x \in [a,b]$ then c exist $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

tangent at $(c, f(c))$ has same slope as secant line at $(a, f(a))$ & $(b, f(b))$

Example 1.3 $f(x) = \sin(x)$ $[0.1, 2.1]$

$$f'(c) = \frac{\sin(2.1) - \sin(0.1)}{2.1 - 0.1}$$

$f'(x) = (\sin(x))' = \cos x$ $\cos x = 0.3816;$ $x = 1.1791$

$y - f(1.1791) = 0.3816(x - 1.1791)$ $y = 0.3816x + 0.4742$

Integrals

Theorem 1.8: (First Fundamental Theorem)

$$\int_a^b f(x)dx = F(b) - F(a) \; ; \quad F'(x) = f(x)$$

Theorem 1.9: f continous over $[a,b]$ and $x \in [a,b]$ then:

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Example 1.4 $f(x) = \cos(x)$ $[0, \pi/2]$

$$\frac{d}{dx}\int^{x^2}\cos(t)\,dt = \frac{d}{dx}\left(\sin t \Big|_0^{x^2}\right) = \frac{d}{dx}(\sin(x^2)) = 2x \cdot \cos(x^2)$$

$$\frac{d}{dx}\int_0^x \cos(t)dt = \frac{d}{dx}\sin(t)\Big|_0^x = \frac{d}{dx}[\sin(x)] = \cos(x)$$

Theorem 1.10 (Mean Value Theorem for Integrals) $f \in C[a,b]$

$c \in [a,b]$

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(c)$$

average value of $f(x)$ in interval $[a,b]$

Example 1.5. $f(x) = \sin(x) + \frac{1}{3}\sin(3x)$ $[0, 2,5]$

$F(x) = \int f(x)\,dx = \int \sin(x)\,dx + \frac{1}{3}\int \sin(3x)\,dx = -\cos(x) + \frac{1}{3\cdot 3}\int \sin($[illegible]$d(3x)$

$= -\cos(x) - \frac{1}{9}\cos(3x)$

THEOREM 1.11 $f, g \in C[a,b]$ $g(x) \geq 0$ for $x \in [a,b]$

$$\int_a^b f(x) g(x)\, dx = f(c) \int_a^b g(x)\, dx$$

EXAMPLE 1.6. $f(x) = \sin(x)$ $g(x) = x^2$ $[0, \pi/2]$

SERIES

DEFINITION 1.5 $\{a_n\}_{n=1}^{\infty}$ sequence; $\sum_{n=1}^{\infty} a_n$ is infinite series.

$S_n = \sum_{k=1}^{n} a_k$; INFINITE series converges if and only if the sequence $\{S_n\}_{n=1}^{\infty}$ converges to limit S

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = S$$

EXAMPLE 1.7 $\{a_n\}_{n=1}^{\infty} = \left\{\frac{1}{n(n+1)}\right\}_{n=1}^{\infty}$

$$\boxed{S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}}$$

$$S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} \cdots \frac{1}{k(k+1)} + \cdots + \frac{1}{N(N+1)}$$

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

$$S_n = \sum_{k=1}^{N} \frac{1}{k} - \frac{1}{k+1}$$

S_1

$$S_1 = \sum_{k=1}^{N} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{N}$$

$$S_2 = \sum_{k=1}^{N} \frac{1}{k+1} = \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} + \frac{1}{N+1}$$

$$\boxed{S_n = 1 - \frac{1}{N+1}} \qquad \boxed{S_n = \frac{N+1-1}{N+1} = \frac{N}{N+1}}$$

$$S = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

THEOREM 1.12 (Taylor's Theorem) $f \in C^{n+1}[a,b]$ $x_0 \in [a,b]$

$c = c(x)$

$f(x) = P_n(x) + \underbrace{R_n(x)}_{\text{remainder}}$; $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

Example 1.8

$f(x) = \sin(x)$ $P_n(x)$ $n = 9$ $x_0 = 0$

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \quad ; \quad f(x_0) = \sin(0) = 0 = f^{(2k)}$$

$$P_n(x) = f(0)x^0 + f'(x_0)\frac{x}{1!} + f''(x_0)\frac{x^2}{2!} + \cdots + f^{(9)}\frac{x^9}{9!}$$

$$P_n(x) = +\cos 0\, x + \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^5}{7!} + \frac{x^9}{9!}$$

$$\boxed{f(x) = \lim_{n\to\infty} P_n(x)}$$

Corollary 1.1

$P_n(x)$ - TYLOR POLINOMIAL OF DEGREE n THEN:

$$P_n^{(k)}(x_0) = f^{(k)}(x_0) \qquad k = 0, 1, 2, \dots n$$

EVALUATION OF POLYNOMIALS

$P(x) = x^5 - 6x^4 + 8x^3 - 8x^2 + 4x - 40$

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_2x^2 + a_1x + a_0$$

Horner method:

$$P_5(x) = (((((a_5x + a_4)x + a_3)x + a_2)x + a_1)x + a_1)$$

Excercises: ① $L = \lim_{n\to\infty} \frac{4n+1}{2n+1}$; $L = \lim_{n\to\infty} \frac{2n^2+6n-1}{4n^2+2n+1} = \frac{1}{2}$

② $\sum_{n=1}^{\infty} x_n$ $\lim_{n\to\infty} x_n = 2$

③ ⓐ $f(x) = -x^2 + 2x + 3$ $[-1, 0]$; $L = 2$

$\boxed{f(x) = [0, 3]}$

ⓑ $f(x) = \sqrt{x^2 - 5x - 2}$ $[6, 8]$; $L = 3$

Linear regression

1980, 82, 84, 86, ... 2000

[338,7; 341,1; 344,4; 347,2; 351,5; 354,2; 356,4; 358,9; 362,6; 366,6; 369,4]

Modeling with differencial equations:

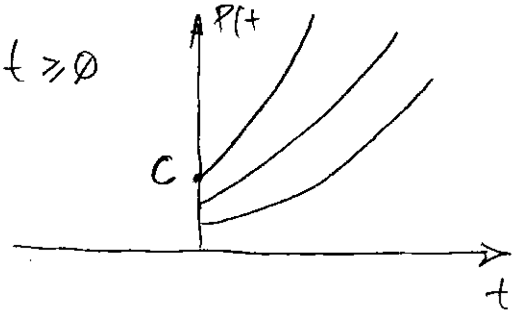
t - time

P - number of individuals in population

RATE OF GROW IS PROPORTIONAL WITH POPULATION SIZE; $\cdot ce^{kt}$

$$\frac{dP}{dt} = kP \; ; \; P(t) > 0 \;\; \forall t \qquad P = ce^{kt} \qquad P' = c \cdot k e^{kt} = kP = cke^{kt}$$

$c > 0$ $t \geq 0$

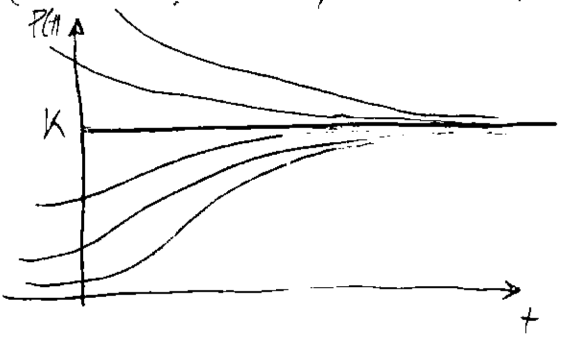


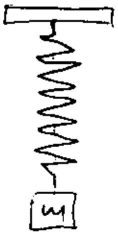
$$\frac{dP}{dt} \approx kP \quad \text{(small population)}$$

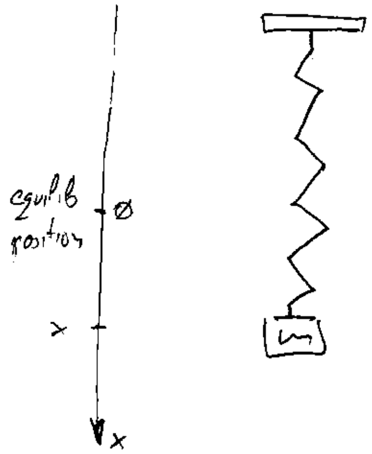
$$\frac{dP}{dt} < 0 \qquad P > K \quad \text{(P decreases if P exceeds K)}$$

logistics differ equation

$$\frac{dP}{dt} = kP\left(1-\frac{P}{K}\right) \; ; \quad \left.\begin{matrix} P(t)=0 \\ P(t)=K \end{matrix}\right\} \begin{matrix} \text{solutions} \\ \text{equilibrum solutions} \end{matrix}$$







k - STRING CONSTANT

restoring force = $-kx$

$\boxed{F = m \cdot a}$ II Newton

$$m \cdot \frac{d^2x}{dt^2} = -kx \Rightarrow \text{second order differential equation}$$

$$x(t) = C_1 \sin\left(\sqrt{\tfrac{k}{m}}\, t\right) + C_2 \cos\left(\sqrt{\tfrac{k}{m}}\, t\right)$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \qquad \omega = \frac{2\pi}{T} = \sqrt{\frac{k}{m}} \qquad T = 2\pi\sqrt{\frac{m}{k}}$$

$y' = xy$; f - SOLUTION $\quad y = f(x)$

$f'(x) = x f(x)$; $\quad y' = x^3 \quad y = \frac{x^4}{4} + C$

EXAMPLE 1 $\quad y = \dfrac{1+ce^t}{1-ce^t}$; $\quad y' = \frac{1}{2}(y^2-1)$

$$y' = \frac{ce^t(1-ce^t) - (1+ce^t)(-ce^t)}{(1-ce^t)^2} = \frac{ce^t - \cancel{c^2e^{2t}} + ce^t + \cancel{c^2e^{2t}}}{(1-ce^t)^2} = \frac{2ce^t}{(1-ce^t)^2}$$

$$y' = \frac{2ce^t}{(1-ce^t)^2} \; ; \quad \frac{1}{2} \cdot \frac{\cancel{1} + 2ce^t + \cancel{c^2e^{2t}} \cancel{- 1} + 2ce^t - \cancel{c^2e^{2t}}}{(1-ce^t)^2} = \frac{2ce^t}{(1-ce^t)^2}$$

EXAMPLE 2 $\quad y' = \frac{1}{2}(y^2-1) \quad y(0)=2$

$$y = \frac{1+ce^t}{1-ce^t} = |t=0| = \frac{1+c}{1-c} = 2 \qquad \begin{matrix} 1+c = 2-2c \\ 3c = 1 \quad \boxed{c = 1/3} \end{matrix}$$

$$\boxed{y = \frac{1+\frac{1}{3}e^t}{1-\frac{1}{3}e^t}}$$

Excercises ① $y = x - \frac{1}{x}$; solution of: $xy' + y = 2x$

$$y' = 1 + x^{-1-1} = 1 + \frac{1}{x^2} \; ; \qquad x\left(1+\frac{1}{x^2}\right) + x - \frac{1}{x} = 2x \; ;$$

$$x + \cancel{\frac{1}{x}} + x - \cancel{\frac{1}{x}} = 2x \quad \text{ok}$$

② $y = \sin x \cos x - \cos x \qquad y' + (\tan(x))y = \cos^2 x \qquad y(0) = -1$

$-\frac{\pi}{2} < x < \frac{\pi}{2}$; $\qquad y(0) = -1 = \sin 0 \cdot \cos(0) - \overset{1}{\cancel{\cos 0}}$; $y(0) = -1$

$$y' = \sin' x \cos x + \sin x \cos' x + \sin x = \cos x \cos x - \sin x \sin x + \sin x$$
$$= \cos^2 x - \sin^2 x + \sin x$$

$\cos^2 x - \sin^2 x + \sin x + \frac{\sin x}{\cos x} \cdot \left(\sin x \cos x - \cos x\right) = \cos^2(x)$

$\cos^2 x - \cancel{\sin^2 x} + \cancel{\sin x} + \cancel{\sin^2 x} - \cancel{\sin x} = \cos^2 x$ ✓

③ $k = ?$ $k \neq 0$ $y = \sin kt$ $y'' + 9y = 0$

ⓐ $y' = k \cdot \cos(kt)$ $y'' = -k^2 \sin(kt)$ $-k^2 \sin(kt) + 9 \cdot \sin kt = 0$

$\sin(kt)\left(9 - k^2\right) = 0$; $k^2 = 9$; $\boxed{k = \pm 3}$

ⓑ $y = A \sin kt + B \cos kt = A \sin 3t + B \cos 3t$

$y' = 3A \cos(3t) - 3B \sin(3t)$; $y' = -9A \sin(3t) - 9B \cos(3t)$

$-9\cancel{A \sin(3t)} - 9\cancel{B \cos(3t)} + 9\cancel{A \sin 3t} + 9\cancel{B \cos 3t} = 0$ ✓

④ $r = ?$ $\boxed{y = e^{rt}}$ $y'' + y' - 6y = 0$

$y' = r e^{rt}$; $y'' = r^2 e^{rt}$ $r^2 e^{rt} + r e^{rt} - 6 e^{rt} = 0$

$e^{rt}\left(r^2 + r - 6\right) = 0$ $r_{1,2} = \frac{-1 \pm \sqrt{1+24}}{2} = \frac{-1 \pm 5}{2} = \begin{cases} -3 \\ 2 \end{cases}$

⑤ $y'' + 2y' + y = 0$

ⓐ $y = e^t$; ⓑ $y = e^{-t}$ ⓒ $y = te^{-t}$ ⓓ $y = t^2 e^{-t}$

ⓐ $e^t + 2e^t + e^t = 0$ ✗

ⓑ $e^{-t} - 2e^{-t} + e^{-t} = 0$ $e^{-t}(1 - 2 + 1) = 0$ ✓

ⓒ $y' = e^{-t} - te^{-t}$; $y'' = -e^{-t} - \left(e^{-t} - te^{-t}\right) = te^{-t} - 2e^{-t} = e^{-t}(t-2)$

$-\underline{\underline{2e^{-t}}} + \cancel{te^{-t}} + \underline{\underline{2e^{-t}}} - 2te^{-t} + \cancel{te^{-t}} = 0$ ✓

ⓓ $y' = 2te^{-t} - t^2 e^{-t}$ $y'' = 2\left(e^{-t} - te^{-t}\right) - \left(2te^{-t} - t^2 e^{-t}\right) =$

$= 2e^{-t} - 2te^{-t} - 2te^{-t} + t^2 e^{-t} =$

$= 2e^{-t} - 4te^{-t} + t^2 e^{-t}$

$2e^{-t} - \cancel{4te^{-t}} + \cancel{t^2 e^{-t}} + \cancel{4te^{-t}} - 2\cancel{t^2 e^{-t}} + \cancel{t^2 e^{-t}} = 0$ $2e^{-t} = 0$ #

ⓑ $y = Ce^{x^2/2}$ $\boxed{y' = xy}$ $y' = c \cdot \frac{2x}{2} \cdot e^{x^2/2} = cx e^{x^2/2}$

$xy = Cx e^{x^2/2}$

$\underline{y(0) = 5}$ $y = C \cdot e^0$; $y(0) = C = 5$ $\boxed{C = 5}$

$\boxed{y = 5 e^{x^2/2}}$

$y' = xy$

ⓓ $y(1) = 2$ $ce^{x^2/2} = 2$; $c = 2 \cdot e^{-x^2/2} = 2 \cdot e^{-1/2} = \frac{2}{\sqrt{e}}$

$y = 2 \cdot e^{-1/2} \cdot e^{\frac{x^2}{2}} = 2 e^{(x^2-1)/2}$

$$⑦ ⓐ\quad y' = -y^2 \qquad y = \frac{x^3}{3} \qquad y' = \left(\frac{x^3}{3}\right)' = \frac{3x^2}{3} = x^2$$

$$\left(\frac{1}{x}\right)' = \left(x^{-1}\right)' = -x^{-2} \quad ; \quad y = \frac{1}{x} \; ; \; y' = -x^{-2} = -\frac{1}{x^2} = -(y)^2$$

$$ⓑ \quad y = \frac{1}{x+c} \Rightarrow \text{FAMILY OF SOLUTION}$$

$$LHS = y' = -x^{-2} = -\frac{1}{x^2} \qquad y = \frac{1}{x} \qquad -y^2 = -\frac{1}{x^2} = RHS \qquad LHS = RHS$$

$$y' = \left(\frac{1}{x+c}\right)' = -(x+c)^{-1-1} = -\frac{1}{(x+c)^2} = LHS$$

$$RHS = -y^2 = -\left(\frac{1}{x+c}\right)^2 \qquad RHS = LHS$$

$$ⓓ \quad y' = -y^2 \qquad y(0) = 0.5 \qquad \frac{1}{x+c} = 0.5 \qquad \frac{1}{c} = \frac{1}{2} \qquad \boxed{c = 2}$$

$$\boxed{y = \frac{1}{x+2}}$$

$$⑧ \quad y' = xy^3 \qquad y = e^{-\frac{2}{x^2}} \qquad y' = \left(-\frac{2}{x^2}\right)' \cdot e^{-\frac{2}{x^2}} = \frac{4}{x^3} \cdot e^{-\frac{2}{x^2}}$$

$$y = \frac{1}{(x+c)^2} \; ; \quad y' = -\frac{2}{(x+c)^3} \; ; \quad y = \frac{1}{\sqrt{x+c}} \; ; \quad y' = \left[(x+c)^{-\frac{1}{2}}\right]'$$

$$y' = -\frac{1}{2}(x+c)^{-\frac{1}{2}-1} = -\frac{1}{2}(x+c)^{-\frac{3}{2}} = -\frac{1}{2}y^3 \qquad \#$$

$$y = \frac{1}{(x^2+c)^{1/2}} \; ; \qquad y' = -\frac{\cancel{2}x}{\cancel{2}}(x^2+c)^{-\frac{1}{2}-1} = -x(x^2+c)^{-3/2} = -xy^3 \;\#$$

$$RHS = y = \frac{1}{(-x^2+c)^{1/2}} \; ; \quad y' = -2x\left(-\frac{1}{2}\right)(-x^2+c)^{-1/2-1} = x\,(c-x^2)^{-3/2} = x \cdot y^3$$

$$LHS \quad xy^3 = x \cdot \frac{1}{(-x^2+c)^{3/2}} = RHS$$

$$ⓓ \quad y' = xy^3 \; ; \quad y(0) = 2 \quad ; \quad y(0) = \frac{1}{(-\underset{\downarrow 0}{x^2} + c)^{1/2}} = \frac{1}{\sqrt{c}} = 2 \; ; \quad \boxed{c = \frac{1}{4}}$$

$$\boxed{y = \frac{1}{\left(\frac{1}{4} - x^2\right)^{1/2}}}$$

$$⑨ \quad \frac{dP}{dt} = 1.2P\left(1 - \frac{P}{4200}\right)$$

ⓐ $P \leq 4200 \quad \frac{dP}{dt} \geq 0$

ⓑ $P > 4200 \quad \frac{dP}{dt} < 0$

ⓒ $P = 4200 \quad \frac{dP}{dt} = 0$

$P = 0$

$$⑩ \quad \frac{dy}{dt} = y^4 - 6y^3 + 5y^2 \; ; \qquad \text{IF: } y = c \quad \frac{dy}{dt} = 0$$

$$y^4 - 6y^3 + 5y^2 = 0 \qquad y^2(y^2 - 6y + 5) = 0 \qquad y_1 = 0$$

$$y_{2,3} = \frac{6 \pm \sqrt{36-20}}{2} = \frac{6 \pm 4}{2} = \begin{cases} 1 \\ 5 \end{cases} \qquad y_2 = 1 \quad y_3 = 5$$

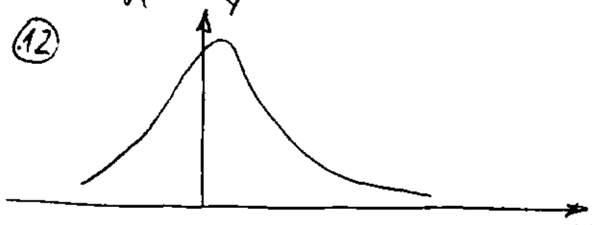
$$\boxed{y^2(y-1)(y-5) = 0} \qquad \frac{dy}{dt} > 0 \quad y \in (-\infty, 0) \cup (0,1) \cup (5,\infty)$$

$$\frac{dy}{dt} < 0 \qquad y \in (1,5)$$

N40

⑪ $\frac{dy}{dt}=e^{t}(y-1)^{2}$

⑫



ⓐ $y' = 1+xy$

ⓑ $y' = -2xy$

ⓒ $\boxed{y' = 1-2xy}$

$y \sim c\,e^{-\frac{x^2}{2}}$

$$y' = -\frac{\cancel{2}x}{\cancel{2}} \cdot c\, e^{-\frac{x^2}{2}} = -x \cdot c\, e^{-x^2/2}$$

$$y \sim e^{-\frac{(x-c)^2}{2}} \qquad y' = -\frac{\cancel{2}(x-c)}{\cancel{2}} \cdot e^{-\frac{(x-c)^2}{2}} = -x\, e^{-\frac{(x-c)^2}{2}} + c\, e^{-\frac{(x-c)^2}{2}}$$

$$y = e^{-(x-c)^2} ; \qquad y' = -2(x-c) \cdot e^{-(x-c)^2} = -2x\, e^{-(x-c)^2} + c\, e^{-(x-c)^2}$$

$$\boxed{\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t/2}\, dt}$$

$$\operatorname{erf}(x) = \left| \begin{array}{l} u = -t/2 \\ du = -\frac{1}{2}\, dt \\ t=0 \quad u=0 \\ t=x \quad u=-\frac{x}{2} \end{array} \right| = \frac{2}{\sqrt{\pi}}(-2)\int_0^{-x/2} e^{+u}\, du$$

$$\operatorname{erf}(x) = -\frac{4}{\sqrt{\pi}}\, e^{u}\Big|_0^{-x/2} = \frac{4}{\sqrt{\pi}}\, e^{u}\Big|_{-x/2}^{0} = \frac{4}{\sqrt{\pi}}\left(1 - e^{-x/2}\right)$$

$$y = \left(-\frac{1}{2}\, i\sqrt{\pi}\, \operatorname{erf}(ix) + 2\right) e^{-x^2}$$

