STEWART Ch. 12.5 Exp. 2 (CONTINUE)

$a = x_1 - x_0 \quad b = y_1 - y_0 \quad c = z_1 - z_0$

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} \qquad \boxed{\frac{x-x_0}{x_1-x_0} = \frac{y-y_0}{y_1-y_0} = \frac{z-z_0}{z_1-z_0}}$$

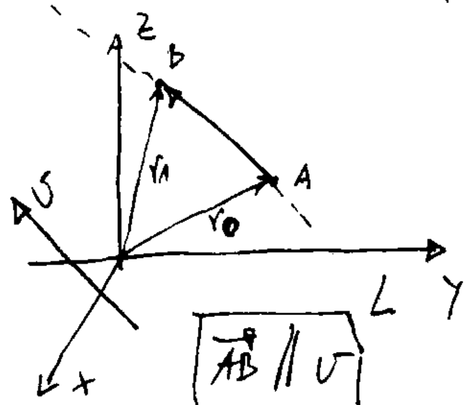
• LINE SEGMENT

$$\boxed{x = 2+t \quad y = 4-5t \quad z = -3+4t \quad 0 \le t \le 1}$$

DEFINITION OF LINE SEGMENT $\overrightarrow{AB}$

$t=0 \quad x = 2 \quad y = 4 \quad z = -3 \quad A(2,4,-3)$

$t=1 \quad x = 2+1 = 3 \quad y = 4-5 = -1 \quad z = -3+4 = 1 \quad B(3,-1,1)$



$\boxed{\overrightarrow{AB} \parallel v}$

$r = r_0 + v \cdot t$

$\overrightarrow{AB} + r_0 = r_1 \quad \overrightarrow{AB} = r_1 - r_0 \parallel v$

$r = r_0 + \overrightarrow{AB} \cdot t = \underline{r_0 + (r_1 - r_0) \cdot t}$

$$\boxed{r = r_0 + r_1 t - r_0 t = r_0(1-t) + r_1 \cdot t}$$

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• DEFINITION (VECTOR) OF LINESEGMENT AB

$$\boxed{r(t) = (1-t) r_0 + r_1 t \quad 0 \le t \le 1}$$

$\boxed{\text{CH13}}$ Exp. 5 $\quad P(1,3,-2) \quad Q(2,-1,3)$ } LINESEGMENT

$r = (1-t) r_0 + r_1 = (1-t)\langle 1,3,-2\rangle + \langle 2,-1,3\rangle =$

$= (1-t)(i+3j-2k) + t(2i-j+3k) = i+3j-2k - t(i+3j-2k) + t(2i-j+3k)$

~~$3i - 2j + k - t(i+3j-2k) = (3-t)i + (2+3t)j + (1+2t)k$~~

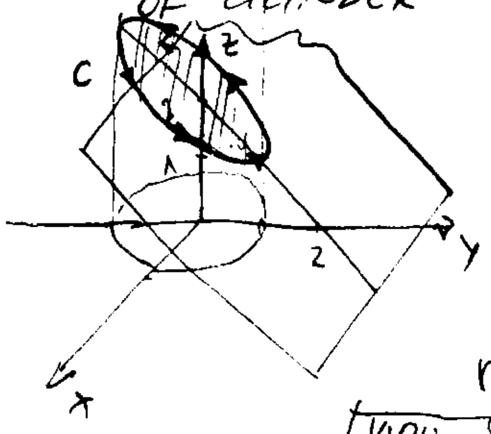
~~$x = 3-t \quad y = -2-3t \quad z = 1+2t$~~

$= i+3j-2k + t(-i-3j+2k) + t(2i-j+3k) = i+3j-2k + t(i-4j+5k)$

$= (1+t)i + (3-4t)j + (-2+5t)k$

$x = 1+t \quad y = 3-4t \quad z = -2+5t \quad 0 \le t \le 1$

$\boxed{\text{Exp. 6}}$ FIND VECTOR FUNCTION REPRESENTING INTERSECTION OF CYLINDER $x^2+y^2=1$ AND PLANE $z+y=2$



$z = 2-y = -y+2 = -(y-2)$

- PROJECTION OF „C" ON x-y PLANE

$x = \cos t \quad y = \sin t \quad 0 \le t \le 2\pi$

$z = 2-y = 2-\sin t$

$\boxed{x = \cos t \quad y = \sin t \quad z = 2-\sin t} \quad 0 \le t \le 2\pi$

$r(t) = \cos t\, i + \sin t\, j + (2-\sin t) k \quad 0 \le t \le 2\pi$

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$x = (4+\sin 20t)\cos t \quad y = (4+\sin 20t)\sin t \quad z = \cos 20t$

• TWISTED CUBIC CURVE

$r(t) = \langle t, t^2, t^3 \rangle \qquad \underbrace{x = t \quad y = t^2}_{} \quad z = t^3$

$y = x^2 \}$ XY PLANE PROJECTION

$\boxed{\text{Exc.1}}$ Domain of vector functions

$r(t) = \langle t^2, \sqrt{t-1}, \sqrt{5-t} \rangle$

$t > 1 \quad t < 5 \qquad t \in (1,5)$

$r(t) = \frac{t-2}{t+2} i + \sin t\, j + \ln(9-t^2) k$

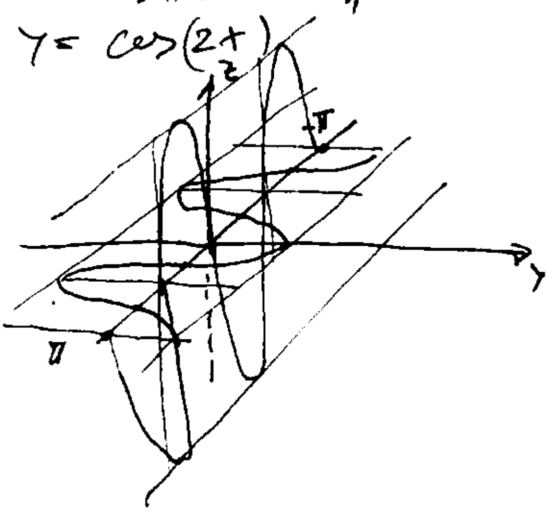
$9 - t^2 > 0 \quad t^2 < 9 \quad |t| < 3 \qquad (-3,-2) \;\&\; (-2,3)$

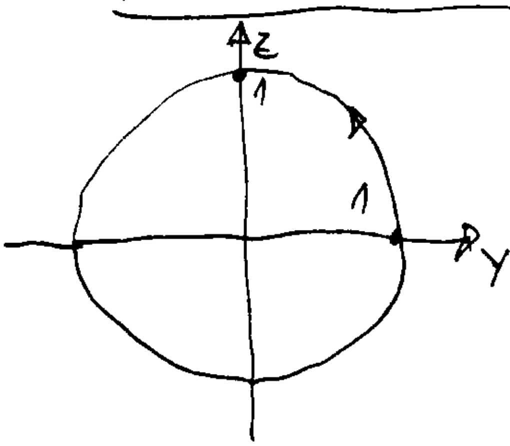
$\boxed{\text{Exc.9}}$ $r(t) = \langle t, \cos 2t, \sin 2t \rangle$

$\underbrace{x = t \quad y = \cos 2t}_{\text{ELIMINATE },t'} \qquad z = \sin 2t$

$y = \cos(2t) \qquad\qquad z = \sin 2t$

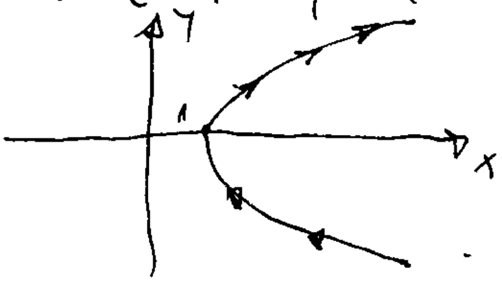
$y^2 + z^2 = \cos^2 2t + \sin^2(2t) = 1$





$\boxed{\text{Exc.7}}$ $r(t) = \langle t^4+1, t \rangle$

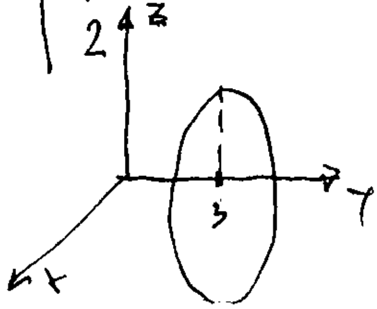
$x = t^4+1 \quad y = t \qquad x = y^4 + 1$



t	$x = t^4+1$	$y = t$
-2	$x = 5$	-2
-1	$x = 2$	-1
0	$x = 1$	0
1	$x = 2$	1
2	5	2

$\boxed{\text{Exc.11}}$ $r(t) = \langle \sin t, 3, \cos t \rangle$

$x = \sin t \quad z = \cos t \qquad x^2 + z^2 = 1$



Ex.18 $P(-2,4,0)$ $Q(6,-1,2)$

$$r(t) = (1-t)r_0 + r_1 t = (1-t)(-2,4,0) + t(6,-1,2) =$$
$$= (-2,4,0) + t(2,-4,0) + t(6,-1,2) = (-2,4,0) + t\langle 8,-5,2\rangle$$
$$\boxed{r(t) = (-2+8t)\,i + (4-5t)\,j + 2t\cdot k} \qquad 0\le t\le 1$$

Ex.35 $z^2 = x^2+y^2$ cone; plane $z = 1+y$

Ex.34 $x^2+y^2=4$ ... $z=xy$

$x = 2\cos t$ $\quad y = 2\sin t$ $\qquad z = 4\cos t\cdot\sin t = \underline{\underline{2\cdot\sin 2t}}$

$x^2+y^2 = 4\cos^2 t + 4\sin^2 t = 4$ ok $\qquad t = 0..2\pi$

$$(1+y)^2 = x^2+y^2 \Rightarrow 1+2y+\cancel{y^2} = x^2+\cancel{y^2}$$
$$x^2 = 1+2y \qquad \boxed{y = \frac{1}{2}(x^2-1) = \frac{1}{2}(x+1)(x-1)}$$

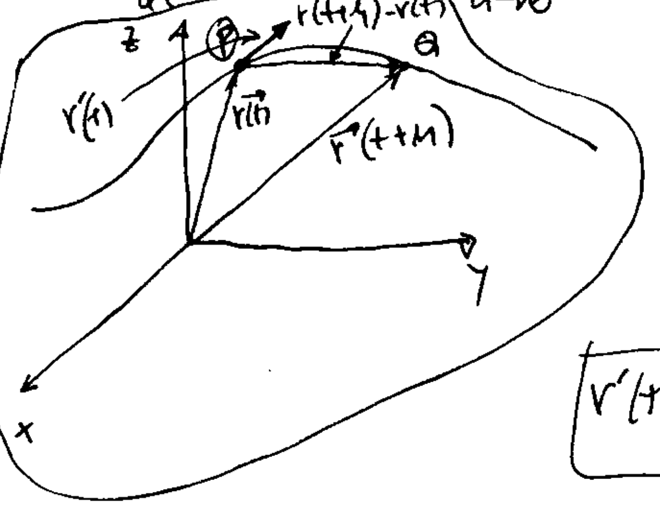
$x=t$ $\qquad y = \frac{1}{2}(t^2-1)$ $\qquad z = 1+\frac{1}{2}(t^2-1)$

$$\boxed{C:\ r(t) = t\cdot i + \frac{1}{2}(t^2-1)\,j + \left[1+\frac{1}{2}(t^2-1)\right]k}$$

13.2 Derivates and Integrals of Vector Functions

• Derivates

$$\frac{d\vec r}{dt} = \vec r\,'(t) = \lim_{h\to 0}\frac{\vec r(t+h)-\vec r(t)}{h}$$



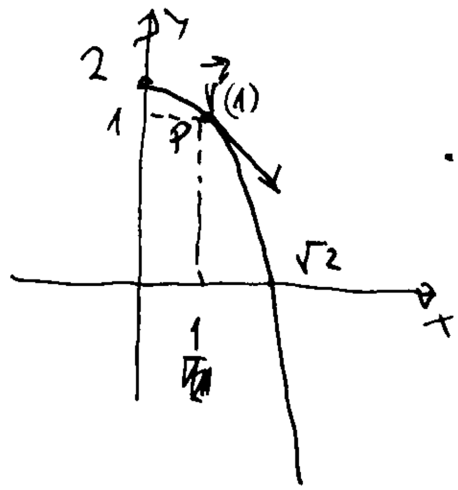
$\vec r(t+h)-\vec r(t)$ → secant vector

$r'(t)$ → tangent vector

Theorem: $\vec r(t) = \langle f(t), g(t), h(t)\rangle$
$$= f(t)\,\vec i + g(t)\,\vec j + h(t)\,\vec k$$
$$\boxed{r'(t) = f'(t)\,\vec i + g'(t)\,\vec j + h'(t)\,\vec k}$$

$\boxed{\text{Exp 2}}$ $\vec r(t)=\sqrt{t}\,\vec i+(2-t)\vec j$ $\quad r(t)=?$

$x=\sqrt{t}$ $\quad y=2-t$ $\quad t=x^2$ $\quad \boxed{y=2-x^2}$ $\quad y=-x^2+2$



$\vec r'(1)=?$

$\vec r'(t)=\frac{1}{2}\frac{1}{\sqrt{t}}\,i+(0-1)j$

$\vec r'(1)=\frac{1}{2}\,i-1\,j$

$x=1$ $\quad y=2-1=1$ $\quad P(1,1)$

$\boxed{\text{Exp.3}}$ $x=2\cos t$ $\quad y=\sin t$ $\quad z=t$

tangent line at $O(0,1,\pi/2)$

$\vec r(t)=2\cos t\,\vec i+\sin t\,\vec j+t\cdot\vec k$

$r'(t)=-2\cdot\sin t\,\vec i+\cos t\,\vec j+k$

$0=2\cos t$ $\quad \cos t=0$ $\quad t_0=\frac{\pi}{2}$

$\sin t=1$ $\quad t_0=\pi/2$

$t=\pi/2$ $\quad t_0=\pi/2$

$r'(t_0)=-2\cdot\sin\frac{\pi}{2}\,i+\cos\left(\frac{\pi}{2}\right)\vec j+k=\underline{\underline{-2\vec i+\vec k}}=\langle-2,0,1\rangle$

$\boxed{r'(t_0)=\langle-2,0,1\rangle \text{ TANGENT VECTOR}}$

TANGENT LINE:

$l(t)=\vec r_0+t\cdot\vec v=\langle 0,1,\frac{\pi}{2}\rangle+t\langle-2,0,1\rangle$

$x=-2t$ $\quad y=1$ $\quad z=\frac{\pi}{2}+t$

$\boxed{\text{Exp5}}$ Show that if $|r(t)|=c$ THEN $\vec r'(t)$ IS ORTHOGONAL TO $\vec r(t)$

SOLUTION: $\vec r\cdot\vec r(t)=|r(t)|^2=c^2$

$\underline{\underline{0}}=\frac{d}{dt}[r(t)\cdot r(t)]=r'(t)\cdot r(t)+r(t)\cdot r'(t)=\underline{\underline{2r(t)\cdot r'(t)}}$

$\boxed{r(t)\cdot r'(t)=0}$ $\Rightarrow$ $\vec r'(t)$ IS ORTHOGONAL TO $\vec r(t)$

- INTEGRALS

$\int_a^b \vec r(t)\,dt=\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x\,\vec i+\lim_{n\to\infty}\sum_{i=1}^{n} g(t_i^*)\Delta t\,\vec j+\lim_{n\to\infty}\sum_{i=1}^{n} h(t_i^*)\Delta t\,\vec k$

$$\int_a^b \vec{r}(t)\,dt = \int_a^b f(t)\,dt\ \vec{i} + \int_a^b g(t)\,dt\ \vec{j} + \int_a^b h(t)\,dt\ \vec{k}$$

- EXTENDED VERSION OF FUNDAMENTAL THEOREM OF CALCULUS

$$\int_a^b r(t)\,dt = \vec{R}(t)\Big|_a^b = \vec{R}(b) - \vec{R}(a)$$

13.3 ARC LENGTH & CURVATURE (CONTINUE ON PP. 16)

LEGHT OF PARAMETRIC CURVE:

$$\boxed{x = f(t);\ y = g(t) \quad a \le t \le b}$$

$$L = \int_a^b \sqrt{f'^2(t) + g'^2(t)}\,dt = \int_a^b \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2}\,dt$$

10.2 CALCULUS WITH PARAMETRIC CURVES

$x = f(t) \quad y = g(t)$

by ELIMINATING „t" IT IS POSIBLE TO GET FOLLOWING FORM:

$$y = F(t)$$

$$y = F(f(t)) \qquad g'(t) = f'(t) \cdot F'(f(t)) = f'(t) \cdot F'(t)$$

$$\boxed{F'(t) = \frac{g'(t)}{f'(t)}} \qquad \boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}} \quad \text{IF } \frac{dx}{dt} \neq 0$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}}$$

Exp 1 $C: x = t^2,\ y = t^3 - 3t$

(a) Show that „C" has 2 tangents at $(3,0)$

(b) Points of „C" = ? where tangen is horizontal and vertical

(c) Where „CURVE IS CONCAVE UPWARDS AND DOWNWARDS

(d) Sketch the curve

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2t}$$

$(3,0) \Rightarrow$

$3 = t^2 \Rightarrow t = \pm\sqrt{3}$

$0 = t^3 - 3t \quad t^{\cancel{3}} = 3\cancel{t} \quad t^2 = 3$

$t = \pm\sqrt{3}$

$T_1:\ t = \sqrt{3} \quad x = 3 \quad y = \sqrt{27} - 3\sqrt{3} = 3\sqrt{3} - 3\sqrt{3} = 0$

$T_2:\ t = -\sqrt{3} \quad x = 3 \quad y = -3\sqrt{3} + 3\sqrt{3} = 0$

ⓐ $\vec{r}(t) = x\vec{i} + y\vec{j}$ $\qquad O(3,0)$

$x = t^2 \qquad y = t^3 - 3t$

$\vec{r}'(t) = 2t\,\vec{i} + (3t^2-3)\vec{j}$ TANGENT VECTOR

TANGENT $\quad t = \pm\sqrt{3}$

$\vec{r}'(t_0) = \pm 2\sqrt{3}\,\vec{i} + (3\cdot 3-3)\vec{j} = \pm 2\sqrt{3}\,\vec{i} + 6\vec{j}$

$r = r_0 + t\cdot\vec{r}'(t_0) = \langle 3,0\rangle + (\pm 2\sqrt{3}\,i + 6j)t = \langle 3,0\rangle + \langle \pm 2\sqrt{3}, 6\rangle t$

$\boxed{x = 3 \pm 2\sqrt{3}\,t \qquad y = 6t}$ PARAMETRIC FORM OF THE TWO TANGENTS IN $(3,0)$

$t = \frac{y}{6} \quad x = 3 \pm \cancel{2}\sqrt{3}\,\frac{y}{\cancel{6}3} = 3 \pm \frac{y}{\sqrt{3}} \quad \pm\sqrt{3}x = 3\sqrt{3} \pm y$

$y = \pm\sqrt{3}x \mp 3\sqrt{3}$

$\boxed{y = \pm\sqrt{3}(x-3)}$

ⓑ $\frac{dy}{dt} = 0 \Rightarrow$ HORIZONTAL TANGENT

$3t^2 - 3 = 0 \qquad t^* = \pm 1$

$\frac{dx}{dt} = 0 \Rightarrow$ VERTICAL TANGENT $\quad 2t = 0 \quad \boxed{t=0}$

ⓒ $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{3t^2-3}{2t}\right)}{2t} =$

$= \frac{1}{2t}\left[\frac{2t\cdot t - (t^2-1)\cdot 1}{t^2}\right]\frac{3}{2} = \frac{3}{4t}\left[\frac{2t^2-t^2+1}{t^2}\right] = \frac{3t^2+3}{4t^3}$

$t>0 \Rightarrow \frac{d^2y}{dx^2} > 0$ CONCAVE UPWARD (MINIMUM)

$t<0 \Rightarrow \frac{d^2y}{dx^2} < 0$ CONCAVE DOWNWARD (MAXIMUM)

$\boxed{\text{EX } 2}$ $\quad x = r(\theta - \sin\theta) \qquad y = r(1-\cos\theta)$

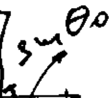
FIND TANGENT AT: $\boxed{\theta = \frac{\pi}{3}}$

$x_0 = r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) \qquad y_0 = r\left(1 - \frac{1}{2}\right) = \frac{r}{2}$

$\boxed{x_0 = r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) \qquad y_0 = r/2} \qquad \boxed{r_0 = \left\langle r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right), \frac{r}{2}\right\rangle}$

$t(t) = r_0 + v\cdot t = r_0 + v\cdot\theta$

C: $\vec{r}(\theta) = r(\theta - \sin\theta)\,\vec{i} + r(1-\cos\theta)\,\vec{j} \qquad \boxed{\theta_0 = \frac{\pi}{3}}$



$\vec{r}'(\theta) = r(1-\cos\theta)\,\vec{i} + r\cdot\sin(\theta)\,\vec{j} \qquad \boxed{v = \langle r(1-\cos\theta_0), r\sin\theta_0\rangle}$

$t(\theta) = \left\langle r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right), \frac{r}{2}\right\rangle + \langle r(1-\cos\theta_0), r\sin\theta_0\rangle\cdot\theta$

$$\vec{t}(\theta) = \left\langle r\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right), \frac{r}{2}\right\rangle + \left\langle \frac{r}{2}, \frac{\sqrt{3}r}{2}\right\rangle \theta = r\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2}+\frac{\theta}{2}\right)\vec{i} + \frac{r}{2}(1+\sqrt{3}\theta)\vec{j}$$

$$\vec{t}(\theta) = \underbrace{r\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2}+\frac{\theta}{2}\right)}_{x}\vec{i} + \underbrace{\frac{r}{2}(1+\sqrt{3}\theta)}_{y}\vec{j}$$

$$x_0 = r\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right) = |r=1| = 0.1812$$

$$y_0 = \frac{r}{2} = |r=1| = 0.5$$

HORIZONTAL TANGENT

$$\frac{dy}{d\theta} = 0 \qquad \frac{dy}{d\theta} = r(1+\sin\theta) = 0 \qquad \sin\theta = 0 \qquad \theta = \pi \qquad \theta = \pi$$

$$x_1 = r\left(\pi - \sin\pi\right) = r\pi$$

$$y_1 = r\left(1 - \underbrace{\cos\pi}_{-1}\right) = 2r$$

$$\boxed{r=1} \quad \boxed{x_1 = \pi \quad y_1 = 2}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r(\sin\theta)}{r(1-\cos\theta)} = \frac{\sin\theta}{1-\cos\theta} = 0$$

$$\sin\theta = 0 \qquad \boxed{\theta_1 = \pi}$$

$$\vec{r}'(\theta) = r(1-\cos\theta)\vec{i} + r\sin(\theta)\vec{j} = \vec{t}(\theta)$$

$$\vec{r}'(\theta_1) = r(1+1)\vec{i} + 0\vec{j} = 2r\vec{i}$$

$$\vec{t}(\theta_1) = \langle r\pi, 2r\rangle + \theta\langle 2r, 0\rangle \qquad \underline{r(\pi+2\theta)\vec{i} + 2r\vec{j}}$$

ALTERNATIVELY:

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \frac{\sin\frac{\pi}{3}}{1-\cos\frac{\pi}{3}} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} = k$$

EQUATION OF LINE

$$y - y_0 = k(x - x_0) \qquad y - \frac{r}{2} = \sqrt{3}\left(x - \frac{r\pi}{3} + \frac{r\sqrt{3}}{2}\right)$$

$$\lim_{\theta\to\pi}\frac{dy}{dx} = \lim_{\theta\to\pi}\frac{\sin\theta}{1-\cos\theta} \qquad \frac{0}{0} \qquad = \frac{0}{2} = 0$$

DOKAZ DEKA VO OVIE TOČKI IMA HORIZONTALNI TANGENTI !!

⊙ AREAS

$y = f(x) \qquad A = \int_a^b f(x)\,dx \qquad f(x) \geq 0$

$x = f(t) \quad \underline{y = g(t)}$

$dx = f'(t)\cdot dt$

$$A = \int_a^b y\,dx = \int_\alpha^\beta g(t) f'(t)\,dt$$

$\boxed{\text{Exp. 3}}$ $A = ?$ UNDER ONE ARC OF CYCLOID

$x = r(\theta - \sin\theta) \qquad y = r(1 - \cos\theta)$

$dx = r(1 - \cos\theta)\,d\theta$

$$A = \int_0^{2\pi} r^2(1-\cos\theta)\cdot(1-\cos\theta)\,d\theta = \int_0^{2\pi} r^2(1-\cos\theta)^2\,d\theta = \underline{\underline{3\pi r^2}}$$

$$\int_0^{2\pi} \cos^2\theta\,d\theta = \int_0^{2\pi} \frac{1}{2}[1+\cos 2\theta]\,d\theta$$

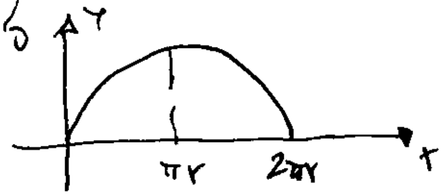
$\cos(\theta+\theta) = \cos^2\theta - \sin^2\theta$

$\qquad 1 \qquad = \cos^2\theta + \sin^2\theta$

$\cos(2\theta) = \cos^2\theta - 1 + \cos^2\theta$

$\qquad = 2\cos^2\theta - 1$

$\cos^2\theta = \frac{1}{2}[1+\cos 2\theta]$



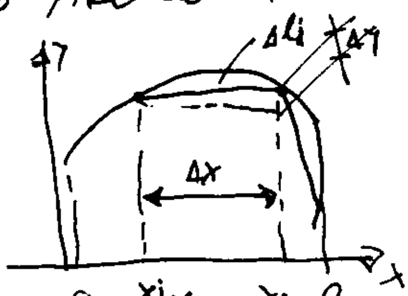
$$\int_0^{2\pi r} y\,dx \qquad x = 2\pi r \Rightarrow 2\pi r = r(\theta - \sin\theta)$$

$\theta - \sin\theta = 2\pi \Rightarrow \boxed{\theta = 2\pi}$

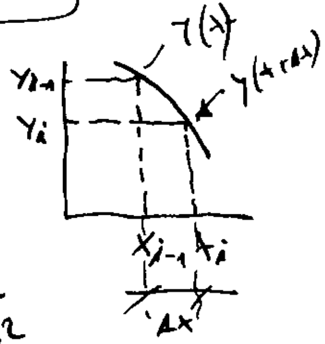
$x = 0 \qquad \theta - \sin\theta = 0 \quad \boxed{\theta = 0}$

$$\int_0^{2\pi} r^2(1-\cos\theta)\cdot(1-\cos\theta)\,d\theta$$

⊙ ARC LENGHT



$$L = \int_a^b dl$$



$$L \doteq \sum_{i=1}^{n} \Delta l_i = \sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2}$$

$y_i - y_{i-1} = f'(x_i^*)\cdot(x_i - x_{i-1}) \qquad \Delta y_i = f'(x_i^*)\cdot\Delta x_i$

$\cancel{\lim_{\Delta x \to 0} \frac{y + h\Delta y}{\Delta x}}$

$$\lim_{h\to 0} \frac{y(x+h) - y(x)}{h} = \lim_{\Delta x \to 0} \frac{y(x+\Delta x) - y(x)}{\Delta x}$$

$$L \doteq \sum_{i=1}^{n} \sqrt{\Delta x_i^2 + f'^2(x_i^*)\cdot\Delta x_i^2} = \sum_{i=1}^{n} \Delta x_i \sqrt{1 + f'^2(x_i^*)}$$

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + f'^2(x_i^*)}\,\Delta x_i$$

$$\boxed{L = \int_a^b \sqrt{1+f'^2(x)}\,dx}$$

LENGHT OF ARC!!!
FOR CURVE GIVEN IN FORM $y = F(x)$

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$x = f(t) \quad y = g(t) \quad \alpha \le t \le \beta$

$\frac{dx}{dt} = f'(t) > 0 \quad f(\alpha) = a \quad f(\beta) = b$

$y = F(x) \qquad g(t) = F(f(t)) \qquad g'(t) = f'(t)\cdot F'(f(t))$

$$\boxed{F'(x) = \frac{g'(t)}{f'(t)} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}} = \frac{dy}{dx}$$

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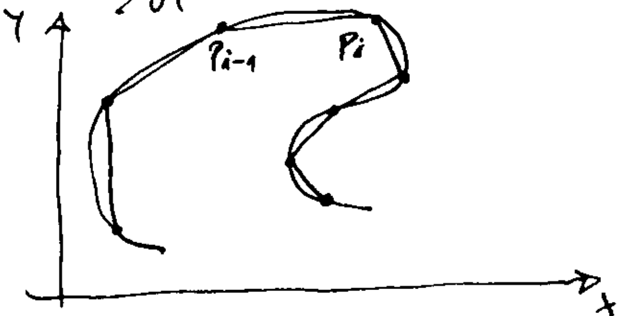
$$L = \int_\alpha^\beta \sqrt{1+\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2}\,\frac{dx}{dt}\cdot dt = \int_\alpha^\beta \frac{\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}}{\cancel{\frac{dx}{dt}}}\cdot\cancel{\frac{dx}{dt}}\,dt \qquad \boxed{\frac{dx}{dt} > 0}$$

$$\boxed{L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt}$$



$$L = \lim_{n\to\infty}\sum_{i=1}^{n}|P_{i-1}P_i| \qquad f(t_i) - f(t_{i-1}) = f'(t_i^*)(t_i - t_{i-1})$$

$$\boxed{\Delta x_i = f'(t_i^*)\,\Delta t}$$

SIMILARLY APPLIED TO $\boxed{g(t) \Rightarrow \Delta y = g'(t_i^*)\,\Delta t}$

$$|P_{i-1}P_i| = \sqrt{\Delta x_i^2 + \Delta y_i^2} = \sqrt{f'^2(t_i^*) + g'^2(t_i^*)}\,\Delta t$$

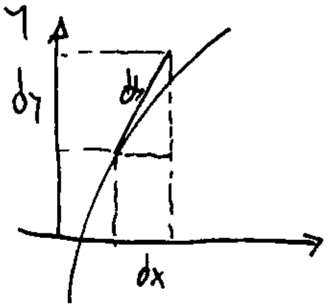
$$L = \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{f'^2(t_i^*)+g'^2(t_i^*)}\,\Delta t = \int_\alpha^\beta \sqrt{f'^2(t)+g'^2(t)}\,dt$$

- ARCLENGHT FUNCTION

$$s(x) = \int_a^x \sqrt{1+f'^2(t)}\,dt \qquad \frac{ds}{dx} = \sqrt{1+f'^2(x)} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\boxed{ds = \sqrt{dx^2+dy^2}}$$

MMV



$$ds^2 = dx^2 + dy^2 \qquad ds = \sqrt{dx^2+dy^2}$$

$$\boxed{L = \int ds = \int_a^b \sqrt{1+\frac{dy^2}{dx^2}}\,dx}$$

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Exp4 $x = \cos t \quad y = \sin t \quad 0 \le t \le 2\pi$

$$L = \int_a^b \sqrt{f'^2(t) + g'^2(t)}\,dt = \int_0^{2\pi} \sqrt{\sin^2(t) + \cos^2(t)}\,dt = 2\pi$$

Exp5 $x = r(\theta - \sin\theta) \quad y = r(1-\cos\theta)$

$$L = \int_0^{2\pi} \sqrt{f'^2(t) + g'^2(t)}\,dt = \int_0^{2\pi} \sqrt{r^2(1-\cos\theta)^2 + r^2\sin^2\theta}\,d\theta$$

MAPLE

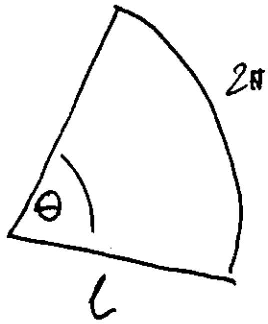
$$= r\int_0^{2\pi} \sqrt{1 - 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{1}}\,d\theta = r\int_0^{2\pi}\sqrt{2-2\cos\theta}\,d\theta = 8r$$

- SURFACE AREA (from 10.2) CONTINUE ON PP.15 ...