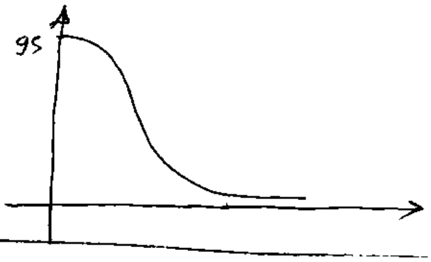
⑬ $\frac{dP}{dt} = k(M-P)$ $\qquad \frac{dP}{dt} = kP\left(\frac{M}{P} - 1\right)$ $\qquad$ M - MAXIMUM LEVEL OF PERFOMANCE

$k=1$; $\quad M=2$; $\qquad y = 2 + e^{-t}$

⑭ coffee $= 95°C$; $\quad$ ROOM TEMPERATURE $= 20°$ $\qquad \frac{T_c = 95°C}{\boxed{C = T_R = 20°C}}$

$$\frac{dT}{dt} = (T_R - T)t \qquad T(0) = 95 \qquad T(0) - T_R = 95 - 20 = 75°$$

$$T(t) = c \cdot e^{-t^2/2} \qquad T(t) = 20 + 75\, e^{-\frac{t^2}{2}}$$



$$\boxed{T(0) = 20 + 75\, e^{0} = 95°}$$

Newton's law of cooling

$$\frac{dT}{dt} = (T_R - T) \qquad \boxed{T = T_R + (C - T_R)\, e^{-t}}$$

$$T(0) = 95 \qquad T(0) = T_R + (C - T_R)\, e^{-0} = \cancel{T_R} + C - \cancel{T_R} = 95 \quad \boxed{C = 95}$$

$$\boxed{T(t) = T_R + (T(0) - T_R)\, e^{-t}}$$

DIRECTION FIELDS & EULER'S METHOD

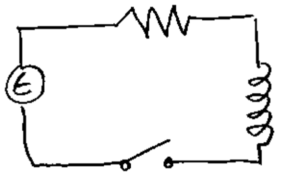
$y' = x + y \qquad y(0) = 1$

$y(x) = -1 - x + 2e^{x}$

$y'(x) = 0 - 1 + 2e^{x} \qquad x + y = \cancel{x} - 1 - \cancel{x} + 2e^{x} = \underline{-1 + 2e^{x}}$

$$Y' = F(x, Y)$$

Example 1 $Y' = x^2 + Y^2 - 1$



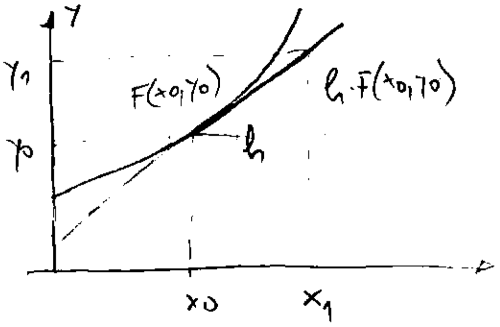
$$\epsilon(t) = R \cdot I + L \frac{dI}{dt}$$

example 2 $R = 12\,\Omega$, $L = 4H$; $\epsilon(t) = U = 60V$

$L \frac{dI}{dt} + RI = V$; $4 \frac{dI}{dt} + 12 \cdot I = 60$; $\boxed{\frac{dI}{dt} = 15 - 3I}$

$i(t) = 5 - 5e^{-3t}$; $\frac{di(t)}{dt} = \frac{V}{L} - \frac{R}{L} i(t)$, $\boxed{i(t) = \frac{V}{R} - \frac{V}{R} e^{-\frac{R}{L}t}}$

EULER'S METHOD



$Y' = x + Y$ $Y(0) = 1$

$F(x, Y) = x + Y$

$F(x_0, Y_0) = x_0 + Y_0$

$Y' = F(x, Y)$ $Y(x_0) = Y_0$

$Y_1 = Y_0 + h\,F(x_0, Y_0)$

$Y_2 = Y_1 + h\,F(x_1, Y_1)$

example 3 $Y' = x + Y$ $Y(0) = 1$ $\boxed{h = 0.1}$

$x_0 = 0$

$Y_0 = 1$

$F(x_0, Y_0) = 1$

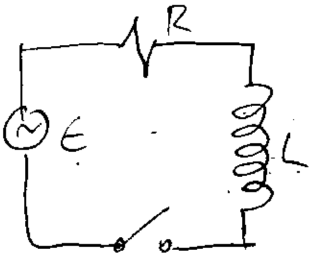
$x_1 = 0.1$

$Y_1 = 1 + F(x_0, Y_0) \cdot 0.1$

$= 1.1$

$F(x_1, Y_1) = x_1 + Y_1 = 2.1$

example 4



$$L \frac{dI}{dt} + RI = \epsilon$$

$\epsilon = 60$; $L = 4H$; $R = 12$

$4 \frac{dI}{dt} + 12 \cdot I = 60$; $h = 0{,}1$

$\boxed{\frac{dI}{dt} = 15 - 3I}$; $F(\overset{I_0}{0}, \overset{t_0}{0}) = 15 - 0 = 15$; $F(I_0, t_0) = 15$

$I_1 = I_0 + 0.1 \cdot F(I_0, t_0) = 0 + 0.1 \cdot 15 = 1{,}5$; $F(I_1, \overset{0,1}{t_1}) = 10{,}5$

$I_2 = I_1 + 0.1 \cdot 10{,}5 = 1{,}5 + 1{,}05 = 2{,}55$ $F(I_2, \overset{0,2}{t_2}) = 7{,}35$

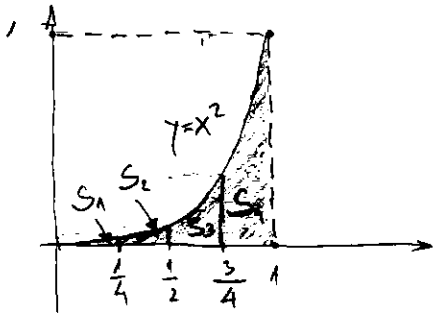
$I_3 = I_2 + 0.1 \cdot 7{,}35 = 2{,}55 + 0{,}735 = 3.285$; $F(I_3, \overset{0,3}{t_3}) = 5{,}145$

$I_4 = I_3 + 0.1 \cdot 5{,}145 = 3{,}285 + 0{,}5145 = 3.7995$; $F(I_4, \overset{0.4}{t_4}) = 3.6015$

$I_5 = I_4 + 0.1 \cdot 3.6015 = 3.7995 + 0.36015 = 4{,}15965$; $F(I_5, \overset{0.5}{t_5}) = 2.52105$

$\boxed{\text{CH 5}}$

$\boxed{\text{EXAMPLE 1}}$ USE RECTANGLES TO ESTIMATE AREA UNDER PARABOLA $\boxed{y = x^2}$ FROM 0 TO 1.



$$R_4 = \frac{1}{4}\left(\left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{4}\right)^2 + 1^2\right) = 0.46875$$

$$A = S_1 + S_2 + S_3 + S_4 < R_4$$

$$L_4 = \frac{1}{4}\left(\left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{4}\right)^2\right) = 0.21875$$

$$0.21875 < A < 0.46875$$

$R_8 = 0.39844$; $L_8 = 0.27344$; $L_{1000} = 0.3328$; $R_{1000} = 0.3338$;

$$\int_0^1 x^2 dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3} = 0.\bar{3}33$$

$\boxed{\text{example 2}}$ $\boxed{\lim_{n\to\infty} R_n = \frac{1}{3}}$

$$R_n = \frac{1}{n}\left(\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \left(\frac{3}{n}\right)^2 + \dots + \left(\frac{n}{n}\right)^2\right) = \frac{1}{n}\cdot\frac{1}{n^2}\Big(\underbrace{1 + 2^2 + 3^2 + \dots + n^2}_{*}\Big)$$

$$* = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(2n^2 + 3n + 1)^{*}}{6}$$

$\textcircled{2}n^2 + 3n + 1 = 0$ $\quad n_1 = -\frac{1}{2}$ $\quad n_2 = -1$

NE VAŽI FORMULATA ZA PRIKAŽ SO ROOT-OVI AKO PRED „X^2" IMA KOEFICIEN (TREBA DA SE IZVLEČE NANAPRED!)

$$2\left(n + \frac{1}{2}\right)(n+1) = \frac{\cancel{2}}{\cancel{2}}(2n+1)(n+1) \;;\quad * = \frac{n(2n+1)(n+1)}{6}$$

$$(2n+1)(n+1) = 2n^2 + 2n + n + 1 = 2n^2 + 3n + 1$$

$$R_n = \frac{n(2n+1)(n+1)}{6n^3}$$

$$(1 - a^{-1}n)(1 - b^{-1}n) = (1 + 2n)(1 + n)$$

$\textcircled{2}x^2 + 5x + 2 = 0 \quad x_1 = -\frac{1}{2}\;; \quad x_2 = -2$

$$2(1 - a^{-1}x)(1 - b^{-1}x) = 2(1 + 2x)\left(1 + \frac{1}{2}x\right) = \frac{\cancel{2}}{\cancel{2}}(1 + 2x)(2 + x) = (1+2x)(2+x)$$

$$(x+2)(1+2x) = x + 2x^2 + 2 + 4x = 2x^2 + 5x + 2$$

$$(x - x_1)(x - x_2) = (x + 0.5)(x + 2) = x^2 + 2x + 0.5x + 1 = x^2 + 2.5x + 1$$

$$2x^2 + 5x + 2 = 0 \qquad x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-5 \pm \sqrt{25 - 16}}{4}$$

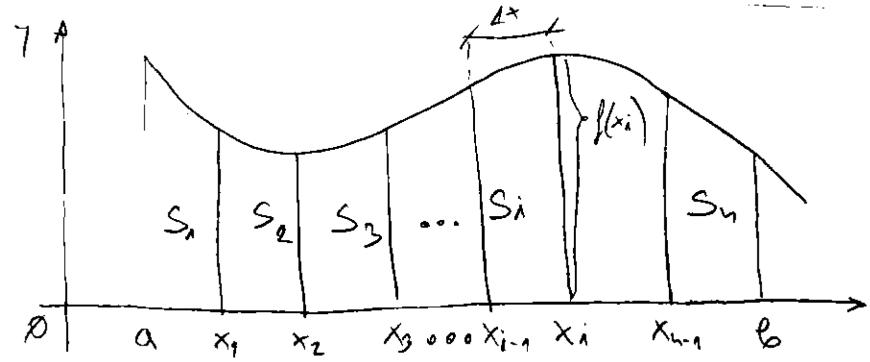
$$x_{1,2} = \frac{-5 \pm \sqrt{9}}{4} = \frac{-5 \pm 3}{4} \qquad \boxed{\begin{matrix} x_1 = -2 \\ x_2 = -\frac{1}{2} \end{matrix}}$$

$$6x^2 + 18x - 24 = 6(x-1)(x+4) = 6(x^2 + 4x - x - 4) = 6x^2 + \dots - 24$$

$$6(x^2 + 3x - 4)$$

$$R_n = \frac{\cancel{5}(2n+1)(n+1)}{6n^{\cancel{3}}} = \frac{(2n+1)(n+1)}{6n^2}$$

$$R_n = \frac{2n^2+2n+n+1}{6n^2} = \frac{2n^2+2n+1}{6n^2} \quad ; \quad \lim_{n\to\infty} \frac{2n^2+3n+1}{6n^2} = \frac{1}{3}$$



$$\Delta x = \frac{b-a}{n}$$

$[x_0, x_1]$; $[x_1, x_2]$; $[x_2, x_3]$; ...

$[x_{n-1}, x_n]$

$x_0 = a$; $x_n = b$

$x_1 = a + \Delta x$; $x_2 = a + 2\Delta x$, $x_3 = a + 3\Delta x$; ... $x_n = a + n\Delta x$

$f(x_i)\Delta x$; $R_n = f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_i)\Delta x + \dots + f(x_n)\Delta x$

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x]$$

$$A = \lim_{n\to\infty} [f(x_0)\Delta x + f(x_1)\Delta x + \dots + f(x_{n-1})\Delta x]$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \quad ; \quad A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x \quad ; \quad A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

---

example 3

$$S = \sum_{i=1}^{N} i^2 \quad ; \quad \int S\, di = \sum_{i=1}^{N} \int i^2\, di = \sum_{i=1}^{N} \frac{i^3}{3} + C$$

A - AREA OF THE REGION UNDER THE GRAPH $f(x) = e^{-x}$

$x = 0 \div 2$

ⓐ Using the right endpoints find A as a limit.

ⓑ Sample points at midpoints

ⓐ $a = 0$ $b = 2$ $\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$

$x_1 = \frac{2}{n}$; $x_2 = a + 2\Delta x = 0 + \frac{4}{n}$, $x_3 = a + 3\Delta x = \frac{6}{n}$, $x_i = \frac{2i}{n}$; $x_n = \frac{2n}{n}$

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_i)\Delta x + \dots + f(x_n)\Delta x$$

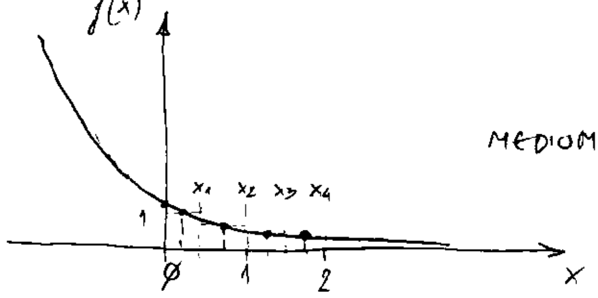
$$= e^{-\frac{2}{n}} \cdot \frac{2}{n} + e^{-\frac{4}{n}} \frac{2}{n} + \dots + e^{-\frac{2i}{n}} \frac{2}{n} + \dots + e^{-\frac{2n}{n}} \frac{2}{n}$$

$$R_n = \sum_{i=1}^{n} e^{-\frac{2i}{n}} \cdot \frac{2}{n} \quad ; \quad \boxed{A = \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} e^{-\frac{2i}{n}}}$$

$$\int_0^2 e^{-x}\,dx = -e^{-x}\Big|_0^2 = -e^{-2} + 1$$

$$= \left| \begin{matrix} -x = y \\ dx = -dy \\ e^{-x} = e^{y} \end{matrix} \;\middle|\; \begin{matrix} x = 0;\ y = 0 \\ x = 2 \quad y = -2 \end{matrix} \right| = -\int_0^{-2} e^{+y}\,dy = -e^{y}\Big|_0^{-2} = -e^{-2} + e^{0} = 1 - e^{-2}$$

ⓑ $\boxed{n=4}$ $\quad x_1 = \frac{2}{n} = 0.5;\quad x_2 = \frac{4}{n} = 1;\quad x_3 = \frac{6}{n} = 1.5;\quad x_4 = \frac{8}{n} = 2$



$$\Delta x = \frac{2}{n} = \frac{2}{4} = 0.5$$

MEDIUM P.: $x_1 = 0 + \frac{\Delta x}{2}$

$x_2 = x_1 + \frac{\Delta x}{\cancel{2}} = 0 + \frac{3\Delta x}{2}$

$x_3 = x_2 + \Delta x = \frac{3\Delta x}{2} + \Delta x = \frac{5\Delta x}{2}$

$x_4 = \frac{7\Delta x}{2} \qquad x_i = \frac{2i-1}{2}\Delta x$

$$M_n = f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x$$

$$\boxed{M_n = \sum_{i=1}^{n} \frac{2}{n} \cdot e^{-\frac{2i-1}{2}}}$$

$$M_4 = 0.25 \cdot e^{-0.25} + 0.25 \cdot e^{-0.75} + 0.25\, e^{1.25} + 0.25\, e^{-1.75} = 0.8557$$

distance problem: $\quad d = \lim_{n\to\infty} \sum_{i=1}^{n} v(t_{i-1})\Delta t = \lim_{n\to\infty} \sum_{i=1}^{n} v(t_i)\Delta t$

The definite integral

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty} \left[f(x_1^*)\Delta x + f(x_2^*)\Delta x + \dots + f(x_n^*)\Delta x \right]$$

~~Left~~ Points

Right Points

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

$\boxed{\varepsilon > 0}$
$$\left| \int_a^b f(x)\,dx - \sum_{i=1}^{n} f(x_i^*)\Delta x \right| < \varepsilon \qquad n > N$$

$\boxed{\text{EXAMPLE 1}}$
$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(x_i^3 + x_i \sin x_i \right) \Delta x$$

$$S = a_1 + a_2 + \dots a_i + \dots + a_n$$

$a_2 = a_1 + d$

$a_3 = a_2 + d = a_1 + 2d$

$$S = a_1 + a_1 + d + a_1 + 2d + \; a_1 + (n-1)d \; + \; a_1 + (n-1)d$$
$$= n \cdot a_1 + d + 2d + \dots + (n-1)d = n a_1 + d(1 + 2 + \dots + n - 1)$$

$$\sum_{i=1}^{n-1} i = \frac{1}{2} n(n-1); \qquad S = n \cdot a_1 + \frac{n d}{2}(n-1) =$$

$$= \left| \begin{matrix} a_1 = 1 \\ d = 1 \end{matrix} \right| = n + \frac{n^2}{2} - \frac{n}{2} = \frac{n^2}{2} + \frac{n}{2} = \frac{n}{2}(n+1)$$

$$a_n = a_1 + (n-1)d$$

$$a_n = a_1 + (n-1)d$$

$$S = a_1 + a_2 + \cdots + a_n$$

$$S = a_n + a_{n-1} + \cdots + a_1 \quad +$$

$$2S = n \cdot (a_1 + a_n)$$

$$\boxed{S = \frac{n}{2}(a_1 + a_n)}$$

$$a_n = a_1 + (n-1)d \quad ; \quad S = \frac{n}{2}\left(a_1 + a_1 + (n-1)d\right) = \frac{n}{2}\left(2a_1 + (n-1)d\right)$$

$$a_1 = 1 \; ; \; d = 1 \quad ; \quad S = \frac{n}{2}(2 + n - 1) = \frac{n}{2}(n+1)$$

$$\boxed{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}}$$

$$S = 1 + 2^2 + 3^2 + \cdots + n^2 = \sum_{i=1}^{n} i^2$$

$$S' = \sum_{i=1}^{n} 2 \cdot i = 2 \sum_{i=1}^{n} i = 2 \cdot \frac{n(n+1)}{2}$$

$$S = \sum_{n=1}^{\infty} n \cdot x^n \quad ; \quad \int S\,dx = \sum_{n=1}^{\infty} n \int x^n\,dx = \sum_{n=1}^{\infty} n \frac{x^{n+1}}{n+1} + C$$

$$S = x \sum_{n=1}^{\infty} n \cdot x^{n-1} \quad ; \quad \int \frac{S}{x}\,dx = \sum_{n=1}^{\infty} \cancel{n} \frac{x^n}{\cancel{n}} = \sum_{n=1}^{\infty} x^n = \frac{1}{1-x} \Big/ (\;)'$$

$$\frac{S}{x} = \left(\frac{1}{1-x}\right)' = -1(1-x)^{-2}(-1) + \frac{1}{(1-x)^2} \qquad S = \frac{x}{(1-x)^2}$$

$$\sum_{i=1}^{n-1} i^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n = n\left(\frac{n^2}{3} + \frac{n}{2} + \frac{1}{6}\right) = \frac{n}{6}\left(2n^2 + 3n + 1\right)$$

$$= \frac{n}{3}\left(n^2 + \frac{3}{2}n + \frac{1}{2}\right) = \frac{n}{3}\left(n + \frac{1}{2}\right)(n+1) = \frac{n}{3}\,\frac{2n+1}{2} \cdot (n+1)$$

$$S = \frac{n(2n+1)(n+1)}{6} \;//$$

$$\boxed{\sum_{i=1}^{n} i^2} \qquad \sum_{i=1}^{n}\left[(1+i)^3 - i^3\right] = (\cancel{2^3} - 1^3) + (\cancel{3^3} - \cancel{2^3}) + (4^3 - \cancel{3^3}) +$$

$$+ (1+n)^3 - \cancel{n^3} = (n+1)^3 - 1^3 =$$

$$= n^3 + 3n^2 + 3n + \cancel{1} - \cancel{1} = \overline{n^3 + 3n^2 + 3n}$$

$$\sum_{i=1}^{n}\left[\cancel{i^3} + 3i^2 + 3i + 1 - \cancel{i^3}\right] = \sum_{i=1}^{n} 3i^2 + 3i + 1 = 3 \cdot S + 3 \cdot \frac{(n+1)n}{2} + n$$

$$3S + \frac{3}{2}n^2 + \frac{3}{2}n + n = 3S + \frac{3n^2}{2} + \frac{5n}{2}$$

$\boxed{075442770}$

$$3S + \frac{3n^2}{2} + \frac{5n}{2} = n^3 + 3n^2 + 3n$$

$$S = \frac{1}{3}\left[n^3 + \frac{6n^2}{2} + \frac{6n}{2} - \frac{3n^2}{2} - \frac{5n}{2}\right] = \frac{1}{3}\left[n^3 + \frac{3n^2}{2} + \frac{n}{2}\right] =$$

$$= \frac{n}{3}\left(n^2 + \frac{3n}{2} + \frac{1}{2}\right) = \frac{n}{3}\left(n + \frac{1}{2}\right)(n+1) = \frac{n}{3}\,\frac{2n+1}{2}\,(n+1) = \underline{\underline{\frac{n(2n+1)(n+1)}{6}}}$$

$$\boxed{\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2} \qquad \boxed{\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}\ ;\ \sum_{i=1}^{n} i = \frac{n(n+1)}{2}}$$

EXAMPLE 2 Evaluate Riemann sum. $f(x) = x^3 - 6x$; right sample points; $a = 0$; $b = 3$; $n = 6$

$$\boxed{\frac{1-q^n}{1-q}}$$

$$S = \sum_{i=1}^{n} f(x_i)\Delta x \ ; \quad \Delta x = \frac{b-a}{n} = \frac{3}{6} = 0.5$$

$x_0 = 0$; $x_1 = \frac{1}{2}$; $x_2 = \frac{2}{2}$; $\boxed{x_i = \frac{i}{2}}$

$$S = \sum_{i=1}^{6}(x_i^3 - 6x_i)\cdot 0.5 = 0.5\left[\sum_{i=1}^{6} x_i^3 - 6\sum_{i=1}^{6} x_i\right] =$$

$$= 0.5\left[\left[\frac{n(n+1)}{2}\right]^2 - 6\,\frac{n(n+1)}{2}\right] = 0.5\left\{\left[\frac{6(7)}{2}\right]^2 - 6\,\frac{6\cdot 7}{2}\right\} =$$

$$= 0.5\left\{(3\cdot 7)^2 - 6\cdot 21\right\} = 157.5$$

$$\boxed{\begin{aligned} S &= \sum_{i=1}^{6}\left[(0.5i)^3 - 6(0.5i)\right]0.5 = 0.5\sum_{i=1}^{6}(0.125i^3 - 3i) = \\ &= 0.5\left[0.125\left(\frac{6\cdot 7}{2}\right)^2 - 3\cdot\frac{6\cdot 7}{2}\right] = 0.5\left[0.125(21)^2 - 3\cdot 21\right] = -3.9375 \end{aligned}}$$

example 3: ⓐ $\int_1^3 e^x\,dx$ $\quad \lim_{n\to\infty}\sum_{i=1}^{n} e^{x_i}\Delta x$ ⓑ use CAS to evaluate the expression

$a = 1$; $b = 3$; $n = 10 \Rightarrow \Delta x = \frac{3-1}{10} = \frac{2}{10} = \frac{1}{5} = 0.2$

$$\Delta x = \frac{2}{n}; \quad \lim_{n\to\infty}\sum_{i=1}^{n}\frac{2\cdot e^{x_i}}{n} = \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n} e^{x_i} = (*)$$

$x_0 = 1$; $x_1 = 1 + \frac{2}{n}$; $x_2 = 1 + \frac{4}{n}$; $x_i = 1 + \frac{2i}{n}$

$$(*) = \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n} e^{\left(1+\frac{2i}{n}\right)} = \lim_{n\to\infty}\frac{2\cdot e}{n}\underbrace{\sum_{i=1}^{n} e^{\frac{2i}{n}}}_{S}$$

$$S = \frac{1 - e^{\frac{2}{n}\cdot n}}{1 - e^{\frac{2}{n}}} = \frac{1 - e^2}{1 - e^{2/n}}$$

$$S = q + q^2 + q^3 + \dots + q^n$$
$$qS = q^2 + q^3 + \dots + q^{n+1}$$
$$(1-q)S = q - q^{n+1} = q(1-q^n) \qquad S = \frac{q(1-q^n)}{1-q}$$

$$\sum_{i=1}^{n} e^{\frac{2i}{n}} = \frac{e^{\frac{2}{n}}\left(1 - e^{\frac{2}{n}\cdot n}\right)}{1 - e^{\frac{2}{n}}} = e^{2/n}\frac{1-e^2}{1-e^{2/n}}$$

$$\lim_{n\to\infty} \frac{2e}{n}\sum_{i=1}^{n} e^{\frac{2i}{n}} = \lim_{n\to\infty}\frac{2e}{n}\,\frac{e^{2/n} - e^{\frac{2+2n}{n}}}{1-e^{+2/n}} =$$

$$= \lim_{n\to\infty}\frac{2}{n}\,\frac{e^{\frac{n+2}{n}} - e^{\frac{2+2n+n}{n}}}{1-e^{2/n}} = \lim_{n\to\infty}\frac{2}{n}\,\frac{e^{\frac{3n+2}{n}} - e^{(n+2)/n}}{e^{2/n}-1}$$

EXAMPLE 4 (a) $\int_0^1 \sqrt{1-x^2}\,dx$ (b) $\int_0^3 (x-1)\,bx$

(a)

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n} \qquad x_0 = 0 \; ; \; x_1 = 0 + \frac{1}{n} \; ; \; x_2 = 0 + \frac{2}{n}$$

$$\boxed{x_i = \frac{i}{n}} \qquad f(x_i)$$

$$f(x_i) = \sqrt{1-\left(\frac{i}{n}\right)^2} \; ; \quad \lim_{n\to\infty}\sum_{i=1}^{n}\overset{\Delta x}{\frac{1}{n}}\cdot\overset{f(x_i)}{\sqrt{1-\left(\frac{i}{n}\right)^2}} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sqrt{1-\left(\frac{i}{n}\right)^2}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\sqrt{n^2-i^2}}{n^2} \; ; \quad \int_0^1 \cos\left(\frac{\pi}{2}x\right)dx = \left|\begin{matrix} u = \frac{\pi}{2}x \\ du = \frac{\pi}{2}dx \\ x=0\,;\, u=0 \\ x=1\,;\, u=\pi/2 \end{matrix}\right| = \int_0^{\pi/2}\cos(u)\cdot\frac{2}{\pi}\,du$$

$$= \frac{2}{\pi}\left(+\sin u \Big|_0^{\pi/2}\right) = \frac{2}{\pi}\left(\sin\frac{\pi}{2} - 0\right) = \frac{2}{\pi}$$

(b)

$$\int_0^3 (x-1)\,dx = \int_0^3 x\,dx - \int_0^3 dx = \left(\frac{x^2}{2} - x\right)\Big|_0^3 = \frac{9}{2} - 3 - 0 = \frac{9-6}{2} = \frac{3}{2}$$

$$\Delta x = \frac{3-0}{n} = \frac{3}{n} \; ; \quad x_0 = 0 \; ; \; x_1 = \frac{3}{n} \; ; \; x_2 = \frac{3}{n}\cdot 2 \; ; \; \dots \; x_i = \frac{3i}{n}$$

$$\lim_{n\to\infty}\underbrace{\sum_{i=1}^{n}\left(\frac{3i}{n} - 1\right)\cdot\frac{3}{n}}_{(*)} = \lim_{n\to\infty}\frac{3}{2}\left(1 + \cancel{\frac{3}{n}}\right) = +\frac{3}{2}$$

[redacted]

$$(*) = \frac{3}{n}\left[\frac{3}{n}\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1\right] = \frac{3}{n}\left[\frac{3}{n}\,\frac{n\cdot(n+1)}{2} - n\right] = \frac{3}{n}\,\frac{3n^2+3n-2n^2}{2n}$$

$$= \frac{3}{n}\,\frac{n^2+3n}{2n} = 3\,\frac{n+3}{2n} = \frac{3}{2}\left(1 + \frac{3}{n}\right)$$

$\boxed{\text{THE MIDPOINT RULE}}$ $$\int_a^b f(x)\,dx \doteq \sum_{i=1}^{n} f(\bar{x}_i)\Delta x = \Delta x\left[f(\bar{x}_1) + f(\bar{x}_2) + \dots + f(\bar{x}_n)\right]$$

$$\Delta x = \frac{b-a}{n} \qquad \bar{x}_i = \frac{1}{2}\left(x_{i-1} + x_i\right) = \text{midpoint}\left[x_{i-1}, x_i\right]$$

FILES : 754
FOLDERS : 64

EXAMPLE 5: USE MIDPOINT RULE FOR APPROXIMATING $\int_1^2 \frac{1}{x}\,dx$ for $n=5$?

$$\sum_{i=1}^{n} f(\bar{x}_i)\Delta x \,; \qquad \Delta x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$$

49213

$$\bar{x}_i = (x_{i-1} + x_i)/2 \,; \qquad f(x) = \frac{1}{x}$$

~~$f(x_1) = 1, f(x_2) = 2$~~

~~[illegible] $=$ $\Delta x = \frac{1}{6} = 0.1666$; $\bar{x}_1 = \frac{1}{n}\cdot\frac{1}{2}$,~~

~~$\bar{x}_2 = x_1 + \frac{1}{n} = \frac{1}{2n} + \frac{1}{n} = \frac{3}{2n}$; $\bar{x}_3 = x_2 + \frac{1}{n} = \frac{3}{2n} + \frac{1}{n} = \frac{5}{2n}$~~

~~$\dots\ x_i = \frac{2i-1}{2n}$; $x_n = \frac{2n-1}{2n}$; $f(\bar{x}_i) = \frac{1}{x_i} = \frac{2n}{2i-1}$~~

~~$\sum_{i=1}^{n}\left(\frac{2i-1}{2n}\right)^{-1}\cdot\frac{1}{n} = \sum_{i=1}^{n}\frac{2\cancel{n}}{2i-1}\cdot\frac{1}{\cancel{n}} = 2\underbrace{\sum_{i=1}^{n}\frac{1}{2i-1}}_{S}$~~

~~$S = \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2n-1}$~~

$\boxed{n=5}$ $\boxed{\Delta x = \frac{1}{5} = 0.2}$ $$\bar{x}_1 = 1 + \frac{1}{n}\cdot\frac{1}{2} = 1 + \frac{1}{2\cdot 5} = 1.1$$

$\boxed{\text{MIDPOINT}}$

$$\bar{x}_2 = x_1 + \frac{1}{n} = 1 + \frac{1}{2n} + \frac{1}{n} = 1 + \frac{3}{2n} \,; \quad \bar{x}_3 = \frac{3}{2n} + \frac{1}{n} = 1 + \frac{5}{2n}$$

$$\bar{x}_i = 1 + \frac{2i-1}{2n} = \frac{2n+2i-1}{2n} \,; \quad f(\bar{x}_i) = \frac{2n}{2n+2i-1} \,;$$

$$S = \sum_{i=1}^{n} f(\bar{x}_i)\cdot\Delta x = \sum_{i=1}^{n}\frac{2\cancel{n}}{2n+2i-1}\cdot\frac{1}{\cancel{n}} = \sum_{i=1}^{n}\frac{2}{2n+2i-1} = \sum_{i=1}^{n}\frac{2}{2i+2n-1}$$

$\boxed{S(5)=0.692}$

$\boxed{\text{RIGHTPOINT}}$

$$x_1 = 1 + \frac{1}{n} \,; \quad x_2 = x_1 + \frac{1}{n} = 1 + \frac{1}{n} + \frac{1}{n} = 1 + \frac{2}{n} \,, \quad x_i = 1 + \frac{i}{n}$$

$$f(x_i) = \left(\frac{n+i}{n}\right)^{-1} \,; \quad R = \sum_{i=1}^{n}\frac{\cancel{n}}{n+i}\cdot\frac{1}{\cancel{n}} = \sum_{i=1}^{n}\frac{1}{n+i}$$

$\boxed{R(5)=0.646}$

$$\int_1^2 \frac{1}{x}\,dx = \ln(x)\Big|_1^2 = \ln(2) = 0.693147$$

PROPERTIES OF DEFINITE INTEGRAL

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx \qquad \Delta x = \frac{a-b}{n} = -\frac{b-a}{n} \qquad \boxed{a>b}$$

② $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$ | ① $\int_a^b c\,dx = c(b-a)$

③ $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$ ④ $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

$$\int_a^b [f(x)+g(x)]\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}[f(x)+g(x)]\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f(x)\Delta x + \lim_{n\to\infty}\sum_{i=1}^{n} g(x)\Delta x$$
$$= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

⑤ $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

EXAMPLE 17

$$\int_0^{10} f(x)\,dx = 17 \qquad \int_0^{8} f(x)\,dx = 12 \qquad \int_8^{10} f(x)\,dx = ?$$

$$\overset{17}{\int_0^{10} f(x)\,dx} = \overset{12}{\int_0^{8} f(x)\,dx} + \int_8^{10} f(x)\,dx \qquad \int_8^{10} f(x)\,dx = 17-12 = 5$$

---

⑥ $f(x) \geq 0 \quad a \le x \le b \quad \int_a^b f(x) \geq 0$

⑦ $f(x) \geq g(x) \quad a \le x \le b \quad \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

⑧ $$\boxed{\begin{array}{l} m \le f(x) \le M \qquad a \le x \le b \\ m(b-a) \le \int_a^b f(x)\,dx \le M(b-a) \end{array}}$$

EXAMPLE 8

$$\int_0^1 e^{-x^2}\,dx \qquad \gamma(x) = \frac{1}{\sqrt{2\pi}\sigma}\, e^{-\frac{(x-\xi)^2}{2\sigma^2}};\quad \left.\begin{matrix}\sigma=1\\ \xi=0\end{matrix}\right\} \Rightarrow \boxed{p(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}}$$

$$\frac{d}{dx}\left(e^{-x^2}\right) = +e^{-x^2}(-2x) = -2x\cdot e^{-x^2} = 0$$

$\gamma(x) = e^{-x^2} \quad \boxed{\gamma(0) = e^{0} = 1}$ max ; $\lim_{x\to\infty} e^{-x^2} = \frac{1}{e^{x^2}\to\infty} \to 0$ $\boxed{\text{min}}$

$$\gamma(1) = e^{-1} \rightarrow e^{-1} \le \int_0^1 e^{-x^2}\,dx \le 1$$

$$\int_0^1 e^{-x^2}\,dx = e^{-x^2}\left(-\frac{x^3}{3}\right) = \left.\frac{x^3 e^{-x^2}}{3}\right|_1^0$$

$$= 0 - \frac{1^3 e^{-1}}{3} = -\frac{e^{-1}}{3}$$

$$\int_0^1 e^{-x^2}dx = \left| \begin{matrix} -x^2 = u \\ -2x\,dx = du \\ x=0 \quad u=0 \\ x=1 \quad u=-1 \end{matrix} \right| = -\int_0^{-1} e^{u} \cdot \frac{du}{2\sqrt{-u}}$$

$$\int_0^1 e^{-x^2}dx = e^{-x^2}\left(-\frac{x^3}{3}\right)\Bigg|_0^1 = e^{-1}\left(-\frac{1}{3}\right) - 0 = \frac{-e^{-1}}{3} \quad \#$$

$$e^{-1} \leq \int_0^1 e^{-x^2}dx \leq 1 \; ; \qquad 0.368 \leq \int_0^1 e^{-x^2} \leq 1$$

$$= \underline{\underline{0.747}}$$

$\boxed{Pr\ 31}$ $\int_0^{\pi} \sin 5x\,dx$ Express the integral as limit of sums!

$$f(x_i) = \sin(5x_i) \qquad x = 0 \div \pi$$

$$n = 10 \qquad \Delta x = \frac{\pi}{10} \qquad x_1 = \frac{\pi}{10} \; ; \; x_2 = x_1 + \Delta x = \frac{\pi}{10} + \frac{\pi}{10} = \frac{2\pi}{10}$$

$$x_i = \frac{i \cdot \pi}{10} \; ; \ldots \; x_n = \frac{n \cdot \pi}{10}$$

$$S_R = \sum_{i=1}^{n} \sin\left(\frac{5i\pi}{n}\right) \Delta x = \sum_{i=1}^{n}\left(\sin\frac{5i\pi}{n}\right)\cdot\frac{\pi}{n} = \frac{\pi}{n}\sum_{i=1}^{n}\sin\left(\frac{5\pi i}{n}\right)$$

MIDLEPOINT:

$$x_1 = \frac{\pi}{10}\cdot\frac{1}{2} \; ; \quad x_2 = x_1 + \frac{\pi}{10} = \frac{\pi}{20} + \frac{\pi}{10} = \frac{3\pi}{20} \; ; \; x_3 = \frac{3\pi}{20} + \frac{\pi}{10} = \frac{5\pi}{20}$$

$$x_i = \frac{(2i-1)\pi}{20}$$

$$S_M = \sum_{i=1}^{n} \sin\left(\frac{(2i-1)5\pi}{2 \cdot n}\right) \cdot \frac{\pi}{n} = \frac{\pi}{n}\sum_{i=1}^{n} \sin\left[\frac{5(2i-1)\pi}{2 \cdot n}\right]$$

$$\int_0^{\pi} \sin(5x)\,dx = -5\cos(5x)\Big|_0$$

$$\int_0^{\pi} \sin(5x)\,dx = \left| \begin{matrix} u = 5x \\ du = 5\,dx \\ x = 0 \; ; \; u = 0 \\ x = \pi \; ; \; u = 5\pi \end{matrix} \right| = \int_0^{5\pi} \sin(u)\,\frac{du}{5} = -\frac{\cos(u)}{5}\Bigg|_0^{5\pi} =$$

$$= \frac{\cos(0)}{5} - \frac{\cos(5\pi)}{5} = \frac{1}{5} + \frac{1}{5} = \frac{2}{5} = 0.4$$

$\boxed{Pr\ 32}$ $\int_2^{10} x^6\,dx \; ;$

$$\Delta x = \frac{10-2}{n} = \frac{8}{n} \; ; \quad x_1 = 2 + \frac{8}{n} \; ; \; x_2 = 2 + \frac{2 \cdot 8}{n} \; ; \; x_i = 2 + \frac{i \cdot 8}{n}$$

$$S_R = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left(2 + \frac{8i}{n}\right)^6 \cdot \frac{8}{n} = \frac{8}{n}\sum_{i=1}^{n}\left(2 + \frac{8i}{n}\right)^6 =$$

$$= \frac{8}{n}\sum_{i=1}^{n}\frac{(2n+8i)^6}{n^6} = \frac{8 \cdot 2^6}{n^7}\sum_{i=1}^{n}(n+4i)^6 = \frac{512}{n}\sum_{i=1}^{n}\left(1 + \frac{4i}{n}\right)^6$$

$\int_2^{10} x^6 dx$; $\Delta x = \frac{8}{n}$ $x_1 = 2 + \frac{8}{n}\cdot\frac{1}{2}$; $x_2 = 2 + \frac{8}{2n} + \frac{8}{n} = 2 + \frac{24}{2n}$

$x_i = 2 + \frac{(2i-1)8}{2n}$

$$\frac{512}{n}\sum_{i=1}^{n}\left(1 + \frac{(2i-1)4}{2n}\right)^6$$

$\boxed{\text{PROBLEM } 9 \div 12}$

$\boxed{\text{Pr.9}}$ $\int_2^{10}\sqrt{x^3+1}\,dx$, $n=4$ $\Delta x = \frac{10-2}{n} = \frac{8}{n}$ $\Delta x = \frac{8}{4} = 2$

$x_1 = 2 + \frac{\Delta x}{2} = 2 + \frac{4}{n}$; $x_2 = x_1 + \Delta x = x_1 + \frac{8}{n} = 2 + \frac{4}{n} + \frac{8}{n} = 2 + \frac{12}{n}$

$x_3 = 2 + \frac{12}{n} + \frac{8}{n} = 2 + \frac{20}{n}$; $x_i = 2 + \frac{(2i-1)\cdot 4}{n}$

$f(\bar{x_i}) = \sqrt{x_i^3+1} = \sqrt{\left[2 + \frac{(2i-1)4}{n}\right]^3 + 1}$

$$S_n = \sum_{i=1}^{n}\frac{8}{n}\sqrt{\left[2 + \frac{(2i-1)4}{n}\right]^3 + 1}$$

$\boxed{\text{Pr.10}}$ $\int_0^{\pi}\sec(x/3)\,dx$; $n=6$, $\Delta x = \frac{\pi}{6} = \frac{\pi}{n}$