8.2 AREA OF SURFACE OF REVOLUTION

CYLINDER $\boxed{A = 2\pi R \cdot H}$

$L = 2\pi l$



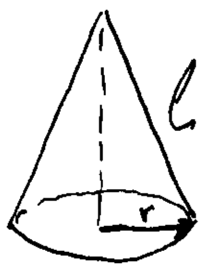
$\cancel{\alpha = \frac{2\pi l}{4}}$

$$\Leftarrow \left\{ L_1 = \frac{2\pi l}{4} \right. = \text{AGOL} \cdot \text{RADIUS}$$

$\theta \cdot l = 2\pi r \qquad \boxed{\theta = \frac{2\pi r}{l}}$

$$\boxed{A = \frac{l^2}{2}\cdot\theta = \frac{l^2}{2}\cdot\frac{2\pi r}{l} = \pi r \cdot l}$$

LATERAL SURFACE AREA OF CONE

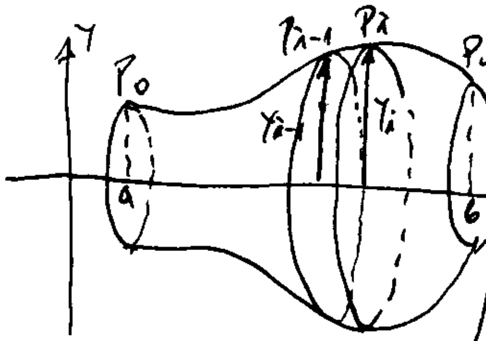


- AREA OF CONE BAND

$$A = \pi r_2(l_1 + l) - \pi r_1 \cdot l_1 = \pi(r_2 - r_1)l_1 + \pi r_2 l$$

$\frac{l_1+l}{r_2} = \frac{l_1}{r_1}$ $\quad l_1\cdot r_1 + l\cdot r_1 = l_1\cdot r_2$ $\quad \boxed{l_1(r_2-r_1) = l\cdot r_1}$

$A = \pi\cdot r_1\cdot l + \pi r_2\cdot l = (r_1+r_2)\pi l = \boxed{2\pi r\cdot l}$ $\quad \boxed{r = \frac{r_1+r_2}{2}}$



$y = f(x)$ $\quad a \le x \le b$

$y_i = f(x_i)$ $\quad P_i(x_i, y_i)$

MMV

$\boxed{A_{CONE\text{-}BAND} = 2\pi r\cdot l \quad r = \frac{r_1+r_2}{2}}$

$S_i = 2\pi \frac{y_i + y_{i-1}}{2}\cdot |P_{i-1}P_i|$ $\quad |P_{i-1}P_i| = \sqrt{1 + f'^2(x_i^*)}\,\Delta x$

$\Delta x \to 0 \Rightarrow y_i = y_{i-1} \doteq f(x_i^*)$

$$\boxed{S_i = 2\pi\cdot f(x_i^*)\cdot\sqrt{1+f'^2(x_i^*)}\,\Delta x}$$

$$S = \lim_{n\to\infty}\sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1+f'^2(x_i)}\,\Delta x = \int_a^b 2\pi f(x)\sqrt{1+f'^2(x)}\,dx$$

$$\boxed{S = \int_a^b 2\pi f(x)\sqrt{1+f'^2(x)}\,dx}$$ MMV

$$S = \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

• IF CURVE IS $x = g(y)$ $\quad c \le y \le d$

$$S = \int_c^d 2\pi y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$ $\quad ds^2 = dx^2 + dy^2$

$$\boxed{S = \int 2\pi y\,ds}$$

ROTATION OVER „x" AXIS

$ds = \sqrt{1+\frac{dy}{dx}}\cdot dx$

$$\boxed{S = \int 2\pi x\,ds}$$

ROTATION OVER „y" AXIS

$ds = \sqrt{1+\frac{dx}{dy}}\,dy$

Ch. 8.2 Exp 1 $y = \sqrt{4-x^2}$ $-1 < x \leq 1$ $x^2+y^2=4$

$$S = \int_{-1}^{1} 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = ?$$

$$y'(x) = \frac{1}{\cancel{2}\sqrt{4-x^2}}\cdot(-\cancel{2}x) \qquad S = 2\pi\int_{-1}^{1}\sqrt{4-x^2}\sqrt{1+\left(\frac{x}{\sqrt{4-x^2}}\right)^2}\, dx$$

$$S = \cancel{2\int_0^1 \sqrt{(4-x^2)-2x\sqrt{4-x^2}}\,dx} = \int_{-1}^{1}\sqrt{4-x^2}\sqrt{1+\frac{x^2}{4-x^2}}\, dx = (*)$$

$$\cancel{4-x^2 = u^2 \quad -2x\,dx = 2u\,du \quad x=0 \; u=\pm\sqrt{4}=\pm 2 \quad x=1 \; u=3}$$

WITH POLAR COORDINATES

$y = \rho\sin\varphi$ $x^2+y^2=4$ $\rho = 2$
$x = \rho\cos\varphi$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\varphi}}{\frac{dx}{d\varphi}} = \frac{\cancel{\rho}\cos\varphi}{-\cancel{\rho}\sin\varphi} = -\operatorname{ctg}\varphi$$

$dx = -\rho\sin\varphi\, d\varphi$ $x=1$ $\varphi = \arccos\frac{1}{\rho} = \arccos\frac{1}{2} = \frac{\pi}{3}$
$x=-1$ $\varphi = \arccos -\frac{1}{2} = -\pi/3$

$$S = \int_{-\pi/3}^{\pi/3} 2\pi\rho\sin\varphi\sqrt{1+\frac{\cos^2\varphi}{\sin^2\varphi}}\; -\rho\sin\varphi\, d\varphi =$$

$$= -2\pi\cdot\cancel{\rho^2}^{4}\int_{-\pi/3}^{\pi/3}\sin^{\cancel{2}}\varphi\sqrt{\frac{1}{\cancel{\sin^2\varphi}}}\, d\varphi = -8\pi\int_{-\pi/3}^{\pi/3}\sin\varphi\, d\varphi = +8\pi\cos\varphi\Big|_{-\pi/3}^{\pi/3}$$
$$= 8\pi\left(\frac{1}{2}+\frac{1}{2}\right) = 8\pi$$

$$(*) = 2\pi\int_{-1}^{1}\sqrt{4-\cancel{x^2}+\cancel{x^2}}\, dx = 2\pi\int_{-1}^{1}\sqrt{4}\, dx = 2\pi\sqrt{4}\,x\Big|_{-1}^{1} = 2\pi\sqrt{4}(1+1) = 2\sqrt{4} = 4\cdot 2\pi = 8\pi$$

Exp. 2 $y = x^2$ $(1,1)\div(2,4)$ $S=?$

$$S = \int_1^4 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy \qquad x = \sqrt{y} \quad \frac{dx}{dy} = -\frac{1}{2\sqrt{y}}$$

$$S = \int_1^4 2\pi x\sqrt{1+\frac{1}{4y}}\, dy = 2\pi\int_1^4\sqrt{y+\frac{1}{4}}\, dy = \pi\int_1^4\sqrt{4y+1}\, dy$$

$$S = \pi \int_1^4 \sqrt{1+4y}\, dy$$ $\boxed{\begin{matrix} 1+4y = u \\ 4dy = du \quad dy = \frac{du}{4} \end{matrix}}$ $\begin{matrix} y=1 & u=5 \\ y=4 & u=17 \end{matrix}$

$$S = \pi \int_5^{17} \sqrt{u}\, \frac{du}{4} = \frac{\pi}{4} \int_5^{17} u^{\frac{1}{2}} du = \frac{\pi}{\cancel{4}2} \frac{u^{\frac{1}{2}+1}}{3/\cancel{2}} \Big|_5^{17} = \frac{\pi}{6} \sqrt{u^3} \Big|_5^{17}$$

$$S = \frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right)$$

$\boxed{\text{Exp 3}}$ $y = e^x \quad 0 \leq x \leq 1$ $\qquad \frac{dy}{dx} = e^x$

$$S = \int_0^1 2\pi y \sqrt{1+e^{2x}}\, dx = 2\pi \int_0^1 \underbrace{e^x \sqrt{1+e^{2x}}\, dx}_{}$$

$u = e^x \qquad du = e^x dx$ $\quad \boxed{x=0 \quad u=1}$ $\boxed{x=1 \quad u=e}$

$$S = 2\pi \int_1^e \sqrt{1+u^2}\, du$$ $\boxed{\begin{matrix} \text{TREBA SMENA} \\ u = tg\theta \end{matrix}}$

$y = \arccos(x)$

$\cos y = x \qquad \sin y\, dy = dx$

$dy = \frac{dx}{\sin y} = \frac{dx}{\sqrt{1-\cos^2 y}} = \frac{dx}{\sqrt{1-x^2}}$

$$\frac{dy}{dx} = \frac{d}{dx}(\arccos(x)) = \frac{1}{\sqrt{1-x^2}}$$

$y = \text{arcctg}(x) \qquad ctg(y) = x \qquad \left(\frac{\cos y}{\sin y}\right)' dy = dx \qquad \frac{-\sin y \cdot \sin y - \cos y \cdot \cos y}{\sin^2 y} dy = dx$

$$\frac{dy}{dx} = -\frac{1}{\frac{\sin^2 y + \cos^2 y}{\sin^2 y}} = -\frac{1}{1+ctg^2 y} = -\frac{1}{1+x^2}$$

$$S = 2\pi \int_1^e \sqrt{1+u^2}\, du = 2\pi\left(\frac{1}{2} u\sqrt{1+u^2} + \frac{1}{2} \text{arcsinh}(u)\right)$$

$$S = -\pi\left(\sqrt{2} + \ln(1+\sqrt{2}) - e\sqrt{1+e^2} + \ln\left(-e+\sqrt{1+e^2}\right)\right)$$ $\boxed{\text{MATLG}}$

$$\ln\left(-e+\sqrt{1+e^2}\right) = \ln\left(\frac{-1+e\sqrt{1+e^2}}{e}\right) = \ln\left(-1+e\sqrt{1+e^2}\right) - \ln e$$

$S = 2\pi \int_1^e \sqrt{1+u^2}\,du \qquad u = tg\,\theta \qquad du = \frac{\cos^2(\theta)+\sin^2(\theta)}{\cos^2(\theta)}\,d\theta$

$du = (1+tg^2(\theta))\,d\theta = (1+u^2)\,d\theta = sec^2(\theta)\,d\theta$

$S = 2\pi \int_a^b \sqrt{1+tg^2\theta}\; sec^2(\theta)\,d\theta$ // $u = 1 \quad tg\,\theta = 1 \quad \theta = \frac{\pi}{4}$; $u = e \quad tg\,\theta = e \quad \theta = arctg(e) = \alpha$

$S = 2\pi \int_{\pi/4}^{\alpha} \sqrt{\frac{1}{\cos^2\theta}}\; sec^2(\theta)\,d\theta = 2\pi \int_{\pi/4}^{\alpha} sec^3(\theta)\,d\theta$

chtzerb

$I = \int sec^3 x\,dx \qquad u = sec\,x \quad dv = sec^2 x\,dx \quad v = \int sec^2 x\,dx = tg(x)$

$du = +\frac{1}{\cos^2 x}\cdot \sin x\,dx = sec(x)\cdot tg(x)\,dx$

$y = tg\,x \qquad \frac{dy}{dx} = \frac{1}{\cos^2 x} = sec^2 x \Rightarrow \int sec^2 x = tg(x)$

$I = sec(x)\cdot tg(x) - \int tg\,x \underbrace{\frac{\sin x}{\cos^2 x}\,dx}_{du} = sec(x)\cdot tg\,x - \int sec(x)\cdot tg^2(x)\,dx$

$= sec(x)\,tg\,x - \int sec(x)\left[\frac{1}{\cos^2(x)} - 1\right]dx = sec(x)\,tg(x) - \int sec(x)\left[sec^2(x) - 1\right]dx$

// $tg^2(x) = \frac{1}{\cos^2 x} - 1 = \frac{1-\cos^2(x)}{\cos^2(x)} = \frac{\sin^2(x)}{\cos^2(x)}$ //

$= sec(x)\,tg\,x - \underbrace{\int sec^3(x)\,dx}_{I} + \int sec(x)\,dx$

$= \ln(sec(x) + tg(x))$

$2I = sec(x)\,tg\,x + \int sec(x)\,dx$

$\boxed{\int f(x)\cdot g'(x)\,dx = f(x)\cdot g(x) - \int f'(x)\,g(x)\,dx}$ POLUCHAVA FORMA NA INTEGRATION BY PARTS

$\boxed{\int u\,dv = u\cdot v - \int v\,du}$

STANDARTNATA FORMA

$\boxed{u = f(x) \quad v = g(x)}$

$\boxed{I = \frac{1}{2} sec(x)\,tg\,x + \frac{1}{2}\ln(sec(x) + tg(x))}\Big|$

$S = \pi\left[sec(x)\,tg\,x + \ln(sec(x) + tg(x))\right]\Big|_{\pi/4}^{\alpha}$

$\boxed{\int sec(x)\,dx = \ln(sec(x) + tg(x))}$ MMV

• SURFACE AREA (CONTINUATION FROM PP. 10) $y=f(t)$ $x=g(t)$ $\alpha\le t\le\beta$

$$S=\int_a^b 2\pi y\cdot ds=\int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx=\int_\alpha^\beta 2\pi y\sqrt{1+\left(\frac{dy/dt}{dx/dt}\right)^2}\cdot g'(t)dt$$

$$=\int_\alpha^\beta 2\pi f(t)\sqrt{1+\left(\frac{f'(t)}{g'(t)}\right)^2}\cdot g'(t)dt=\int_\alpha^\beta 2\pi f(t)\sqrt{g'^2(t)+f'^2(t)}\,dt$$

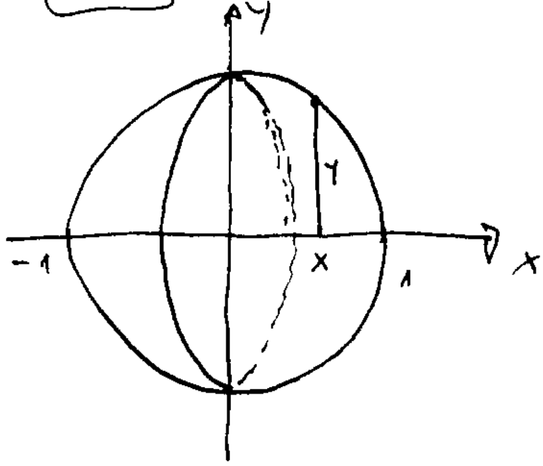
$$S=\int_\alpha^\beta 2\pi f(t)\sqrt{f'^2(t)+g'^2(t)}\,dt=\int_\alpha^\beta 2\pi f(t)\sqrt{\left(\frac{dy}{dt}\right)^2+\left(\frac{dx}{dt}\right)^2}dt$$

$$\boxed{S=\int_\alpha^\beta 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt}$$

DIREKTNO:

$$\boxed{S=\int_0^\pi 2\pi R\sin\theta\sqrt{R^2\sin^2\theta+R^2\cos^2\theta}}$$

• $\boxed{\text{Exp6}}$ Show that the surface of sphere is $4\pi r^2$



$$S=2\int^1 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$$

NE TREBA VRTI REVOLUTION − 1

$x=\cos(\theta)$ $\quad dx=-\sin\theta\,dt$

$y=\sin(\theta)$

$x^2+y^2=1$

$x=-1$ $\quad \theta=\arccos(-1)=\pi$

$x=1$ $\quad \theta=\arccos(1)=\cancel{2\pi}$

$$S=\cancel{2}-\int_\pi^0 2\pi\sin\theta\sqrt{\sin^2(\theta)+\cos^2(\theta)}\,d\theta$$

NAMESTO VAKA MOŽE I INTUITIVNO DA SVATIŠ DEKA $\boxed{\theta=0..\pi}$

$$S=-2\cdot 2\pi\,(-\cos\theta)\Big|_\pi^0=-4\pi(\underbrace{\cos\pi-\cos 0}_{-2})=+8\pi$$

$x^2+y^2=R^2$ $\quad x=R\cos\theta$ $\quad y=R\sin\theta$

$x=-R$ $\quad \cos\theta=-1$ $\quad \theta=\pi$

$x=R$ $\quad \cos\theta=1$ $\quad \theta=0$

$\theta(R)=0$

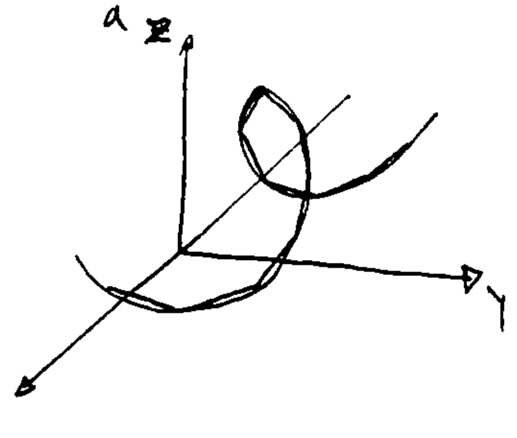
$$S=\int_\pi^0 2\pi R\sin\theta\sqrt{R^2\sin^2\theta+R^2\cos^2\theta}\,d\theta=-\int_\pi^0 2\pi R^2\sin\theta\,d\theta$$

$\theta(-R)=\pi$

$$=-2\pi R^2(-\cos\theta)\Big|_\pi^0=-2\pi R^2\cos\theta\Big|_0^\pi=-2\pi R^2(\underbrace{\cos\pi-\cos\theta}_{-2})=+4\pi R^2$$

• ARC LENGHT AND CURVATURES (13.7)   CONTINUATION FROM pp.5

$$L = \int_a^b \sqrt{f'^2(t) + g'^2(t)}\, dt = \int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$$



$r(t) = \langle f(t), g(t), h(t) \rangle \quad a \le t \le b$

$$\boxed{L = \int_a^b \sqrt{f'^2(t) + g'^2(t) + h'^2(t)}\, dt}$$

$$L = \int_a^b \sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$\boxed{L = \int_a^b |r'(t)|^2\, dt}$$

CARM RIGINS

$|r'(t)| = \left| f'(t)\vec{i} + g'(t)\vec{j} \right| =$

• FOR PLANE CURVES

$\sqrt{f'^2(t) + g'^2(t)}$

EXP 1 LENGHT OF THE ARC OF CIRCULAR HELIX

$$\boxed{\vec{r}(t) = \cos t\, \vec{i} + \sin t\, \vec{j} + t\, \vec{k}}$$

$P(1,0,0) \qquad Q(1,0,2\pi)$

$$\left.\begin{array}{l} x = \cos t \\ y = \sin t \\ z = t \end{array}\right\} P(1, 0\ 0)$$

$t = \arccos(1) = 0$
$t = \arcsin(0) = 0$
$t = 0$

$Q(1,0,2\pi)$

$t = \arccos(1) = 2\pi$
$t = \arcsin(0) = 2\pi$
$t = 2\pi$

"1" GREIXT!

$$L = \int_{a=0}^{b=2\pi} \sqrt{f'^2(t) + g'^2(t) + h'^2(t)}\, dt = \int_0^{2\pi} \sqrt{\underbrace{\sin^2 t + \cos^2 t + t^2}_{\sqrt{1+t^2}}}\, dt$$

$$\int \sqrt{1+t^2}\, dt = \frac{1}{2}\sec(t)\cdot tg\, t + \frac{1}{2}\ln\left(\sec(t) + tg(t)\right)$$

$$L = \left. \frac{1}{2}\sec(t)\, tg(t) + \frac{1}{2}\ln\left(\sec(t) + tg(t)\right)\right|_0^{2\pi} = \pi\sqrt{1+4\pi^2} - \frac{1}{2}\ln\left(-2\pi + \sqrt{1+4\pi^2}\right)$$

$$L = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t + 1}\, dt = \int_0^{2\pi} \sqrt{2}\, dt = \underline{\underline{2\pi\sqrt{2}}}$$

- PARAMETRIZATIONS OF CURVES

$r_1(t) = (t, t^2, t^3)$ twisted cubic

$r_2(t) = (e^{u}, e^{2u}, e^{3u})$ —ıı—

- Arc length function

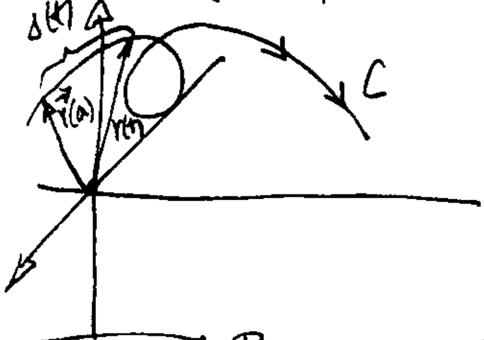
$$s(t) = \int_a^t |r'(u)|\, du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du$$

$$\boxed{\frac{ds(t)}{dt} = |r'(t)|}$$

$\Rightarrow t = t(s)$

$\vec{r}(t) = \vec{r}\,(t(s))$

Ex: $\vec{r}(t(3)) \Rightarrow$ 3 UNITS OF LENGHT ALONG THE CURVE FROM ITS STARTING POINT.



$\boxed{\text{EXP 2}}$ PARAMETRIZE HELIX $\boxed{r(t) = \cos t\, \vec{i} + \sin t\, \vec{j} + t\vec{k}}$ WITH RESPECT TO ARC LENGTH MEASURED FROM $(1, 0, 0)$

$$s(t) = \int_0^t \sqrt{2}\, du = \sqrt{2}\, t \qquad \boxed{t = \frac{s}{\sqrt{2}}}$$

$$\boxed{\vec{r}(t) = \cos\left(\frac{s}{\sqrt{2}}\right)\vec{i} + \sin\left(\frac{s}{\sqrt{2}}\right)\vec{j} + \frac{s}{\sqrt{2}}\vec{k}} = \vec{r}\left(\frac{s}{\sqrt{2}}\right)$$

- <u>CURVATURES</u>

$C: \vec{r}(t) \Rightarrow$ smooth curve

$\vec{T}(t) = \dfrac{r'(t)}{|r'(t)|} \Rightarrow$ UNIT TANGENT VECTOR

CURVATURE OF C AT GIVEN POINT IS A MEASURE OF HOW quickly the curve changes direction.

$$\kappa = \left|\frac{d\vec{T}}{ds}\right| \qquad \frac{dT}{dt} = \frac{dT}{ds}\,\frac{ds}{dt} \qquad \boxed{\frac{d\vec{T}}{ds} = \frac{\frac{dT}{dt}}{\frac{ds}{dt}}}$$

$$\frac{ds}{dt} = |r'(t)| \qquad \boxed{\kappa = \frac{d\vec{T}}{ds} = \frac{|T'(t)|}{|r'(t)|}}$$

Exp. 3 Curvature of circle with $r = a$ is $\frac{1}{a}$

$$r(t) = a\cos\theta\,\vec{i} + a\sin\theta\,\vec{j}$$

$$r'(t) = -a\cdot\sin\theta\,\vec{i} + a\cdot\cos\theta\,\vec{j} \qquad |r'(t)| = a\sqrt{\cos^2+\sin^2} = a$$

$$\vec{T}(t) = \frac{r'(t)}{|r'(t)|} = \frac{-\cancel{a}\sin\theta\,\vec{i} + a\cos\theta\,\vec{j}}{a} = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j}$$

$$\kappa = \frac{|T'(t)|}{|r'(t)|} = \frac{\sqrt{\sin^2\theta + \cos^2\theta}}{a} = \frac{1}{a}$$

---

10 THEOREM: The curvature of curve given by vector function $r =$ is given

$$\boxed{\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}}$$

Proof: $\vec{T} = \frac{r'}{|r'|} \qquad |r'| = \frac{ds}{dt} \Rightarrow r' = \vec{T}\cdot|r'| = \vec{T}\cdot\frac{ds}{dt}$

$$r'' = (r')' = \vec{T}'\frac{ds}{dt} + \vec{T}\cdot\frac{d^2s}{dt^2} \qquad \vec{T}\cdot\vec{T} = 0$$

VIP! pp. 19

$$r'\times r'' = \left(\vec{T}\frac{ds}{dt}\right)\times\left(\vec{T}'\frac{ds}{dt} + \vec{T}\frac{d^2s}{dt^2}\right) = \vec{T}\times\vec{T}'\left(\frac{ds}{dt}\right)^2 + \vec{T}\times\vec{T}\cdot\frac{d^2s}{dt^2}\cdot\frac{ds}{dt}$$

$$\boxed{r'\times r'' = \vec{T}\times\vec{T}'\cdot\left(\frac{ds}{dt}\right)^2}$$

CONTINUE ON PP. 28

- Cross product of two vectors!