$x_1 = \frac{\pi}{n}\cdot\frac{1}{2}$; $x_2 = x_1 + \frac{\pi}{n} = \frac{\pi}{2n} + \frac{\pi}{n} = \frac{3\pi}{2n}$ $x_i = \frac{(2i-1)\pi}{2n}$

$$S_n = \sum_{i=1}^{n}\frac{\pi}{n}\cdot\sec\frac{(2i-1)\pi}{6n}$$

$\boxed{\text{Pr.11}}$ $\int_0^1 \sin(x^2)\,dx$; $\boxed{n=5}$ $x_1 = \frac{1}{n}\cdot\frac{1}{2}$; $x_2 = \frac{1}{2n} + \frac{1}{n} = \frac{3}{2n}$; $x_i = \frac{(2i-1)}{2n}$

$$S = \sum_{i=1}^{n}\sin\left(\frac{2i-1}{2n}\right)^2\cdot\frac{1}{n}$$

$\boxed{\text{Pr.12}}$ $\int_1^5 x^2 e^{-x}\,dx$; $\boxed{n=4}$ $\Delta x = \frac{4}{n}$; $x_1 = 1 + \frac{4}{n}\cdot\frac{1}{2}$; $x_2 = \frac{4}{2n} + \frac{4}{n} = 1\,\frac{12}{2n}$;

$x_i = 1 + \frac{(2i-1)\cdot 4}{2n}$; $S = \frac{4}{n}\sum_{i=1}^{n}\left[1 + \frac{(2i-1)\cdot 4}{2n}\right]e^{-1-\frac{(2i-1)\cdot 4}{2n}}$

$\boxed{\text{Pr.27}}$ $\int_a^b x\,dx$, $\Delta x = \frac{b-a}{n}$; $x_1 = a + \frac{b-a}{n}$, $x_2 = a + \frac{2(b-a)}{n}$;

$x_i = a + \frac{i(b-a)}{n}$; $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{b-a}{n}\left[a + \frac{i(b-a)}{n}\right] = \lim_{n\to\infty}\left\{\frac{(b-a)a}{n}\sum_{i=1}^{n}1 + \right.$

$\left. + \frac{(b-a)^2}{n^2}\sum_{i=1}^{n} i\right\} = (b-a)a + \lim_{n\to\infty}\frac{(b-a)^2}{n^2}\frac{\cancel{n}(n+1)}{2} = (b-a)a + \frac{(b-a)^2}{2}\lim_{n\to\infty}\frac{n+1}{n} =$

$$= ba - a^2 + \frac{b^2-2ba+a^2}{2} = \frac{b^2}{2} - \cancel{ba} + \frac{a^2}{2} + \cancel{ba} - a^2 = \frac{b^2}{2} - \frac{a^2}{2} = \frac{b^2-a^2}{2}$$

$$\int_a^b x\,dx = \frac{x^2}{2}\Big|_a^b = \frac{b^2}{2} - \frac{a^2}{2}$$

$2n^3+n^2+2n^2+n$
$=2n^3+3n^2+n$

$\boxed{\text{Pr. 28}}$

$$\int_a^b x^2\,dx = \frac{b^3-a^3}{3};\qquad \Delta x = \frac{b-a}{n};\quad x_1 = a + \frac{b-a}{n};$$

$$x_i = a + \frac{(b-a)i}{n},\qquad \sum_{i=1}^{n} \frac{b-a}{n}\cdot\left[a+\frac{(b-a)i}{n}\right]^2 = \sum_{i=1}^{n}\frac{b-a}{n^3}\left(a\overset{n}{\vee}+(b-a)i\right)^2$$

$$= \frac{b-a}{n^3}\sum_{i=1}^{n} a^2n^2 \underset{(*)}{+} (b-a)^2 i^2 + 2an(b-a)i = \frac{b-a}{n^3}\left[a^2n^2\cdot n + (b-a)^2\,\frac{n(n+1)(2n+1)}{6} + (*)\right.$$

$$\left. + \boxed{2an(b-a)\frac{n(n+1)}{2}}\right] = \frac{b-a}{n^3}\left[a^2n^3 + (b^2-2ab+a^2)\,\frac{2n^3+3n^2+n}{6}\right] =$$

$$\frac{b-a}{n^3}\left[\frac{6a^2n^3 + 2n^3b^2 + 3n^2b^2 + nb^2 - 4abn^3 - 6abn^2 - 2abn + 2n^3a^2 + 3n^2a^2 + a^2n}{6} + (*)\right]$$

$$= \frac{b-a}{n^3}\left[\frac{n^3(6a^2+2b^2-4ab+2a^2) + n^2(3b^2-6ab+3a^2) + n(b^2-2ab+a^2)}{6} + (*)\right]$$

$$= \frac{b-a}{n^3}\left[\frac{\overbrace{n^3(8a^2+2b^2-4ab)}^{2n^3(b^2-2ab+4a^2)} + 3n^2(b^2-2ab+a^2) + n(b^2-2ab+a^2)}{6} + (*)\right]$$

$$(*) = \frac{6an(b-a)n(n+1)}{6} = \frac{6an^2(bn+b-an-a)}{6} = \frac{6abn^3+6abn^2-6a^2n^3-6a^2n^2}{6}$$

$$\to \frac{b-a}{n^3}\;\frac{2n^3(b^2-2ab+4a^2+3ab-3a^2) + 3n^2(b^2-\cancel{2ab}+a^2+\cancel{2ab}-2a^2) + n(b^2-2ab+a^2)}{6} =$$

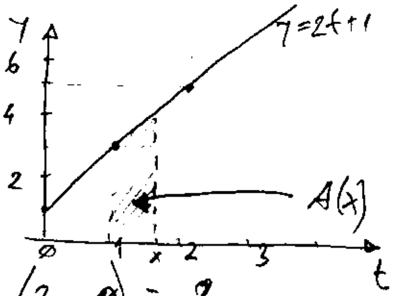
$$= \frac{b-a}{n^{\cancel{3}}}\;\frac{2n^{\overset{2}{\cancel{3}}}(b^2+ab+a^2) + 3n^{\overset{1}{\cancel{2}}}(b^2-a^2) + \cancel{n}(b^2-2ab+a^2)}{6} =$$

$$= \frac{(b-a)\left[2n^2b^2+2n^2ab+2n^2a^2 + 3nb^2-3na^2 + b^2-2ab+a^2\right]}{6n^2}$$

$$\lim_{n\to\infty}\frac{(b-a)\left[2n^2(b^2+ab+a^2)+3n(b^2-a^2)+b^2-2ab+a^2\right]}{6n^2} =$$

$$= \frac{\cancel{2}(b-a)(b^2+ab+a^2)}{\cancel{6}_3} = \frac{b^3+ab^2+\cancel{ba^2}-ab^2-\cancel{a^2b}-a^3}{3} = \frac{b^3-a^3}{3}$$

DISCOVERY PROJECT



① ⓐ $y = 2t + 1$; $P = P_1 + P_2$

$P_1 = (x_2 - x_1)\cdot(y_1 - \cancel{0})$ $x_1 = 1$; $x_2 = 2$

$y_1 = 2x_1 + 1 = 2 + 1 = 3$; $P_1 = 1\cdot(3 - \cancel{0}) = 3$

$P_2 = (x_2 - x_1)(y_2 - y_1)/2$; $y_2 = 2x_2 + 1 = 2.2 + 1 = 5$

$P_2 = 1\cdot(5 - 3)/2 = 1\cdot 2/2 = 1$; $P_1 + P_2 = 3 + 1 = 4$

ⓑ $x > 1$; $A_1(x) = (x-1)(2\cdot x_1 + 1 - 0) = (x-1)(2\cdot 1 + 1) = 3(x-1)$

$A_2(x) = (x_2 - x_1)(y_2 - y_1)/2 = (x-1)(2x+1-3)/2 = (x-1)(2x-2)/2 = \cancel{2}(x-1)^2/\cancel{2}$

$A(x) = A_1(x) + A_2(x) = 3(x-1) + (x-1)^2 = 3x - 3 + (x^2 - 2x + 1) =$

$= \underline{3x} \underline{-3} + x^2 \underline{-2x} + \underline{1} = x^2 + x - 2$

ⓒ $\boxed{A(x) = x^2 + x - 2}$ $\boxed{\frac{d}{dx} A(x) = 2x + 1}$ $y(t) = 2t + 1$

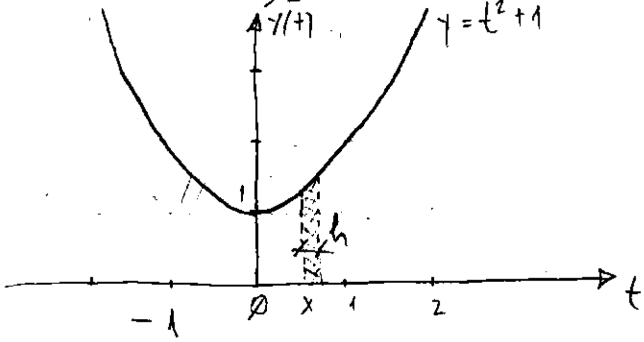
$$\int_1^x (2t+1)\,dt = \left(\cancel{2}\frac{t^2}{\cancel{2}} + t\right)\Bigg|_1^x = x^2 + x - 1 - 1 = x^2 + x - 2$$

② $\boxed{x \geq -1}$ $A(x) = \int_{-1}^{x} (1 + t^2)\,dt = \left(t + \frac{t^3}{3}\right)\Bigg|_{-1}^{x} =$

ⓐ $y(t) = t^2 + 1$

$= x + \frac{x^3}{3} + 1 + \frac{1}{3} = \frac{x^3}{3} + x + \frac{4}{3}$; $y = t^2 + 1$

ⓑ $A'(x) = \cancel{3}\frac{x^2}{\cancel{3}} + 1 = x^2 + 1$



$$\frac{A(x+h) - A(x)}{h} = \Delta A(x)/h$$

$$\Delta A(x) \approx h \cdot y(x) \doteq h\cdot(x^2 + 1)$$

$$\frac{\Delta A(x)}{h} = \frac{h(x^2+1)}{h} = x^2 + 1 \approx y(x)$$

③ ⓐ $f(x) = \cos(x^2)$ [0.2] [-1.25, 1.25]

ⓑ $g(x) = \int_0^x \cos(t^2)\,dt$ $f(x) = 0 \Rightarrow x^2 = \frac{\pi}{2}$ $x = \pm\sqrt{\frac{\pi}{2}} = \pm\frac{\sqrt{2\pi}}{2}$

ⓘ $g(x) = \int_a^x f(t)\,dt$; $g'(x) = ?$; $g'(x) = f(x)$

FUNDAMENTAL THEOREM OF CALCULUS

$$g(x) = \int_a^x f(t)\,dt \qquad a \le x \le b$$

$$\left.\begin{matrix} f(t) = t \\ a = 0 \end{matrix}\right] \Rightarrow \quad g(x) = \int_0^x t\,dt = \left.\frac{t^2}{2}\right|_0^x = \frac{x^2}{2} \; ; \quad g'(x) = x \; ; \quad \boxed{g' = f}$$

Fundamental Theorem of Calculus (FTC 1) Part 1:

$$g(x) = \int_a^x f(t)\,dt \qquad a \le x \le b$$

f continuous in $[a, b]$ then g is continuous in $[a, b]$ and differentiable : $g'(x) = f(x)$

Proof

$$g(x+h) - g(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \left[\cancel{\int_a^x f(t)\,dt} + \int_x^{x+h} f(t)\,dt\right] - \cancel{\int_a^x f(t)\,dt}$$

$$= \int_x^{x+h} f(t)\,dt$$

$$\frac{g(x+h) - g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt$$

$$m \cdot h \le \int_x^{x+h} f(t)\,dt \le M \cdot h \; ; \quad f(u) \cdot h \le \int_x^{x+h} f(t)\,dt \le f(v) \cdot h \; ; \; f(u) \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le f(v)$$

$$f(u) \le \frac{g(x+h) - g(x)}{h} \le f(v) \qquad \lim_{h \to 0} f(u) = f(x) \; ; \; \lim_{h \to 0} f(v) = f(x)$$

$$f(x) \le \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \le f(x) \; ; \qquad f(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = g'(x)$$

$$\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x)}$$

EXAMPLE 3

$$g(x) = \int_a^x f(t)\,dt \; ; \qquad \text{FRESNEL FUNCTION: } S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt$$

$$S'(x) = \sin\left(\frac{\pi t^2}{2}\right)$$

EXAMPLE 4

$$\frac{d}{dx}\int_1^{x^4} \sec t\,dt \qquad \int \sec u\,du = \ln(\sec(u) + \tan(u))$$

$$\frac{1}{\cos u} + \frac{\sin u}{\cos u} = \frac{1 + \sin u}{\cos u} = \frac{1 + \sin\left(\frac{u}{2} + \frac{u}{2}\right)}{\cos u}$$

$$\sin\left(\frac{u}{2} + \frac{u}{2}\right) = \sin\frac{u}{2} \cdot \cos\frac{u}{2} + \sin\frac{u}{2} \cdot \cos\frac{u}{2} = 2\sin\frac{u}{2} \cdot \cos\frac{u}{2}$$

$$\frac{\sin^2\frac{u}{2} + \cos^2\frac{u}{2} + 2\sin\frac{u}{2} \cdot \cos\frac{u}{2}}{\cos u} = \frac{\left(\sin\frac{u}{2} + \cos\frac{u}{2}\right)^2}{\cos u} =$$

$$= \frac{\left(\sin\frac{u}{2} + \cos\frac{u}{2}\right)^2}{\cos^2\frac{u}{2} - \sin^2\frac{u}{2}} = \frac{\left(\sin\frac{u}{2} + \cos\frac{u}{2}\right)^2}{\left(\cos\frac{u}{2} - \sin\frac{u}{2}\right)\cancel{\left(\cos\frac{u}{2} + \sin\frac{u}{2}\right)}}$$

$$\int \sec(t)\,dt = \ln(\sec(t)+\tan(t)) = \ln\left[\tan\left(\frac{t}{2}+\frac{\pi}{4}\right)\right]$$

$$\frac{d}{dx}\int_1^{x^4}\sec(t)\,dt \qquad g(x) = \int_a^x f(t)\,dt \qquad g'(x) = f(x)$$

$$g(x^4) = \int_1^{x^4}\sec(t)\,dx \qquad g'(x^4) = \sec(\cancel{x^4})\cdot 4x^3$$

$$\int x^k\cdot\ln(x)\,dx = \left|\begin{array}{l}\boxed{\int u\,dv = u\cdot v - \int v\,du} \\ u = \ln(x) \\ v = \frac{x^{k+1}}{k+1}\end{array}\right| = \int \ln(x)\,d\left(\frac{x^{k+1}}{k+1}\right) =$$

$$= \frac{x^{k+1}\ln(x)}{k+1} - \int\frac{x^{k+1}}{k+1}\,d\ln(x) = \frac{x^{k+1}\ln(x)}{k+1} - \int\frac{x^{k+1}}{x(k+1)}\,dx =$$

$$= \frac{x^{k+1}\ln(x)}{k+1} - \frac{1}{k+1}\int x^k\,dx = \frac{x^{k+1}\ln(x)}{k+1} - \frac{x^{k+1}}{(k+1)^2}$$

$$e^{\ln x} \qquad y = \ln(x) \qquad x = e^y \qquad x = e^{\ln(x)} \qquad \boxed{\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}}$$

$$f(x) = x^4 \qquad \frac{d}{dx}f(x) = \left|u = x^4\right| = \frac{d}{du}\left[f(u)\right]\,\frac{du}{dx} = 1.\ 4x^3$$

CHAIN RULE

$$\frac{d}{dx}\int_1^{x^4}\sec(t)\,dt = \left[\frac{d}{du}\int_1^{u}\sec(t)\,dt\right]\frac{du}{dx} = \sec(u)\,\frac{d}{dx}(x^4) = 4x^3\sec(u)$$

MORE I SO SMENA !!!

Fundamental Theorem of Calculus part 2

If „f" is continuous on $[a,b]$ then

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b}$$

where F is antiderivative of „f" ie function such: $\boxed{F' = f}$ ∀0

Proof

$$\int x\,dx = \frac{x^2}{2} + C \qquad \left(\frac{x^2}{2}+C\right)' = \frac{2x}{2} + \cancel{C'} = x$$

$$g(x) = \int_a^x f(t)\,dx \qquad g'(x) = f(x)$$

g - antiderivative of „f"

F - other antiderivative of „f"

$$\boxed{F(x) = g(x) + C} \qquad F(b) - F(a) = g(b) + \cancel{c} - g(a) - \cancel{c} = g(b) - \cancel{g(a)} =$$

$$= g(b) = \int_a^b f(t)\,dt$$

$$g(a) = \int_a^a f(t)\,dt = 0 \rightarrow 0$$

$$v(t) = s'(t) \qquad \int_a^b v(t)\,dt = s(b) - s(a)$$

$$\int_{-1}^{3}\frac{dx}{x^2} = \int_{-1}^{3} x^{-2}dx = \frac{x^{-1}}{-1}\Big|_{-1}^{3} = x^{-1}\Big|_{3}^{-1} = \cancel{x}\frac{1}{x}\Big|_{3}^{-1} = \cancel{x}\frac{1}{-1}-\frac{1}{3} = -1-\frac{1}{3}$$

$$= \frac{-4}{3}\Big\}$$ $f(x)=\frac{1}{x^2} \to$ DISCONTINUOS AT $x=0$, HENCE FTC CAN'T BE USED !!!

FTC $g(x)=\int_a^x f(\cancel{t})\,dt$ $\quad g'(x)=f(x)$; $\int_a^b f(x)dx = F(b)-F(a)$ $\boxed{F'=f}$

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \qquad \int_a^b F'(x)dx = F(\cancel{b})-F(a)$$

EXERCISES!

⑥ $g(x) = \int_0^x (1+\sqrt{t})dt = t\Big|_0^x + \int_0^x t^{\frac{1}{2}}dt = t\Big|_0^x + \frac{t^{\frac{3}{2}}}{\frac{3}{2}}\Big|_0^x = \left(t+\frac{2}{3}t\sqrt{t}\right)\Big|_0^x$

$g(x) = \cancel{x} + \frac{2}{3}\cancel{x}\sqrt{\cancel{x}}$; $\quad g'(x) = 1 + \frac{2}{3}\left(x^{\frac{3}{2}}\right) = 1+\sqrt{x}$

(11) $F(x) = \int_x^2 \cos(t^2)dt = \left| \cancel{\begin{matrix} v=-t \\ t=x \to v=-x \\ t=2 \to v=-2\end{matrix}} \right| = -\int_2^x \cos(t^2)dt \quad F'(x) = -\cos(\cancel{x}^2)$

(13) $h(x) = \int_2^{1/x} \arctan(t)\,dt$. $\quad \frac{dh(x)}{dx}\int_2^{1/x} \text{arctg}(t)\,dt =$

$= \frac{dh(\cancel{x})}{du}\frac{du}{dx} = \left| \begin{matrix} u=\frac{1}{\cancel{t}} & du=-t^{-2}dt \\ t=2 & u=\frac{1}{2} \\ t=\frac{1}{x} & u=x \end{matrix} \right| = -\int_{1/2}^{\cancel{x}} \text{arctg}\left(\frac{1}{u}\right)\frac{1}{u^2}du$

$= -\int_{1/2}^{x} u^{-2}\,\text{arctg}\left(\frac{1}{u}\right)du$ $\quad\boxed{u=1/x}$ $\quad h(x) = -\cancel{x}^{2}\,\text{arctg}\left(\frac{1}{x}\right)$

$$\frac{dh(x)}{dx} = \frac{d}{du}\left[\int_2^u \text{arctg}(t)\,dt\right]\frac{du}{dx} = \text{arctg}\left(\frac{1}{x}\right)\cdot\left(\frac{1}{x}\right)' = -x^{-2}\,\text{arctg}\left(\frac{1}{x}\right)$$

---

(14) $h(x) = \int_0^{x^2}\sqrt{1+r^3}\,dr \Rightarrow \frac{dh}{du}\frac{du}{dx} = \left|u=x^2\right| = \frac{d}{du}\int_0^u \sqrt{1+r^3}dr\,\frac{d}{dx}x^2$