$a = \langle a_1, a_2, a_3\rangle \qquad b = \langle b_1, b_2, b_3\rangle$

$$a\times b = \langle a_2b_3 - b_2a_3,\ a_3b_1 - b_3a_1,\ a_1b_2 - b_1a_2\rangle$$

- DETERMINANT OF 2 ORDER

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- DETERMINANT

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$a = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k} \qquad b = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$$

$$a \times b = \vec{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = a \times b$$

$\boxed{\text{Exp 2}}$ $a \times a = 0$

$$\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = i \begin{vmatrix} a_2 & a_3 \\ a_2 & a_3 \end{vmatrix} - j \begin{vmatrix} a_1 & a_3 \\ a_1 & a_3 \end{vmatrix} + k \begin{vmatrix} a_1 & a_2 \\ a_1 & a_2 \end{vmatrix}$$

$$= (a_2 a_3 - a_2 a_3) i - j (a_1 a_3 - a_1 a_3) + k (a_1 a_2 - a_1 a_2) = 0$$

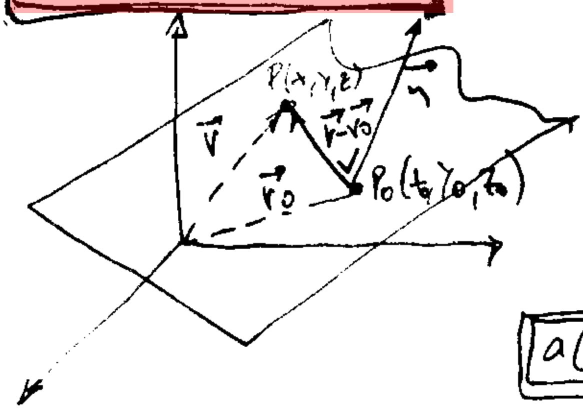
$$|r(t)|^2 = c^2 \qquad r(t) \cdot r(t) = c^2$$

$$\frac{d(r(t) \cdot r(t))}{dt} = r'(t)\, r(t) + r(t) \cdot r'(t) = 0 \qquad \boxed{2 r'(t) \cdot r(t) = 0}$$

$$r'(t) \perp r(t)$$

$$|\vec{T}| = 1 \Rightarrow \vec{T}(t) \perp \vec{T}'(t)$$

- CH. 12.5 PLANES



DOT PRODUCT

$$\boxed{\begin{aligned} \vec{n} \cdot (\vec{r} - \vec{r}_0) &= 0 \\ \vec{n} \cdot \vec{r} &= \vec{n} \cdot \vec{r}_0 \end{aligned}}$$

VECTOR EQUATIONS OF PLANES!!

MMV

$$n = \langle a, b, c \rangle \quad r = \langle x, y, z \rangle \quad r_0 = \langle x_0, y_0, z_0 \rangle$$

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$ MMV

$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ ⇒ SCALAR EQUATION OF PLANE THROUGH $P(x_0,y_0,z_0)$ WITH NORMAL VECTOR $n=\langle a,b,c\rangle$

EXP 4 EQUATION OF PLANE THROUGH $(2,4,-1)$ AND $n=\langle 2,3,4\rangle$

$\vec{n}(\vec{r}-\vec{r_0})=0 \qquad 2(x-2)+3(y-4)+4(z+1)=0$

$2x-\cancel{4}+3y-12+4z+\cancel{4}=0$

$$\boxed{2x+3y+4z=12}$$

x - INTERCEPT $(y,z)=(0,0)$ $x=(6,0,0)$

y - INTERCEPT $(x,z)=(0,0)$ $y=(0,4,0)$

z - INTERCEPT $(x,y)=(0,0)$ $z=(0,0,3)$

$ax+by+cz-(ax_0+by_0+cz_0)=0$

$$\boxed{ax+by+cz+d=0 \qquad d=-ax_0-by_0-cz_0}$$

EXP. 5 $P(1,3,2)$, $Q(3,-1,6)$ AND $R(5,2,0)$

$\boxed{\vec{n}(\vec{r}-\vec{r_0})=0}$ $\boxed{a=\overrightarrow{PQ}}$ $b=\overrightarrow{PR}$

$\boxed{a\times b=0}$

$$\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i\begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} - j\begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} + k\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$$

$a=\overrightarrow{PQ}=(1-3)\vec{i}+(3+1)\vec{j}+(2-6)\vec{k}=\langle -2,4,-4\rangle$

$(3-1)\vec{i}+(-1-3)\vec{j}+(+6-2)\vec{k}=\langle 2,-4,4\rangle$

$b=\overrightarrow{PR}=(5-1)\vec{i}+(2-3)\vec{j}+(0-2)\vec{k}=\langle 4,-1,-2\rangle$

$$\vec{n}=\begin{vmatrix} i & j & k \\ 2, & -4, & 4 \\ 4, & -1, & -2 \end{vmatrix} = i\begin{bmatrix} -4, & +4 \\ -1, & -2 \end{bmatrix} - j\begin{bmatrix} 2 & 4 \\ 4 & -2 \end{bmatrix} + k\begin{bmatrix} 2, & -4 \\ 4, & -1 \end{bmatrix}$$

$$= 12i + 20j + 14k$$

$\vec{n}\cdot(\vec{r}-\vec{r_0})=0$

P: $\langle 12,20,14\rangle\langle x,y,z\rangle-\langle 12,20,14\rangle\langle 1,3,2\rangle$

$12x+20y+14z-(12+60+28)=12x+20y+14z-100=0$

$\boxed{12x+20y+14z=100}$ $\boxed{6x+10y+7z=50}$

$\boxed{\text{Exp 6}}$ L: $x=2+3t$ $y=-4t$ $z=5+t$ P: $4x+5y-2z=18$

$$\frac{x-2}{3}=\frac{y}{-4}=\frac{z-5}{1}$$

$\vec{r}=\vec{r_0}+v\cdot t=\langle 2+3t,-4t,5+t\rangle$

$4(2+3t)+5(-4t)-2(5+t)=18$

$8+12t-20t-10-2t=18$ $-2-8t-2t=18$

$-10t=20$ $\boxed{t=-2}$

$P(x_0,y_0,z_0)=(2-6,8,5-2)=\underline{\underline{(-4,8,3)}}$

$\boxed{\text{Exp 7}}$ P_1: $x+y+z=1$ P_2: $x-2y+3z=1$

(a) $\theta=?$

(b) SYMETRIC EQUATIONS FOR LINE OF INTERSECTION "L"

$\vec{n_1}=\langle 1,1,1\rangle$ $\vec{n_2}=\langle 1,-2,3\rangle$

---

ch12.3 PROPERTIES OF DOT PRODUCT $(a\cdot b=a_1b_1+a_2b_2+a_3b_3)$

(1) $a\cdot a=|a|^2$ (2) $a\cdot b=b\cdot a$ (3) $a(b+c)=ab+ac$ ...

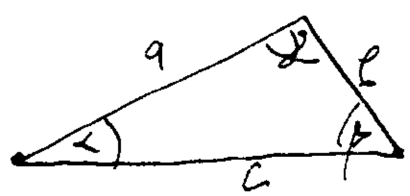
$\boxed{\text{Theorem 3}}$ ANGLE BETWEEN VECTORS a&b

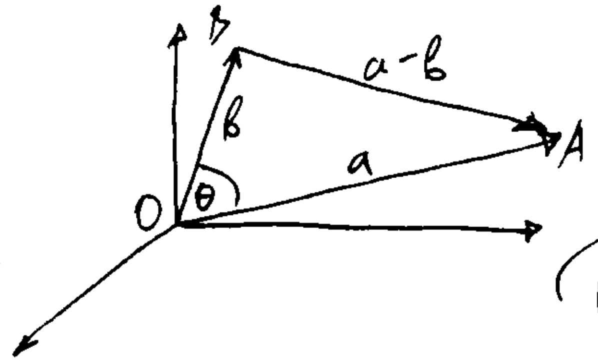
$a\cdot b=|a|\cdot|b|\cdot\cos\theta$ $\boxed{\cos\theta=\frac{a\cdot b}{|a|\cdot|b|}}$ MMV

- LOW OF COSINES

$\boxed{c^2=a^2+b^2-2ab\cos\theta}$

MMV





$$|AB|^2 = |OA|^2 + |OB|^2 - 2|OA||OB|\cos\theta$$

$$|a-b|^2 = |a|^2 + |b|^2 - 2|a||b|\cos\theta$$

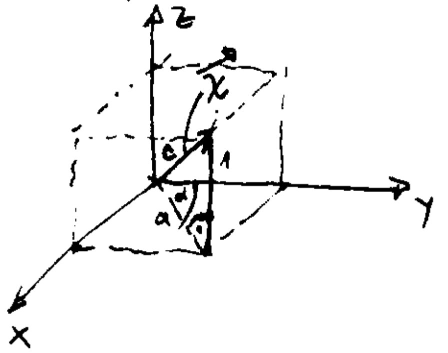
$$|a-b|^2 = (a-b)\cdot(a-b) = a\cdot a - a\cdot b - a\cdot b + b\cdot b$$

$$= |a|^2 - 2ab + |b|^2$$

$$-2\cdot a\cdot b = -2|a|\cdot|b|\cos\theta$$

$$\boxed{\cos\theta = \frac{a\cdot b}{|a|\cdot|b|}}$$ DOKAZANO !!!

$$x = \langle 1, 1, 1\rangle \qquad |x|^2 = 1^2 + 1^2 + 1^2 = 3 \qquad \boxed{x = \sqrt{3}}$$



$$a = \sqrt{1^2+1^2} = \sqrt{2}$$

$$c^2 = a^2 + 1^2 \qquad c^2 = 2+1 = 3 \qquad \boxed{c = \sqrt{3}}$$

ⓐ $\vec{n_1} = \langle 1, 1, 1\rangle \qquad \vec{n_2} = \langle 1, -2, 3\rangle$

$$\cos\theta = \frac{n_1\cdot n_2}{|n_1||n_2|} = \frac{1+(-2)+3}{\sqrt{3}\cdot\sqrt{1+4+9}} = \frac{2}{\sqrt{3}\sqrt{14}} = \frac{2}{\sqrt{42}}$$

$$\theta = \arccos\left(\frac{2}{\sqrt{42}}\right) = 1.257 \text{ rad} \doteq 72^\circ$$

ⓑ $P1: x + y + z = 1 \qquad P2: x - 2y + 3z = 1$

$$z = 1 - x - y \qquad x - 2y + \overset{2}{\cancel{3}} - 3x - 3y = \cancel{1} \qquad -2x - 5y + 2 = 0$$

$$5y = -2x + 2 \qquad \boxed{y = -\frac{2}{5}x + \frac{2}{5}} \Rightarrow y = -\frac{2}{5}(x-1)$$

$$\boxed{\frac{y}{-2} = \frac{x-1}{5}}$$

$$\vec{r}(t) = \vec{r_0} + t\cdot\vec{v} \quad \text{LINE}$$

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\overbrace{\langle a, b, c\rangle}^{v}$$

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

$$\vec{n_1}\times\vec{v} = 0 \qquad (c-b)i + (a-c)j + (b-a)k = 0$$

$$\vec{n_2}\times v = 0 \qquad (-2c-3b)i + (3a-c)j + (b+2a)k = 0$$

• POINT ON L: $z=0$ [redacted]

$x+y=1 \qquad y=1-x \qquad y=1-1=0$

$x-2y=1 \qquad x-2+2x=1 \qquad 3x=3 \qquad x=1$

$\boxed{P(1,0,0)}$ ON LINE "L"

$$\boxed{v = n_1 \times n_2 = \begin{bmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{bmatrix} = 5\vec{i} - 2\vec{j} - 3\vec{k} = \langle a, b, c\rangle}$$ MMV

$$\boxed{\frac{x-1}{5} = \frac{y}{-2}} = \frac{z}{-3} \qquad \frac{y}{-2} = \frac{x-1}{5}$$

$$y = -\frac{2}{5}(x-1) = -\frac{2}{5}x + \frac{2}{5}$$ (VIDI pg.22)

• $x = 1-y-z \qquad x-2y+3z=1 \qquad \cancel{1}-y-z-2y+3z=\cancel{1}$

$-3y + 2z = 0 \qquad -3y = -2z \qquad \boxed{\frac{y}{-2} = \frac{z}{-3}}$

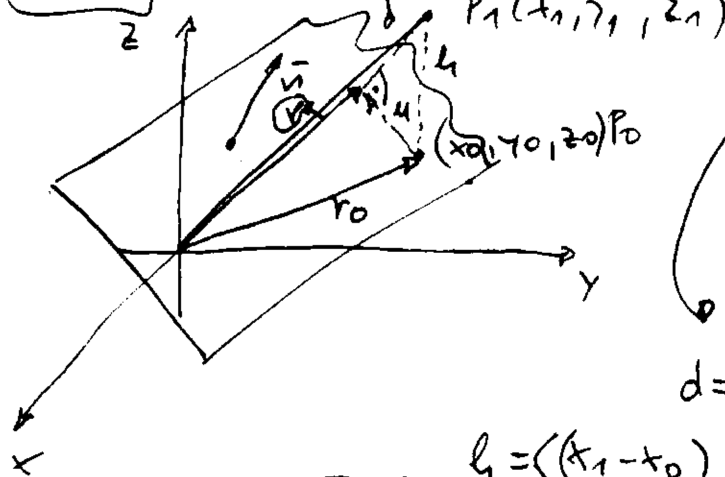
• LINE IN SIMETRIC FORM

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

IS LINE OF INTERSECTION OF TWO PLANES

$$\boxed{\frac{x-x_0}{a} = \frac{y-y_0}{b} \qquad \frac{y-y_0}{b} = \frac{z-z_0}{c}}$$ MMV

$\boxed{\text{Cap 8}}$ DISTANCE BETWEEN $P_1(x_1,y_1,z_1)$ TO $ax+by+cz+d=0$



$r_0 + u = r$

$\boxed{u = r - r_0}$

$\vec{n}\cdot(\vec{r}-\vec{r}_0) = 0$

$ax + by + cz + d = 0$

$n = a\vec{i} + b\vec{j} + c\vec{k}$

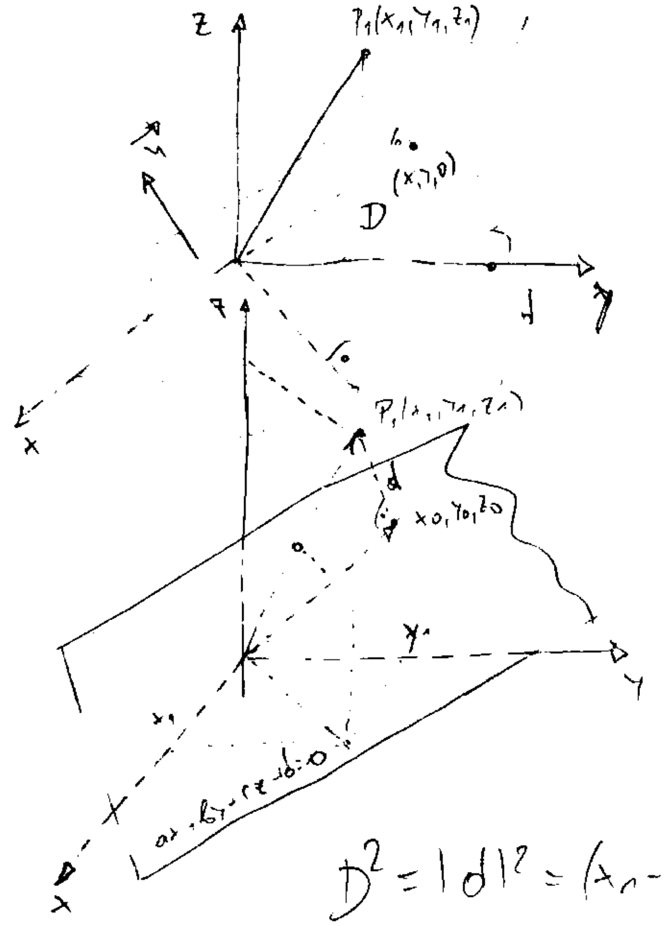
$\langle a,b,c\rangle\left[\langle x,y,z\rangle - \langle x_0,y_0,z_0\rangle\right]$

$d = -ax_0 - bx_0 - cx_0$

$h = \langle (x_1-x_0),(y_1-y_0),(z_1-z_0)\rangle = \overline{P_0P_1}$

$d = \sqrt{|h|^2 - |u|^2} = \sqrt{|x_1-x_0|^2 + (y_1-y_0)^2 + (z_1-z_0)^2 - (x-x_0)^2 - (y-y_0)^2 - (z-z_0)^2}$

$|\vec{n}| = \sqrt{a^2+b^2+c^2}$



$D: \; z = 0$

$\boxed{D = z_1}$

$P: \; z = +d$

$D = z_1 - d$

$a x_0 + b y_0 + c y_0 + d = 0$

$\vec{d} = \langle x_1, y_1, z_1 \rangle - \langle x_0, y_0, z_0 \rangle$

$|d| = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}$

$\vec{d} = (x_1 - x_0)\, \vec{i} + (y_1 - y_0)\, \vec{j} + (z_1 - z_0)\, \vec{k}$

$\left.\dfrac{\vec{d}}{|d|} = a\vec{i} + b\vec{j} + c\vec{k}\right\}$ NORMAL VECTOR

$$D^2 = |d|^2 = (x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2$$

$$\frac{(x_1 - x_0)}{D} = \frac{a}{\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}} = a$$

$$\frac{(y_1 - y_0)}{D} = \frac{b}{\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}} = b$$

$$\frac{(z_1 - z_0)}{D} = c$$

$$x_1 - x_0 = aD$$
$$y_1 - y_0 = bD$$
$$z_1 - z_0 = cD$$

$$x_0 = \frac{x_1 - aD}{\cancel{\;}} = x_1 - aD$$
$$y_0 = y_1 - bD$$
$$z_0 = z_1 - cD$$

$$D = \frac{a}{x_1 - x_0} = \frac{b}{y_1 - y_0} = \frac{c}{z_1 - z_0}$$

$$a x_0 + b y_0 + c y_0 = -d$$

$$a \cdot \frac{(x_1 - aD)}{\cancel{\;}} + b\,\frac{(y_1 - bD)}{\cancel{\;}} + c\,\frac{(z_1 - cD)}{\cancel{\;}} = -d$$

$$a x_1 - a^2 D + b y_1 - b^2 D + c z_1 - c^2 D = -d$$

$$(a x_1 + b y_1 + c z_1 + d) = (a^2 + b^2 + c^2) D$$

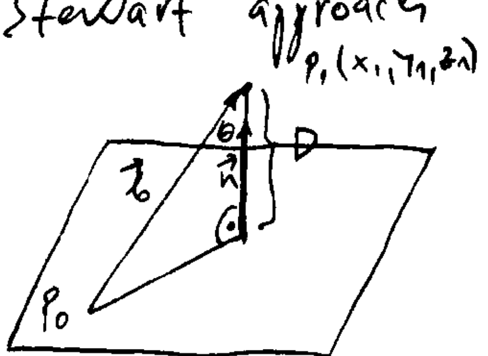
$$\boxed{D^{-1} = \frac{a^2 + b^2 + c^2}{a x_1 + b y_1 + c z_1 + d}}$$

$$\boxed{D = \frac{a x_1 + b y_1 + c z_1 + d}{a^2 + b^2 + c^2}}$$

MMV

DOKAZANO SO CELOSNO OD PUSTAT!!!
(SAMO ŠTO SOGLASNO IZVEDENOTO VO STEWART VO IMENITELOT TREBA $\sqrt{\;}$)

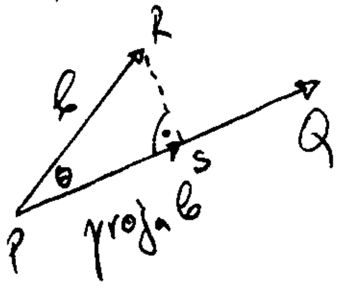
• Stewart approach

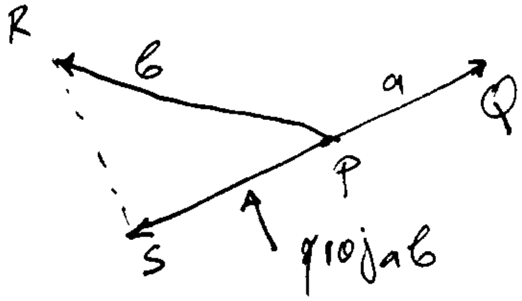


„D" IS ABSOLUTE VALUE OF SCALAR PROJECTION OF VECTOR $\vec{b}$ onto normal VECTOR $\vec{n}$

$$\boxed{n = \langle a, b, c \rangle}$$

12.3 PROJECTIONS





$$\vec{PS} = |b| \cos\theta = comp_a b$$

$$\underbrace{a \cdot b = |a| \cdot |b| \cdot \cos\theta}_{\text{THEOREM 3 (LOW OF COSINE USAGE)}}$$

$$a \cdot b = |a| \left(|b| \cdot \cos\theta \right) \Rightarrow \boxed{comp_a b = \frac{\vec{a} \cdot \vec{b}}{|a|}}$$

$$|b| \cos\theta = \frac{|a| \cdot |b| \cos\theta}{|a|} = \frac{a \cdot b}{|a|} = \frac{\vec{a}}{|a|} \vec{b}$$

} DOT PRODUCT BTW. UNIT VECTOR WITH $\vec{a}$ DIRECTION AN „$\vec{b}$"

$$\boxed{comp_a b = \frac{a \cdot b}{|a|}} \qquad \boxed{comp_b a = \frac{a \cdot b}{|b|}}$$

SCALAR PROJECTION

• VECTOR PROJECTION

$$proj_a b = \left(\frac{a \cdot b}{|a|} \right) \odot \left(\frac{a}{|a|} \right) = \frac{a \cdot b}{|a|^2} a$$

UNIT VECTOR IN $\vec{a}$ DIRECTION

OBIČNO „PO"

• Stewart approach continue...

$$D = |comp_n b| = \frac{|n \cdot b|}{|n|} = \frac{\langle a, b, c \rangle \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle}{\sqrt{a^2 + b^2 + c^2}} =$$

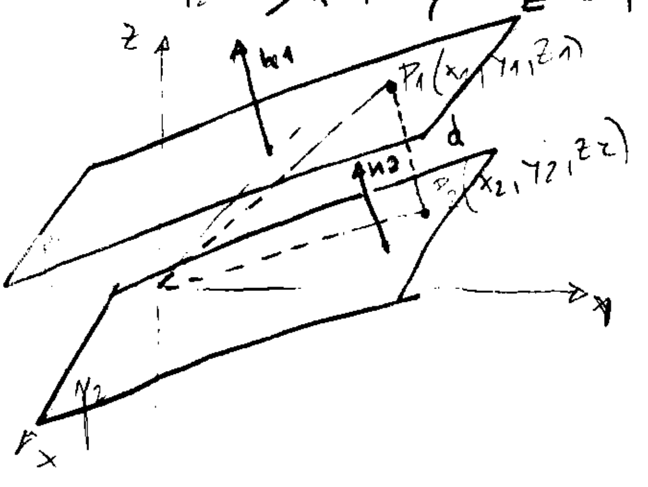
$$= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|a x_1 + b y_1 + c z_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

$d = -(a x_0 + b y_0 + c z_0)$

EXP 9 DISTANCE BETWEEN PARALEL PLANES

$P_1: 10x + 2y - 2z = 5$ $\qquad n_1 = \langle 10, 2, -2 \rangle = \langle a_1, b_1, c_1 \rangle$

$P_2: 5x + y - z = 1$ $\qquad n_2 = \langle 5, 1, -1 \rangle = \langle a_2, b_2, c_2 \rangle$



$y = \emptyset$

$$D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

$$-\left(\begin{array}{l} 10x_1 + 2y_1 - 2z_1 = 5 \\ 5x_2 + y_2 - z_2 = 1 \quad / \cdot 2 \\ 10x_2 + 2y_2 - 2z_2 = 2 \end{array}\right.$$

$$10(x_1 - x_2) + 2(y_1 - y_2) - 2(z_1 - z_2) = 3 \quad (*)$$

12.4 THEOREM 6

$$|a \times b| = |a||b| \sin\theta$$

$$a \times b = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

$$= \underbrace{(a_2 b_3 - a_3 b_2)}_{c_1} \vec{i} + \underbrace{(a_1 b_3 - a_3 b_1)}_{c_2} \vec{j} + \overbrace{(a_1 b_2 - a_2 b_1)}^{c_3} \vec{k}$$

$$|a \times b|^2 = c_1^2 + c_2^2 + c_3^2 = (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2$$

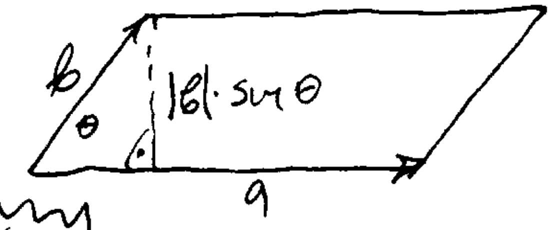
$$= |a|^2 |b|^2 - (|a| \cdot |b|)^2 \cos^2\theta = |a|^2 |b|^2 - |a|^2 |b|^2 \cdot \cos^2\theta = |a|^2 |b|^2 \sin^2\theta$$

$$|a \times b|^2 = |a|^2 |b|^2 \sin^2\theta \qquad \boxed{|a \times b| = |a||b| \sin\theta}$$

$\boxed{\bar{a} \times \bar{b}}$ - NORMAL VECTOR TO $\bar{a}$ & $\bar{b}$ WITH DIRECTION ACORDING RIGHT HAND RULE AND WITH LENGTH $|a||b| \cdot \sin\theta$

COROLLARY: TWO NON-ZERO VECTORS a AND b ARE PARALEL IF AND ONLY IF: $\boxed{a \times b = 0}$

$\theta = 0 \vee \theta = \pi \qquad |a \times b| = |a||b| \underbrace{\sin\theta}_{0} \qquad |a \times b| = 0 \quad \underline{a \times b = 0}$



$A = |a| \cdot (|b| \sin\theta) = |a||b| \sin\theta = |a \times b|$

LENGHT OF $a \times b$ IS EQUAL TO AREA OF THE PARALELOGRAM DEFINED BY $\vec{a}$ AND $\vec{b}$.

$n_1 \times n_2 = \emptyset$

$\vec{d} = (x_1 - x_2)\vec{i} + (y_1 - y_2)\vec{j} + (z_1 - z_2)\vec{k}$

$\frac{\vec{d}}{D} = a_1 \vec{i} + b_1 \vec{j} + a_1 \vec{k}$

$\frac{\vec{d}}{D} = a_2 \vec{i} + b_2 \vec{j} + c_2 \vec{k}$

$10(x - x_0) + 2(y - y_0) + 2(z - z_0) = 0$

$10(x - \frac{1}{2}) + 2y + 2z = 0$

- VIDI PlanePlot PRIMER VO: StewartSe Worksheet.mw

: $\boxed{MMV}$

$\boxed{(x_1 - x_2)/D = a_1 = 10}$ $\boxed{(y_1 - y_2)/D = 2}$ $\boxed{(z_1 - z_2)/D = -2}$

$x_1 - x_2 = a_2 = 5$ $y_1 - y_2 = 1$ $z_1 - z_2 = -1$

$\cancel{x_1 = 10 + x_2}$

$\cancel{10 + x^2 - x^2 = 5}$

$\boxed{x_1 - x_2 = 10D \quad (y_1 - y_2) = 2D \quad (z - z_2) = -2D}$

VO $\circledast$

$10 \cdot 10D + 2 \cdot 2D + 2 2D = 3$ $\quad 20D + 4D + 4D = 3$

$\boxed{D = \frac{3}{28}}$

- SVEDI GO NA PROBLEMOT RASTOJANIE MEGJU TOČKA I RAMNINA

$\langle a, b, c, d \rangle = \langle a_2, b_2, c_2, d_2 \rangle$

$$\boxed{D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}}$$

MMV

DISTANCE BETWEEN POINT AND PLANE (VIDI 11.25)

$y = z = 0 \quad 10x = 5 \quad x = \frac{1}{2} \quad P_1(\frac{1}{2}, 0, 0)$

$$D = \frac{5 \cdot \frac{1}{2} - 1}{\sqrt{25 + 1 + 1}} = \frac{5/2 - 1}{\sqrt{27}} = \frac{5 - 2}{2\sqrt{27}} = \frac{3}{2 \cdot 3\sqrt{3}} = \frac{3}{6\sqrt{3}}$$

$$\boxed{D = \frac{\sqrt{3}}{6}}$$

$\boxed{Ex 10}$ $L_1: x = 1 + t \quad y = -2 + 3t \quad z = 4 - t$

$r_1(t) = \langle 1, -2, 4 \rangle + t \underbrace{\langle 1, 3, -1 \rangle}_{v_1}$

$L_2: x = 2s \; ; \; y = 3 + s \; ; \; z = -3 + 4s$

$r_2(t) \; \langle 0, 3, -3 \rangle + t \underbrace{\langle 2, 1, 4 \rangle}_{v_2}$

$$\vec{n} = v_1 \times v_2 = \begin{vmatrix} i & j & k \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} i & j & k \\ 1 & 3 & -1 \\ 2 & 1 & 4 \end{vmatrix} = 13i - 6j - 5k$$

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}}$$

$\vec{v} = \langle \overset{a}{13}, \overset{b}{-6}, \overset{c}{-5} \rangle$

$d = ax_0 + b$

$\boxed{s=0}$ $x_0 = 0$ $y_0 = 3$ $z = -3$ $\boxed{P_0(0, 3, -3)}$ → OVAJ TOČKU ZA RAZGLEDUVAM.

$$D = \frac{|0 + (-18) + 15 + d|}{\sqrt{a^2+b^2+c^2}} = \frac{|-3+d|}{\sqrt{13^2+36+25}}$$

$\boxed{t=0}$ $\boxed{x_1 = 1 \quad y_1 = -2 \quad z_1 = 4}$ ⇒ OVAA RAVNINA JA RAZGLEDUVAM!!

$(-d) = a \cdot x_1 + by_1 + cz_1 = 13 + (-6)(-2) + (-5)4 = 13+12-20$
$= 25-20 = 5$

$d = -5$

$$D = \frac{-5-3}{\sqrt{230}} = \frac{|-8|}{\sqrt{230}} = \frac{8}{15\sqrt{5}}$$

- CURVATURE (CONTINUATION FROM 11.18)

$k = \left|\frac{d\vec{T}}{ds}\right|$ $\quad T(t) = \frac{r'(t)}{|r'(t)|}$

$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds}\frac{ds}{dt}$

$$\boxed{k = \left|\frac{\frac{d\vec{T}(t)}{dt}}{\frac{ds}{dt}}\right| = \left|\frac{d\vec{T}}{ds}\right|}$$

MMV

$$\boxed{k = \frac{|T'(t)|}{|r'(t)|}} \qquad \boxed{T(t) = \frac{r'(t)}{|r'(t)|}}$$

$\boxed{\text{Theorem 10}}$ $\quad \boxed{k(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}}$ MMV

<u>Proof:</u> $T = \frac{r'}{|r'|}$ $\quad |r'| = \frac{ds}{dt}$ $\quad \vec{r}' = (|\vec{r}'|)(\vec{T}) = \frac{ds}{dt}(\vec{T})$

$\vec{r}'' = \frac{d^2s}{dt^2}(\vec{T}) + \frac{ds}{dt}(\vec{T}')$

$$r' \times r'' = \frac{ds}{dt}\vec{T} \times \left(\frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\vec{T}'\right) = \frac{ds}{dt}\frac{d^2s}{dt^2}\,\cancel{\vec{T}\times\vec{T}}^{\,0} + \left(\frac{ds}{dt}\right)^2 \vec{T}\times\vec{T}'$$

$$\boxed{r' \times r'' = \left(\frac{ds}{dt}\right)^2 \vec{T}\times\vec{T}'}$$

$$|r' \times r''| = \left(\frac{ds}{dt}\right)^2 |\vec{T}\times\vec{T}'|$$

$$\boxed{|a\times b| = |a|\cdot|b|\cdot \sin\theta}$$

$$\vec{T} \perp \vec{T}' \Rightarrow \theta = \frac{\pi}{2} \quad \boxed{\sin\theta = 1}$$

$$|r' \times r''| = \frac{d^2s}{dt^2}\,|a|\cdot|b| = \left(\frac{ds}{dt}\right)^2 |\vec{T}|\,|\vec{T}'| = \left(\frac{ds}{dt}\right)^2 \cdot |\vec{T}'| \Rightarrow$$

1 (unit vector)

$$|\vec{T}'| = \frac{|r' \times r''|}{(ds/dt)^2} = \frac{|r' \times r''|}{|r'(t)|^2}$$

$$\boxed{k = \frac{|\vec{T}'|}{|r'(t)|} = \frac{|r' \times r''|}{|r'(t)|^2}}$$

DOKAZANO!!

$\boxed{\text{Exp 4}}$ CURVATURE OF: $r(t) = \langle t, t^2, t^3\rangle = ?$

$$r'(t) = \langle 1, 2t, 3t^2\rangle$$

$$r''(t) = \langle 0, 2, 6t\rangle$$

$$r' \times r'' = \begin{vmatrix} i & j & k \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \begin{bmatrix} 2t & 3t^2 \\ 2 & 6t \end{bmatrix}\vec{i} - \begin{bmatrix} 1 & 3t^2 \\ 0 & 6t \end{bmatrix}\vec{j} + \begin{bmatrix} 1 & 2t \\ 0 & 2 \end{bmatrix}\vec{k}$$

$$= (12t^2 - 6t^2)\vec{i} - 6t\,\vec{j} + 2\vec{k} = \langle 6t^2, -6t, 2\rangle$$

$$k(t) = \frac{\sqrt{36t^4 + 36t^2 + 4}}{\left(\sqrt{1 + 4t^2 + 36t^2}\right)^3} = \frac{2\sqrt{1 + 9t^2 + 9t^4}}{(1 + 4t^2 + 36t^2)^{3/2}}$$

$$\langle 0, 0, 0\rangle \qquad k(t) = \frac{2\sqrt{1}}{(1)^{3/2}} = 2 \qquad k(0) = 2$$

- SPECIAL CASE FOR PLANE CURVE

$y = f(x) \qquad r(x) = x\,i + f(x)\,j \qquad r'(x) = i + f'(x)\,j$

$r''(x) = f''(x)\,j \qquad i \times j = k \;\&\; j \times j = 0$

$$r'(x) \times r''(x) = (\vec{i} + f'(x)\vec{j}) \times f''(x)\vec{j} = f''(x)\,\vec{i}\times\vec{j} + f'(x)\cdot f''(x)\,\vec{j}\times\vec{j}$$

$$r'(x) \times r''(x) = f''(x)\cdot\vec{k} \qquad L = \int_\alpha^\beta |r'(t)|\,dt = \int_a^b \sqrt{1 + f'^2(x)}\,dx$$

$|r'(x)|$

$$\boxed{|r'(t)| = \sqrt{1+f'^2(x)}}$$

$$\boxed{k(x) = \frac{|f''(x)|}{\left(1+f'^2(x)\right)^{3/2}}}$$

Exp 5 CURVATURE OF PARABOLA $y = x^2$ AT $(0,0)$; $(1,1)$; $(2,4)$

$y = f(x)$ $\quad y' = 2x$ $\quad y'' = 2$

$$k(x) = \frac{2}{\left(1+4x^2\right)^{3/2}}$$

$k(0) = 2$

$k(1) = \dfrac{2}{5^{3/2}}$

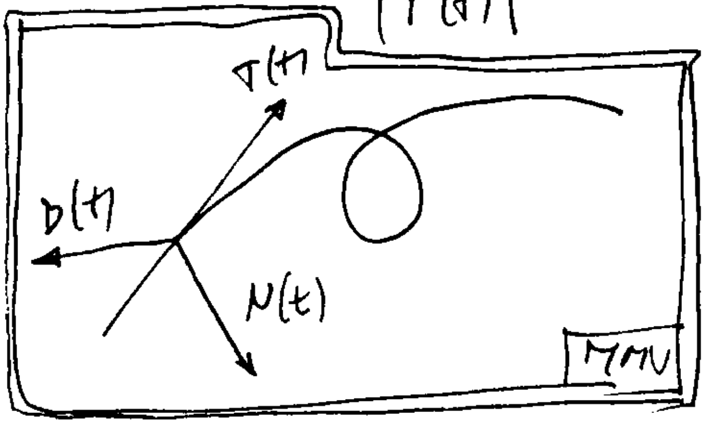
$$\boxed{k(2) = \frac{2}{(17)^{3/2}}}$$

⊙ NORMAL AND BINORMAL VECTORS

- UNIT NORMAL VECTOR

$$N(t) = \frac{T'(t)}{|T'(t)|}$$

$T(t) \Rightarrow$ UNIT TANGENT VECTOR



$$\boxed{B(t) = T(t) \times N(t)}$$

↳ BINORMAL VECTOR

Exp 6 FIND UNIT NORMAL AND BINORMAL VECTORS FOR:

$$r(t) = \cos t\, \vec{i} + \sin t\, \vec{j} + t k$$

$$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{-\sin t\,\vec{i} + \cos t\,\vec{j} + \vec{k}}{\sqrt{\underbrace{\sin^2 t + \cos^2 t}_{} + 1}} = \frac{1}{\sqrt{2}}\left(-\sin t\, i + \cos t\, j + k\right)$$

$$N(t) = \frac{T'(t)}{|T'(t)|} = \frac{-\cos t\,\vec{i} - \sin t\,\vec{j}}{\sqrt{2}\sqrt{\frac{1}{2}\cos^2 t + \frac{1}{2}\sin^2 t}} = \left(-\cos t\,\vec{i} - \sin t\,\vec{j}\right)$$

$$B(t) = \begin{vmatrix} i & j & k \\ -\sin t/\sqrt{2} & \cos t/\sqrt{2} & 1/\sqrt{2} \\ -\cos t & -\sin t & 0 \end{vmatrix} = \left(\sin t\,\vec{i} - \cos(t)\,\vec{j} + \vec{k}\right)\frac{\sqrt{2}}{2}$$

$t = 0 \Rightarrow r(0) = \langle 1, 0, 0 \rangle$

✓ $T(0) = \dfrac{1}{\sqrt{2}}\left(j + k\right) = 0.707(j+k)$

$N(0) = (-i) = -i$ ✓

✓ $B(0) = \dfrac{\sqrt{2}}{2}\left(-j + k\right) = 0.70(-j+k)$

MMV: VIDI Stewart5e Worksheet6
MNOGU VRAVI ILUSTRACII !!!

Exp. 7 FIND EQUATIONS OF NORMAL & OSCULATING PLANES OF THE HELIX IN Exp. 6, AT POINT $P(0, 1, \frac{\pi}{2})$

$r(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\vec{k}$ $\qquad N(t) = -\cos t\,\vec{i} - \sin t\,\vec{j}$

$T(t) = \frac{1}{\sqrt{2}}(-\sin t\, i + \cos t\, j + k)$ $\qquad B(t) = \frac{\sqrt{2}}{2}(\sin t\,\vec{i} - \cos(t)\,\vec{j} + \vec{k})$

[illegible] $P(0, 1, \frac{\pi}{2})$ $\qquad \boxed{t = \frac{\pi}{2}}$

$T(t) = \frac{1}{\sqrt{2}}(-i + k)$ $\qquad N(t) = -\vec{j}$ $\qquad B(t) = \frac{\sqrt{2}}{2}\vec{i} + \frac{\sqrt{2}}{2}\vec{k}$

$\vec{n}(\bar{r} - \bar{r}_0) = 0$ $\qquad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$\qquad ax + by + cz + d = 0$

- NORMAL PLANE

$\vec{n} \equiv T(t) = \langle -\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2} \rangle = \langle a, b, c \rangle$

$P(0, 1, \frac{\pi}{2}) = \langle x_0, y_0, z_0 \rangle \Rightarrow \underline{x_0 = 0,\ y_0 = 1,\ z_0 = \frac{\pi}{2}}$

$-\frac{\sqrt{2}}{2}(x) + 0(y - 1) + \frac{\sqrt{2}}{2}(z - \frac{\pi}{2}) = -\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}z - \frac{\sqrt{2}\pi}{4} = 0$

$\boxed{z = x + \frac{\pi}{2}}$

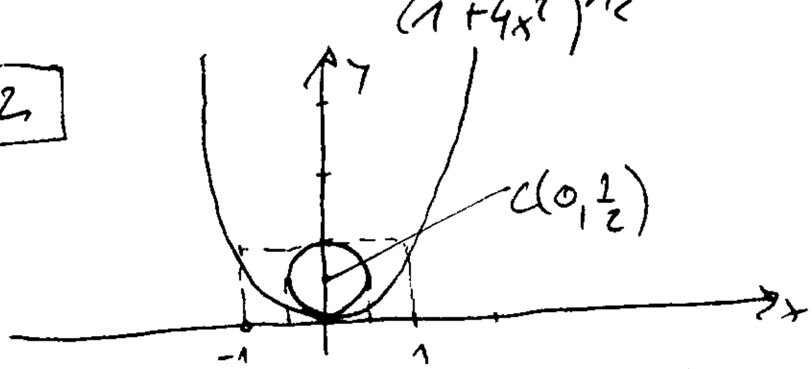
- OSCULATING PLANE

$\vec{n} \equiv B(t) = \langle \frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2} \rangle$ $\qquad P(0, 1, \frac{\pi}{2}) = P(x_0, y_0, z_0)$

$\boxed{\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}z - \frac{\sqrt{2}}{4}\pi = 0}$ $\qquad \boxed{z = -x + \frac{\pi}{2}}$

Exp. 8 $y = x^2$ $\qquad K(t) = \frac{2}{(1 + 4x^2)^{3/2}}$ $\qquad$ OSCULATING CIRCLE = ?

$\boxed{K(0) = 2}$ $\qquad r = \frac{1}{K(0)} = \frac{1}{2}$



$\boxed{x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}}$

- PARAMETRIC $\quad x = \frac{1}{2}\cos\theta \quad y = \frac{1}{2} + \frac{1}{2}\sin t$

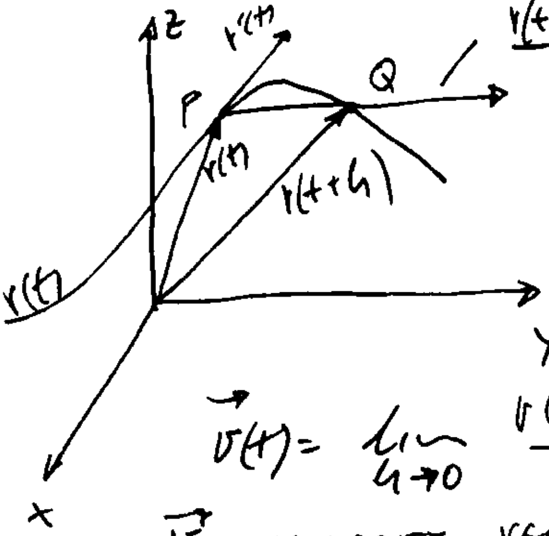
$x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$ $\qquad \frac{1}{4}\cos^2\theta + \frac{1}{4}\sin^2 t = \frac{1}{4}$ $\qquad \boxed{\frac{1}{4} = \frac{1}{4}}$

$T(t) = \frac{r'(t)}{|r'(t)|}$ $\qquad N(t) = \frac{T'(t)}{|T'(t)|}$ $\qquad B(t) = T(t) \times N(t)$

$K = \left|\frac{dT}{ds}\right| = \left|\frac{dT/dt}{ds/dt}\right| = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$

$\boxed{MMV}$

13.4 MOTION IN SPACE: VELOCITY AND ACCELERATION



$\frac{r(t+h)-r(t)}{h}$ APROXIMATES THE DIRECTION OF PARTICLE MOVING ALONG $r(t)$.

- MAGNITUDE IS DISPLACEMENT VECTOR PER UNIT TIME.

$$\vec{v}(t) = \lim_{h\to 0} \frac{r(t+h)-r(t)}{h} = r'(t) \triangleq \underbrace{T(t)}_{\text{TANGENT VECTOR}} |r'(t)|$$

$\vec{v}$ - VELOCITY VECTOR

$|\vec{v}|$ - SPEED OF PARTICLE AT TIME „t" IS MAGNITUDE OF THE VELOCITY VECTOR.

$|v(t)| = |r'(t)| = \frac{ds}{dt}$ ⇒ RATE OF CHANGE OF DISTANCE WITH RESPECT TO TIME

$\boxed{\text{EXP 1}}$ $r(t) = t^3\vec{i} + t^2\vec{j}$ POSITION VECTOR OF MOVING OBJECT IN THE PLANE

velocity, speed and acceleration = ?

$v(t) = r'(t) = 3t^2\vec{i} + 2t\vec{j}$ ⇒ VELOCITY

$|v(t)| = \sqrt{9t^4+4t^2} = t\sqrt{9t^2+4}$ ⇒ SPEED

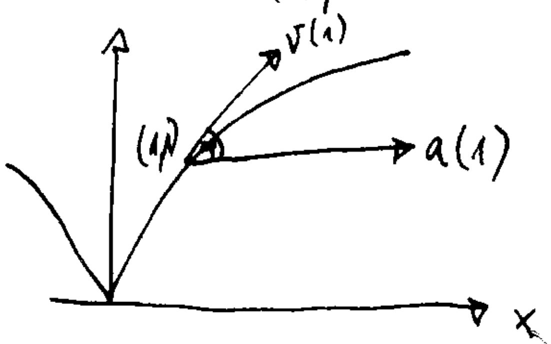
$\vec{a} = \frac{d\vec{v}}{dt} = 6t\vec{i} + 2\vec{j}$ $\quad |\vec{a}| = \sqrt{36t^2+4}$

$$\frac{d}{dt}(|v(t)|) = \frac{1}{2\sqrt{9t^4+4t^2}} \cdot (9\cdot 4\cdot t^3 + 8t) = \frac{2\cancel{4}\cancel{t}(9t^2+2)}{\cancel{2}\cancel{t}\sqrt{9t^2+4}}$$

$$= \frac{2(9t^2+2)}{\sqrt{9t^2+4}}$$

$\boxed{t=1}$ $v(1) = 3\vec{i} + 2j$ $\quad |v(1)| = \sqrt{9+4} = \sqrt{13}$

$a(1) = 6\vec{i} + 2j$



$\boxed{\text{Exp 2}}$ $r(t) = \langle t^2, e^t, te^t \rangle$

$v(t) = r'(t) = \langle 2t, e^t, e^t + te^t \rangle = 2t\vec{i} + e^t\vec{j} + e^t(1+t)\vec{k}$

$|v(t)| = \sqrt{4t^2 + e^{2t} + e^{2t}(1+t)^2}$

$a(t) = 2\vec{i} + e^t\vec{j} + [e^t(1+t) + e^t]\vec{k} = 2\vec{i} + e^t\vec{j} + e^t(2+t)\vec{k}$

$\boxed{t=1}$ $v(1) = \langle 2, e, 2e \rangle$ $r(1) = \langle 1, e, e \rangle$

$a(1) = \langle 2, e, 3e \rangle$

$v(t) = ?$ $r(t) = ?$

$\boxed{\text{Exp 3}}$ $r(0) = \langle 1, 0, 0 \rangle$ $v(0) = i - j + k$ $a(t) = 4ti + 6tj + k$

$a(t) = v'(t)$ $v(t) = \int_0^t a(t)\,dt = \int_0^t (4x\vec{i} + 6x\vec{j} + \vec{k})\,dx$

$v(t) = \left(4\frac{x^2}{2}\vec{i} + \frac{6x^2}{2}\vec{j} + x\vec{k}\right)\Big|_0^t = 2t^2\vec{i} + 3t^2\vec{j} + t\vec{k}$

$v(t) = v(0) + 2t^2\vec{i} + 3t^2\vec{j} + t\vec{k} = i - j + k + 2t^2 i + 3t^2\vec{j} + t\vec{k}$

$v(t) = (1 + 2t^2)\vec{i} + (-1 + 3t^2)\vec{j} + (1+t)\vec{k}$

$v(t) = r'(t)$ $r(t) = r(0) + \int_0^t v(t)\,dt = \left(t + \frac{2t^3}{3}\right)\vec{i} + \left(-t + \frac{3t^3}{3}\right)\vec{j} + \left(t + \frac{t^2}{2}\right)\vec{k}$

$r(t) = r(0) + r(t)$ $\boxed{r(0) = \vec{i}}$

$r(t) = \left(\frac{2t^3}{3} + t + 1\right)\vec{i} + (t^3 - t)\vec{j} + \left(\frac{t^2}{2} + t\right)\vec{k}$

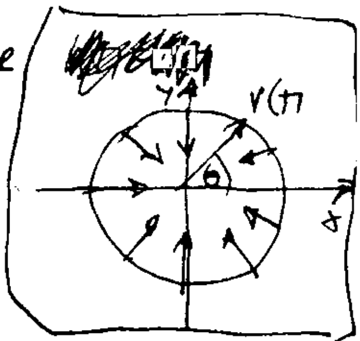
$v(t) = v(0) + \int_{t_0}^t a(u)\,du$ $r(t) = r(0) + \int_{t_0}^t v(u)\,du$

- Newton Second Law of Motion

$\boxed{F(t) = m\,a(t)}$

$\boxed{\text{Exp. 4}}$ constant angular ~~speed~~; position vector

$\vec{r}(t) = +a\sin\omega t\,\vec{i} + a\cos(\omega t)\,\vec{j}$



- FIND THE FORCE ACTING ON THE OBJECT

$r'(t) = +a\omega\cdot\cos(\omega t)\vec{i} - a\omega\sin(\omega t)\vec{j} = v(t)$

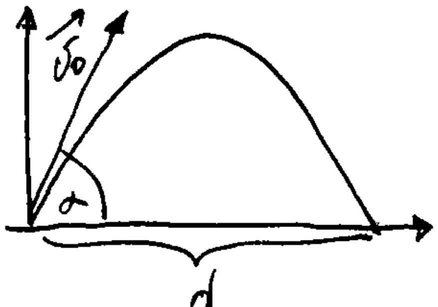
$a(t) = v'(t) = -a\omega^2\sin(\omega t)\vec{i} - a\omega^2\cos(\omega t)\vec{j}$

$F = m\cdot a(t) = -am\omega^2\sin(\omega t)\vec{i} - am\omega^2\cos(\omega t)\vec{j} = -m\omega^2\cdot\vec{r}(t)$

↳ OPPOSITE DIRECT.

Exp. 5

MMV



$r(t) = ?$

$F = m \cdot a = -mg\,j \Rightarrow \boxed{a = -g j}$

$a = \frac{1}{m} \int F(t)\,dt$

$v'(t) = a \Rightarrow v = \int -g j\,dt = -gt j + C$

$C = v_0$

$\boxed{\vec{v} = \vec{v}_0 - g t \vec{j}}$

$r(t) = \vec{v}_0 t - g \frac{t^2}{2} j + D \qquad D = r(0) = 0$

$\boxed{\vec{r}(t) = \vec{v}_0 t - g \frac{t^2}{2} \vec{j}}$

$\vec{v}_0 = v_0 \cos(\alpha)\, i + v_0 \sin(\alpha)\, j$

$\vec{r}(t) = v_0 t \cos(\alpha)\, i + \left(v_0 t \sin\alpha - g \frac{t^2}{2} \right) j$

- PARAMETRIC EQUATION OF TRAJECTORY IS:

$\boxed{x = t v_0 \cos(\alpha)} \qquad \boxed{y = t v_0 \sin\alpha - g \frac{t^2}{2}}$

$y = 0 \qquad g \frac{t^2}{2} = t v_0 \sin\alpha \qquad \cancel{t = \frac{2 v_0 \sin\alpha}{g}}$

$\boxed{t_0 = \frac{2 v_0}{g} \sin\alpha}$

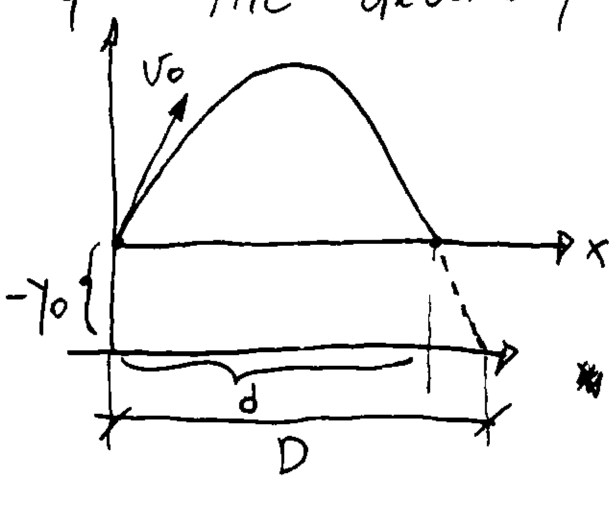
$x(t_0) = \frac{2 v_0}{g} \sin\alpha \cdot v_0 \cdot \cos\alpha$

$x(t_0) = d = \frac{2 v_0^2}{g} \frac{1}{2} \sin(2\alpha) = \frac{v_0^2}{g} \cdot \sin(2\alpha)$

$\sin(2\alpha) = 1 \Rightarrow 2\alpha = \frac{\pi}{2} \qquad \boxed{\alpha = \frac{\pi}{4}}$

MAXIMUM DISTANCE

Exp. 6 $v_0 = 150\,m/s \quad \alpha = 45^\circ$ 10 m ABOVE GROUND LEVEL. Where DOES THE PROJECTILE HITS THE GROUND, AND WITH WHAT SPEED.