$= \sqrt{1+u^3}\cdot 2x = \underline{2x\sqrt{1+\cancel{x}^6}}$

$\int \frac{dx}{\cos(x)} = \left| \cancel{\begin{matrix} u=\cos x \\ du=-\sin(x)dx \end{matrix}} \right| = \Big| \begin{matrix} u=\frac{1}{\cos(\cancel{x})} \\ du = \ln[\sec(x)]\cdot -\sin \;dx \end{matrix}$

$$⑶1\ \boxed{\int_0^{\pi/4}\sec^2(t)\,dt} \qquad \circledast \qquad I=\int \sec(t)\,dt$$

$$I=\frac{\sec^2(x)+\sec(x)\cdot\tan(x)}{\sec(x)+tg(x)} = \frac{\sec(x)\left[\sec(x)\,\frac{\sin(x)}{\cos x}\right]}{\cancel{\sec(t)+\frac{\sin x}{\cos x}}} = \underline{\underline{\sec(x)}}$$

$$\circledast = \frac{1}{\cos^2(x} + \frac{1}{\cos(x)}\cdot\frac{\sin(x)}{\cos(x)} = \frac{1+\sin(x)}{\cos^2(x)} = \frac{1+\sin(x)}{1-\sin^2 x} =$$

$$= \frac{\cancel{1+\sin(x)}}{\cancel{(1+\sin(x))}(1-\sin(x))} = \frac{1}{1-\sin(x)}$$

$$\frac{1}{\sec(t)+tg(t)}\quad\frac{1}{1-\sin(t)} = \frac{1}{\frac{1}{\cos(t)}+\frac{\sin(t)}{\cos(t)}}\cdot\frac{1}{1-\sin(t)} =$$

$$= \frac{\cos(t)}{(1+\sin(x))(1-\sin(t))} = \frac{\cos(t)}{1-\sin^2(t)} = \frac{\cos(t)}{\cos^2(t)} = \frac{1}{\cos(t)}$$

$$\boxed{\sec(t)+tg(t)=u}$$

$$du = \left(\frac{1}{\cos(t)}\right)' + tg'(x) = \left[\frac{-1}{\cos^2(x)}(-\sin(x)) + \frac{1}{\cos^2(x)}\right]dx$$

$$\frac{d}{dt}\,\frac{f(t)}{g(t)} = \frac{f'(t)g(t) - f(t)g'(t)}{g^2(t)}$$

$$\frac{d}{dx}\left[\frac{\sin(x)}{\cos(x)}\right] = \frac{\sin'(x)\cos(x) - \cos'(x)\sin(x)}{\cos^2(x)} = \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} =$$

$$= \frac{1}{\cos^2(x)}\ ;\ // \qquad 1+\frac{\sin^2(x)}{\cos^2(x)} = \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)}$$

$$du = \left[\frac{\sin(x)}{\cos^2(x)} + \frac{1}{\cos^2(x)}\right]dx = \left[\sec(x)\cdot tg(x) + \sec^2(x)\right]dx$$

$$u=\sec^2(t)\ ;\quad u=\frac{1}{\cos^2(t)}\ ;\quad du = -2\,\frac{1}{\cos^3(t)}\cdot(-\sin(t))\,dt$$

$$du = \frac{\sin(t)}{\cos^2(t)\cos(t)}\,dt = \frac{tg(t)}{\cos^2(t)}\,dt = \underset{u}{\sec^2(t)}\,tg(t)\,dt$$

$$\boxed{u=tg(t)}\qquad du=\frac{1}{\cos^2(t)}\,dt\ ,\quad \int\sec^2(t)\,dt = \int\frac{1}{\cos^2(t)}\,dt =$$

$$= \int 1.\,du = u = tg(t)\ ;\quad \int_0^{\pi/4}\sec^2(t)\,dt = tg(t)\Big|_0^{\pi/4} = 1-0 = 1\ //$$

(33) $\int_{\pi}^{2\pi} \csc^2\Theta \, d\Theta$ $\qquad \csc^2\Theta = \frac{1}{\sin^2\Theta}$

$ctg\,\Theta = \frac{\cos\Theta}{\sin\Theta} \qquad ctg'\Theta = \frac{\cos'\Theta \cdot \sin\Theta - \sin'\Theta \cdot \cos\Theta}{\sin^2\Theta} = \frac{-\sin^2\Theta - \cos^2\Theta}{\sin^2\Theta}$

$= -\frac{1}{\sin^2\Theta}$

$u = ctg\,\Theta \qquad du = -\frac{1}{\sin^2\Theta} d\Theta = -\csc^2\Theta \, d\Theta$

$$\int_{\pi}^{2\pi} \csc^2\Theta \, d\Theta = -\int_{\pi}^{2\pi} du = u \Big|_{2\pi}^{\pi} = ctg\,\pi - ctg\,2\pi = \frac{\overset{-1}{\cos\pi}}{\underset{0}{\sin\pi}} - \frac{\overset{1}{\cos 2\pi}}{\underset{0}{\sin 2\pi}} =$$

$-\infty$

$$= \frac{\cos\pi \cdot \sin 2\pi - \cos 2\pi \cdot \sin\pi}{\sin\pi \cdot \sin 2\pi} =$$

$\sin\alpha \cdot \sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$

$\sin 2\pi \cdot \sin\pi = \cos(2\pi - \pi) - \cos(2\pi + \pi) = \cos\pi - \cos(3\pi) = -1 + 1 = 0$

(34) $\int_0^{\pi/6} \csc\Theta \cot\Theta \, d\Theta \qquad \frac{1}{\sin\Theta} \cdot \frac{\cos\Theta}{\sin\Theta}$

$u = \csc(\Theta) \qquad du = \left(\frac{1}{\sin\Theta}\right)' d\Theta = -\frac{1}{\sin^2\Theta} \cdot \cos\Theta \, d\Theta$

$= -\frac{1}{\sin\Theta} \cdot \frac{\cos\Theta}{\sin\Theta} d\Theta = -\underbrace{\csc\Theta \cdot \cot\Theta \, d\Theta}$

$$\int \csc\Theta \cdot \cot\Theta = \int -1 \, du = -u = \underline{\underline{-\csc\Theta}}$$

(35) $\int_1^9 \frac{1}{2x} dx = \frac{1}{2} \ln(x) \Big|_1^9 = \frac{1}{2} [\ln(9) - \ln(1)]$

(36) $\int_0^1 10^x dx$ $\qquad e^{x\ln 10} = 10^x \quad /\ln$

$x \ln 10 \, \cancel{\ln e} = \ln 10^x$

$\ln 10$

$$\int_0^1 10^x dx = \int_0^1 e^{x \ln 10} dx = \left| \begin{array}{ll} x \cdot \ln 10 = u & \\ \ln 10 \cdot dx = du & \\ x = 0 & u = 0 \\ x = 1 & u = \ln 10 \end{array} \right| = \int_0 e^u \frac{du}{\ln 10} =$$

$$= \frac{e^u}{\ln 10} \Big|_0^{\ln 10} = \frac{e^{\ln 10}}{\ln 10} - \frac{1}{\ln 10} = \frac{10 - 1}{\ln 10} = \frac{9}{\ln 10}$$

(37) $I = \int_{1/2}^{\sqrt{3}/2} \frac{6\,dt}{\sqrt{1 - t^2}}$; $(\arcsin(t))' = \frac{1}{\sqrt{1 - t^2}}$; $u = \arcsin(t)$, $du = \frac{dt}{\sqrt{1 - t^2}}$,

$$I = 6 \int_{1/2}^{\sqrt{3}/2} 1 \cdot du = 6u \Big|_{1/2}^{\sqrt{3}/2} = 6\Big[\underset{\pi/4}{\arcsin\left(\tfrac{\sqrt{3}}{2}\right)} - \underset{0}{\arcsin\left(\tfrac{1}{2}\right)}\Big] = 6\left[\frac{\pi}{3} - \frac{\pi}{6}\right] = \frac{\cancel{6}\pi}{\cancel{6}}$$

(38) $\int_0^1 \frac{4}{1 + t^2} dt = 4\,\mathrm{arctg}(t) \Big|_0^1 = 4 \cdot [\mathrm{arctg}(1) - \mathrm{arctg}(0)]$

$= \pi$

(39) $$\int_{-1}^{1} e^{u+1}\,du = e\int_{-1}^{1} e^{u}\,du = e^{u+1}\Big|_{-1}^{1} = \underline{e^2-1}$$

(40) $$\int_1^2 \frac{4+u^2}{u^3}\,du = \int_1^2 \frac{4}{u^3}\,du + \int_1^2 \frac{du}{u} = 4\,\frac{u^{-3+1}}{-3+1}\Big|_1^2 + \ln u\Big|_1^2$$
$$= \frac{4}{(-2)u^2}\Big|_1^2 + \ln 2 = \ln 2 - \frac{2}{2^2} + \frac{2}{1} = \ln 2 - \frac{1}{2} + 2 = \ln 2 + \frac{3}{2}$$

(41) $$\int_0^1 x^4\,dx + \int_1^2 x^5\,dx = \frac{x^5}{5}\Big|_0^1 + \frac{x^6}{6}\Big|_1^2 = \frac{1}{5} + \frac{2^6}{6} - \frac{1}{6} = \frac{1}{5} + \frac{64-1}{6}$$

(42) $$\int_{-\pi}^{0} x\,dx + \int_0^{\pi} \sin x\,dx = \frac{x^2}{2}\Big|_{-\pi}^{0} + -\cos x\Big|_0^{\pi} = \frac{\cancel{0}^2}{2} - \frac{\pi^2}{2} \overset{-1}{-\cancel{\cos\pi}} + \overset{1}{\cancel{\cos 0}}$$
$$= -\frac{\pi^2}{2} + 2$$

$\boxed{43 \div 46}$ $$\int_0^{27} \sqrt[3]{x}\,dx = \int_0^{27} x^{\frac{1}{3}}\,dx = \frac{x^{\frac{4}{3}}}{\frac{4}{3}}\Big|_0^{27} = \frac{3}{4}\sqrt[3]{27^4}$$

(49) $$g(x) = \int_{2x}^{3x} \frac{u^2-1}{u^2+1}\,du = \int_{2x}^{0} \frac{u^2-1}{u^2+1}\,du + \int_0^{3x} \frac{u^2-1}{u^2+1}\,du =$$
$$= \int_0^{2x} \frac{1-u^2}{1+u^2}\,du + \int_0^{3x} \frac{u^2-1}{1+u^2}\,du\ ; \quad \boxed{\frac{d}{dx}g(x) = \frac{dg}{dv}\,\frac{dv}{dx} \quad v = 2x}$$

$2x$

$$\frac{1-v^2}{1+v^2}\cdot(2x)' + \frac{-1+v^2}{1+v^2}(3x)' = 2\,\frac{1-4x^2}{1+4x^2} - 3\,\frac{1-9x^2}{1+9x^2}$$

$$\left(\frac{1}{2x+1}\right)' = \frac{-1}{(2x+1)^2}\cdot(2x+1)' = \frac{-2}{(2x+1)^2}$$

g_2

(50) $$g(x) = \int_{\tan(x)}^{x^2} \frac{1}{\sqrt{2+t^4}}\,dt = -\,g_1\underbrace{\int_0^{\tan(x)} \frac{1}{\sqrt{2+t^4}}\,dt}_{u=\tan(x)} + \underbrace{\int_0^{x^2} \frac{1}{\sqrt{2+t^4}}\,dt}_{v=x^2}$$

$$\frac{dg}{dx} = \frac{dg_1}{du}\,\frac{du}{dx} + \frac{dg_2}{du}\,\frac{du}{dx} = -\frac{1}{\sqrt{2+u^4}}\,\frac{d}{dx}(\tan(x)) + \frac{1}{\sqrt{2+x^8}}\cdot 2x$$
$$= -\frac{1}{\sqrt{2+\tan^4(x)}}\sec^2(x) + \frac{2x}{\sqrt{2+x^8}}$$

(53) $$F(x) = \int_1^x f(t)\,dt \qquad f(t) = \int_1^{t^2} \frac{\sqrt{1+u^4}}{u}\,du \qquad F''(2) = ?$$

$$\frac{dF(x)}{dx} = f(x)\ ; \qquad F''(x) = f'(x) = \frac{d}{dx}\int_1^{x^2} \frac{\sqrt{1+u^4}}{u}\,du = \frac{\sqrt{1+x^8}}{x^2}\cdot 2x$$
$$= 2\cdot\frac{\sqrt{1+x^8}}{x}$$

$$F''(2) = \sqrt{257}$$

(54) $Y = \int_0^x \frac{1}{1+t+t^2}\,dt \qquad \frac{dY}{dx} = \frac{1}{1+\cancel{x}+\cancel{x}^2}$

~~$x^2+x+1=0 \qquad x_{1,2} = \frac{-1 \pm \sqrt{1-4}}{2} = -0.5(1 \pm i\sqrt{3})$~~

$$\left(\frac{1}{1+x+x^2}\right)' = \frac{-1}{(1+x+x^2)^2}(2x+1) = \frac{2x+1}{(1+x+x)^2} = 0 \qquad \boxed{x=-0.5}$$

(55) $\int_a^b f(x)\,dx = F(b)-F(a) \qquad f(1)=12 \qquad f'(x)$ continuous

$$\int_1^4 f'(x)\,dx = 17 \qquad \int_1^4 f'(x)\,dx = f(4)-f(1) = 17 \qquad f(4) = 17+12 = 29$$

(56) $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \qquad P(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{t^2}{2}}dt$ NORMAL PDF

$\frac{t^2}{2} = u^2 \qquad t=x \quad u = \frac{x}{\sqrt{2}}$

$t^2 = 2u^2$

$2t\,dt = 4u\,du$

$$P(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2}\frac{\overset{\sqrt{2}}{\overset{2}{\cancel{4}}}u\,du}{\cancel{2}\sqrt{2}u} = \frac{\sqrt{2}}{\sqrt{2\pi}}\int_{-\infty}^{x/\sqrt{2}} e^{-u^2}du = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{x/\sqrt{2}} e^{-u^2}du$$

~~$\boxed{P(x) = \frac{1}{2}\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)}$ $P(2) = \frac{1}{2}\mathrm{erf}\left(\frac{2}{\sqrt{2}}\right) = \frac{1}{2}\mathrm{erf}(\sqrt{2})$~~

$$P(x) = \frac{1}{2}\,\frac{2}{\sqrt{\pi}}\left[\int_{-\infty}^{0} e^{-u^2}du + \int_0^{x/\sqrt{2}} e^{-u^2}du\right] = \cancel{\frac{1}{2}\frac{2}{\sqrt{\pi}}\left[\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)+1\right]}$$

$$= \frac{1}{2}\left[\frac{2}{\sqrt{\pi}}\int_0^{x/\sqrt{2}} e^{-u^2}du - \frac{2}{\sqrt{\pi}}\int_0^{-\infty} e^{-u^2}du\right] = \frac{1}{2}\left[\mathrm{erf}(x/\sqrt{2}) - \cancelto{1}{\mathrm{erf}(-\infty)}\right]$$

$$P(x) = \frac{1}{2}\left[\mathrm{erf}(x/\sqrt{2})+1\right]$$

(a) $$\int_a^b e^{-t^2}dt = \int_a^0 e^{-t^2}dt + \int_0^b e^{-t^2}dt = \frac{\sqrt{\pi}}{2}\left[\frac{2}{\sqrt{\pi}}\int_0^b e^{-t^2}dt - \frac{2}{\sqrt{\pi}}\int_0^a e^{-t^2}dt\right] =$$

$$= \frac{\sqrt{\pi}}{2}\left[\mathrm{erf}(b) - \mathrm{erf}(a)\right]$$

(b) $y = e^{x^2}\mathrm{erf}(x) \qquad y' = 2xy + \frac{2}{\sqrt{\pi}}$

$$y' = (e^{x^2})'\mathrm{erf}(x) + e^{x^2}\mathrm{erf}'(x) = e^{x^2}\cdot 2x\cdot\mathrm{erf}(x) + e^{x^2}\frac{d}{dx}\frac{2}{\sqrt{\pi}}\int^{x} e^{-t^2}dt =$$

$$= 2xe^{x^2}\cdot\mathrm{erf}(x) + \cancel{e^{x^2}}\cdot\frac{2}{\sqrt{\pi}}\cdot\cancel{e^{-x^2}} = 2xe^{x^2}\mathrm{erf}(x) + \frac{\sqrt{2}}{\pi}$$

$$2xy + \frac{\sqrt{2}}{\pi} = 2x\cdot e^{x^2}\mathrm{erf}(x) + \frac{\sqrt{2}}{\pi} \quad //$$

(57) $S(x) = \int_{0}^{x} \sin\left(\frac{\pi t^2}{2}\right) dt$

$\int_0^2 x^2\,dx = \frac{x^3}{3}\Big|_0^2 = \frac{8}{3}$ $\qquad$ $\int_{-2}^{0} x^2\,dx = \frac{x^3}{3}\Big|_{-2}^{0} = -\frac{(-2)^3}{3} = \frac{8}{3}$

$\int_0^{-2} x^2\,dx = \frac{x^3}{3}\Big|_0^{-2} = -\frac{8}{3}$

(a.) $S'(x) = \sin\left(\frac{\pi x^2}{2}\right) = 0$ ~~$\frac{\pi x^2}{2} = \pi$ $x^2 = 2$ $x = \sqrt{2}$~~

$S''(x) = \left(\sin\frac{\pi x^2}{2}\right)' = \frac{\pi}{2}(2x)\cos\left(\frac{\pi x^2}{2}\right) = \pi x \cos\left(\frac{\pi x^2}{2}\right) = 0$

$\frac{\pi x^2}{2} = \frac{\pi}{2}$ $\quad x^2 = 1$ $\quad \boxed{x = \pm 1} \rightarrow \boxed{\text{MAX } \sin\left(\frac{\pi x^2}{2}\right)}$

$\pi x \cos\left(\frac{\pi x^2}{2}\right) < 0$ $\qquad \begin{cases} x < 0 \\ \cos\left(\frac{\pi x^2}{2}\right) > 0 \end{cases}$ $\qquad \begin{cases} x > 0 \\ \cos\left(\frac{\pi x^2}{2}\right) < 0 \end{cases}$

$-\frac{\pi}{\cancel{2}} \leq \frac{\pi x^2}{\cancel{2}} \leq \frac{\pi}{2}$ $\qquad -1 \leq x^2 \leq 1$

$S'(x) = \sin\left(\frac{\pi x^2}{2}\right) = 0$ $\qquad \frac{\pi x^2}{2} = \pi$ $\qquad \boxed{x = \pm\sqrt{2}}$ $\qquad$ MAXIMUM / MINIMUM } FRESNEL

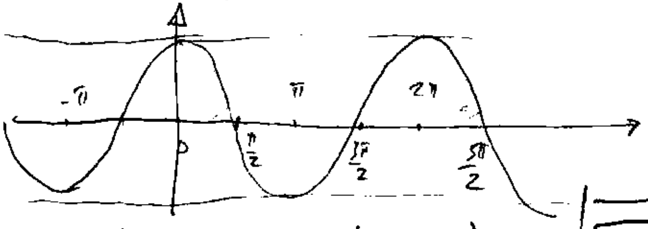
(b.) UPWARD CONCAVE:
$(0, 1)$, $(-\sqrt{3}, -1)$, $(\sqrt{3}, \sqrt{5})$ …

$S''(x)$ $\boxed{\pi x \cos\frac{\pi x^2}{2} > 0}$

- $\boxed{x > 0}$; $S''(x) > 0$ KADE ŠTO $\cos\frac{\pi x^2}{2} > 0$

$-\frac{\pi}{2} < \frac{\pi x^2}{2} \leq \frac{\pi}{2}$ $\qquad \left[(2n+1)\frac{\pi}{2}, (2n+1)\frac{\pi}{2}\right]$ $\quad n = 0 \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$n = 1 \left[\frac{3\pi}{2}, \frac{5\pi}{2}\right]$; $\quad n = 2 \; [\,..$



$[2n-1, 2n+1]\frac{\pi}{2}$ $\quad n = 2k$ $\quad k = 0, 1, 2, 3$
$n = 0, 2, 4,$
$[4k-1, 4k+1]\frac{\pi}{2} = \left[2k-\frac{1}{2}, 2k+\frac{1}{2}\right]\pi$

$\left(2n-\frac{1}{2}\right)\pi < \frac{\pi x^2}{2} < \left(2n+\frac{1}{2}\right)\pi$ $\qquad \boxed{\sqrt{4n-1} < x < \sqrt{4n+1}}$

$x > 0$
$n = 0$ $\quad (0 < x < 1)$
$n = 1$ $\quad \sqrt{3} < x < \sqrt{5}$

- $\boxed{x < 0}$ $S''(x) > 0$ KADE ŠTO $\cos\left(\frac{\pi x^2}{2}\right) < 0$

$[4k-3, 4k-1]\frac{\pi}{2} = \left[2k-\frac{3}{2}, 2k-\frac{1}{2}\right]\pi$ $\quad k = \cancel{0}, 1, 2, 3$
$k > 0$

$\left(2n-\frac{3}{2}\right)\pi < \frac{\pi x^2}{2} < \left(2n-\frac{1}{2}\right)\pi \qquad 4n-3 < x^2 < 4n-1 \qquad \sqrt{4n-3} < -x < \sqrt{4n-1}$

$\boxed{-\sqrt{4n-1} < x < -\sqrt{4n-3}} \qquad n=1 \quad x \in [-1, -\sqrt{3}] \qquad n=2 \quad x \in [-\sqrt{7}, -\sqrt{5}]$

ⓒ $s(x) = 0.2 \Rightarrow \boxed{x = 0.73}$

(58) $S(x) = \int_0^x \frac{\sin t}{t}\, dt$

ⓑ $S'(x) = \left(\frac{\sin t}{t}\right)' = \frac{\sin'(t)\cdot t - t'\cdot \sin t}{t^2} = \frac{t\cdot\cos(t) - \sin t}{t^2} = 0 = w(t)$

$t\cos t = \sin t$; $\boxed{\tan t - t = 0}$ $\qquad \frac{\sin t}{t} = 0 \Rightarrow \boxed{t = \pi}$ crosses zero hence + start to decrease