$x = v_0 \cdot t \cdot \cos(\alpha)$

$y = v_0 \cdot t \sin\alpha - g \frac{t^2}{2}$

$\alpha = 45^\circ = \frac{\pi}{4}$

$x = \frac{\sqrt{2} v_0 t}{2} \qquad y = \frac{\sqrt{2}}{2} v_0 t - g \frac{t^2}{2}$

$-10 = \frac{\sqrt{2}}{2} \cdot v_0 t_0 - g \frac{t_0^2}{2} = t_0 \left(\frac{\sqrt{2}}{2} v_0 - \frac{g t_0}{2} \right)$

$g t_0^2 - \sqrt{2} v_0 t_0 - 20 = 0$

$9{,}8\, t_0^2 - \sqrt{2} \cdot 150\, t_0 - 20 = 0 \qquad \boxed{t_0 = 21{,}74\,s}$

$x(t_0) = D = t_0 v_0 \cdot \cos\left(\frac{\pi}{4}\right) = 21.74 \cdot 150 \cdot \frac{\sqrt{2}}{2} = 2305{,}9\,m \approx 2306\,m$

$$\vec{v} = \vec{v}_0 - gt\,j = v_0\cos(\alpha)\,i + v_0\sin(\alpha)\,j - gt\,j = v_0\cos(\alpha)\,\vec{i} + (v_0\sin\alpha - gt)\,\vec{j}$$

$$|v(t_0)| = \sqrt{\left(150\cdot\frac{\sqrt{2}}{2}\right)^2 + \left(150\cdot\frac{\sqrt{2}}{2} - 9.8\cdot 21{,}74\right)^2} = \boxed{150{,}65\ \frac{m}{s}}$$

☐ TANGENTIAL AND NORMAL COMPONENTS OF ACCELERATION

$v = |\vec{v}|$ $\qquad \vec{T}(t) = \frac{r'(t)}{|r'(t)|} = \frac{\vec{v}(t)}{|v(t)|} = \frac{\vec{v}}{v}$

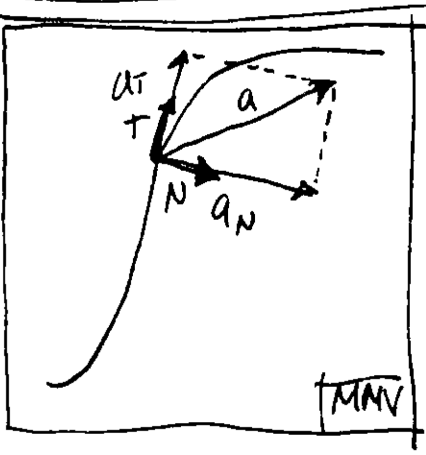
$$\boxed{\vec{v} = v\cdot\vec{T}(t)} \qquad \boxed{\vec{a} = v'\,\vec{T}(t) + v\cdot\vec{T}'(t)}$$

$$\kappa = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{|\vec{T}'|}{v} \qquad \boxed{|\vec{T}'| = \kappa\cdot v}$$

$$\vec{N} = \frac{T'}{|T'|} \qquad \boxed{\vec{T}' = \kappa\cdot v\cdot\vec{N}}$$

MMV

$$\boxed{\vec{a} = v'\,\vec{T} + \kappa\cdot v^2\cdot\vec{N} = a_T\cdot\vec{T} + a_N\cdot\vec{N}} \qquad \boxed{\begin{array}{l} a_T = v' \\ a_N = \kappa\cdot v^2 \end{array}}$$



$$\vec{v} \cdot \vec{a} = (v\cdot\vec{T}(t))\cdot(v'\,\vec{T} + \kappa\cdot v^2\,\vec{N})$$

DOT PROD

$$= v v'|\vec{T}|^2 + \kappa v^3\,\underbrace{\vec{T}\cdot\vec{N}}_{0} = v\cdot v'|\vec{T}|^2 \quad (|\vec{T}| \to 1)$$

$$\vec{v}\cdot\vec{a} = v\cdot v'$$

$$\boxed{a_T = v' = \frac{\vec{v}\cdot\vec{a}}{v} = \frac{\vec{r}'\cdot\vec{r}''}{|\vec{r}'|}}$$

MMV

$$\boxed{a_N = \kappa\cdot v^2 = \frac{|\vec{r}'(t)\times\vec{r}''(t)|}{|\vec{r}'(t)|^3}\cdot v^2 = \frac{|\vec{r}'(t)\times\vec{r}''(t)|}{|\vec{r}'(t)|}} \qquad v^2 = |\vec{r}'|^2$$

MMV

Exp. 7 $\quad r(t) = \langle t^2, t^2, t^3\rangle \qquad a_T = ? \qquad a_N = ?$

$$a_T = \frac{\vec{r}'\cdot\vec{r}''}{|\vec{r}'|} = \frac{\langle 2t, 2t, 3t^2\rangle\cdot\langle 2, 2, 6t\rangle}{\sqrt{4t^2+4t^2+9t^4}} = \frac{4t+4t+18t^3}{\sqrt{8t^2+9t^4}} = \frac{8t+18t^3}{\sqrt{8t^2+9t^4}}$$

$$a_T = \frac{\cancel{t}(8+18t^2)}{\cancel{t}\sqrt{8+9t^2}} \qquad \boxed{a_T = \frac{8+18t^2}{\sqrt{8+9t^2}}}$$

$$a_N = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|} = \frac{|\langle 2t, 2t, 3t^2\rangle \times \langle 2, 2, 6t\rangle|}{\sqrt{8t^2+9t^4}} = \frac{|\langle 6t^2, -6t^2, 0\rangle|}{\sqrt{8t^2+9t^4}}$$

$$\boxed{a_N = \left|\frac{\langle 6t, -6t, 0\rangle}{\sqrt{8+9t^2}}\right|} \qquad a_N = \frac{\sqrt{36t^2+36t^2}}{\sqrt{8+9t^2}} = \frac{6t\sqrt{2}}{\sqrt{8+9t^2}}$$

- KEPLERS LAW OF PLANETARY MOTION (MMV)

1. A PLANET REVOLVES AROUND SUN IN AN ELLIPTICAL ORBIT WITH THE SUN AT ONE FOCUS
2. THE LINE JOINING SUN WITH A PLANET SWEEPS EQUAL AREAS AT EQUAL TIMES
3. THE SQUARE OF THE PERIOD OF REVOLUTION IS PROPORTIONAL TO THE CUBE OF THE LENGTH OF MAJOR AXIS OF ORBIT

I MMV

- PROOF OF FIRST KEPLER LAW:

$\vec{r} = r(t) \qquad \vec{v} = \vec{r}' \qquad \vec{a} = \vec{r}''$

NEWTON 2 LAW OF MOTION: $\vec{F} = \vec{a} \cdot m$

LAW OF GRAVITATION: $\vec{F} = -\frac{G \cdot M \cdot m}{r^3} \cdot \vec{r} = -\frac{G M \cdot m}{r^2} \cdot \vec{u}$

G – GRAVITATIONAL CONSTANT

$r = |\vec{r}| \qquad \boxed{u = \frac{\vec{r}}{r}}$

$$\vec{a} \cdot m = -\frac{G \cdot M \cdot m}{r^2} \cdot \vec{u} \qquad \vec{a} = -\frac{GM}{r^2} \cdot \vec{u} = -\frac{GM}{r^3} \cdot \vec{r} \Rightarrow$$

$\vec{a} \parallel \vec{r}$

$\hookrightarrow \vec{a} \times \vec{r} = 0$

ANALOGNO $\vec{a} \times \vec{a} = 0$

$$\boxed{\frac{d}{dt}(\vec{r} \times \vec{v}) = \vec{r}' \times \vec{v} + \vec{r} \times \vec{v}'}$$

$$\frac{d}{dt}(\vec{r} \times \vec{v}) = \underbrace{\vec{v} \times \vec{v}}_{0} + \underbrace{\vec{r} \times \vec{a}}_{0 \text{ (PARALELNI SE)}}$$

$\vec{r} \times \vec{v} = \vec{h}$ — $\vec{h}$ KONSTANTEN VEKTOR ZATOA IZVODOT E $= 0$

$$\vec{h} = \vec{r} \times \vec{v} = \vec{r} \times \vec{r}' = r \cdot \vec{u} \times (r \cdot \vec{u})' = r \cdot \vec{u} \times (r'\vec{u} + r\vec{u}') =$$

$$= r \cdot r' \underbrace{\vec{u} \times \vec{u}}_{0} + r^2 \vec{u} \times \vec{u}' = r^2 \vec{u} \times \vec{u}' \qquad \boxed{\vec{h} = r^2 \vec{u} \times \vec{u}'}$$

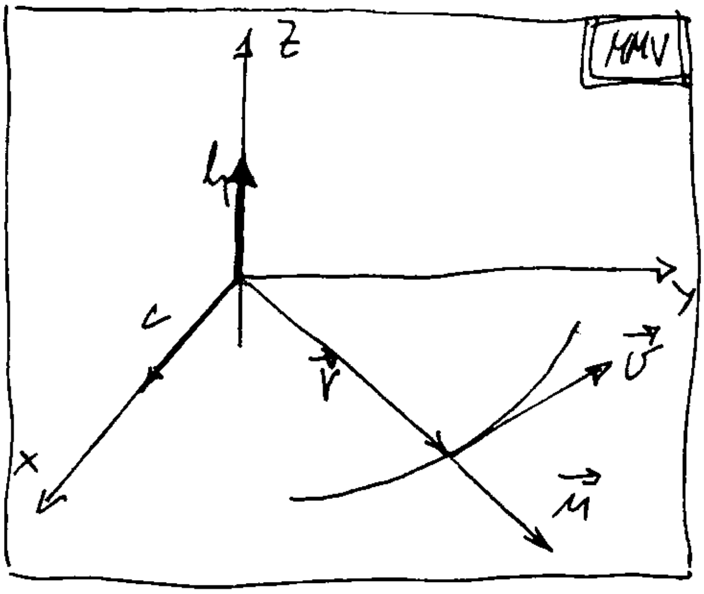
$$\vec{a} \times \vec{h} = -\frac{GM}{r^2} \cdot \vec{u} \times (r^2 \vec{u} \times \vec{u}') = -GM\, \vec{u} \times (\vec{u} \times \vec{u}') = -GM[(\underbrace{\vec{u} \cdot \vec{u}'}_{0})\vec{u} - (\vec{u} \cdot \vec{u})\vec{u}']$$

$|\vec{u}|^2$

$$\vec{a}\times\vec{h} = +GM/|\vec{u}|^2\cdot\vec{u} \qquad \boxed{\vec{a}\times\vec{h} = GM\cdot\vec{u}'}$$

$$(\vec{v}\times\vec{h})' = \vec{v}'\times\vec{h} + \underbrace{\vec{h}'\times\vec{v}}_{0} = \vec{v}'\times\vec{h} = \vec{a}\times\vec{h} = GM\cdot\vec{u}' \Big/ \int$$

$$\boxed{\vec{v}\times\vec{h} = GM\vec{u} + C}$$



$$\vec{r}\cdot(\vec{v}\times\vec{h}) = \vec{r}\cdot(GM\vec{u}+\vec{c}) =$$
$$= GM\vec{r}\cdot\vec{u} + \vec{r}\cdot\vec{c} =$$
$$= GM\cdot r\,\vec{u}\cdot\vec{u} + (|\vec{r}|\cdot|\vec{c}|\cdot\cos\theta) =$$
$$= GM\cdot r + r\cdot c\cdot\cos\theta$$

$$\boxed{r = \frac{\vec{r}\cdot(\vec{v}\times\vec{h})}{GM + c\cdot\cos\theta}}$$

$$\boxed{r = \frac{\vec{r}\cdot(\vec{v}\times\vec{h})}{1 + e\cdot\cos\theta}\cdot\frac{1}{GM}} \qquad \boxed{e = c/GM}$$

$$\vec{r}\cdot(\vec{v}\times\vec{h}) = (\vec{r}\times\vec{v})\cdot\vec{h} = \vec{h}\cdot\vec{h} = |\vec{h}|^2 = h^2$$

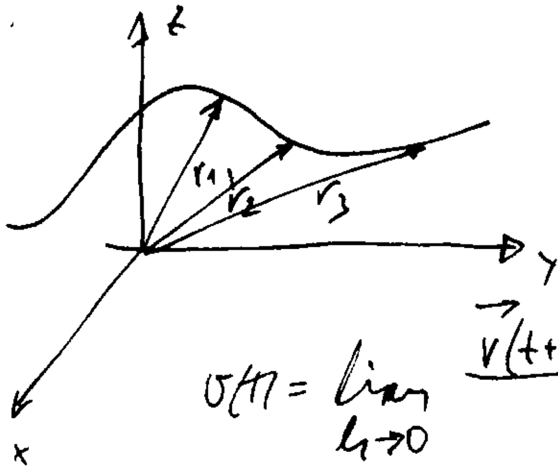
$$r = \frac{h^2/GM}{1+e\cos\theta} = \frac{h^2\cdot e/c}{1+e\cos\theta} \qquad \boxed{d = \frac{h^2}{c}}$$

$$\boxed{r = \frac{de}{1+e\cos\theta}} \to \text{POLAR EQUATION OF CONIC SECTION}$$

Exercises (13.4)

Exc. 1

	t	x	y	z
r_1	0	2.7	9.8	3.7
r_2	0.5	3.5	7.2	3.3
r_3	1.0	4.5	6.0	3.0
	1.5	5.9	6.4	2.8
	2.0	7.3	7.8	2.7



$$v(t) = \lim_{h\to 0}\frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

$$r(0) = 2.7\vec{i} + 9.8\vec{j} + 3.7\vec{k}$$
$$r(0.5) = 3.5\vec{i} + 7.2\vec{j} + 3.3\vec{k}$$
$$r(1) = 4.5\vec{i} + 6.0\vec{j} + 3.0\vec{k}$$

$$v_1 = \frac{r(0.5) - r(0)}{h} = \frac{\langle 3.5-2.7,\ 7.2-9.8,\ 3.3-3.7\rangle}{0.5}$$

$$\vec{v}_1 = \frac{\langle 0.8, -2.6, -0.4\rangle}{0.5} = 1.6\vec{i} - 5.2\vec{j} - 0.8\vec{k}$$

$$\vec{v}_2 = \frac{\vec{r}(1) - \vec{r}(0.5)}{h} = \frac{\langle 4.5-3.5, 6.0-7.2, 3.0-3.3 \rangle}{0.5} = \frac{\langle 1, -1.2, -0.3 \rangle}{0.5}$$

$$\boxed{\vec{v}_2 = 2\vec{i} - 2.4\vec{j} - 0.6\vec{k}}$$

$$\vec{v}_{avg} = \frac{\vec{v}_1 + \vec{v}_2}{2} = \frac{1.6\vec{i} - 4.2\vec{j} - 0.8\vec{k} + 2\vec{i} - 2.4\vec{j} - 0.6\vec{k}}{2}$$

$$\boxed{v_{avg} = 1.8\vec{i} - 3.8\vec{j} - 0.7\vec{k}}$$

$$v_a\Big|_{0 \div 1} = \frac{r(1) - r(0)}{h} = \frac{\langle 4.5-2.7, 6.0-9.8, 3.0-3.7 \rangle}{1} = \underline{\underline{1.8\vec{i} - 3.8\vec{j} - 0.7\vec{k}}}$$

LT ~~RESULTAT~~

$$|v(1)| = \sqrt{1.8^2 + 3.8^2 + 0.7^2} = \sqrt{3.24 + 14.44 + 0.49} = \underline{\underline{4.2 \ m/s}}$$

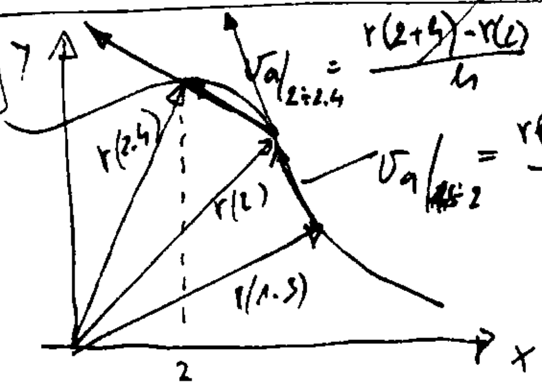
$$v_a\Big|_{0.5 \div 1} = \frac{r(1) - r(0.5)}{h} = 2.0\vec{i} - 2.4\vec{j} - 0.6\vec{k}$$

$$v_a\Big|_{1 \div 1.5} = \frac{r(1.5) - r(0.5)}{h} = 2.8\vec{i} + 0.8\vec{j} - 0.4\vec{k}$$

$$v(1) = \frac{v_a|_{0.5 \div 1} + v_a|_{1 \div 1.5}}{2} = 2.4\vec{i} - 0.8\vec{j} - 0.5\vec{k}$$

$$|v(1)| = \sqrt{2.4^2 + 0.8^2 + 0.5^2} = \sqrt{5.76 + 0.64 + 0.25} \doteq 2.58$$

Exc 2



$$v_a\Big|_{2 \div 2.4} = \frac{r(2+h) - r(2)}{h} = 2.5\,(r(2.4) - r(2))$$

0.4, 0.5

$$v_a\Big|_{1.5 \div 2} = \frac{r(1.5+h) - r(1.5)}{h} = 2\,[r(2) - r(1.5)]$$

TWO TIMES LARGER THAN $[r(2) - r(1.5)]$

$$v(2) = \frac{v_a|_{2 \div 2.4} + v_a|_{1.5 \div 2}}{2}$$

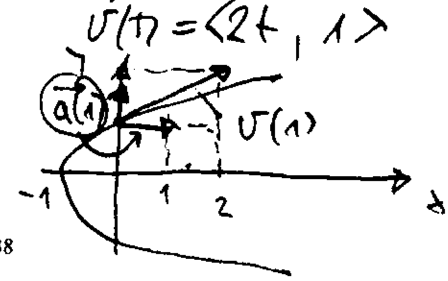
SPEED = $|v(2)|$

$y = x^2 - 1$

Exc 3 $\quad \vec{r}(t) = \langle t^2 - 1, t \rangle \quad \boxed{t = 1}$

$\vec{v}(t) = \langle 2t, 1 \rangle$

$x = t^2 - 1 \quad y = t \quad \Rightarrow t = y \quad x = y^2 - 1$



$$\vec{v}(1) = 2\vec{i} + \vec{j}$$

$$|v(1)| = \sqrt{4 + 1} = \boxed{\sqrt{5}}$$

$$\vec{a}(t) = \langle 2, 0 \rangle = 2\vec{i} \qquad \boxed{\vec{a}(1) = 2\vec{i}}$$

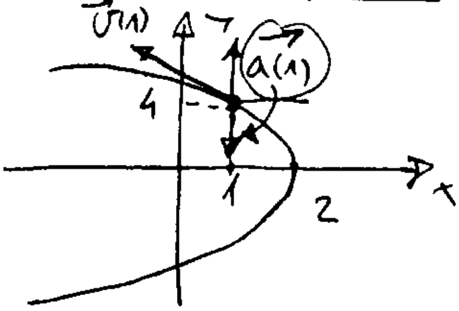
$\boxed{\text{Exc. 4}}$ $\vec{r}(t) = \langle 2-t, 4\sqrt{t} \rangle \qquad t = 1$

$\vec{v}(t) = \vec{r}'(t) = \langle -1, + \frac{2}{\sqrt{t}} \rangle \qquad \vec{a}(t) = \vec{r}''(t) = \langle 0, -\frac{1}{\sqrt{t^3}} \rangle$

$x = 2-t \qquad y = 4\sqrt{t} \qquad t = 2-x \qquad y = 4\sqrt{2-x}$

$\frac{y^2}{16} = 2-x \qquad \boxed{x = -\frac{y^2}{16} + 2} \qquad |\vec{v}(t)| = \sqrt{1 + \frac{4}{t}}$

$t = 1 \Rightarrow \boxed{x = 1 \quad y = 4}$



$\vec{v}(1) = -\vec{i} + 2\vec{j}$

$\vec{a}(1) = \langle 0, -1 \rangle = -\vec{j}$

$|\vec{v}(1)| = \sqrt{1+4} = \underline{\underline{\sqrt{5}}}$

$(\sqrt{t})' = (t^{\frac{1}{2}})' = \frac{t^{\frac{1}{2}-1}}{2} = \frac{1}{2\sqrt{t}} \qquad (t^{-\frac{1}{2}})' = \underline{\underline{-\frac{1}{2} \frac{1}{\sqrt{t^3}}}}$

$\boxed{\text{Exc. 7}}$ $\vec{r}(t) = \sin(t)\vec{i} + t\vec{j} + \cos t\,\vec{k} \qquad t = 0$ $\qquad \boxed{x^2 + z^2 = 1,\ y = t}$

HELIX ABOUT Y-AXIS

$\vec{v}(t) = \vec{r}'(t) = \cos(t)\vec{i} + \vec{j} - \sin(t)\vec{k}$

$t = 0 \quad x = 0, y = 0, z = 1 \qquad \boxed{\vec{v}(0) = \vec{i} + \vec{j}}$

$\vec{a}(t) = \vec{v}'(t) = -\sin(t)\vec{i} - \cos(t)\vec{k} \qquad a(0) = -\vec{k}$

$|\vec{v}(0)| = \sqrt{1+1} = \sqrt{2} \qquad \boxed{|\vec{v}(t)| = \sqrt{\cos^2 + \sin^2 + 1} = \sqrt{2}}$

$\boxed{\text{Exc. 11}}$ $r(t) = \sqrt{2}t\vec{i} + e^t\vec{j} + e^{-t}\vec{k}$

$\vec{v}(t) = \sqrt{2}\vec{i} + e^t\vec{j} - e^{-t}\vec{k} \qquad \vec{a}(t) = e^t\vec{j} + e^t\vec{k}$

$\boxed{|\vec{v}(t)| = \sqrt{2 + e^{2t} + e^{-2t}}}$

$t = 0 \quad x = 0 \quad y = 1 \quad z = 1$

$v(0) = \langle \sqrt{2}, 1, -1 \rangle$

$a(0) = \langle 0, 1, 1 \rangle$

$|\vec{v}(t)| = \sqrt{(e^t + e^{-t})^2} = \underline{\underline{e^t + e^{-t}}}$

Exc. 15 $\vec{v}, \vec{r} = ?$ $a(t) = \vec{k}$ $v(0) = i - j$ $r(0) = 0$

$$\vec{v} = \vec{v}(0) + \int \vec{a}(t)\,dt = i - j + \int \vec{k}\,dt = \vec{i} - \vec{j} + t\vec{k}$$

$$\vec{r}(t) = r(0) + \int \vec{v}(t)\,dt = 0 + \int \vec{i} - \vec{j} + t\vec{k}\,dt$$

$$\boxed{\vec{r}(t) = t\,\vec{i} - t\,\vec{j} + \frac{t^2}{2}\vec{k}}$$

$$v(t) = \int \vec{a}\,dt = t\vec{k} + c_1 \quad v(0) = 0 \cdot \vec{k} + c_1 = i - j \quad c_1 = i - j$$

$$\boxed{v(t) = t \cdot k + i - j}$$

Exc. 19 $r(t) = \langle t^2, 5t, t^2 - 16t \rangle$ $|v_{min}| = ?$

$$\vec{v}(t) = \vec{r}'(t) = \langle 2t, 5, 2t - 16 \rangle$$

$$|v(t)| = \sqrt{(2t)^2 + 25 + (2t-16)^2} = \sqrt{4t^2 + 25 + 4t^2 - 64t + 256}$$

$$|v(t)| = \sqrt{8t^2 - 64t + 281} = v$$

$$\frac{dv}{dt} = \frac{1}{2\sqrt{8t^2 + 64t + 281}}(16t - 64) = 0$$

$$t = +\frac{64}{16} = 4 \qquad \boxed{t = +4}$$

Exc. 20 $|\vec{v}| = c$ $|\vec{v}|^2 = c^2$ $\vec{v} \cdot \vec{v} = c^2$ $\Downarrow$

$$\frac{d}{dt}(\vec{v} \cdot \vec{v}) = \vec{v} \cdot \vec{v}' + \vec{v}' \cdot \vec{v} = 2\vec{v} \cdot \vec{v}' = 0$$

$$\boxed{\vec{v} \cdot \vec{a} = 0} \Rightarrow \boxed{\vec{v} \perp \vec{a}}$$

$\vec{a}$ & $\vec{v}$ are ORTHOGONAL VECTORS!

Exc. 25 $a(t) = -g\vec{j}$ $\vec{v} = v_0 - gt\vec{j}$ $r = v_0 t - g\frac{t^2}{2}\vec{j}$

$$\vec{v}_0 = v_0 \cos(\alpha)\,\vec{i} + v_0 \sin\alpha\,\vec{j} = \frac{\sqrt{2}}{2}\vec{i} + \frac{\sqrt{2}}{2}\vec{j}$$

$$r(t) = v_0 t - g\frac{t^2}{2}\vec{j}$$

$$r(t) = \frac{\sqrt{2}v_0}{2}t\,\vec{i} + \frac{\sqrt{2}}{2}t v_0\vec{j} - g\frac{t^2}{2}\vec{j} = \underbrace{v_0\frac{\sqrt{2}}{2}t}_{x}\vec{i} + \underbrace{\left(\frac{\sqrt{2}}{2}t v_0 - g\frac{t^2}{2}\right)}_{y}\vec{j}$$

$$t_0 \Rightarrow y(t_0) = 0 \quad v_0 \cdot \frac{\sqrt{2}}{\cancel{2}}\cancel{t} = g\frac{t^2}{\cancel{2}} \qquad \boxed{t_0 = \frac{v_0\sqrt{2}}{g}} \quad t_0 = 0.144 \cdot v_0$$

~~$|r(t_0)| = 90m$~~ ~~$90 = V_0 \frac{\sqrt{2}}{2} t_0 \vec{r}$~~

$|r(t_0)| = \frac{\sqrt{2}}{2} V_0 \cdot t_0 = 90m$ $\quad \frac{\sqrt{2}}{2} \cdot V_0 \cdot \frac{V_0\sqrt{2}}{g} = 90 \quad V_0^2 = 90g$

$$\boxed{V_0 = \sqrt{90g}}$$

$\boxed{33}$ Exc FIND a_T and a_N?

$a = a_T \cdot \vec{T} + a_N \cdot \vec{N}$ $\quad \boxed{a_N = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|r'(t)|}}$

$a_T = \frac{\vec{r}' \cdot \vec{r}''}{|\vec{r}'|} = v'$ $\quad \boxed{v = |\vec{v}|}$

$r(t) = \cos t\, \vec{i} + \sin t\, \vec{j} + t k$

$r'(t) = -\sin t\, \vec{i} + \cos t\, \vec{j} + \vec{k}$ $\quad |\vec{r}'| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

$r''(t) = -\cos t\, \vec{i}$ ~~$+$~~ $\sin t\, \vec{j}$

$r'(t) \times r''(t) = \langle +\sin(t), -\cos t, 1 \rangle$

$|r' \times r''| = \sqrt{\sin^2(t) + \cos^2 t + 1} = \sqrt{2}$ $\quad a_N = \frac{\sqrt{2}}{\sqrt{2}} = 1$

$r'(t) \cdot r''(t) = \sin t \cdot \cos t$ ~~k~~ $- \cos t \sin t = 0$ $\quad a_T = 0$

$\boxed{39. Exc}$ $r(t) = (3+t)\vec{i} + (2 + \ln t)\vec{j} + \left(7 - \frac{4}{t^2+1}\right)\vec{k}$ $\quad \boxed{b\langle 6, 4, 9 \rangle}$

$\vec{v}(t) = (3+1)\vec{i} + \left(\frac{1}{t}\right)\vec{j}$ ~~$+$~~ $\frac{4}{(t^2+1)^2}(2t)\vec{k}$

$\vec{v}(t) = 4\vec{i} + \frac{1}{t}\vec{j} + \frac{8t}{(t^2+1)^2}\vec{k}$

$$\left.\begin{array}{lll} 3+t=6 & t=3 & \\ 2+\ln t = 4 & 2+\ln 3 = 4 & 2 + 1.099 \neq 4 \\ 7 - \frac{4}{t^2+1} = 9 & & \end{array}\right\}$$

NE SE NAOGJA NA TRAEKTORIJATA

$r(t) - b(t) = (3+t-6)\vec{i} + (2 + \ln t - 4)\vec{j} + \left(7 - \frac{4}{t^2+1} + 9\right)k$

$\boxed{d(t)} \Rightarrow$ distance

$\frac{d}{dt}[|d(t)|] = 0$ $\quad \frac{d}{dt}\left[\sqrt{(t-3)^2 + (\ln t - 2)^2 + \left(2 - \frac{4}{t^2+1}\right)^2}\right]$

$$\frac{d}{dt}\left\{\left[(t-3)^2+(\ln t-2)^2+\left(2-\frac{4}{t^2+1}\right)^2\right]^{1/2}\right\}=0$$

$$2(t-3)+2(\ln t-2)\cdot\frac{1}{t}+2\left(2-\frac{4}{t^2+1}\right)\left(+\frac{4}{(t^2+1)^2}\cdot 2t\right)=0$$

$$2t-6+\frac{2}{t}\ln t-\frac{4}{t}+\left(4-\frac{8}{t^2+1}\right)\left(\frac{8t}{(t^2+1)^2}\right)=0$$

$$2t-6+\frac{2}{t}\ln t-\frac{4}{t}+\frac{\overset{4t^2-4}{\cancel{4t^2+4-8}}}{t^2+1}\,\frac{8t}{(t^2+1)^2}=0$$

$$2t-6+\frac{2\ln t}{t}-\frac{4}{t}+\frac{4(t^2-1)}{t^2+1}\,\frac{8t}{(t^2+1)^2}=0$$

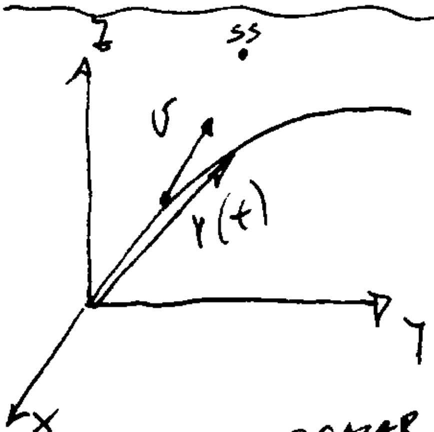
$$\cancel{2}t-\overset{3}{\cancel{6}}+\frac{\cancel{2}\ln t}{t}-\frac{\overset{2}{\cancel{4}}}{t}+\frac{\overset{16}{\cancel{32}}(t^2-1)\,t}{(t^2+1)^3}=0$$

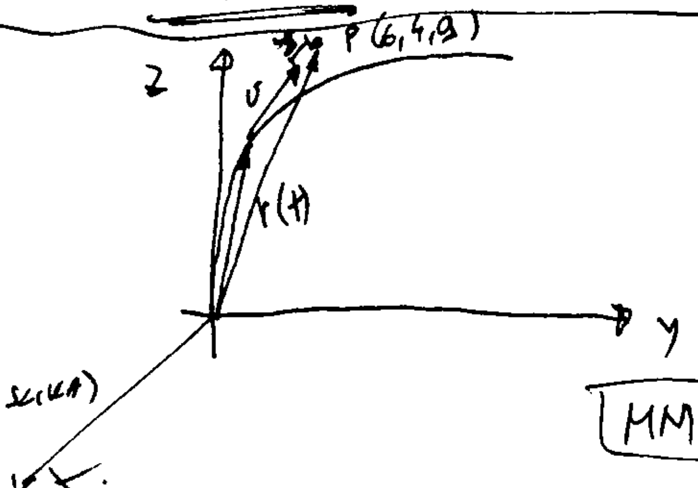
$$t-3+\frac{1}{t}\ln t-\frac{2}{t}+\frac{16(t^2-1)\,t}{(t^2+1)^3}=0 \quad /\cdot t$$

$$\boxed{t^2-3t+\ln t-2+\frac{16(t^2-1)\,t^2}{(t^2+1)^3}=0} \qquad t_0=2.9106$$

$$\boxed{\begin{aligned} r(t_0) &= (3+2.9106)\vec{i}+(2+\ln 2.9106)\vec{j}+\left(7-\frac{4}{t^2+1}\right)\vec{k}\\ &= 5.9\,\vec{i}+3.07\,\vec{j}+6.58\,\vec{k}\end{aligned}}$$

→ NAJBLISKA TOCKA DO SPACE STATION !
VRAKETATOT DA IZLEZI VO $t_0=2.9106$





(VIDI SLIKA)

$\boxed{\text{HMV}}$

SKALAR

$$\boxed{\overrightarrow{OP}=r(t)+s\cdot v(t)=6\vec{i}+4\vec{j}+9\vec{k}}$$

$$v(t)=r'(t)=i+\frac{1}{t}j+\frac{4\cdot 2t}{(t^2+1)^2}\vec{k}$$

$$r(t)+s\cdot v(t) = (3+t)\vec{i} + (2+\ln t)\vec{j} + \left(7-\frac{4}{t^2+1}\right)\vec{k} + s\vec{i} + \frac{s}{t}\vec{j} + \frac{8st}{(t^2+1)^2}\vec{k}$$

$$= (3+t+s)\vec{i} + \left(2+\ln t+\frac{s}{t}\right)\vec{j} + \left(7-\frac{4}{t^2+1}+\frac{8st}{(t^2+1)^2}\right)\vec{k} = \langle 6, 4, 9\rangle$$

$$3+t+s=6 \qquad \boxed{s=3-t}$$

$$2+\ln t+\frac{3-t}{t}=4 \qquad 2t+t\ln t+3-t=4t$$

$$t\ln(t) = -3+3t \qquad \boxed{t\ln(t)=3(t-1)} \; (*)$$

$$7-\frac{4}{t^2+1}+\frac{8(3-t)t}{(t^2+1)^2}=9 \qquad 7(t^2+1)^2-4(t^2+1)+8(3t-t^2) = 9(t^2+1)^2$$

$$-\cancel{2}^{\,2}(t^2+1)^2-\cancel{4}^{\,2}(t^2+1)+\cancel{8}^{\,4}(3t-t^2)=0$$

$$-t^4-2t^2-1-2t^2-2+12t-4t^2=0 \qquad -t^4-8t^2+12t-3=0$$

$$t^4+8t^2-12t+3=0$$

MALE $\boxed{t=1}$

$(*)$ $1\cdot \cancel{\ln(1)}^{\,0} = 3\cancel{(1-1)}^{\,0}$ $\qquad 2+\ln 1+\frac{3-t}{t} = 2+0+2 = 4$

• KORISNICI : 1200

$$\text{PAYBACK PERIOD} = \frac{\text{INVESTMENT REQUIRED}}{\text{NET ANUAL CASH INFLOW}}$$

EXAMPLE: MACHINE A COSTS 15000 \$

REDUCTION OF OPERATING COSTS PER YEAR: 5000 \$

MACHINE B COSTS 12000 \$

REDUCTION OF OP. COSTS PER YEAR: 5000 \$

$$\text{PAY BACK PERIOD A} = \frac{15000}{5000} = 3$$

$$\text{PAY BAC PERIOD B} = \frac{12000}{5000} = \boxed{2.4}$$

IF CAPEX TOTAL < FREE CASH FLOW CUMULATIVE

$$PB = \frac{\text{CAPEX} - \text{FREE CASH FLOW}}{D28 - C28} \Bigg\} \text{OD EXCEL}$$

NPV = INVESTICIJA − SEDAŠNJA VREDNOST ↗ 16.5%

$$-R_0 + \sum_{t=1}^{N} \frac{R_t}{(1+\text{WACC})^t}$$

WACC – DISCOUNT RATE (WACC)

t – TIME OF CASH INFLOW

R_t – NET CASH FLOW (AMOUNT OF CASH, INFLOW – OUTFLOW) AT TIME t

R_0 – INVESTMENT (CAPEX)

N = 4	1	2	3	4
$(1+\text{WACC})^t$	$(1+0.165)^1 = 1.165$	$(1+0.165)^2 = 1.357$	$(1.165)^3 = 1.581$	$(1.165)^4 = 1.842$
CAPEX	10.000	0	0	0
Revenues	5.000	5.000	5.000	5.000
Outgoings	1.000	0	0	0
Depreciation 20%	2.000	2.000	2.000	2000
EBITDA = Reven. − Outgoings	4.000	4000	4.000	4.000
EBIT = EBITDA − Deprec	2.000	2.000	2.000	2.000
NET INCOME EBIT − Tax (10%) [0.1 EBIT]	1.800	1.800	1.800	1.800
FREE CASH FLOW = R_t = DEPREC. + NET INCOME	3800	3800	3800	3800
NPV = −CAPEX + $\sum_{t=1}^{4} \frac{R_t}{(1+\text{WACC})^t}$	3.621	3.462	2.972	2.551

• PAYBACK PERIOD

$$time(i) + \frac{CAPEX - FreeCashFlowPVCumulative(i+1)}{(FreeCashFlowPVCum(i+1) - FreeCashFlowPVCum(i))}$$

IF CAPEX > FreeCashFlowPVCum(i+1)

FreeCashFlowPV (i+1)

4 GODINI 12.248€
CAPEX 10.000€

ZA KOLKU VREME SE VRAĆA INVESTICIJATA

$\boxed{4 \times 12 = 48}$ $\quad 12.248/48 = 255,16$ €/MONTH

$$\frac{10.000€}{255,16} = 39{,}2 \text{ MESECI} = 3.2 \text{ YEARS}$$

MOD SEZSKI PRISTAP

	x_1	x_2	x_3	x_4
CASH FLOW PER MONTH	271.82	288.58	247.71	212.62
CASH FLOW YEARLY	3261	3462	2972	2551

$$12x_1 + 12 \cdot x_2 + 12 \cdot x_3 + b \cdot x_4 = CAPEX$$

$$b = \frac{CAPEX - (12x_1 + 12x_2 + 12x_3)}{x_4} = \frac{10000 - (3261 + 3462)}{x_4} =$$

$$= \frac{10.000 - 9697,24}{212,62} = 1.424$$

$$\boxed{\frac{b}{12} = 0.1187}$$

$$\boxed{\text{PAYBACK} = t + b/12}$$

YEARLY FORMULA — CASHFLOWS PV

$$x_1 + x_2 + x_3 + a \cdot x_4 = CAPEX \rightarrow t_1 + t_2 + t_3$$

1711

1 $$a = \frac{CAPEX - CashFlowCumulative(i-1)}{CashFlow(i)}$$

696
695

"TOPLIČNA FORMULA !!"

ALTERNATIVNO
$$a = \frac{CAPEX - CashFlowPVCum(i-1)}{CashFlowPVCum(i) - CashFlowPVCum(i-1)}$$

• FINALNA FORMULA:

$$Payback = \#\left[\left(\frac{CAPEX}{CashFlowPVCum(i)}\right)\right]^{-1} (t-1) + \frac{CAPEX - CashFlowPVCum(i-1)}{CashFlowPV(i)}$$

NMV $t = 1, 2, 3, 4 \ldots$

$$Payback = \#\left[\frac{CashFlowPVCum(i)}{CAPEX}\right] (t-1) + \frac{CAPEX - CashFlowPVCum(i-1)}{CashFlowPV(i)}$$

NMV

IRR (Internal Rate of Return)

$$NPV = -CAPEX + \sum_{t=1}^{N} \frac{R_t}{(1+IRR)^t} = 0$$

KEPLER'S LAWS (CONTINUE)

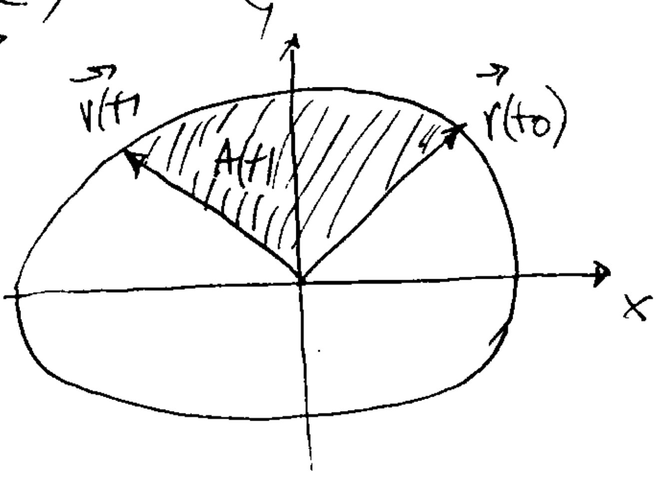
$$\vec{r} = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j}$$

a.) $\vec{h} = r^2 \frac{d\theta}{dt}$

b.) DEDUCE THAT $r^2 \frac{d\theta}{dt} = h$

c.) $\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$

d.) $\frac{dA}{dt} = \frac{1}{2}\vec{h} = const.$



(a) $\vec{h} = \vec{r}\times\vec{v} = r^2 \cdot \vec{u}\times\vec{u}'$

$$\vec{u} = \frac{\vec{r}}{|r|} = \frac{r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j}}{r} = \underline{\cos\theta\,\vec{i} + \sin\theta\,\vec{j}}$$

$$|r| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = r$$

$$\boxed{\vec{u}' = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j}}$$

$$\vec{u}\times\vec{u}' = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{vmatrix} = \begin{vmatrix} \sin\theta & 0 \\ \cos\theta & 0 \end{vmatrix} i - \begin{vmatrix} \cos\theta & 0 \\ -\sin & 0 \end{vmatrix} j + \overbrace{\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}}^{\cos^2+\sin^2\theta} k$$

$$= k$$

$$\vec{u}(\theta) = \cos(\theta)\,\vec{i} + \cos\theta\,\vec{j} \qquad \theta = \theta(t)$$

$$\frac{d}{dt}\vec{u}(\theta) = \frac{d}{d\theta}[u(\theta)]\,\frac{d\theta}{dt} = \left(-\sin\theta\,\vec{i} + \cos\theta\,\vec{j}\right)\frac{d\theta}{dt} \rightarrow a$$

$$\vec{u}\times\vec{u}' = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & 0 \\ -a\sin\theta & a\cos\theta & 0 \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -a\sin\theta & a\cos\theta \end{vmatrix}\vec{k}$$

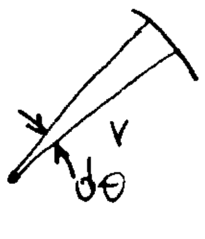
$$= \left(a\cos^2\theta + a\sin^2\theta\right)\vec{k} = \underline{a\,\vec{k}} = \frac{d\theta}{dt}\cdot\vec{k}$$

$$\boxed{\vec{h} = r^2\cdot\vec{u}\times\vec{u}' = r^2\,\frac{d\theta}{dt}\cdot\vec{k}}$$

(b) $|\vec{h}| = \vec{h}\cdot\vec{h} = \left(r^2\,\frac{d\theta}{dt}\right)^2\cdot\underbrace{\cancel{\vec{k}}\cdot\vec{k}}_{\text{DOT PRODUCT} = 1} = r^2\,\frac{d\theta}{dt} = h$

$$\boxed{h = r^2\,\frac{d\theta}{dt}}$$

(c) AREA OF ARC

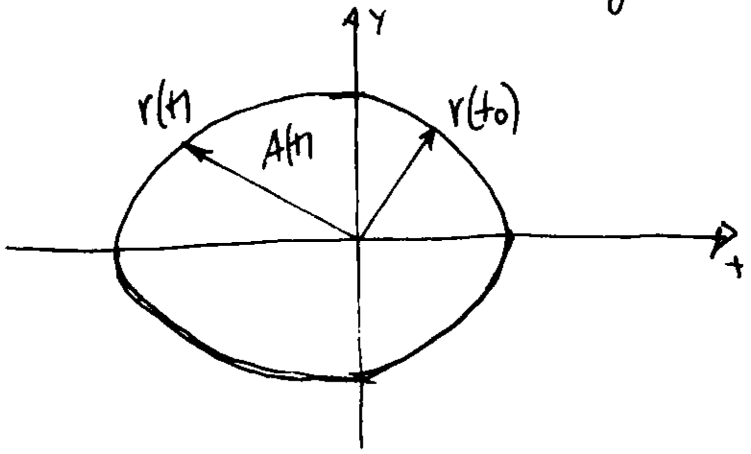


$$dA = \frac{r^2}{2}\,d\theta \qquad A = \frac{r^2}{2}\cdot\theta$$

KRUG: $A = r^2\pi = \frac{r^2}{2}\,(2\pi)$

$$\frac{dA}{d\theta} = \frac{r^2}{2} \Big/ \frac{d\theta}{dt} \qquad \boxed{\frac{dA}{\cancel{d\theta}}\,\frac{\cancel{d\theta}}{dt} = \frac{r^2}{2}\,\frac{d\theta}{dt}} \qquad \frac{dA}{dt} = \frac{r^2}{2}\,\frac{d\theta}{dt} \quad \boxed{\text{MHV}}$$

$$\vec{r}(t) = \cos(\theta)\,\vec{i} + \sin\theta\,\vec{j} \qquad \boxed{\theta = \theta(t)}$$



$$A = \int_{t_0}^{t} \left(\cos(\theta)\,\vec{i} + \sin\theta\,\vec{j}\right) dt$$

$$g(t) = \int_{b}^{t} f(x)\,dt \qquad \frac{dg(t)}{dt} = f(t)$$

$$\frac{dA}{dt} = \cos(\theta)\,\vec{i} + \sin\theta\,\vec{j} \qquad ?$$

$$A = \int_{\theta_0}^{\theta} \cos\varphi\,\vec{i} + \sin\varphi\,\vec{j}\,d\varphi \qquad \frac{dA}{d\theta} = \cos(\theta)\,\vec{i} + \sin\theta\,\vec{j}$$

$$\varphi = \theta(t) \qquad d\varphi = \theta'(t)\,dt$$

$$\varphi = \theta_0 \qquad \theta_0 = \theta(t_0) \qquad t = t_0$$

$$A = \int_{t_0}^{t} \left(\cos\theta(t)\,\vec{i} + \sin\theta(t)\,\vec{j}\right)\theta'(t)\,dt \qquad ?$$

$$\boxed{\frac{dA}{dt} = \left(\overrightarrow{\cos\theta(t)\,\vec{i} + \sin\theta(t)\,\vec{j}}\right)\frac{d\theta(t)}{dt}} \qquad \frac{dA}{dt} = \vec{r}\,\frac{d\theta(t)}{dt}$$

(4.) $A = \frac{r^2}{2}\theta \qquad \boxed{\frac{dA}{dt} = \frac{r^2}{2}\frac{d\theta}{dt}}$

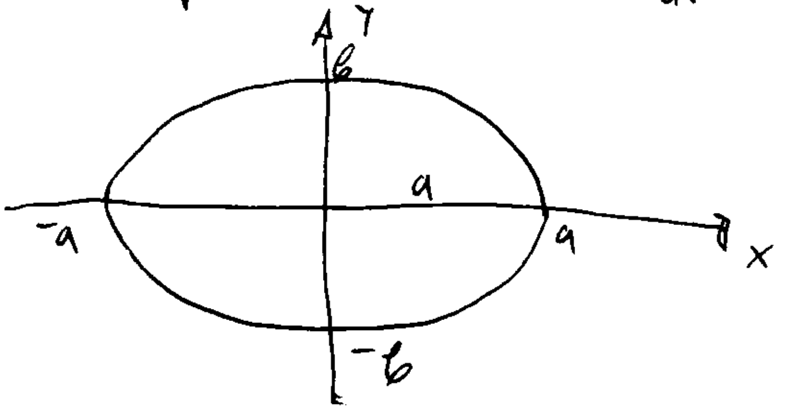
$$h = r^2\frac{d\theta}{dt} \quad \Rightarrow \quad \boxed{\frac{dA}{dt} = \frac{h}{2}} \qquad \boxed{A = \frac{h}{2}\cdot t}$$

KEPLER'S II LAW

② T – PERIOD OF A PLANET ABOUT THE SUN

$$\vec{v}(t) = \vec{r}'(t) = \left(-\sin\theta\,\vec{i} + \cos\theta\,\vec{j}\right)\frac{d\theta}{dt}$$

$$|v| = \sqrt{\sin^2\theta + \cos^2\theta}\;\frac{d\theta}{dt} = \frac{d\theta}{dt}$$



$$\boxed{\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1}$$

ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad x = a\cos\theta \quad y = b\sin\theta$$

$$\frac{a^2\cos^2\theta}{\cancel{a^2}} + \frac{b^2\sin^2\theta}{\cancel{b^2}} = 1 \qquad \sqrt{a^2\cos^2\theta + b^2\sin^2\theta = 1}$$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$L = 4\int_0^{\pi/2} \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\;d\theta = \cancel{\sqrt{a^2+b^2}\,\frac{\pi}{4}}$$

$$\cancel{\boxed{L = 2\pi\sqrt{a^2+b^2}}} \qquad \cancel{\text{IF } a = b \to \text{CIRCLE} \quad L = 2\pi\sqrt{a^2+a^2} =}$$

- AREA OF ELLIPSE

$$\boxed{A = a \cdot b \cdot \pi}$$

$$\cancel{4\int_0^{\pi/2} b\sin\theta\,d\theta}$$

$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right) \qquad y = b\sqrt{\left(1 - \frac{x^2}{a^2}\right)}$$

$$A = 4\int_0^a b\sqrt{1 - \frac{x^2}{a^2}}\,dx = \cancel{4}\cdot\frac{ab\pi}{\cancel{4}} = a \cdot b\,\pi$$

$$A = \frac{h}{2} \cdot t \qquad A = a \cdot b \cdot \pi \qquad \boxed{a \cdot b \cdot \pi = \frac{h}{2} \cdot T}$$

$$\boxed{T = \frac{2a \cdot b \cdot \pi}{h}}$$

- $$r = \frac{e \cdot d}{1 + e\cos\theta} \qquad d = \frac{h^2}{c}$$ e - eccentricity $$\boxed{e = \frac{c}{G \cdot M}}$$

$$e \cdot d = \frac{h^2}{c} \cdot \frac{c}{GM} = \frac{h^2}{GM}$$

$$e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{c}{a} \qquad c = \sqrt{a^2 - b^2} \qquad h \overset{?}{=} b$$

$$e \cdot d = \frac{h^2}{GM} = \underbrace{\frac{c}{a}}_{e} \cdot \frac{h^2}{c} = \frac{h^2}{a} = \frac{b^2}{a}$$

$$T = \frac{2a \cdot b \cdot \pi}{\sqrt{GM \cdot \frac{b^2}{a}}} \qquad \boxed{\frac{h^2}{GM} = ed = \frac{b^2}{a}}$$ → DIFERENTNO OD Stewart NE ZNAM OD KUDA?

$$h^2 = GM \cdot \frac{b^2}{a}$$

$$T = \frac{2a \cdot \cancel{b} \cdot \pi \sqrt{a}}{\cancel{b}\sqrt{GM}} \qquad T^2 = \frac{4a^2\pi^2 \cdot a}{GM} = \frac{4a^3\pi^2}{GM}$$

$$\boxed{T^2 = \frac{4\pi^2}{GM} a^3}$$ ⟹ THIRD KEPLER LAW

- MASS OF THE SUN $M = 1.99 \cdot 10^{30}$ kg $(T = 365,25)$

GRAVITATIONAL CONSTANT: $G = 6.67 \cdot 10^{11}\ N m^2/kg^2$

$$a = \sqrt[3]{\frac{G \cdot M}{4\pi^2} T^2} = \cancel{7.659 \cdot 10^7} = \cancel{76,550 \cdot 10^6} = \cancel{76550\ \text{km}}$$

$$T = 365,25 \cdot 24 \cdot 60 \cdot 60\ \text{sec} \qquad \cancel{a = 9,76 \cdot 10^9 = 9.760.000\ \text{km}}$$

$$a = 1.496 \cdot 10^{11} = 14,96 \cdot 10^9 = 14,96 \cdot 10^9 \cdot = 149600.000\ \text{km}$$

4 $a = \sqrt[3]{\dfrac{G\cdot M}{4\pi^2} T^2} = \sqrt[3]{\dfrac{6.67\cdot 10^{-11}\cdot 5.98\cdot 10^{24}}{4\cdot\pi^2}(24\cdot 60\cdot 60)^2}$

$a = 4.225\cdot 10^7$

satelite altitute – $H = a - R$

$R = 6.37\cdot 10^6 m$ ⇒ earth radius

$H = 42,25\cdot 10^6 - 6,37\cdot 10^6 = \underline{35,880474\cdot 10^6} = \underline{35880 km}$

VECTOR FIELDS

① DEFINITION: Let D be a set in R^2 (a plane region). A vector field on R^2 is a function F that assigns to each point in D two-dimensional vector $F(x,y)$

$$F(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j} = \langle P(x,y), Q(x,y)\rangle \quad P\vec{i} + Q\vec{j}$$

② DEFINITION: E subset of R^3

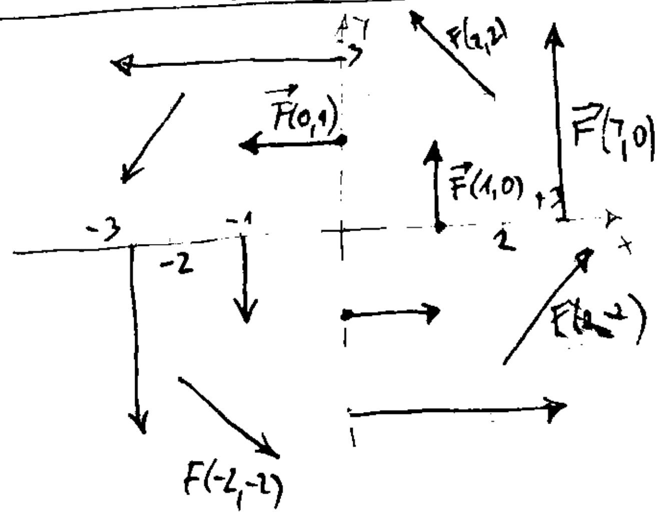
$$F(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$$

$F(x,y,z)$ is continuous if and only if P, Q, R are continuous

$$\boxed{\vec{F}(\vec{x}) = \vec{F}(x,y,z)} \qquad \vec{x} = \langle x,y,z\rangle \text{ position vector}$$

Ex. 1 $\vec{F}(x,y) = -y\vec{i} + x\vec{j}$ **(MMV)**

$$\boxed{\vec{F}(1,0) = \vec{j} \qquad \vec{F}(0,1) = -\vec{i}}$$



(x,y)	$\vec{F}(x,y)$	(x,y)	$\vec{F}(x,y)$
(1,0)	<0,1>	(-1,0)	<0,-1>
(2,2)	<-2,2>	(-2,-2)	<2,-2>
(3,0)	<0,3>	(-3,0)	<0,-3>
(0,1)	<-1,0>	(0,-1)	<1,0>
(-2,2)	<-2,-2>	(2,-2)	<2,2>
(0,3)	<-3,0>	(0,-3)	<3,0>

$\vec{x} = x\vec{i} + y\vec{j}$ $\qquad$ $\vec{F}(\vec{x}) = \vec{F}(x,y)$

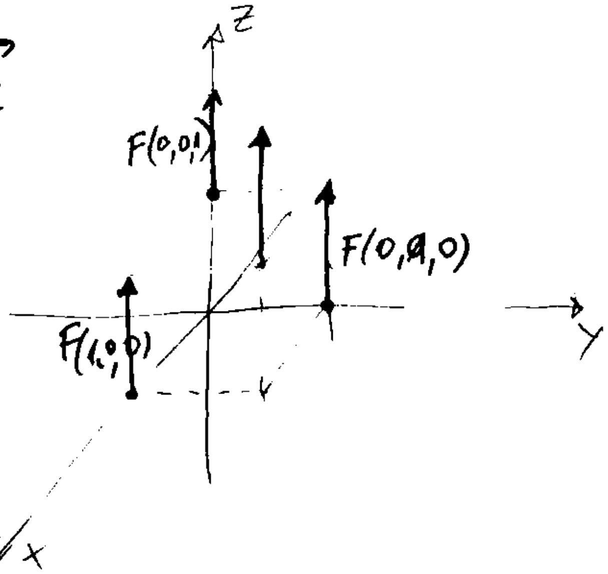
$$\vec{x}\cdot\vec{F}(\vec{x}) = (x\vec{i}+y\vec{j})\cdot(-y\vec{i}+x\vec{j}) = -xy+xy=0$$

$\vec{F}(x,y)$ IS PERPENDICULAR TO $\vec{x}=\langle x,y\rangle$ AND IS THEREFORE TANGENT TO CIRCLE WITH CENTER ~~AND~~ THE ORIGIN AND RADIUS:

$$|\vec{x}| = \sqrt{x^2+y^2}$$

Exp 2 $\quad F(x,y,z) = z\vec{k}$

(x,y,z)	$F(x,y,z)$
$(1,0,0)$	$\langle 0,0,1\rangle$
$(0,1,0)$	$\langle 0,0,1\rangle$
$(0,0,1)$	$\langle 0,0,1\rangle$
$(1,1,1)$	$\langle 0,0,1$



Exp. 4 NEWTON'S LAW OF GRAVITATION

$$\boxed{F = \frac{m\cdot M\cdot G}{r^2}}$$

$\vec{x} = \langle x,y,z\rangle \Rightarrow$ POSITION VECTOR OF OBJECT WITH MASS "m"

$r = |\vec{x}| \qquad r^2 = |\vec{x}|^2$

$$\boxed{\text{UNIT VECTOR} = \frac{\vec{x}}{|\vec{x}|}}$$

$$\vec{F}(\vec{x}) = -\frac{m\cdot M\cdot G}{|\vec{x}|^3}\cdot\vec{x}$$

$$\boxed{\vec{F}(\vec{r}) = -\frac{m\cdot M\cdot G}{r^3}\cdot\vec{r}} \Rightarrow \text{GRAVITATIONAL FIELD}$$

$$\vec{x} = x\vec{i}+y\vec{j}+z\vec{k}$$

$$\boxed{\vec{F}(\vec{x}) = -\frac{m\cdot M\cdot G}{(x^2+y^2+z^2)^{3/2}}(x\vec{i}+y\vec{j}+z\vec{k})}$$

Exp. 5 Coulomb's law (KULONOV ZAKON)

$$\boxed{\vec{F}(\vec{x}) = \frac{\varepsilon q Q}{|\vec{x}|^3}\vec{x}}$$

$qQ > 0$ like charges (ODBIJA SE)

$qQ < 0$ unlike charges (PRIVLAČNA)

- FORCE PER UNIT CHARGE

$$\boxed{\vec{E}(\vec{x}) = \frac{1}{q}\vec{F}(\vec{x}) = \frac{\varepsilon Q}{|x|^3}\vec{x}}$$ MMV

$\Rightarrow$ ELECTRIC FIELD

⊡ GRADIENT FIELDS

⊙ DIRECTIONAL DERIVATES AND GRADIENT VECTOR

$$f_x(x_0, y_0) = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h}$$

$$f_y(x_0, y_0) = \frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0,y_0)}{h}$$

$\to$ RATE OF change OF $z = f(x,y)$ IN THE DIRECTION OF THE UNIT VECTORS $\vec{i}$ AND $\vec{j}$

- RATE OF CHANGE IN THE DIRECTION OF ARBITRARY UNIT VECTOR: $\boxed{u = \langle a, b\rangle}$?

$14.3 + 3\cdot 1.3 = 14.3 + 3.9 = 18.2^\circ C$

- VIDI SLIKA Stewart 5e (pp. 967)

$\vec{PQ'} = h\vec{u} = \langle h\cdot a, h\cdot b\rangle$ $\quad x - x_0 = h\cdot a$, $\quad y - y_0 = h\cdot b$

$x = x_0 + ha \qquad y = y_0 + hb$

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}$$

• If $h \to 0$ WE OBTAIN THE RATE OF CHANGE OF „z" (WITH RESPECT TO DISTANCE) IN DIRECTION „$\vec{u}$", CALLED $\boxed{\text{DIRECTIONAL DERIVATE}}$ OF „f" IN DIRECTION „u".

$$\boxed{D_u f(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}}$$

$\boxed{u = \langle a, b\rangle}$ MMV

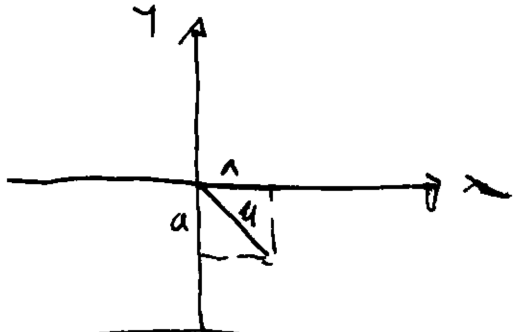
$\vec{u} = \vec{i} = \langle 1, 0\rangle \qquad D_i f = f_x = \frac{\partial f}{\partial x}$

$\vec{u} = \vec{j} = \langle 0, 1\rangle \qquad D_j f = f_y = \frac{\partial f}{\partial y}$

PARTIAL DERIVATES ARE SPECIAL CASE OF DIRECTIONAL DERIVATES.