ⓔ $\int_0^x \frac{\sin t}{t}\, dt = 1 \Rightarrow x \doteq 1.12 \qquad$ fsolve(S(x) = 1) $\boxed{x = 1.0648}$

ⓓ $\lim_{x \to \infty} \int_0^x \frac{\sin t}{t}\, dt = \frac{\pi}{2}$; ⓔ ~~$\frac{\sin t}{t} = 0 \Rightarrow t = \pi$ first inflection~~

ⓒ $\frac{t\cos(t) - \sin(t)}{t^2} = 0$; $\boxed{t = 4.934}$ fsolve(w(t), t, 4 .6)
S(4.934) = 1.6556

---

(60) ⓐ local max/min $\quad \underset{\text{max}}{2}, \underset{\text{min}}{4}, \underset{\text{max}}{6}, \underset{\text{min}}{8}, \underset{\text{max}}{10}$; ⓑ MAX 10
ⓒ concave downwards
(1,3) (5,7) (9,10)

(61) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^4} \qquad \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\cdot \Delta x \qquad \Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$

$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^3}\cdot\frac{1}{n} \qquad f(1) = \frac{1}{n^3} \quad f(2) = \left(\frac{2}{n}\right)^3$; $\boxed{x_i = \frac{i}{n}}$

$S = \lim_{n\to\infty} \frac{1}{n^4} \sum_{i=1}^{n} i^3 = \lim_{n\to\infty} \frac{1}{n^4}\left(\frac{n(n+1)}{2}\right)^2 = \lim_{n\to\infty} \frac{1}{n^4}\frac{n^2(n+1)^2}{4}$

$= \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2} = \lim_{n\to\infty}\left(\frac{1}{4} + \cancelto{0}{\frac{1}{n}} + \cancelto{0}{\frac{1}{2n^2}}\right) = \frac{1}{4}$

$\int_0^1 x^3\, dx = \frac{x^4}{4}\Big|_0^1 = \frac{1}{4} - 0 = \frac{1}{4}$

(62) $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \sqrt{\frac{i}{n}} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i \qquad x_i = \sqrt{\frac{i}{n}}$

$S = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\sqrt{\frac{i}{n}}$ $\qquad \int_0^1 \sqrt{x}\,dx = \frac{x^{\frac{1}{2}+1}}{\frac{x}{2}+1} = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}x\sqrt{x}\Big|_0^1 = \boxed{\frac{2}{3}}$

(64) $$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = \frac{d}{dx}\left[-\int_0^{g(x)} f(t)\,dt + \int_0^{h(x)} f(t)\,dt\right] =$$

$$= -f(g(x))\cdot g'(x) + f(h(x))\,h'(x)$$

---

(65) (a) $1 \le \sqrt{1+x^3} \le 1+x^3 \quad x \ge 0$

(b) $1 \le \int_0^1 \sqrt{1+x^3}\,dx \le 1.25$

$$\left(\sqrt{1+x^3}\right)' = \frac{1}{2}\left(1+x^3\right)^{-\frac{1}{2}} \cdot 3x^2 = \frac{3x^2}{2\sqrt{1+x^3}}$$

---

(66) $$f(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 2-x & 1 < x \le 2 \\ 0 & x > 2 \end{cases} \qquad g(x) = \int_0^x f(t)\,dt$$

(67) $$I = \int_a^x \frac{f(t)}{t^2}\,dt = 2\sqrt{x} - 6$$

$$\int \frac{dx}{\sqrt{x}} = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = \underline{\underline{2\sqrt{x}}} \qquad \frac{f(t)}{t^2} = \frac{1}{\sqrt{t}} \quad f(t) = \frac{t^2}{\sqrt{t}} = t\sqrt{t}$$

$$\int_a^x \frac{t\sqrt{t}}{t^2}\,dt = \int_a^x \frac{dt}{\sqrt{t}} = 2\sqrt{t}\,\Big|_a^x = 2\sqrt{x} - \boxed{2\sqrt{a}}$$

$2\sqrt{a} = 6$; $\sqrt{a} = 3$; $a = 3^2 = 9$

$$I = \int_9^x \frac{f(t)}{t^2}\,dt = \left| f(t) = t\sqrt{t} \right| = 2\sqrt{x} - 6$$

---

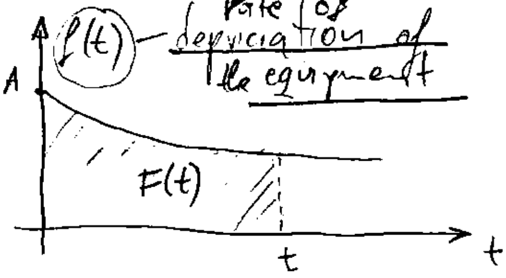
(68) $B = 3A$; $A = \int_0^a e^x\,dx$; $B = \int_0^b e^x\,dx$

$$\int_0^b e^x\,dx = 3\int_0^a e^x\,dx \Rightarrow e^b - 1 = 3e^a - 3 \; , \; 3e^a = e^b + 2$$

$e^a = \frac{e^b+2}{3}$ $\qquad \boxed{a = \ln\frac{e^b+2}{3}}$ $\qquad e^b = 3e^a - 2$

$b = \ln(3e^a - 2)$

⑥⑨



ⓐ $\text{Value} = A \cdot f(t) \cdot t$

$$F(t) = \int_0^t f(s)\,ds \qquad F'(t) = f(t)$$

ⓑ $C = C(t)$; $\quad C(t) = \frac{1}{t}\left[A + \int_0^t f(s)\,ds\right]$; $\quad C(t) = \frac{A + F(t)}{t}$

$$C'(t) = -\frac{1}{t^2}\left[A + \int_0^t f(s)\,ds\right] + \frac{1}{t} f(t) = 0$$

$$f(t) = \frac{1}{t}\left[A + \int_0^t f(s)\,ds\right] = C(t)$$

A - fixed cost for repairment of the machine

$$C = \frac{1}{t}\left[A + \int_0^t f(s)\,ds\right]$$

ⓐ $F(t) = \int_0^t f(s)\,ds$ — LOSS IN VALUE OF THE MACHINE OVER THE PERIOD OF TIME SINCE THE LAST OVERHOUL

F(t) - Loss of VALUE IN INTERVAL $[0, t]$

ⓑ $C = \frac{1}{t}\left[A + \int_0^t f(s)\,ds\right] = \frac{A + F(t)}{t}$ $\quad$ AVERAGE EXPENDITURE PER UNIT OF TIME ⓣ

$$C'(t) = -\frac{1}{t^2}\left[A + \int_0^t f(s)\,ds\right] + \frac{1}{t} f(t) = 0 \qquad f(t) = \frac{1}{t}\left[A + \int_0^t f(s)\,ds\right] = C(t)$$

$$\left.\begin{matrix} t = T \\ C'(t) = 0 \end{matrix}\right\} \Rightarrow \quad \boxed{f(T) = C(T)}$$

⑦⓪ - New COMPUTING SYSTEM WITH INITIAL VALUE : Ⓥ
- System DEPRECIATE AT RATE $f(t)$
- Accumulate maintenance cost at rate: $g(t)$
- t - TIME MEASURED IN MONTHS

ⓐ $C(t) = \frac{1}{t}\int_0^t [f(s) + g(s)]\,ds$ -

$$C'(t) = -\frac{1}{t^2}\int_0^t [f(s) + g(s)]\,ds + \frac{1}{t}[f(t) + g(t)] = 0$$

$$\underline{\underline{f(t) + g(t)}} = \frac{1}{t}\int_0^t [f(s) + g(s)]\,ds = \underline{\underline{C(t)}} \qquad t = T$$

ⓑ $f(t) = \begin{cases} \frac{V}{15} - \frac{V}{450}t, & 0 \le t \le 30 \\ 0, & t > 30 \end{cases}$ $\qquad g(t) = \frac{Vt^2}{12{,}900} \quad t > 0$

T = ? FOR TOTAL DEPRECIATION

$$D(t) = \int_0^t f(s)\,ds = V$$

$$D(t) = \int_0^t f(s)\,ds = \int_0^{30}\left(\frac{V}{15} - \frac{V}{450}t\right)dt + \int_{30}^{t} \overset{0}{\cancel{f(s)}}\,ds =$$

$$= \left(\frac{V}{15}t - \frac{V}{450}\frac{t^2}{2}\right)\Big|_0^{30} = \frac{V.30}{15} - \frac{V}{450}\cdot\frac{900}{2} = \underline{\underline{2V - V = V}}$$

$x \le 30$

$$\int_0^x \left(\frac{V}{15} - \frac{V}{450}t\right)dt = V \ ; \quad \frac{\cancel{V}\cdot x}{15} - \frac{\cancel{V}}{450}\,\frac{x^2}{2} = \cancel{V}$$

$$2\cdot 30x - x^2 = 900 \qquad x^2 - 60x + 900 = 0$$

$$x_{1,2} = \frac{60 \pm \sqrt{3600 - 3600}}{2} = \frac{60}{2} = 30 \qquad \boxed{T = x = 30}$$

6. MINIMUM OF „c" on interval $[0, T]$

$$c(t) = \frac{1}{t}\int_0^t [f(s) + g(s)]\,ds$$

$$c'(t) = \underbrace{-\frac{1}{t^2}\int_0^t [f(s) + g(s)]\,ds}_{I} + \frac{1}{t}[f(s) + g(s)] = 0$$

$$I = -\frac{1}{t^2}\int_0^t \left[\frac{V}{15} - \frac{V}{450}s + \frac{V s^2}{12.900}\right] ds =$$

$$= -\frac{1}{t^2}\,\frac{Vt}{15} + \frac{Vt^2}{2\cdot 450} - \frac{Vt^3}{12.900}$$

$$\frac{Vt}{\cancel{15}} - \frac{Vt^2}{900} + \frac{Vt^3}{3\cdot 12.900} = \frac{\cancel{Vt}}{\cancel{15}} - \frac{Vt^2}{450} + \frac{Vt^3}{12.900}$$

$$\frac{2Vt^2}{900} - \frac{Vt^2}{900} + \frac{Vt^3}{3\cdot 12900} - \frac{3Vt^3}{3\cdot 12900} = 0$$

$$\frac{Vt^2}{900} - \frac{2Vt^3}{3\cdot 12900} = 0 \qquad \frac{43\cancel{V}t^2 - 2\cancel{V}t^3}{38700} = 0$$

$$(43 - 2t)t^2 = 0 \qquad \boxed{t = \frac{43}{2} = \underline{\underline{21.5}}}$$

$$\underline{\underline{c(t)}} = \frac{1}{t}\left[\frac{Vt}{15} - \frac{Vt^2}{2\cdot 450} + \frac{Vt^3}{3\cdot 12.900}\right] = \underline{\frac{V}{15} - \frac{Vt}{2\cdot 450} + \frac{Vt^2}{3\cdot 12.900}}$$

$$c(30) = \frac{73[illegible]}{127[illegible]}V = 0.05659\cdot V \qquad c(21.5) = 0.05472V$$

$$c(t) = \cancel{\frac{V}{15}} - \frac{Vt}{900} + \frac{Vt^2}{38700} = \cancel{\frac{V}{15}} \underbrace{- \frac{Vt}{450} + \frac{Vt^2}{12.900}}_{f+g}$$

$$-\frac{Vt}{900} + \frac{2Vt}{900} + \frac{Vt^2 - 3Vt^2}{38700} = 0 \qquad \cdot\frac{Vt}{900} * \frac{2Vt^2}{38700} = 0$$

$$43\cancel{V}t - 2\cancel{V}t^2 = 0 \qquad (43-2t)t = 0 \qquad \boxed{t = \frac{43}{2} = 21.5}$$

INDEFINITE INTEGRALS AND NET CHANGE THEOREM

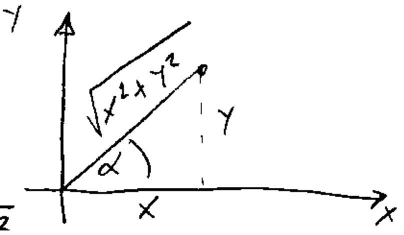
$$\boxed{\begin{array}{l}\int f(x)\,dx = F(x)\\ F'(x) = f(x)\end{array}}$$

→ ANTIDERIVATIVE OF $f(x)$ CALLED INDEFINITE INTEGRAL

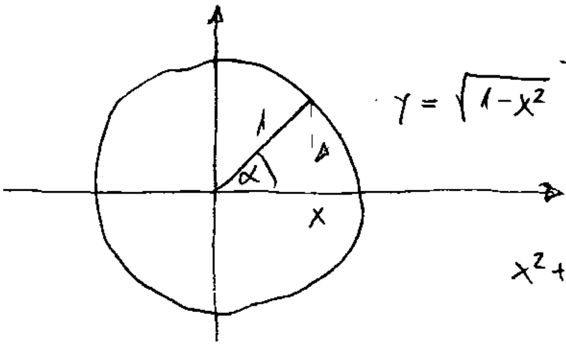
$$\int x^2\,dx = \frac{x^3}{3} + C \qquad \frac{d}{dx}\left(\frac{x^3}{3} + C\right) = x^2$$

$$\left(\frac{1}{\sin(x)}\right)' = -\frac{1}{\sin^2(x)}\cdot\cos(x)$$

$\arcsin(\alpha)$



$$\sin\alpha = \frac{y}{\sqrt{x^2+y^2}}$$



$y = \sqrt{1-x^2}$

$$\sin\alpha = \frac{\sqrt{1-x^2}}{1} \qquad \alpha = \arcsin\left(\sqrt{1-x^2}\right)$$

$$x^2 + y^2 = 1$$
$$y = \sqrt{1-x^2}$$

$$\arcsin(ⓧ) = \arcsin\left(\arcsin\sqrt{1-x^2}\right) = \left(\sqrt{1-x^2}\right)' = \frac{1}{2}\left(1-x^2\right)^{\frac{1}{2}-1}(-2x)$$

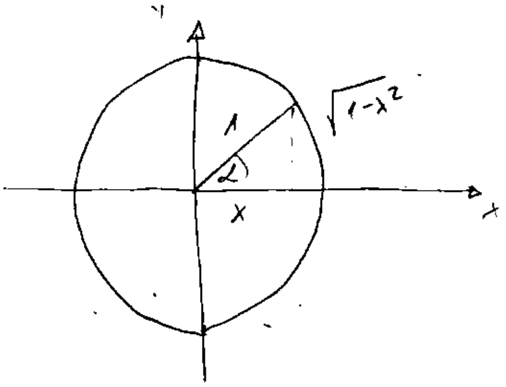
$$= \frac{1}{\cancel{2}}\,\frac{1}{\sqrt{1-x^2}}\cdot -\cancel{2}x = \frac{x}{\sqrt{1-x^2}}$$

$$\sin\alpha = \frac{y}{1} = \sqrt{1-x^2}\ ; \quad \alpha = \arcsin\left(\sqrt{1-x^2}\right)$$

$$\arctan(x) = \frac{\sqrt{1-x^2}}{x} \qquad \left(\frac{\sqrt{1-x^2}}{x}\right)' = \frac{\frac{1}{2}\,\frac{-2x\cdot x}{\sqrt{1-x^2}} - \sqrt{1-x^2}}{x^2} =$$

$$= \frac{1}{x^2}\left[-\frac{x\cdot x}{\sqrt{1-x^2}} - \sqrt{1-x^2}\right] = \frac{1}{x^2}\,\frac{\cancel{-x^2} - 1 + \cancel{x^2}}{\sqrt{1-x^2}} = \frac{-1}{x^2\sqrt{1}}$$

$$\frac{d}{d\alpha}\left(\arctan'(\alpha)\right) = \left(\alpha = \right.$$



$$tg\,\alpha = \frac{\sqrt{1-x^2}}{x}$$

$$\left(\frac{\sqrt{1-x^2}}{x}\right) = \frac{1}{x^2}\ \frac{x^2-x-1}{\sqrt{1-x^2}}$$

$$-\frac{1}{\sqrt{1-x^2}} - \frac{\sqrt{1-x^2}}{x^2} = \frac{-\cancel{x^2}-1+\cancel{x^2}}{x^2\sqrt{1-x^2}} = -\frac{1}{x^2\sqrt{1-x^2}}$$

$$\mathrm{arctg}\ \frac{\sqrt{1-x^2}}{x} = \alpha$$

$$\mathrm{arctg}(x) = ? \qquad \cos\alpha = \frac{1}{x} \qquad \sin\alpha = \frac{1}{\sqrt{1-x^2}}$$

$$\boxed{\arcsin \frac{1}{\sqrt{1-x^2}} = \alpha}$$

DIFFERENTIATING INVERSE TRIGONOMETRIC FUNCTION

$$y = \sin^{-1}x \qquad \sin y = x \qquad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

$$\frac{d}{dx}(\sin y) = \frac{dx}{dx} \qquad \cos y\ \frac{dy}{dx} = 1 \qquad \frac{dy}{dx} = \frac{1}{\cos y}$$

$$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2} \qquad \boxed{\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}}$$

$$y = tg^{-1}x \qquad tg(y) = x \qquad tg'(y)\frac{dy}{dx} = 1 \qquad \frac{1}{\cos^2 y}\ \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \cos^2 y \qquad \left(\frac{\sin y}{\cos y}\right)' = \frac{\sin' y\cos y - \sin y\cos' y}{\cos^2 y} =$$

$$= \frac{\cos^2 y + \sin^2 y}{\cos^2 y} = 1 + tg^2 y = \frac{1}{\cos^2 y}$$

$$(1+tg^2 y)\ \frac{dy}{dx} = 1 \qquad \boxed{\frac{dy}{dx} = \frac{1}{1+tg^2 y} = \frac{1}{1+x^2}}$$

$$\left[tg^{-1}(x)\right]' = \frac{1}{1+x^2}$$

$$(a^x)' = \left(e^{x\ln a}\right)' = e^{x\ln a}\ln a = \underline{\underline{a^x \ln a}}$$

$$\int a^x dx = \int e^{x\ln a}dx = \left|\begin{matrix} u = x\ln a \\ du = \ln a\, dx\end{matrix}\right| = \int e^u \frac{du}{\ln a} = \frac{e^{x\ln a}}{\ln a} = \frac{a^x}{\ln a}$$

$\ln_a x = y$; $a^y = x$; $\frac{d}{dx}(a^y) = \frac{dx}{dx}$; $\frac{d}{dx}\left(e^{y\ln a}\right)$

$= e^{y\ln a}\cdot\ln a\,\frac{dy}{dx} = \frac{dx}{dx}$ $a^y\cdot\ln a\,\frac{dy}{dx} = 1$

$$\boxed{\frac{dy}{dx} = \frac{1}{a^y\ln(a)} = \frac{1}{x\ln(a)}}$$

$\boxed{a=e}$ $\ln x = y$ $e^y = x$ $e^y\frac{dy}{dx} = 1$ $\frac{dy}{dx} = \frac{1}{x}$

$(\ln x)' = \frac{1}{x}$

EXAMPLE 4 $\int_0^2\left(2x^3 - 6x + \frac{3}{x^2+1}\right)dx = \cancel{2}\frac{x^4}{\cancel{4}_2} - \overset{3}{\cancel{6}}\frac{x^2}{\cancel{2}} + 3\,\text{arctg}\,x\Big|_0^{2} =$

$= \frac{x^4}{2} - 3x^2 + 3\,\text{arctg}(x)\Big|_0^2 = \frac{16}{2} - 3\cdot4 + 3\cdot1.1071 = 8 - 12 + 3.3214$
$= -0.6786$

$y = tg^{-1}x$ $tg(y) = x$

EXAMPLE 5 $\int_1^9\frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt = \int_1^9\left(2 + \sqrt{t} - \frac{1}{t^2}\right)dt =$

$= \left(2t + \frac{t^{3/2}}{3/2} - \frac{t^{-2+1}}{-2+1}\right)\Big|_1^9 = \left(2t + \frac{2}{3}t\sqrt{t} + \frac{1}{t}\right)\Big|_1^9 =$

$= \left(2\cdot9 + \frac{2}{\cancel{3}}9\sqrt{9} + \frac{1}{9} - 2 - \frac{2}{3} - 1\right) = 18 + 18 + \frac{1}{9} - \frac{2}{3} - 3 =$

$= 33 + \frac{1-6}{9} = 35 - \frac{5}{9} = \frac{9\cdot35 - 5}{9} = \frac{292}{9} = 32{,}44$

APLICATIONS $\int_a^b f(x)\,dx = F(b) - F(a)$ $F'(x) = f(x) = y$

$\int_a^b F'(x)\,dx = F(b) - F(a)$

The Net Change Theorem $\int_a^b F'(x)\,dx = F(b) - F(a)$

ECONOMICS $C(x)$ - TOTAL cost of PRODUCING x units of certain commodity COST FUNCTION

ΔC ADITIONAL cost for increasing production from x_1 to x_2

$$\boxed{\frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1} = \frac{C(x_1 + \Delta x) - C(x_1)}{\Delta x}}$$

$$\boxed{\text{marginal cost} = \lim_{\Delta x\to0}\frac{C(x_1 \quad C(x_1)}{\Delta x} = \frac{dC}{dx}}$$

$\Delta x = 1$; n large

$$\boxed{c'(n) \doteq c(n+1) - c(n)}$$

a - overhead cost
b - cost of raw material
c - labour cost

$C(x) = a + bx + cx^2 + dx^3$

- Suppose the cost of company to produce „x" items is:

$$\boxed{C(x) = 10.000 + 5x + 0.01x^2}$$

MARGINAL cost : $\frac{dC(x)}{dx} = 5 + 0.02x$

$C'(x)\Big|_{5000} = 5 + 100,0 = 10$[illegible]$5$ $\qquad C'(x)\Big|_{500} = 5 + 0.02 \cdot 500 = \underline{15\$}$ / item

$\frac{1440}{900} = 1.6 \qquad \frac{1680}{1050} = 1.6 \qquad C(501) - C(500) = 15,01\$$

Net Change theorem:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

- $V(t)$ VOLUME OF WATER IN RESERVOIR AT TIME „t"

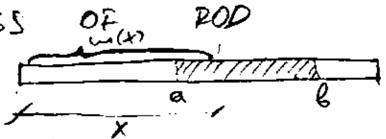
$V'(t)$ Rate at which WATTER FLAWS IN THE RESERVOIR at time „t"

$$\int_{t_1}^{t_2} V'(t)\,dt = V(t_2) - V(t_1)$$

$\frac{dn}{dt}$ - RATE of chage of population

$$\int_{t_1}^{t_2} \frac{dn}{dt}\,dt = n(t_2) - n(t_1)$$ NET CHANGE IN POPULATION

- MASS OF ROD



$m(x)$ - mass of rod with length x

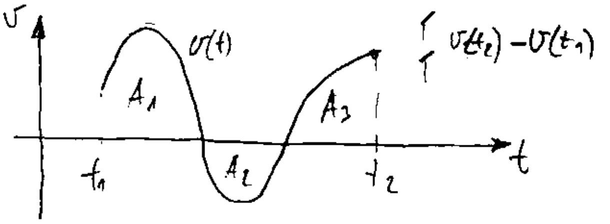
$\rho(x) = m'(x) \Rightarrow$ LINEARNA GUSTINA (linear density)

$$\int_a^b \rho(x)\,dx = m(b) - m(a)$$

- $\int_{x_1}^{x_2} C'(x)\,dx = C(x_2) - C(x_1)$ increase of cost when the production is increased from x_1 to x_2

$v(t) = s'(t) \qquad \int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$ NET CHANGE IN POSITION (displacement)

$\int_{t_1}^{t_2} |v(t)|\,dt =$ total distance traveled



$$\int_{t_1}^{t_2} v(t)\,dt = A_1 - A_2 + A_3$$

$$\int_{t_1}^{t_2} |v(t)|\,dt = A_1 + A_2 + A_3$$

$$a(t) = \frac{d}{dt}[v(t)] = v'(t) \qquad \int_{t_1}^{t_2} a(t)\,dt = \int_{t_1}^{t_2} v'(t)\,dt = v(t_2) - v(t_1)$$

acceleration is change of velocity from time t_1 to time t_2

$\boxed{\text{Ex6}}$ $v(t) = t^2 - t - 6$ [m/s]

ⓐ displacement of the particle during $\boxed{1 \leq t \leq 4}$

ⓑ distance traveled

ⓐ $$\int_1^4 (t^2 - t - 6)\,dt = s(t_2) - s(t_1) = \left(\frac{t^3}{3} - \frac{t^2}{2} - 6t\right)\Big|_1^4 = -\frac{9}{2} = -4.5\text{ m}$$

ⓑ $$\int_1^4 |v(t)|\,dt = \frac{61}{6} = 10{,}1667\ ; \qquad -\int_1^3 v(t)\,dt + \int_3^4 v(t)\,dt = \frac{61}{6} \doteq 10{,}17$$

$$|v(t)| = \begin{cases} -v(t) & 1 \leq t < 3 \\ v(t) & 3 \leq t \end{cases}$$

$\boxed{\text{Ex7}}$ $$\int_0^{24} P(t)\,dt = \int_0^{24} E'(t)\,dt = E(24) - E(0)$$

$$\int_0^{24} P(t)\,dt = P(1) + P(2) + \ldots P(24) = \sum_{i=1}^{12} P(i)\,\cancel{dt}^{\;2h} = \underline{\underline{15840\ \text{MW/h}}}$$

$\boxed{\text{5.4 EXERCISES}}$

① $$\left(\sqrt{x^2+1}\right)' = \frac{1}{\cancel{2}}(x^2+1)^{\frac{1}{2}-1} \cdot \cancel{2}x = \frac{x}{\sqrt{x^2+1}}$$

② $$\int x\cos(x)\,dx = x\sin x + \cos x + C \qquad (x\sin x + \cos x)' = \cancel{(x)'}^{\,1} \cdot \cancel{\sin x} + \underline{\underline{x \cdot \cos x}} - \cancel{\sin x}$$

③ $$\left(\frac{x}{a^2\sqrt{a^2-x^2}}\right)' = \frac{1}{a^2} \frac{x'\sqrt{a^2-x^2} - x \cdot \frac{1}{\cancel{2}} \frac{1}{\sqrt{a^2-x^2}} \cdot (-\cancel{2}x)}{a^2 - x^2} =$$

$$= \frac{\sqrt{a^2-x^2} + \frac{x^2}{\sqrt{a^2-x^2}}}{a^2(a^2-x^2)} = \frac{\cancel{a^2} - \cancel{x^2} + \cancel{x^2}}{\cancel{a^2}\sqrt{a^2-x^2}\,(a^2-x^2)} = \frac{1}{\sqrt{(a^2-x^2)^3}}$$

④ $$\left(-\frac{\sqrt{x^2+a^2}}{a^2x}\right)' = -\frac{1}{a^2} \frac{\frac{x^2}{\sqrt{x^2+a^2}} - \sqrt{x^2+a^2}}{x^2} = -\frac{1}{a^2} \frac{\cancel{x^2} - \cancel{x^2} - a^2}{x^2\sqrt{x^2+a^2}} = \frac{1}{x^2\sqrt{x^2+a^2}}$$

⑩ $$\int \left(x^2 + 1 + \frac{1}{x^2+1}\right)dx = \frac{x^3}{3} + x + \int \frac{1}{x^2+1}\,dx = I$$

$$y = \tan^{-1}(x) \qquad \frac{dy}{dx} = (\tan(x))' = \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$

$$\frac{dy}{dx} = 1 + \tan^2(x) = 1 + y^2$$

$y = \operatorname{arctg}(x)$ $\quad \underline{\underline{x = \tan(y)}}$ $\quad tg(y) = x$ $\quad \underline{\left(\frac{d}{dx} tg(y)\right)\frac{dy}{dx} = 1}$

$$(*) = tg'(y) = 1 + tg^2(y) = \underline{\underline{1 + x^2}} \qquad \underline{\frac{dy}{dx} = \frac{\overset{(*)}{1}}{1+x^2}}$$

$$\boxed{\int \frac{1}{1+x^2}\,dx = \operatorname{arctg}(x)}$$

$$\boxed{I = \frac{x^3}{3} + x + \operatorname{arctg}(x)}$$

⑫ $\int (3e^u + \sec^2 u)\,du = 3e^u + tg\,u + C$

$$(tg(u))' = \frac{\sin u}{\cos u} = \frac{\sin' u \cos u - \cos' u \cdot \sin u}{\cos^2 u} = \frac{1}{\cos^2 u}$$

$$= \sec^2 u$$

⑬ $\int \frac{\sin x}{1 - \sin^2 x}\,dx = \int \frac{\sin x}{\cos^2(x)}\,dx = \int tg(x) \cdot \sec x\,dx = \sec(x) + C$

$$tg'(x) = \frac{1}{\cos^2 x} = \sec^2(x)$$

$$(tg(x) \cdot \sec(x))' = \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \left(\frac{\sin x}{\cos^2 x}\right)' = \frac{\cos x \cdot \cos^2 x + \overset{2\cancel{\sin x} \cdot \sin x}{}}{\cos^4 x}$$

$$(\sec x)' = \left(\frac{1}{\cos x}\right)' = (-1)\frac{-\sin(x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = tg(x)\sec(x)$$

⑭ $\int \frac{\sin(2x)}{\sin x}\,dx = \int \frac{\sin x \cdot \cos x + \sin x \cdot \cos x}{\sin x}\,dx = \int \frac{2\cancel{\sin x}\cos x}{\cancel{\sin x}} =$

$$= 2\int \cos x\,dx = 2\sin(x) + C$$

⑮ $\int x\sqrt{x}\,dx = \int x^{\frac{3}{2}}\,dx = \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} = \frac{x^{\frac{5}{2}}}{\frac{5}{2}} = \frac{2}{5}\sqrt{x^5} = \underline{\underline{\frac{2}{5}x^2\sqrt{x}}}$

⑯ $G(x) = \int (\cos x - 2\sin x)\,dx = +\sin x + 2\cos x$

㉞ $\int_0^{\pi/3} \frac{\sin\theta + \sin\theta\tan^2\theta}{\sec^2\theta}\,d\theta = \int_0^{\pi/3} \cos^2\theta\,(\sin\theta + \sin\theta\,tg^2\theta)\,d\theta$

$$= \underbrace{\int_0^{\pi/3} \sin\theta \cdot \cos^2\theta\,d\theta}_{I_1} + \underbrace{\int_0^{\pi/3} \sin^3\theta\,d\theta}_{I_2}$$

$$I_1 = \int_0^{\pi/3} \sin\theta\left(1-\sin^2\theta\right)d\theta = \int_0^{\pi/3}\left(\sin\theta - \sin^3\theta\right)d\theta$$

$$I_2 = \int_0^{\pi/3} \sin^3\theta\, d\theta = \Bigg| \begin{array}{l} x=\sin\theta \\ dx = -\cos\theta\, d\theta \end{array}$$

466999

PAPANO
IDENTIFIKACIJA
TECEUS
RENTA

$$\left(\sin^4\theta\right)' = 4\sin^3\theta - \cos\theta$$

$$\int_0^{\pi/3} \sin^2\theta\, d(\cos\theta) = \left| \begin{array}{l} u = \cos\theta \\ du = -\sin\theta\, d\theta \\ \theta = 0 \Rightarrow u = 1 \\ \theta = \frac{\pi}{3} \Rightarrow u = \frac{1}{2} \end{array} \right| = -\int_1^{1/2}\left(1-u^2\right)du =$$

$$= -u\Big|_1^{1/2} + \frac{u^3}{3}\Big|_1^{1/2} = -\left(\frac{1}{2}-1\right) + \frac{1}{3}\left(\frac{1}{8}-1\right) = +\frac{1}{2} - \frac{1}{3}\,\frac{7}{8} = \frac{-7+12}{24} = +\frac{5}{24}$$

$$\int \sin^3\theta\, d\theta = \left| \begin{array}{l} u = \cos\theta \\ du = -\sin\theta\, d\theta \\ \sin^2\theta = 1-\cos^2\theta \\ \quad = 1-u^2 \end{array} \right| = -\int \sin^2\theta\left(-\sin\theta\right)d\theta =$$

$$= -\int\left(1-u^2\right)du = -u + \frac{u^3}{3} = \underline{\frac{\cos^3\theta}{3} - \cos\theta} =$$

$$= \frac{1}{3}\left(1-\sin^2\theta\right)\cos\theta - \cos\theta = \frac{1}{3}\cos\theta - \frac{1}{3}\sin^2\theta\cos\theta - \cos\theta$$

$$= -\frac{1}{3}\sin^2\theta\cos\theta - \frac{2}{3}\cos\theta$$

$$I_2 = \frac{5}{24}\,; \qquad I_1 = \int_0^{\pi/3}\sin\theta\, d\theta - \frac{5}{24} = -\cos\theta\Big|_0^{\pi/3} - I_2 = -\left(\frac{1}{2}-1\right) - I_2$$

$$I_1 = \frac{1}{2} - I_2 \qquad I = I_1 + I_2 = \frac{1}{2} - \cancel{I_2} + \cancel{I_2} = \frac{1}{2}$$

$$I = \int_0^{\pi/3}\left(\sin\theta - \cancel{\sin^3\theta}\right)d\theta + \int_0^{\pi/3}\cancel{\sin^3\theta}\, d\theta = \int_0^{\pi/3}\sin\theta\, d\theta = \frac{1}{2}$$

---

(39) $$\int_{-1}^{2}\left(x - 2|x|\right)dx = \frac{x^2}{2}\Big|_{-1}^{2} - \int_{-1}^{0}(-2x)\,dx - 2\int_0^2 x\, dx =$$

$$= \frac{1}{2}\left(4-1\right) + \cancel{2}\,\frac{x^2}{\cancel{2}}\Big|_{-1}^{0} - \cancel{2}\,\frac{x^2}{\cancel{2}}\Big|_0^2 = \frac{3}{2} + \left(0-1\right) - \left(4-0\right) = \frac{3}{2} - 5 = -\frac{7}{2}$$

(40) $$\int_0^{3\pi/2}|\sin x|\, dx = \int_0^{\pi}\sin x\, dx + \int_{\pi}^{3\pi/2} -\sin x\, dx = -\cos x\Big|_0^{\pi} + c \ \Big|^{3\pi/2}$$

$$= -\left(-1-1\right) + \left(0+1\right) = 2+1 = 3$$

㊷ $y = 2x + 3x^4 - 2x^6$

㊸ $x = 2y - y^2$ $\quad \int_0^2 (2y - y^2)\, dy$

㊹ $y = \sqrt[4]{x}$ $\quad \boxed{x = y^4}$

$$A = \int_0^1 y^4\, dy = \frac{y^5}{5}\Big|_0^1 = \frac{1}{5}$$

$$A = 1 - \int_0^1 \sqrt[4]{x}\, dx = 1 - \frac{x^{\frac{1}{4}+1}}{\frac{1}{4}+1}\Big|_0^1 = 1 - \frac{x^{5/4}}{5/4}\Big|_0^1 = 1 - \frac{4}{5} x \sqrt[4]{x}\Big|_0^1 = 1/5$$

㊺ $w'(t)$ RATE OF GROWTH PER YEAR

$$\int_5^{10} w'(t)\, dt = w(10) - w(5)$$

$\boxed{\text{FTC1}}$ $g(x) = \int_a^x f(\cancel{t})\, d\cancel{t}$ $\quad \boxed{g'(x) = f(x)}$

$\boxed{\text{FTC2}}$ $\int_a^b f(x)\, dx = F(b) - F(a)$ $\quad F'(x) = f(x)$

㊼ GALONS PER MINUTE $= r(t)$

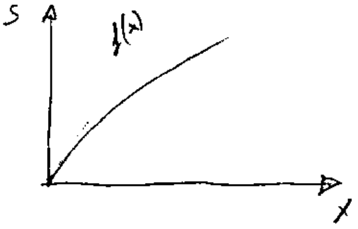
$$\int_0^{120} r(t)\, dt \; ; \quad r(t) = -\frac{dV(t)}{dt} = -V'(t)$$

$$-\int_0^{120} V'(t)\, dt = -V(t)\Big|_0^{120} = -V(120) + V(0) = V(0) - V(120)$$

㊽ 100 Bees increases at $n'(t)$ Bees per week

$$100 + \int_0^{15} n'(t)\, dt = 100 + n(15) - n(0)$$ } TOTAL POPULATION OF BEES AFTER 15 WEEKS

㊾ $R'(x)$ – MARGINAL REVENUE ; $\int_{1000}^{5000} R'(x)\, dx$ } INCREASE OF REVENUE IF SALES INCREASE FROM 1000 TO 5000 UNITS



$$f(x) = \frac{ds}{d\cancel{x}} = s'(x) \; ; \quad \int_3^5 s'(x)\, dx = s(5) - s(3) = \Delta s \; .$$

Δs – change of altitude if x change from 3 to 5

(52) x [feets]
a [pounds/foot]

$$\frac{da}{dx} (=) \frac{\text{pounds/foot}}{\text{feets}} = [\text{pouns}]/(\text{foot})^2$$

$$\int_2^8 a\, dx (=) \frac{\text{pound}}{\text{foot}}\, \text{feet} (=) \text{pounds}$$

$$\frac{da}{dx} (=) \frac{kg}{m^2}$$

$$\int_1^3 a(x)\, dx \,(=)\, kg$$

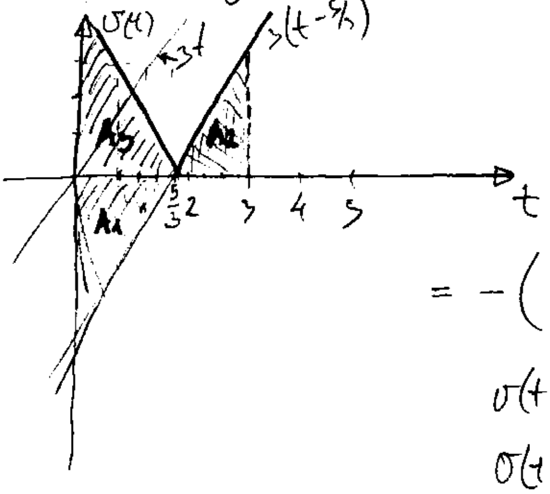
(53) $v(t) = 3t - 5 \quad 0 \le t \le 3$

DISP $-A_1 + A_2 = \int_0^3 v(t)\, dt = \int_0^3 (3t - 5)\, dt = \left(3\frac{t^2}{2} - 5t\right)\Big|_0^3 = 3\,\frac{9}{2} - 15 = \frac{27 - 30}{2} = -\frac{3}{2}$

[illegible] $\int_0^3 v'(t)\, dt = v(3) - v(0) = 9 - \cancel{5} + \cancel{5} = 9$

$$v'(t) = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2 s}{dt^2}$$

$\boxed{\text{DISTANCE}}$ $\int_0^3 |3t-5|\,dt = A_2 + A_3 = A_2 + A_1$



$$|3t-5| = -(3t-5) \quad 0 \le t \le 3$$

$$\int_0^3 |3t-5|\,dt = -\int_0^3 (3t-5)\,dt =$$

$$= -\left(\frac{3t^2}{2} - 5t\right)\Big|_0^3 = -\frac{27-30}{2} = \frac{3}{2}$$

$$v(t) = 3t-5 \qquad v(t) = 0 = 3t-5 \quad t = \frac{5}{3}$$

$$v(t) = 3\left(t - \frac{5}{3}\right)$$

$$|v(t)| = \begin{cases} -3\left(t-\frac{5}{3}\right) & 0 \le t \le 5/3 \\ 3\left(t-\frac{5}{3}\right) & 5/3 \le t \le 3 \end{cases}$$

$$\int_0^3 v(t) = \int_0^{5/3} -3(t-5/3)\,dt + \int_{5/3}^3 3(t-5/3)\,dt = -\left(\frac{3t^2}{2} - 5t\right)\Big|_0^{5/3} + \left(\frac{3t^2}{2} - 5t\right)\Big|_{5/3}^3 =$$

$$= -\left(\frac{3}{2}\frac{25}{9} - \frac{5 \cdot 5}{3}\right) + \left(\frac{3 \cdot 9}{2} - \frac{15 \cdot 2}{2}\right) - \left(\frac{3}{2}\frac{25}{9} - \frac{25}{3}\right) =$$

$$= -2\left(\frac{75 - 25 \cdot 6}{18}\right) + \frac{27-30}{2} = -2\,\frac{75}{18} - \frac{3}{2} = \frac{150-27}{18} = \frac{123}{18} = \frac{41}{6}$$

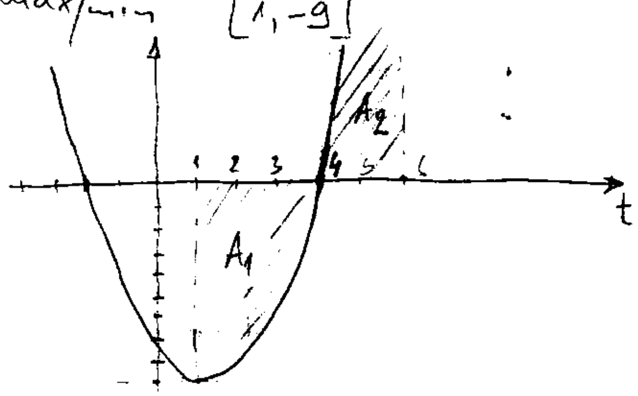
---

(54) $v(t) = t^2 - 2t - 8 \quad 1 \le t \le 6$.