Exp. 1 View MAP ON pp. 968 Stewart 5e

$u = (\vec{i} - \vec{j})/\sqrt{2} \qquad |u| = \sqrt{\frac{1}{2} + \frac{1}{2}} = \sqrt{1} = 1$



$a^2 + a^2 = 1 \qquad 2a^2 = 1$

$$\boxed{a = \frac{1}{\sqrt{2}}} \qquad u = \frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}$$

$D_u(T) = ?$ $\qquad D_u T \doteq \frac{60-50}{75} = \frac{10}{75}$

$$\boxed{D_u T = 0{,}1333 \ F^\circ/mi}$$

Theorem 3 $\quad u = \langle a, b\rangle \qquad D_u f(x,y) = f_x(x,y)\,a + f_y(x,y)\cdot b$

Proof: $\boxed{g(h) = f(x_0 + ha, y_0 + hb)}$

$$g'(0) = \lim_{h \to 0} \frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h} = \lim_{h\to 0} \frac{g(h)-g(0)}{h}$$

$= D_u f(x_0, y_0)$

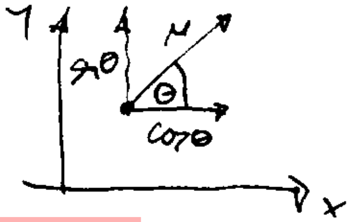
• ALTERN:

$g(h) = f(x,y) \qquad x = x_0 + ha \quad y = y_0 + hb$

$$g'(h) = \frac{\partial g}{\partial x}\frac{dx}{dh} + \frac{\partial g}{\partial y}\frac{dy}{dh} = f_x(x,y)\cdot a + f_y(x,y)\cdot b$$

$$\boxed{g'(0) = f_x(x_0,y_0)\,a + f_y(x_0,y_0)\cdot b}$$

$$\boxed{D_u f(x_0,y_0) = f_x(x_0,y_0)\,a + f_y(x_0,y_0)\cdot b}$$



$a = \cos\theta \qquad b = \sin\theta$

$$D_u f(x_0,y_0) = f_x(x_0,y_0)\cdot\cos\theta + f_y(x_0,y_0)\cdot\sin\theta$$

Exp. 2 DIRECTIONAL DERIVATE $D_u f(x,y)$ if

$f(x,y) = x^3 - 3xy + 4y^2 \qquad \vec{u} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j} \quad \theta = \frac{\pi}{6}$

$\vec{u} = \cos\frac{\pi}{6}\vec{i} + \sin\frac{\pi}{6}\vec{j} = \frac{\sqrt{3}}{2}\vec{i} + \frac{1}{2}\vec{j}$

$f_x(x,y) = \frac{\partial f(x,y)}{\partial x} = 3x^2 - 3y \qquad f_y(x,y) = \frac{\partial f(x,y)}{\partial y} = \underline{-3x + 8y}$

$$D_u f(x,y) = f_x(x,y)\cdot a^{\sqrt{3}/2} + f_y(x,y)\cdot b^{1/2} = \frac{\sqrt{3}}{2}(2x-3y) + \frac{1}{2}(-3x+8y)$$

$$D_u f(x,y) = \frac{\sqrt{3}}{2}(3x^2-3y) + \frac{1}{2}(-3x+8y) = \frac{3\sqrt{3}}{2}x^2 - \frac{3\sqrt{3}}{2}y - \frac{3}{2}x + 4y$$

$$\boxed{D_u f(x,y) = \left(\frac{3\sqrt{3}}{2}x - \frac{3}{2}\right)x + \left(4 - \frac{3\sqrt{3}}{2}\right)y}$$

$$D_u f(1,2) = \left(\frac{3\sqrt{3}}{2} - \frac{3}{2}\right) + \left(4 - \frac{3\sqrt{3}}{2}\right)\cdot 2 = \frac{3\sqrt{3}}{2} - \frac{3}{2} + 8 - 3\sqrt{3}$$

$$= \frac{3\sqrt{3} - 3 + 16 - 6\sqrt{3}}{2} = \frac{13 - 3\sqrt{3}}{2}$$

⊙ GRADIENT VECTOR

$$D_u f(x,y) = f_x(x,y)\cdot a + f_y(x,y)\, b = \langle f_x(x,y), f_y(x,y)\rangle \langle a, b\rangle$$

$$\boxed{D_u f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle \cdot \vec{u}}$$

$$\boxed{\begin{aligned} \operatorname{grad} f = \nabla f &= \langle f_x(x,y), f_y(x,y)\rangle \\ \nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle &= \frac{\partial f(x,y)}{\partial x}\vec{i} + \frac{\partial f(x,y)}{\partial y}\vec{j} \end{aligned}}$$

MMU

EXP 3 $\quad f(x,y) = \sin x + e^{xy}$

$$\nabla f(x,y) = \langle \cos x + y\cdot e^{yx},\ x e^{xy} \rangle$$

$$\nabla f(0,1) = \langle 2, 0 \rangle$$

DOT PRODUCT

$$\boxed{D_u f(x,y) = \nabla f(x,y) \odot \vec{u}}$$

EXPRESSION FOR DIRECTIONAL DERIVATIVE AT USING GRADIENT VECTOR NOTATION

EXP 2 MAKE PRESENTATION (Stewart Worksheet 7.wxw)

$$\vec{n}\circ(\vec{r} - \vec{r_0}) = 0 \qquad a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

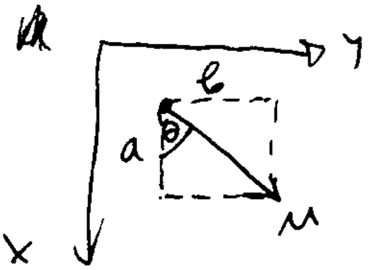
$$P(x_0, y_0, z_0) = (1,2,0) \qquad u = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}, 0 \right\rangle$$

$$Q(x_1, y_1, z_1) = \left\langle 1 + \frac{\sqrt{3}}{2}, 2 + \frac{1}{2}, 0 \right\rangle = \left\langle \frac{2+\sqrt{3}}{2}, \frac{5}{2}, 0 \right\rangle$$

$c = 0$

$a(x - x_0) + b(y - y_0) = 0 \qquad a(x-1) + b(y-2) = 0$

$a(x - x_1) + b(y - y_1) = 0 \qquad a\left(x - \frac{2+\sqrt{3}}{2}\right) + b\left(y - \frac{5}{2}\right) = 0$

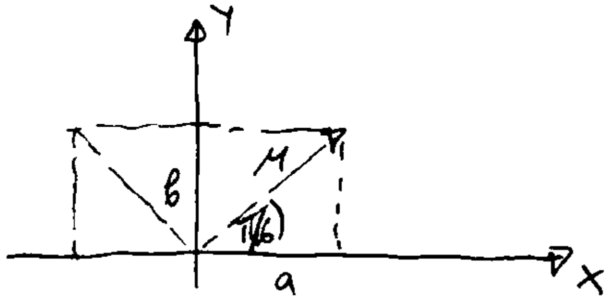


$\boxed{a = \cos\theta} \quad b = \sin\theta =$

$\frac{\sqrt{3}}{2}(x-1) + \frac{1}{2}(y-2) = 0$

$\frac{\sqrt{3}}{2}x - \frac{\sqrt{3}}{2} + \frac{1}{2}y - 1 = 0 \qquad \frac{\sqrt{3}}{2}x + \frac{1}{2}y = 1 + \frac{\sqrt{3}}{2}$

$\sqrt{3}x + y = 2 + \sqrt{3}$



$\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0$

$\vec{n} = \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$

$\boxed{\vec{n} = \langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \rangle}$

$\vec{n} \cdot (\vec{r} - \vec{r_0}) = \langle -\frac{\sqrt{3}}{2}, \frac{1}{2}, 0 \rangle \cdot \left(\langle x, y, z \rangle - \langle 1, 2, 0 \rangle \right)$

$-\frac{\sqrt{3}}{2}(x-1) + \frac{1}{2}(y-2) = 0$

PLANE

$-\sqrt{3}x + \sqrt{3} + y - 2 = 0 \qquad \boxed{-\sqrt{3}x + y = 2 - \sqrt{3}}$

---

exp 4 Directional derivate

$f(x,y) = x^2y^3 - 4y \quad$ at $(2,-1) \quad \boxed{v = 2\vec{i} + 5\vec{j}}$

$D_v f(x,y) = \nabla f(x,y) \cdot \vec{v}/|\vec{v}|$

$\nabla f(x,y) = \frac{\partial f(x,y)}{\partial x}\vec{i} + \frac{\partial f(x,y)}{\partial y}\vec{j} = (2xy^3)\vec{i} + (3x^2y^2 - 4)\vec{j}$

$D_v f(x,y) = \langle 2xy^3, 3x^2y^2 - 4 \rangle \langle 2, 5 \rangle \frac{1}{|\vec{v}|} = \cancel{\langle 4xy^3, 15x^2y^2 - 20 \rangle \frac{1}{|\vec{v}|}}$

$\nabla f(2,-1) = \langle -2\cdot 2, 3\cdot 4\cdot 1 - 4 \rangle = \langle -4, 8 \rangle$

$|\vec{v}| = \sqrt{4+25} = \sqrt{29}$

$D_u f(2,-1) = \langle -4, 8\rangle \cdot \langle 2, 5\rangle \frac{1}{\sqrt{29}} = \cancel{\frac{-8}{\sqrt{29}}} \cancel{\frac{40}{\sqrt{29}}}$

$\frac{1}{\sqrt{29}}(-8+40) = \frac{32}{\sqrt{29}}$ //

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FUNCTIONS OF THREE VARIABLES

$D_u f(x,y,z)$

- DIRECTIONAL DERIVATIVE of f AT (x_0,y_0,z_0) in the direction OF UNIT VECTOR $u = \langle a, b, c\rangle$

$$D_u f(x_0,y_0,z_0) = \lim_{h\to 0} \frac{f(x_0+h, y_0+h, z_0+h) - f(x_0,y_0,z_0)}{h}$$

VECTOR NOTATION

$$\boxed{D_u f(\vec{x_0}) = \lim_{h\to 0} \frac{f(\vec{x_0} + h\vec{u}) - f(\vec{x_0})}{h}}$$

$$D_u f(x,y,z) = f_x(x,y,z)a + f_y(x,y,z)b + f_z(x,y,z)\cdot c$$

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle$$

$$\nabla f(x,y,z) = \frac{\partial f(x,y,z)}{\partial x}\vec{i} + \frac{\partial f(x,y,z)}{\partial y}\vec{j} + \frac{\partial f(x,y,z)}{\partial z}\vec{k}$$

$$\boxed{D_u f(x,y,z) = \nabla f(x,y,z)\cdot\vec{u}}$$

$\boxed{\text{Exp. 5}}$ $f(x,y,z) = x\sin(yz)$

(a) $\nabla f = ?$

(b) $D_u f(x,y,z)$ $(1,3,0)$ $v = i + 2j - k$

$|v| = \sqrt{1+4+1} = \sqrt{6}$ $u = \frac{v}{|v|} = \frac{1}{\sqrt{6}}(i+2j-k)$

$$\boxed{\nabla f(x,y,z) = \sin(y,z)\vec{i} + z\cdot x\cdot\cos(yz)\vec{j} + xy\cos(yz)\vec{k}}$$

$\nabla f(x,y,z) = \sin(yz)\,\vec{i} + zx\cos(y,z)\,\vec{j} + xy\cos(yz)\,\vec{k}$

$\nabla f(1,3,0) = 3\vec{k}$

$D_u f(x,y,z) = \nabla f(1,3,0)\cdot\vec{u} = 3\vec{k}\cdot\left(\vec{i} + 2\vec{j} - \vec{k}\right)\frac{1}{\sqrt{6}} =$

$= \frac{3}{\sqrt{6}} = \frac{\cancel{3}\sqrt{3}}{\sqrt{2}\cancel{\sqrt{3}}} = \frac{\sqrt{3}}{\sqrt{2}}$

- MAXIMIZING THE DIRECTIONAL DERIVATE
- IN WHICH DIRECTION DOES f changes fastest and what is the maximum rate of change

$\boxed{\text{Theorem 15}}$ MAXIMUM VALUE of DIRECTIONAL DERIVATE $D_u f(\vec{x})$ is $|\nabla f(x)|$ AND OCCURS WHEN $\vec{u}$ has same direction as the gradient VECTOR $\nabla f(x)$.

PROOF:

$$\boxed{D_u f = \nabla f\cdot\vec{u} = |\nabla f||\vec{u}|\cos\theta = |\nabla f|\cdot|\vec{u}|\cdot\cos\theta}$$ MMV

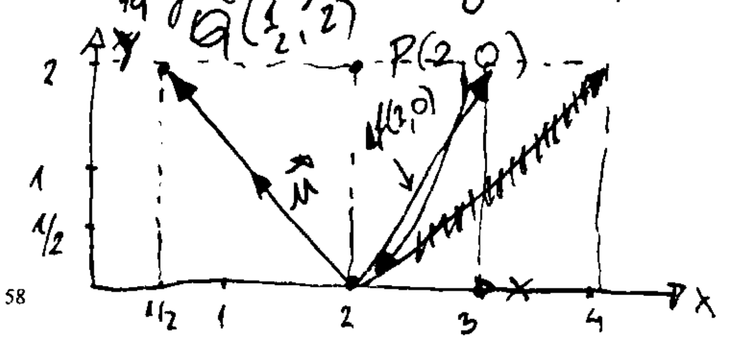
$\theta = 0 \Rightarrow (\cos\theta = 1) \to$ MAXIMUM VALUE $|\nabla f|$

$\hookrightarrow \vec{u}$ has same direction as ∇f

$\boxed{\text{Exp 6}}$ $f(x,y) = xe^{y}$ (a) RATE OF CHANGE OF $f(x,y)$ AT $P(2,0)$ IN DIRECTION $\vec{PQ}$ $\boxed{Q(\frac{1}{2},2)}$

(b) In which direction f have maximum rate of change. What is maximum rate of change?

(a) $D_u f(x,y) = \nabla f(x,y)\cdot\vec{u}$



$\vec{v} = \langle \frac{1}{2}, 2\rangle - \langle 2, 0\rangle$

$\vec{v} = \langle -\frac{3}{2}, 2\rangle = -\frac{3}{2}\vec{i} + 2\vec{j}$

$|\vec{v}| = \sqrt{\frac{9}{4} + 4} = \sqrt{\frac{25}{4}} = \frac{5}{2}$

$|\vec{v}| = 2.5$

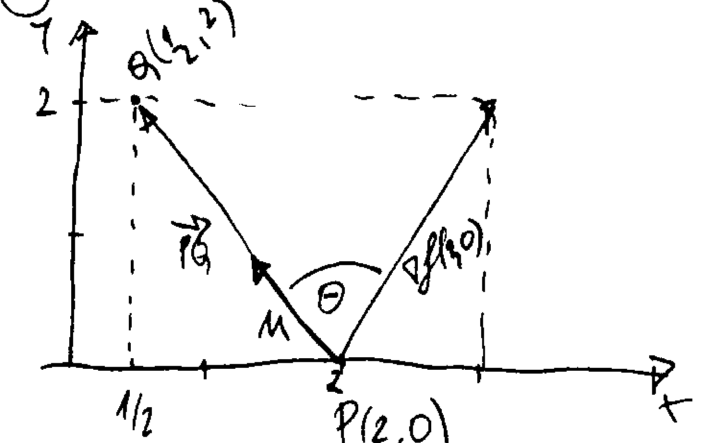
$$\boxed{\vec{u} = \frac{2}{5}\left(-\frac{3}{2}\vec{i} + 2\vec{j}\right) = -\frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}}$$ MMV

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} = e^{y}\vec{i} + x e^{y}\vec{j}$$

$$D_u = \nabla f(2,0)\cdot\vec{u} \; ; \; \nabla f(2,0) = \vec{i} + 2\vec{j} = \langle 1, 2 \rangle$$

$$D_u f(2,0) = \left(\vec{i} + 2\vec{j}\right)\cdot\left(-\frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}\right) = -\frac{3}{5} + \frac{8}{5} = \frac{5}{5} = 1$$

(b) MAXIMUM RATE OF CHANGE



$$\left|\nabla f(x,y)\right| = \left|\nabla f(2,0)\right| =$$

$$= \sqrt{1+4} = \sqrt{5}$$

Exp. 7 TEMPERATURE AT (x,y,z) IN SPACE IS GIVEN BY:

$$T(x,y,z) = 80/(1+x^2+2y^2+3z^2) \qquad T\,(=)\,{}^{\circ}C$$
$$x,y,z\,(=)\,m$$

$P(1,1,-2)$ IN WHICH DIRECTION AT THIS POINT THE TEMPERATURE IS GROWING FASTEST? MAXIMUM RATE OF INCREASE?

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

$$\frac{\partial f}{\partial x} = 80(-1)\frac{2x}{(1+x^2+2y^2+3z^2)^2} = -\frac{160x}{(1+x^2+2y^2+3z^2)^2};$$

$$\frac{\partial f}{\partial y} = -\frac{320y}{(1+x^2+2y^2+3z^2)^2}; \quad \frac{\partial f}{\partial z} = -\frac{480z}{(1+x^2+2y^2+3z^2)^2}$$

$$\nabla f(1,1,-2) = -\frac{160}{(1+1+2+12)^2}\vec{i} - \frac{320}{(16)^2}\vec{j} + \frac{960}{256}\vec{k}$$

$$\nabla f(1,1,-2) = -\frac{160\vec{i}}{256} - \frac{320\vec{j}}{256} + \frac{960\vec{k}}{256} = \frac{160}{256}\left(-\vec{i} - 2\vec{j} + 6\vec{k}\right)$$

$$\left|\nabla f(1,1,-2)\right| = \frac{160}{256}\sqrt{41} = \frac{5}{8}\sqrt{41}$$ MMV $$\boxed{\vec{u} = \frac{-\vec{i} - 2\vec{j} + 6\vec{k}}{\sqrt{41}}}$$

MAXIMUM RATE OF INCREASE IS $5\sqrt{41}/8$ °C/m

TANGENT PLANES TO LEVEL SURFACES

ОПРИА ОСНОВЕН ОЦД

$S: F(x,y,z)=k \Rightarrow$ SURFACE $\quad P(x_0,y_0,z_0)$ LIES ON "S"

C – CURVE LIEING ON "S" PASSING THROUGH (P)

$C: \vec{r}(t)=\langle x(t), y(t), z(t)\rangle \quad t_0 \Rightarrow x_0, y_0, z_0$

$r(t_0)=\langle x_0, y_0, z_0\rangle$

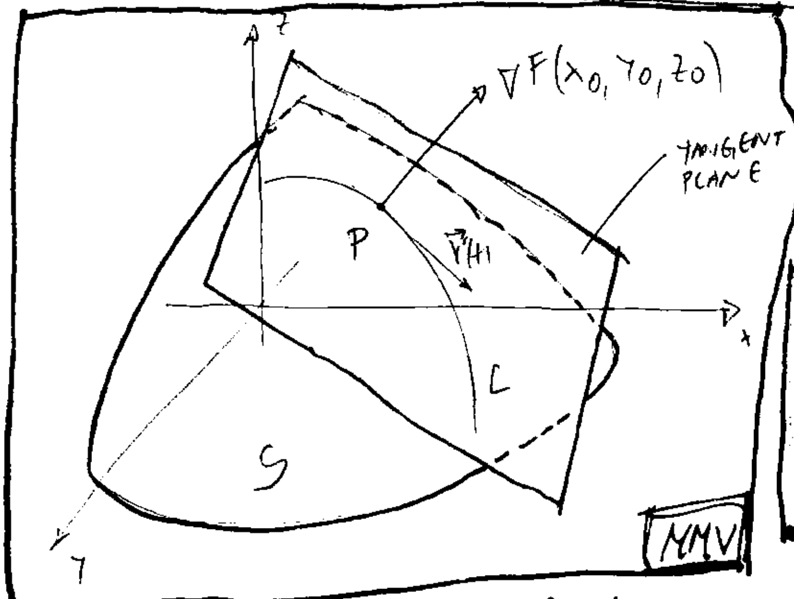
$$F(x(t), y(t), z(t))=k$$

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial F}{\partial y}\frac{\partial y}{\partial t}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial t}=0$$

$$\nabla F=\langle F_x, F_y, F_z\rangle=\frac{\partial F}{\partial x}\vec{i}+\frac{\partial F}{\partial y}\vec{j}+\frac{\partial F}{\partial z}\vec{k}$$

$$\nabla\vec{F}\cdot\vec{r}'=0 \qquad \vec{r}'=\langle x'(t), y'(t), z'(t)\rangle \quad x'(t)=\frac{dx}{dt}$$

$$t=t_0 \Rightarrow \boxed{\vec{r}(t_0)=\langle x_0, y_0, z_0\rangle}$$



$$\vec{n}\cdot(\vec{r}-\vec{r}_0)=0$$

VECTOR EQUATION OF PLANE

$$\boxed{F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0}$$

(*) TANGENT PLANE

• VECTOR EQUATION OF LINE: $\vec{r}_0+t\cdot\vec{v}=0$

$$(x_0+at)\vec{i}+(y_0+bt)\vec{j}+(z_0+ct)\vec{k}=0$$

$x=x_0+at \qquad y=y_0+bt. \qquad z=z_0+ct$

$t=\frac{x-x_0}{a} \qquad t=\frac{y-y_0}{b} \qquad t=\frac{z-z_0}{c}$

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$$

NORMAL LINE

$$\boxed{\frac{x-x_0}{F_x(x_0,y_0,z_0)}=\frac{y-y_0}{F_y(x_0,y_0,z_0)}=\frac{z-z_0}{F_z(x_0,y_0,z_0)}}$$

• SPECIAL CASE: $F(x,y,z)=f(x,y)-z=0$

$F_x(x_0,y_0,z_0)=f_x(x_0,y_0)$

$F_y(x_0,y_0,z_0)=f_y(x_0,y_0)$

$F_z(x_0,y_0,z_0)=-1$

$\Rightarrow \; f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0) - (z-z_0)=0$

$$\boxed{(z-z_0) = f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)} \quad \text{TANGENT PLANE}$$

Exp. 8 FIND EQUATIONS OF TANGENT PLANE AND NORMAL LINE FOR ELLIPSOID:

$$\overbrace{\frac{x^2}{4}+y^2+\frac{z^2}{9}}^{F(x,y,z)}=3 \;, \quad \text{AT} \quad P(-2,1,-3)$$

- TANGENT PLANE:

$$\left.\frac{\partial F(x,y,z)}{\partial x}\right|_{x_0,y_0,z_0}=\frac{x_0}{2} \qquad \left.\frac{\partial F}{\partial y}\right|_{x_0,y_0,z_0}=2y_0 \qquad \left.\frac{\partial F}{\partial z}\right|_{x_0,y_0,z_0}=\frac{2}{9}z_0 \qquad -\frac{6}{9}=-\frac{2}{3}$$

$$(x+2)\cdot\frac{x_0}{2}+(y-\cancel{y_0}^{\,1})2y_0+(z-\cancel{z_0}^{\,-3})\frac{2}{9}z_0=0$$

$$-(x+2)+2(y-1) - \frac{6}{9}(z+3)=0$$

$$\boxed{(x+2) - 2(y-1)+\frac{2}{3}(z+3)=0}$$

PLANE

$x+2-2y+2+\frac{2}{3}z+2=0$

$x-2y+\frac{2}{3}z+6=0$

$3x-6y+2z+18=0$

- NORMAL LINE

$$\boxed{\frac{x+2}{-1}=\frac{y-1}{2}=\frac{z+3}{-2/3}}$$

▫ SIGNIFICANCE OF GRADIENT VECTOR

Exc. 4 $f(x,y)=x^2y^3-y^4 \quad P(2,1) \quad \theta=\pi/4$

$D_u f(x,y)=?\qquad D_u f(x,y)=\nabla f(x,y)\cdot\vec{u}$

$D_u f=|\nabla f(x,y)|\cdot|\vec{u}|\cdot\cos\theta=|\nabla f(x,y)|\cdot\cos\left(\frac{\pi}{4}\right) \quad \frac{\sqrt{2}}{2}$

$\nabla f(x,y)=2xy^3\vec{i} + (3x^2y^2-4y^3)\vec{j} \qquad (3\cdot4\cdot1-4)^2=8^2=64$

$\sqrt{16\cdot2}=4\sqrt{2}$

$\left.|\nabla f|\right|_{x_0y_0}=\sqrt{4x^2y^6+(3x^2y^2-4y^3)^2}=\sqrt{4\cdot4\cdot1+64}=\sqrt{80}$

$\sqrt{80}=4\cdot\sqrt{5}$

$D_u f=\cancel{4}^{\,2}\sqrt{5}\cdot\frac{\sqrt{2}}{\cancel{2}}=2\sqrt{5}$ // (?)