$$t^2 - 2t - 8 = 0 \qquad t_{1,2} = \frac{+2 \pm \sqrt{4+32}}{2} = \frac{2 \pm \sqrt{36}}{2} = \frac{2 \pm 6}{2} = \begin{cases} 4 \\ -2 \end{cases}$$

$$v'(t) = 2t - 2 = 0 \quad \boxed{t=1} \qquad \boxed{v(1) = 1 - 2 - 8 = -9}$$

max/min $[1, -9]$



$$v(t) = (t-4)(t+2) = t^2 + 2t - 4t - 8$$
$$= t^2 - 2t - 8$$

$$\text{DISPL} = \int_1^6 v(t)\,dt = \int_1^6 (t^2 - 2t - 8)\,dt = -\frac{10}{3}\,\text{m}$$

$$\text{DISPL} = A_2 - A_1$$

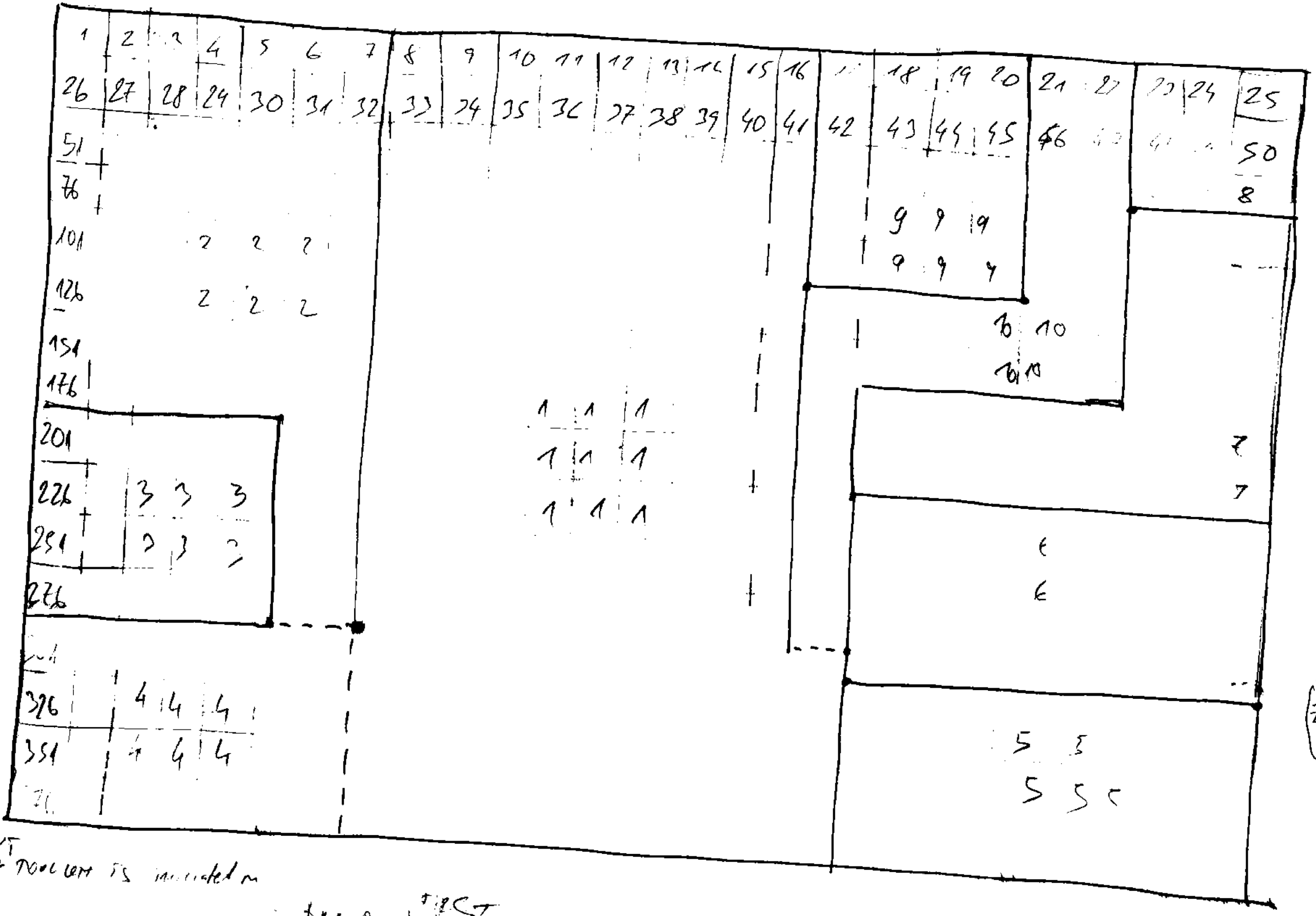
$$\text{DIST} = A_1 + A_2 = -\int_1^4 v(t)\,dt + \int_4^6 v(t)\,dt = \frac{98}{3}\,\text{m}$$

(55) $a(t) = t + 4, \quad v(0) = 5, \quad 0 \le t \le 10 \qquad v(t) = ? \quad s(t) = ?$

$$a(t) = \frac{dv}{dt} \qquad v(10) = v(0) + \int_0^{10} a(t)\,dt = 5 + \int_0^{10} (t+4)\,dt = 95 \,\frac{\text{m}}{\text{s}}$$

$$v(t) = v(0) + \int a(t)\,dt = v(0) + \int (t+4)\,dt = v(0) + \frac{t^2}{2} +$$

$$s(t) = \int_0^{10} |v(t)|\,dt = \int_0^{10} \left|\frac{t^2}{2} + 4t + 5\right| dt = \frac{1250}{3}\,\text{m}$$



zoran-hw@yahoo.com

IDVT
– All room is included in

creating to database

DRAWN ~~[illegible]~~ FIRST ROOM

(56) $a(t) = 2t+3$ $\quad v(t) = v(0) + \int (2t+3)\,dt = -4 + \cancel{2}\frac{t^2}{\cancel{2}} + 3t = t^2+3t-4$

$v(0) = -4$ $\quad s(t) = \int_0^3 |v(t)|\,dt = \int_0^3 |t^2+3t-4|\,dt = \frac{89}{6}$

$0 \le t \le 3$

(57) $\rho(x) = \frac{dm}{dx} = 9 + 2\sqrt{x}$; $l = 4m$

$M = \int_0^4 \rho(x)\,dx = \int_0^4 (9+2\sqrt{x})\,dx = \frac{140}{3}\ kg$

(58) $r(t) = 200-4t$ $[l/min]$; $0 \le t \le 50$

$r(t) = -\frac{dV}{dt}$ $\quad V = -\int_0^{10} (200-4t)\,dt = 1800\,l$

152

(59) $t = \{0, 10, 20, 30, \quad 100\}$ $[sec]$

$v = \{0, \underline{38}, 52, \underline{58}, 55, \underline{51}, 56, \underline{53}, 50, \underline{47}, 45\}$ $[miles/h]$

$S = \sum_{i=1}^{10} v(t_i)\Delta t = 1,4028$ [miles] $\quad \boxed{t_i = 10(i-1)\ [sec]}$ $\boxed{t_i = \frac{10(i-1)}{3600}\ [h]}$ $\Delta t = \frac{1}{360}\ [h]$ } RIGHT POINT RULE

$1\ sec = \frac{1}{3600}\ h$ $\quad 10\ sec = \frac{1}{360}$;

- MIDPOINT RULE

$\Delta t = \frac{100}{n}$; $n = 5$; $\Delta t = 20\ sec = \frac{2\cancel{0}}{360\cancel{0}} = \frac{1}{180}\ [h]$

$S = \sum_{i=1}^{5} v(t_i)\Delta t = \frac{1}{180}(38+58+51+53+47) = \frac{247}{180} = 1.37$ miles

$\boxed{t_i = \frac{1\cancel{0}\cdot(2i-1)}{360\cancel{0}}}$; $\quad t_1 = \frac{10}{3600}$; $t_2 = \frac{30}{3600}$ $\cdots$ $t_n = \frac{90}{3600}$

(60) $t = [0, 1, 2, 3, 4, 5, 6]$

$r(t) = [\overset{0}{2}, \overset{1}{10}, \overset{2}{24}, \overset{3}{36}, \overset{4}{46}, \overset{5}{54}, \overset{6}{60}]$

ⓐ upper/lower of quantity $Q(6)$

upper: $6 \times 60 = 360$ tons

lower: $6 \times 2 = 18$ tons

ⓑ $n = 3$ $\quad Q(6) = \sum_{i=1}^{3} r(t_i)\cdot\Delta t$ $\quad \Delta t = \frac{6}{n} = \frac{6}{3} = 2$ $\quad t_1 = 1$; $t_2 = 3$; $t_3 = 5$ $\quad \boxed{t_i = 2i-1}$

$Q(6) = 2\cdot\sum_{i=1}^{3} r(t_i) = 2(10+36+54) = 2\cdot100 = 200$ tons $\quad \boxed{t_i = i}$

$R(6) = \sum_{i=1}^{6} r(t_i)\Delta t = \left| \Delta t = \frac{b-a}{n} = \frac{6-0}{6} = 1 \right| = \sum_{i=1}^{6} r(i) = 10+24+36+46+54+60$

$= 230$ tons

$L(6) = \sum_{i=1}^{6} r(t_i)\Delta t = 2+10+24+36+46+54 = 172$ tons $\quad \boxed{t_i = i-1}$

(61) $C'(x) = 3 - 0.01x + 0.000006x^2$ ($/yard)

INCREASE OF cost if PRODUCTION RISE FROM 2000 TO 4000 yards

$$C = \int_{2000}^{4000} (3 - 0.01x + 0.000006x^2)\,dx = 58000\$$$

$$\int_0^4 r(t)\,dt = V(4) - V(0) ; \quad V(4) = V(0) + \int_0^4 r(t)\,dt$$

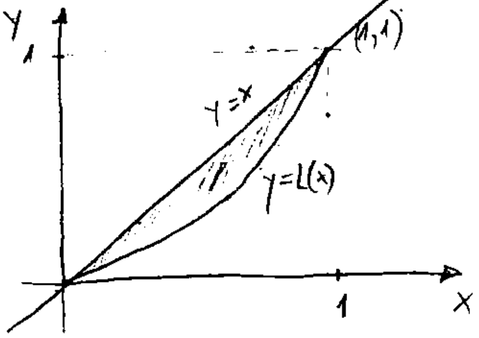
(62) $V(0) = 25.000$ L

$$V(4) = V(0) + \sum_{i=1}^{4} r(t_i)\,\Delta t ; \quad \Delta t = \frac{4}{n} = \frac{4}{4} = 1 ; \quad \boxed{t_i = \frac{2i-1}{2}}$$

$$V(4) = 25.000 + [\overset{1500}{r(0.5)} + \overset{1750}{r(1.5)} + \overset{750}{r(2.5)} + \overset{-750}{r(3.5)}] = 25.000 + 1500 + 1750$$

$$\boxed{V(4) = 25.000 + 3250 = 28.250}$$

(63) Lorenz curve – distribution of income between households



x – housholds (percentage of total housholds)

y – percentage of total income of the country

$\left(\frac{a}{100}, \frac{b}{100}\right)$ a% of housholds receive less or equal to b% of total income

- Absolute equality
  a% housholds receive a% of income
  y=x

- coefficient of inequality

(a) $$\frac{\int_0^1 (x - L(x))\,dx}{\int_0^1 x\,dx} = \frac{\int_0^1 (x - L(x))\,dx}{\frac{x^2}{2}\Big|_0^1} = 2\int_0^1 (x - L(x))\,dx$$

$$L\left(\tfrac{1}{2}\right) = \frac{5}{12}\frac{1}{4} + \frac{7}{12}\frac{2}{4} = \frac{19}{48} \doteq 0.3$$

(b) • $L(x) = \frac{5}{12}x^2 + \frac{7}{12}x$; $I = \int_0^{0.5} \left(\frac{5}{12}x^2 + \frac{7}{12}x\right)dx = 0.0903$ $\boxed{I = 9{,}03\%}$

39%

Bottom 50% of housholds receive 9,03% of total income in the country

• $$2\int_0^1 \left(\frac{5}{12}x^2 + \frac{7}{12}x\right)dx = 2 \cdot \frac{31}{72} = \frac{31}{36} = \boxed{0.86}$$ *

$$2\int_0^1 (x - L(x))\,dx = 2 \cdot \frac{1}{2} - 2\int_0^1 L(x)\,dx = 1 - 0.86 = 0.14 \left(1 - \frac{31}{36} = \frac{36-31}{36} = \frac{5}{36} \doteq 0.14\right)$$

(64) $t = [0, 10, 15, 20, 32, 59, 62, 125]$ [s]

$v = [0, 185, 319, 447, 742, 1325, 1445, 4151]$ [ft/s]

(a) $v(t) = 0.00146t^3 - 0.11553t^2 + 24.98169t - 21.26872$

(b) $a = \frac{dv}{dt} = 0.00438t^2 - 0.23106t + 24.98169$

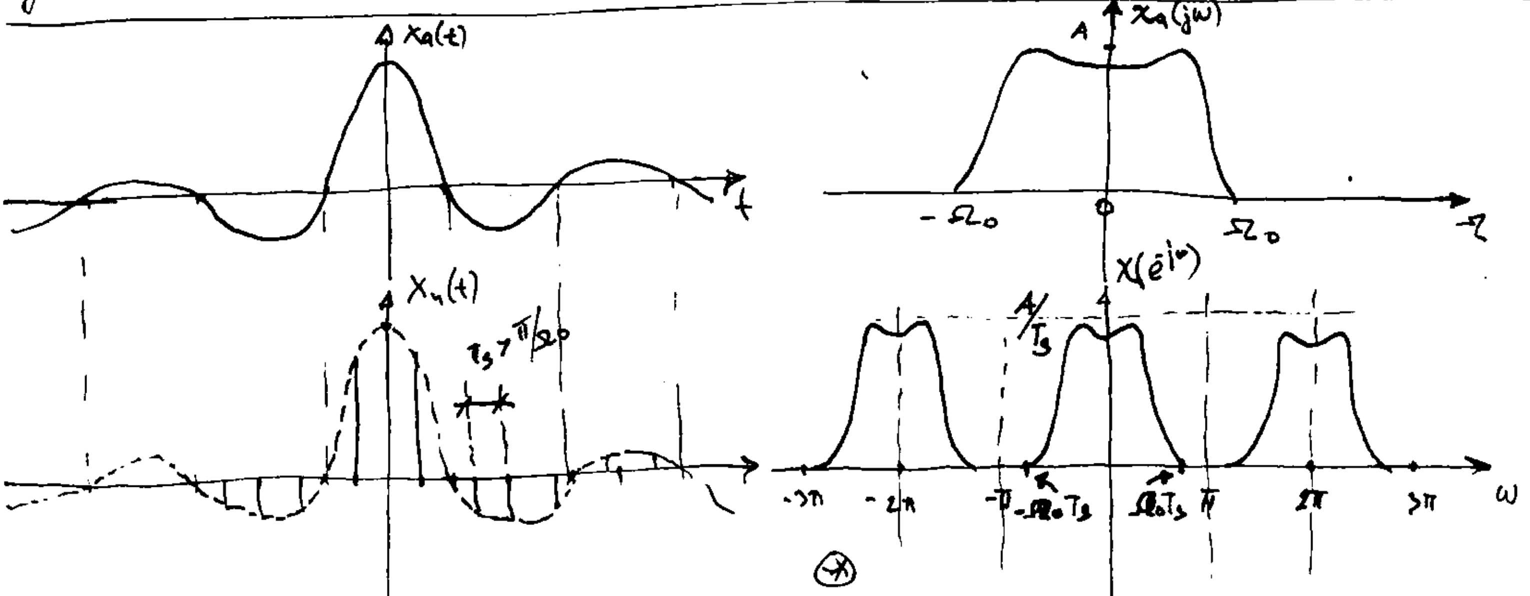
$\boxed{\text{SAMPLING}}$

$\boxed{f_s = \frac{1}{T_s}}$ $\boxed{\omega = \Omega T_s}$ MPD

$$\mathcal{X}_a(j\Omega) = \int_{-\infty}^{\infty} x_a(t) \cdot e^{-j\Omega t} dt \; ; \quad x_a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{X}_a(j\Omega) e^{j\Omega t} d\Omega$$

$x_a(n) \overset{\Delta}{=} x_a(n \cdot T_s)$, $\boxed{X(e^{j\omega t}) = \frac{1}{T_s} \sum_{l=-\infty}^{\infty} \mathcal{X}_a\left[j\left(\frac{\omega}{T_s} - \frac{2\pi}{T_s} l\right)\right]}$

DTFT $X(e^{j\omega t})$ is sum of ~~scaled~~ amplitude-scaled, frequency scalled and translated versions of fourier transform $\mathcal{X}_a(j\Omega)$



$\boxed{\text{DEFN 2:}}$ BAND LIMITED SIGNAL:

$\Omega_0 \Rightarrow X_a(j\Omega) = 0$ FOR $|\Omega| > \Omega_0$; $\boxed{F_0 = \frac{\Omega_0}{2\pi}} \Rightarrow$ SIGNAL BANDWIDTH IN Hz

⊛ $\Rightarrow$

$\pi > \Omega_0 T_s \Rightarrow T_s < \left(\frac{\Omega_0}{\pi}\right)^{-1} \Rightarrow T_s < \left(\frac{2\Omega_0}{2\pi}\right)^{-1} \overset{F_0}{=} \frac{1}{2F_0}$; $2F_0 < \frac{1}{T_s} \Rightarrow \boxed{F_s > 2F_0}$

$\boxed{\Omega_0 < \frac{\pi}{T_s}}$

IF $F_s > 2F_0$:

$$\boxed{X(e^{j\omega}) = \frac{1}{T_s} \mathcal{X}_a\left(j\frac{\omega}{T_s}\right) \quad -\frac{\pi}{T_s} \leq \frac{\omega}{T_s} \leq \frac{\pi}{T_s} \quad -\frac{\pi}{T_s} \leq \Omega \leq \frac{\pi}{T_s}}$$

$\boxed{\text{THEOREM 3}}$ Sampling Principle $x_a(t)$ can be reconstructed from its sample values $x(n) = x_a(nT_s)$ if : $\boxed{F_s > 2F_0 \text{ Nyquist RATE}}$

$\boxed{\Omega_0 < \frac{\pi}{T_s}}$ $\boxed{\Delta t << T_s}$ $x_G(m) \overset{\Delta}{=} x_a(m\Delta t)$ $\boxed{\Delta t \text{ GRID INTERVAL}}$

SIMULATION OF ANALOG SIGNAL

$$\boxed{X_a(j\Omega) \doteq \sum_m x_G(m) \cdot e^{-j\Omega m\Delta t} \cdot \Delta t = \Delta t \sum_m x_G(n) \cdot e^{-j\Omega m\Delta t}}$$

APROXIMATION OF FOURIER TRANSFORM.

$\boxed{\text{EXAMPLE 3.17}}$ $x_a(t) = e^{-1000|t|}$; Determine Fourier Transform

MA④

$$X_a(j\Omega) = \int_{-\infty}^{\infty} x_a(t) \cdot e^{-j\Omega t} dt = \int_{-\infty}^{\infty} e^{-1000|t|} \cdot \cancel{dt}\, e^{-j\Omega t} dt$$

$$\boxed{\int e^{ax} dx = \left| \begin{matrix} u = ax \\ du = a\,dx \\ dx = \frac{du}{a} \end{matrix} \right| = \frac{1}{a} \int e^{u} du = \frac{e^{ax}}{a}}$$

$$\boxed{\log_a x = \frac{\ln x}{\ln a}} = \frac{1}{500 \log[e]}$$

$$X_a(j\Omega) = \int_{-\infty}^{\infty} e^{-1000|t|} e^{-j\Omega t} dt = \underbrace{\int_{-\infty}^{0} e^{+1000t} e^{-j\Omega t} dt}_{①} + \underbrace{\int_{0}^{\infty} e^{-1000t} e^{-j\Omega t} dt}_{②}$$

$$= \left| |t| = \begin{cases} t & t \geq 0 \\ -t & t < 0 \end{cases} \right| = \frac{1000 \sqrt{\frac{2}{\pi}}}{w^2 + 10^6} = \frac{\sqrt{\frac{2}{\pi}}\ 0}{\frac{w^2}{1000} + 10^3} = \frac{\frac{1}{1000}\sqrt{\frac{2}{\pi}}}{1 + \left(\frac{w}{1000}\right)^2}$$

$$= \frac{0{,}000798}{1 + \left(\frac{w}{1000}\right)^2}$$

$$① = \int_{-\infty}^{0} e^{(1000 - j\Omega)t} dt = \frac{1}{(1000 - j\Omega)} \cdot e^{(1000 - j\Omega)t} \Big|_{-\infty}^{0} =$$

$$= \frac{1}{(1000 - j\Omega)} \left[e^{0} - \cancel{e^{-\infty}} \to 0 \right. = \frac{1}{1000 - j\Omega}$$

$$② = -\frac{1}{(1000 + j\Omega)} \cdot e^{-(1000 + j\Omega)t} \Big|_{0}^{\infty} = -\frac{1}{(1000 + j\Omega)} \cdot \left[0 - e^{0} \right] = \frac{1}{1000 + j\Omega}$$

$$① + ② = \frac{1}{1000 - j\Omega} + \frac{1}{1000 + j\Omega} = \frac{1000 \cancel{+ j\Omega} + 1000 \cancel{- j\Omega}}{1000^2 + \Omega^2} = \frac{2\cancel{000} \cdot 10^{\cancel{3}}}{10^{\cancel{6}3}\left(1 + \left(\frac{\Omega}{1000}\right)^2\right)}$$

$$= \frac{2 \cdot 10^{-3}}{1 + \left(\frac{\Omega}{1000}\right)^2} \doteq \frac{0.002}{1 + \left(\frac{\Omega}{1000}\right)^2}$$