$D_u f(x_0, y_0) = f_x(x_0, y_0)\cdot\cos\theta + f_y(x_0, y_0)\cdot\sin\theta$

$= 2x_0 y_0^3 \cos\theta + (\underbrace{3x_0^2 y_0^2}_{12} - \underbrace{4y_0^3}_{4})\sin\theta = \frac{\sqrt{2}}{2}(2\cdot 2 + 8) = 6\sqrt{2}$

$f(x,y) = x^2y^3 - y^4 \qquad P(2,1)$

$f_x(x) = 2xy^3 \qquad f_y(x) = 3x^2y^2 - 4y^3$

$f_x(x) = 4 \qquad f_y(x) = 3\cdot 4\cdot 1 - 4\cdot 1 = 12 - 4 = 8$

$|\nabla f(x,y)| = \sqrt{4^2 + 8^2} = \sqrt{80}$

$\boxed{\text{Ex.8}}$ $f(x,y) = y\ln x \quad P(1,-3) \quad \vec{u} = \langle -\frac{4}{5}, \frac{3}{5}\rangle$

$f_x(x,y) = \left(\frac{y}{x}\right) \qquad f_y(x,y) = \ln(x)$

$f_x(1,-3) = \frac{-3}{1} = -3 \qquad f_y(1,-3) = \ln(1) = 0$

$$\boxed{D_u f(x,y) = \left(f_x(x,y)\,\vec{i} + f_y(x,y)\,\vec{j}\right)\cdot\vec{u}}$$

$D_u f(x,y) = (-3\vec{i} + 0\vec{j})\langle -\frac{4}{5}, \frac{3}{5}\rangle = +\frac{12}{5}$

$\boxed{\text{Ex.23}}$ MAX RATE OF CHANGE

$P(1,0)$

$f(x,y) = \sin(xy) \qquad f_x = y\cos(xy) = 0 \qquad f_y = x\cdot\cos(xy) = 1$

$\nabla f = \sqrt{f_x^2 + f_y^2} = \sqrt{1} = 1 \qquad \nabla f = \langle 0, 1\rangle$

$\boxed{\text{Ex.33}}$ $V(x,y,z) = 5x^2 - 3xy + xyz$

(a) $D_u V(x,y,z) \qquad v = \vec{i} + \vec{j} - \vec{k} \qquad \boxed{P(3,4,5)}$

$\nabla V(x,y,z) = (10x - 3y + yz)\vec{i} + (-3x + xz)\vec{j} + xy\vec{k}$

$\nabla V(3,4,5) = (\underbrace{30 - 12 + 20}_{38})\vec{i} + (\underbrace{-9 + 15}_{6})\vec{j} + 12\vec{k}$

$= 38\vec{i} + 6\vec{j} + 12\vec{k}$

$$D_u V(x,y,z) = \langle 38, 6, 12 \rangle \cdot \frac{\langle 1, 1, -1 \rangle}{|v|} = (38 + 6 + -12)\frac{1}{\sqrt{3}} = \frac{32}{\sqrt{3}}$$

$$|v| = \sqrt{1+1+1} = \sqrt{3}$$

$$|\nabla V(x,y,z)| = \sqrt{38^2 + 6^2 + 12^2} = \sqrt{1444 + 36 + 144} = 2\sqrt{406}$$

Ex 39 $x^2 + 2y^2 + 3z^2 = 21$ $P(4, -1, 1)$

$$\frac{\partial F}{\partial x}(x - x_0) + \frac{\partial F}{\partial y}(y - y_0) + \frac{\partial F}{\partial z}(z - z_0) = 0$$

$$z = \pm\sqrt{7 - \tfrac{1}{3}x^2 - \tfrac{2}{3}y^2}$$

$F_x = 2x$ $F_y = 4y$ $F_z = 6z$

$2 \cdot 4 = 8$ $= -4$ $= 6$

$$8(x-4) - 4(y+1) + 6(z-1) = 0$$

$$8x - \underline{\underline{32}} - 4y - \underline{\underline{4}} + 6z - \underline{\underline{6}} = 0 \qquad 8x - 4y + 6z - 42 = 0$$

$$\boxed{4x - 2y + 3z - 21 = 0}$$ TANGENT PLANE

• NORMAL LINE

$$\frac{x-4}{8} = \frac{y+1}{-4} = \frac{z-1}{6}$$

$$\boxed{x = 4 + 8t \qquad y = -4t - 1 \qquad z = 1 + 6t}$$ PARAMETRIC EQUATIONS of line

∇_0 MAPAE SE CRTA SO SPACECURVE

$$\vec{r} = \vec{r_0} + \vec{v} \cdot t = \langle 4, -1, 1 \rangle + \underbrace{\langle 8, -1, 1 \rangle}_{\vec{v}} t$$

Ex. 41 $z + 1 = x e^y \cos z$ $x e^y \cos z - z = +1$ $(1, 0, 0)$

$F_x = e^y \cos z$ $F_y = x \cos z \, e^y$ $F_z = -x e^y \sin z - 1$

$F_{x_0} = 1$ $F_{y_0} = 1 \cdot 1 \cdot 1 = 1$ $F_z = -1 \cdot e^0 \cdot \sin 0 - 1 = -1$

$$\nabla F = \langle 1, 1, -1 \rangle = \vec{i} + \vec{j} - \vec{k}$$

$(x - 1) + (y - 0) - z = 0$ $x + y - z = 1$ } TANGENT PLANE

$\frac{x-1}{1} = \frac{y}{1} = \frac{z}{-1}$ $x = 1 + t$ $y = t$ $z = -t$ } NORMAL LINE

Exc. 45 $xy + yz + zx = 3$ $P(1,1,1)$

$\nabla F = F_x \vec{i} + F_y \vec{j} + F_z \vec{k} = (y+z)\vec{i} + (x+z)\vec{j} + (x+y)\vec{k}$

• TANGENT PLANE

$(x-1) F_x\big|_{x_0,y_0,z_0} + (y-1) F_{y_0} + (z-1) F_{z_0} = 0$

$2(x-1) + 2(y-1) + 2(z-1) = 0$ $\underline{x + y + z - 3 = 0}$

$f(x,y) = z$? $z(y+x) = 3 - xy$ $\boxed{z = \frac{3-xy}{x+y}}$

• NORMAL LINE

$\cancel{x}-1 = y-1 = z-1$ $x = 1+t$ $y = 1+t$ $z = 1+t$

Exc. 47 $f(x,y) = x^2 + 4y^2$ $\nabla f(x,y) = ?$

TANGENT LINE AT $(2,1)$ OF LEVEL CURVE $f(x,y) = 8$

$\frac{\partial f}{\partial x} = f_x = 2x$ $\frac{\partial f}{\partial y} = f_y = 8y$

$\nabla f(x,y) = 2x\vec{i} + 8y\vec{j}$

$\nabla f(2,1) = 4\vec{i} + 8\vec{j}$

$D_u f = \nabla f \cdot \vec{u} =$ ~~xxxx~~ $D_x f = \nabla f \cdot \vec{i} = (4\vec{i} + 8\vec{j}) \cdot \vec{i}$

~~xxxx~~ $\frac{x - x_0}{f_x} = \frac{y - y_0}{f_y} \Rightarrow \frac{x-2}{4} = \frac{y-1}{8}$

$\frac{x-2}{\cancel{4}} = \frac{y-1}{2}$ $2x - 4 = y - 1$ $\boxed{y = 2x - 3}$

$\boxed{2x - y = 3}$

• $r = r_0 + \vec{v}t$

$\vec{v} \times \nabla f = \begin{vmatrix} i & j & k \\ a & b & 0 \\ 4 & 8 & 0 \end{vmatrix} = (4b - 8a) e_z = (4b-8a)\vec{k} = 0$

$4b = 8a$ $\boxed{\frac{b}{a} = 2}$

$\frac{x - x_0}{a} = \frac{y - y_0}{b}$ $y - y_0 = \frac{b}{a}(x - x_0)$

$y - 1 = 2(x-2)$ $y - 1 = 2x - 4$ $y = 2x - 3$

$\vec{v} \cdot \nabla f = 0$ $\langle a, b \rangle \cdot \langle 4, 8 \rangle = 0$

$4a + 8b = 0$ $\frac{b}{a} = \frac{4}{-8} = -\frac{1}{2}$

$\frac{x-x_0}{a} = \frac{y-y_0}{b}$ $\qquad y-y_0 = \frac{b}{a}(x-x_0)$

$y-1 = -\frac{1}{2}(x-2)$ $\qquad y-1 = -\frac{1}{2}x+1$ $\qquad \boxed{y=-\frac{1}{2}x+2}$

$f_x\cdot(x-x_0)+f_y(y-y_0)=0$ $\Rightarrow$ FROM TANGENT PLANE EQUATION

$4(x-2)+8(y-y_0)=0$ $\qquad x-2+2y-2=0$

$2y=-x+4$ $\qquad \boxed{y=-\frac{1}{2}x+2}$

ALTERNATIVE APPROACH

---

49 $\qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$\nabla F = \langle \frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \rangle$ $\qquad \vec{\nabla} F\cdot(\vec{r}-\vec{r}_0)=0$

$\langle \frac{2x_0}{a^2}, \frac{2y_0}{b^2}, \frac{2z_0}{c^2} \rangle \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0$

$\frac{2x_0}{a^2}(x-x_0)+\frac{2y_0}{b^2}(y-y_0)+\frac{2z_0}{c^2}(z-z_0)=0$

$\frac{2x_0x}{a^2}-\frac{2x_0^2}{a^2}+\frac{2y_0y}{b^2}-\frac{2y_0^2}{b^2}+\frac{2zz_0}{c^2}-\frac{2z_0^2}{c^2}=0$

$\frac{2x_0x}{a^2}+\frac{2y_0y}{b^2}+\frac{2zz_0}{c^2}=2\underbrace{\left(\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2}\right)}_{1}$

$$\boxed{\frac{\cancel{2}xx_0}{a^2}+\frac{\cancel{2}yy_0}{b^2}+\frac{\cancel{2}zz_0}{c^2}=\cancel{2}\ 1}$$

Ex. 52 $\qquad x^2+2y^2+3z^2=1$ $\qquad$ POINTS = ?

TANGENT PLANE PARALLEL TO: $3x-y+3z=1$

$\nabla F(x,y,z)=2x\vec{i}+4y\vec{j}+6z\vec{k}$ $\qquad \nabla F(x_0,y_0,z_0)=\langle 3,-1,3\rangle$

• TANGENT PLANE

$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

$a=2x=3$ $\qquad b=4y=-1$ $\qquad c=6z=3$

$2x_0=3$ $\qquad \boxed{x_0=\frac{3}{2}}$ $\qquad 4y_0=-1$ $\qquad \boxed{y_0=-1/4}$ $\qquad 6z_0=3$ $\qquad \boxed{z_0=\frac{1}{2}}$

$3(x-\frac{3}{2})-(y+\frac{1}{4})+3(z-\frac{1}{2})=0$ $\qquad \frac{9}{2}+\frac{1}{4}+\frac{3}{2}=\frac{18+1+6}{4}=\frac{25}{4}$

$3x-9/2-y-\frac{1}{4}+3z-\frac{3}{2}=0$ $\qquad 3x-y+3z=\frac{25}{4}$

$12x-4y+12z=25$

$\nabla F(x_0, y_0, z_0) \cdot \langle 3, -1, 3 \rangle = 0$

$\langle \underset{a}{2x_0}, \underset{b}{4y_0}, \underset{c}{6z_0} \rangle \cdot \langle 3, -1, 3 \rangle = 0$

$6x_0 - 4y_0 + 18z_0 = 0$

$\boxed{x_0^2 + 2y_0^2 + 3z_0^2 = 1}$

NE BIVA VAKA ZOSTO $\langle 3, -1, 3 \rangle \in \parallel \langle a, b, c \rangle$ A NE $\perp$!!!

$\langle 2x_0, 4y_0, 6z_0 \rangle \times \langle 3, -1, 3 \rangle =$

$2x_0(x - x_0) + 4y_0(y - y_0) + 6z_0(z - z_0) = 0$

$2x_0x - 2x_0^2 + 4y_0y - 4y_0^2 + 6z_0z - 6z_0^2 = 0$

$\underbrace{2x_0}_{a}x + \underbrace{4y_0}_{b}y + \underbrace{6z_0}_{c}z = \underbrace{2x_0^2 + 4y_0^2 + 6z_0^2}_{2}$?

$2x_0x + 4y_0y + 6z_0z = 2$

$f = \langle 3, -1, 3 \rangle$ $\boxed{f \times t = 0}$

$t = \langle a, b, c \rangle$

$\boxed{f \times t = (-c - 3b)\vec{i} + (3a - 3c)\vec{j} + (3b + a)\vec{k} = 0}$

$-c - 3b = 0$ $3a - 3c = 0$ $3b + a = 0$

$\boxed{c = -3b}$ $3a - 3(-3b) = 0$ $\Downarrow$ $\boxed{a = -3b}$

$3a + 9b = 0$

$-9b + 9b = 0$

$\boxed{x_0 = \frac{a}{2} = \frac{-3b}{2}; \quad y_0 = \frac{b}{4} \quad z_0 = \frac{c}{6} = -\frac{3b}{6} = -\frac{b}{2}}$

$\left(\frac{3b}{2}\right)^2 + 2\left(\frac{b}{4}\right)^2 + 3\left(\frac{-b}{2}\right)^2 = 1$ $\frac{9b^2}{4} + \cancel{2}\frac{b^2}{\cancel{16}\,8} + 3\frac{b^2}{4} = 1$

$18b^2 + b^2 + 6b^2 = 8$ $25b^2 = 8$ $\boxed{b = \frac{\pm\sqrt{8}}{5} = \frac{\pm 2\sqrt{2}}{5}}$

① $\boxed{c = -\frac{6\sqrt{2}}{5} \quad a = -\frac{6\sqrt{2}}{5}}$ $\boxed{x_0 = -\frac{3\sqrt{2}}{5};\ y_0 = \frac{\sqrt{2}}{10};\ z_0 = -\frac{\sqrt{2}}{5}}$

$-\frac{6\sqrt{2}}{5}\left(x + \frac{3\sqrt{2}}{5}\right) + \frac{2\sqrt{2}}{5}\left(y - \frac{\sqrt{2}}{10}\right) - \frac{6\sqrt{2}}{5}\left(z + \frac{\sqrt{2}}{5}\right) = 0$

$\Rightarrow -3\left(x + \frac{3\sqrt{2}}{5}\right) + \left(y - \frac{\sqrt{2}}{10}\right) - 3\left(z + \frac{\sqrt{2}}{5}\right) = 0$

$$-3x - \frac{9\sqrt{2}}{5} + y - \frac{\sqrt{2}}{10} - 3z - \frac{3\sqrt{2}}{5} = 0$$

$$-3x + y - 3z = \frac{12\sqrt{2}}{5} + \frac{\sqrt{2}}{10} = \frac{24\sqrt{2}+\sqrt{2}}{10} = \frac{25\sqrt{2}}{10} = \frac{5\sqrt{2}}{2}$$

$$\boxed{-3x + y - 3z = \frac{5\sqrt{2}}{2}} \quad P2$$

(2) $b = -\frac{2\sqrt{2}}{5} \quad a = -3b = \frac{6\sqrt{2}}{5} \quad c = -3b = \frac{6\sqrt{2}}{5}$

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$\boxed{x_0 = \frac{a}{2} = \frac{3\sqrt{2}}{5} \quad y_0 = \frac{b}{4} = -\frac{\sqrt{2}}{10} \quad z_0 = \frac{c}{4} = -\frac{b}{2} = \frac{\sqrt{2}}{5}}$$

$$P3: \quad \frac{6\sqrt{2}}{5}\left(x - \frac{3\sqrt{2}}{5}\right) - \frac{2\sqrt{2}}{5}\left(y + \frac{\sqrt{2}}{10}\right) + \frac{6\sqrt{2}}{5}\left(z - \frac{\sqrt{2}}{5}\right) = 0$$

$$3\left(x - \frac{3\sqrt{2}}{5}\right) - \left(y + \frac{\sqrt{2}}{10}\right) + 3\left(z - \frac{\sqrt{2}}{5}\right) = 0$$

$$3x - 9\sqrt{2}/5 - y - \sqrt{2}/10 + 3z - 3\sqrt{2}/5 = 0$$

$$3x - y + 3z = \frac{12\sqrt{2}}{5} + \frac{\sqrt{2}}{10} = \frac{25\sqrt{2}}{10} = \frac{5\sqrt{2}}{2}$$

$$P3: \quad \boxed{3x - y + 3z = \frac{5\sqrt{2}}{2}}$$

$$3a + b + 3c = 0 \qquad 3 \cdot \frac{6\sqrt{2}}{5} + \frac{2\sqrt{2}}{5} + 3\frac{6\sqrt{2}}{5} = 0$$

$$3 \cdot 3 + 1 + 3 \cdot 3 = 0 \qquad 19 = 0 \;?$$

DOTPRODUCT ?
NE MOŽE
$\langle 3, -1, 3 \rangle \parallel \langle a, b, c \rangle$

• ALTERNATIVEN PRISTUP: (Stewart Solutions)

$$\langle 2x_0, 4y_0, 6z_0 \rangle = \langle x_0, 2y_0, 3z_0 \rangle = c\langle 3, -1, 3 \rangle$$

$$x_0 = 3c \qquad y_0 = -\frac{c}{2} \qquad z_0 = c$$

$$x_0^2 + 2y_0^2 + 3z_0^2 = 1 \qquad 9c^2 + 2\frac{c^2}{4} + 3c^2 = 1$$

$$36c^2 + 2c^2 + 12c^2 = 4 \qquad 50c^2 = 4 \qquad c = \frac{4}{50} = \pm\frac{\sqrt{2}}{5}$$

$$\boxed{x_0 = \pm\frac{3\sqrt{2}}{5} \quad y_0 = \mp\frac{\sqrt{2}}{10} \quad z_0 = \pm\frac{\sqrt{2}}{5}}$$

57 SUM OF x-, y-, z-intercepts OF TANGENT LINE IS $\emptyset$!

$$\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{c} \qquad F_{x_0} = \frac{1}{2\sqrt{x_0}} \quad F_{y_0} = \frac{1}{2\sqrt{y_0}} \quad F_{z_0} = \frac{1}{2\sqrt{z_0}}$$

$$\boxed{F_{x_0} = F_x\big|_{x=x_0}}$$

TANGENT PLANE $\quad \frac{x-x_0}{2\sqrt{x_0}} + \frac{y-y_0}{2\sqrt{y_0}} + \frac{z-z_0}{2\sqrt{z_0}} = 0$

$$\frac{x}{\sqrt{x_0}}+\frac{y}{\sqrt{y_0}}+\frac{z}{\sqrt{z_0}}=\frac{x_0}{\sqrt{x_0}}+\frac{y_0}{\sqrt{y_0}}+\frac{z_0}{\sqrt{z_0}}=\sqrt{x_0}+\sqrt{z_0}+\sqrt{z_0}=\sqrt{c}$$

$$\frac{x}{\sqrt{x_0}}+\frac{y}{\sqrt{y_0}}+\frac{z}{\sqrt{z_0}}=\sqrt{c}$$

X - INTERCEPT ($y=z=0$)	Y - INTERCEPT	Z - INTERCEPT
$x=\sqrt{x_0\cdot c}$	$y=\sqrt{y_0 c}$	$z=\sqrt{z_0 c}$

$$x+y+z=\sqrt{c}\left(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}\right)=\sqrt{c}\cdot\sqrt{c}=c$$ } SUM OF THE INTERCEPTS.

[59] PARAMETRIC EQUATIONS OF TANGENT LINE AT $P(-1,1,2)$

$$\left.\begin{aligned} z&=x^2+y^2\\ 4x^2+y^2+z^2&=9\end{aligned}\right\}$$ INTERSECTION CURVE

$$z^2=x^4+2x^2y^2+y^4$$

$$4x^2+y^2+x^4+2x^2y^2+y^4=9 \qquad f(x,y,z)=4x^2+y^2+x^4+2x^2y^2+y^4$$

$$f(x,y)=(4+x^2)x^2+y^2(1+2x^2+y^2)$$

$$f_x=\frac{\partial f(x,y)}{\partial x}=8x+4x^3+4xy^2 \qquad f_y=2y+4yx^2+4y^3$$

$$f_{x_0}(-1,1)=-8-4-4=\underline{\underline{-16}} \qquad f_{y_0}(-1,1)=2+4+4=\underline{\underline{10}}$$

TANGENT LINE = ?

$$f_{x_0}(x-x_0)+f_{y_0}(y-y_0)=0 \qquad -16(x+1)+10(y-1)=0$$

$$-16x-16+10y-10=0 \qquad 10y=16x+26 \qquad \boxed{y=1.6x+2.6}$$

$$\frac{x-x_0}{1/f_{x_0}}=\frac{y-y_0}{-1/f_{y_0}} \qquad x=x_0+\frac{1}{f_{x_0}}t \qquad y=y_0-\frac{1}{f_{y_0}}t$$

$$\boxed{x=-1+\frac{1}{16}t \qquad y=1-\frac{1}{10}t} \qquad \boxed{z=2}$$

$$\frac{x-x_0}{1/16}=\frac{y-y_0}{-1/10} \qquad \frac{x-x_0}{f_{y_0}}=\frac{y-y_0}{f_{x_0}}$$

$$\frac{x-x_0}{10}=\frac{y-y_0}{-12} \qquad \boxed{\begin{aligned}x&=-1+10t\\ y&=1-12t\end{aligned}} \Rightarrow$$ (ISTO KAKO VO Stewart SOLUTIONS)

ZGREŠIV ŠTO ZAMISLUVAV DEKA PRESEKOT E „2D"!!

• ALTERNATIVE APPROACH (STEWART SOLUTIONS)

$f(x,y,z) = z - x^2 - y^2 \qquad g(x,y,z) = 4x^2 + y^2 + z^2$

$\nabla f = -2x\vec{i} - 2y\vec{j} + \vec{k} = \langle -2x, -2y, 1 \rangle$

$\nabla g = 8x\vec{i} + 2y\vec{j} + 2z\vec{k} = \langle 8x, 2y, 2z \rangle$

$\Delta f_0 = \langle 2, -2, 1 \rangle \qquad \Delta g_0 = \langle -8, 2, 4 \rangle$

$\vec{v} = \Delta f \times \Delta g = \begin{vmatrix} i & j & k \\ 2 & -2 & 1 \\ -8 & 2 & 4 \end{vmatrix} = \underline{-10\vec{i} - 16\vec{j} - 12\vec{k}}$

$\vec{r} = \vec{r_0} + t \cdot \vec{v} = \langle -1, 1, 2 \rangle + t\langle -10, -16, -12 \rangle$

$\boxed{x = -1 - 10t \qquad y = 1 - 16t \qquad z = 2 - 12t}$

$\boxed{\text{Ex.}60}$ $y + z = 3$ INTERSECTS $x^2 + y^2 = 5$ $P(1,2,1)$

$f(x,y,z) = y + z \qquad \nabla f = \langle 0, 1, 1 \rangle$

$g(x,y,z) = x^2 + y^2 \qquad \nabla g = \langle 2x, 2y, 0 \rangle = \langle 2, 4, 0 \rangle$

$\vec{v} = \Delta f \times \Delta g = -4\vec{i} + 2\vec{j} - 2\vec{k}$

$\vec{r} = \vec{r_0} + t \cdot \vec{v} = \langle 1, 2, 1 \rangle + t\langle -4, 2, -2 \rangle$

$\boxed{x = 1 - 4t \qquad y = 2 + 2t \qquad z = 1 - 2t}$

$\boxed{61, \text{Exc}}$ $F(x,y,z) = 0 \qquad G(x,y,z) = 0$

(a) $\nabla F \cdot \nabla G = \left(F_x\vec{i} + F_y\vec{j} + F_z\vec{k}\right) \cdot \left(G_x\vec{i} + G_y\vec{j} + G_z\vec{k}\right) = 0$

$\boxed{F_x G_x + F_y G_y + F_z G_z = 0}$

(b) $z^2 = x^2 + y^2 \qquad F: x^2 + y^2 - z^2 = 0$

$G: x^2 + y^2 + z^2 - r^2 = 0$

$\nabla F = \langle 2x, 2y, -2z \rangle \qquad \nabla G = \langle 2x, 2y, 2z \rangle$

- INTERSECTION $\quad z^2 = x^2 + y^2 \qquad 2x^2 + 2y^2 = r^2 \qquad y^2 = \frac{1}{2}(r^2 - 2x^2)$

$\boxed{y^2 = \frac{1}{2}r^2 - x^2}$

$P\left(x, \sqrt{\frac{1}{2}r^2 - x^2}, \sqrt{x^2 + \frac{1}{2}r^2 - x^2}\right)$

$P\left(x, \sqrt{\frac{1}{2}r^2 - x^2}, r/\sqrt{2}\right)$

$\nabla F \cdot \nabla G = \langle 2x, 2y, 2z \rangle \langle 2x, 2y, -2z \rangle =$

$= 4x^2 + 4y^2 - 4z^2 = 4\underbrace{(x^2+y^2)}_{z^2} - 4z^2 = 4z^2 - 4z^2 = 0$

ALTERN: $\nabla F \cdot \nabla G = 4(x^2+y^2-z^2) = 4 \cdot \cancel{F} = 0$

(62. Exc.) $f(x,y) = \sqrt[3]{xy}$

$f_x = \left((xy)^{\frac{1}{3}}\right)' = y \cdot \frac{1}{3}(xy)^{-\frac{2}{3}} = \frac{1}{3} \frac{y}{\sqrt[3]{x^2y^2}}$

$f_y = \frac{x}{3\sqrt[3]{x^2y^2}}$ $\qquad f_z = 0$

$D_u f = \langle f_x, f_y \rangle \cdot \langle a, b \rangle \qquad u = \langle a, b \rangle$

$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{(h \cdot 0)^{1/3} - 0}{h} = 0$

$f_y(0,0) = \lim_{h \to 0} \frac{f(0, 0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{(0 \cdot h)^{1/3} - 0}{h} = 0$

$u = \langle a, b \rangle$

$D_u f(0,0) = \lim_{h \to 0} \frac{f(0+ha, 0+hb) - f(0,0)}{h} = \lim_{h \to 0} \frac{\sqrt[3]{ha\,hb}}{h}$

$D_u f(0,0) = \lim_{h \to 0} \frac{h^{2/3}\sqrt[3]{ab}}{h^{3/3}} = \lim_{h\to 0} \frac{\sqrt[3]{ab}}{h^{1/3}}$ doesn't exist!!

(67. Exc) $f(x,y)$ $\qquad u = \langle a, b \rangle$, $v = \langle c, d \rangle$

$D_u f = f_x \cdot a + f_y b$ $\qquad f_x = \frac{D_u f - f_y b}{a}$

$D_v f = f_x c + f_y d$

$D_v f = \frac{D_u f - f_y b}{a} \cdot c + f_y d = \frac{c D_u f - f_y b \cdot c + a f_y d}{a}$

$a \cdot D_v f - c D_u f = f_y(-b \cdot c + ad)$

$\boxed{f_y = \frac{a D_v f - c D_u f}{ad - bc}}$ $\qquad f_x = \frac{D_u f - \frac{ba D_v f - bc D_u f}{ad - bc}}{a}$

$$f_x = \frac{\cancel{a}dD_uf - \cancel{bcD_uf} - b\cancel{a}D_vf + \cancel{bcD_uf}}{\cancel{a}(ad-bc)} = \frac{dD_uf - bD_vf}{ad-bc}$$

$$\boxed{\nabla f = \langle f_x, f_y \rangle = \left\langle \frac{dD_uf - bD_vf}{ad-bc}, \frac{aD_vf - cD_uf}{ad-bc} \right\rangle}$$

Ex.64 $z = f(x,y)$ $\vec{x_0} = \langle x_0, y_0 \rangle$

Definition 14.4.7 If $z = f(x,y)$ is diferentiable at (a,b) if Δz can be expressed in form:

$$\boxed{\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y}$$

$\varepsilon_1, \varepsilon_2 \to 0$ $(\Delta x, \Delta y) \to (0,0)$

$\Delta z = f(\vec{x}) - f(\vec{x_0})$ $\langle \Delta x, \Delta y \rangle = \vec{x} - \vec{x_0}$ $\boxed{(\Delta x, \Delta y) \to (0,0) \Leftrightarrow \vec{x} \to \vec{x_0}}$

$$\langle f_x(x_0,y_0), f_y(x_0,y_0) \rangle = \nabla f(\vec{x_0})$$

$$f(\vec{x}) - f(\vec{x_0}) = \nabla f(\vec{x_0}) \cdot \underbrace{(x - x_0)}_{(\vec{x}-\vec{x_0})} + \langle \varepsilon_1, \varepsilon_2 \rangle \langle \Delta x, \Delta y \rangle$$

$$\langle \varepsilon_1, \varepsilon_2 \rangle \cancel{\langle \Delta x, \Delta y \rangle} = f(\vec{x}) - f(\vec{x_0}) - \nabla f(\vec{x_0}) \underbrace{(x - x_0)}_{(\vec{x}-\vec{x_0})}$$

$$\frac{f(\vec{x}) - f(\vec{x_0}) - \nabla f(\vec{x_0})(\vec{x}-\vec{x_0})}{|\vec{x}-\vec{x_0}|} = \frac{\langle \varepsilon_1, \varepsilon_2 \rangle \cancel{\langle \Delta x, \Delta y \rangle}}{|\vec{x}-\vec{x_0}|}$$

$$\boxed{\frac{\vec{x}-\vec{x_0}}{|\vec{x}-\vec{x_0}|} = \vec{u}}$$ $$\lim_{\vec{x} \to \vec{x_0}} \frac{\langle \varepsilon_1, \varepsilon_2 \rangle (\vec{x}-\vec{x_0})}{|\vec{x}-\vec{x_0}|} = 0$$

$\varepsilon_1, \varepsilon_2 \to 0$
koga: $\vec{x} \to \vec{x_0}$

$$\Rightarrow \lim_{\vec{x} \to \vec{x_0}} \frac{f(\vec{x}) - f(\vec{x_0}) - \nabla f(\vec{x_0})(\vec{x}-\vec{x_0})}{|\vec{x}-\vec{x_0}|} = 0$$

55 l → 670 km 55 l → 6.7 · 100 $\boxed{8.2 l / 100 km}$

- GRADIENT FIELDS (Ch. 16.1)

$$\nabla f(x,y) = f_x(x,y)\,\vec{i} + f_y(x,y)\,\vec{j}$$ → VO REALNOSTA PRETSTAVUVA VECTORSKO POLE!!

$$\nabla f(x,y,z) = f_x(x,y,z)\,\vec{i} + f_y(x,y,z)\,\vec{j} + f_z(x,y,z)\,\vec{k}$$

$\boxed{\text{Exp. 6}}$ $f(x,y) = x^2y - y^3$ $f_x = 2xy$ $f_y = x^2 - 3y^2$

$z = x^2y - y^3$ $x^2y - y^3 - z = 0$

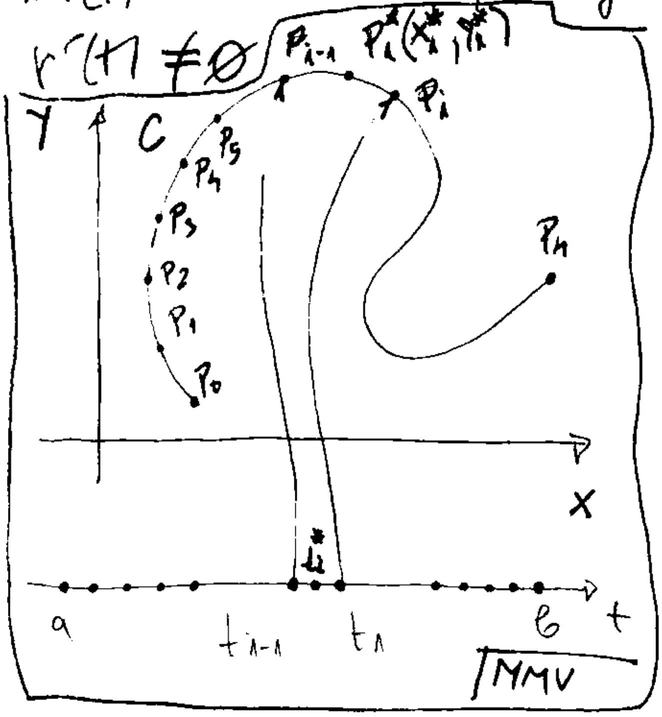
$F(x,y,z) = x^2y - y^3 - z$ $\boxed{f_z = -1}$

LINE INTEGRALS (CURVE INTEGRALS)

$x = x(t)$ $y = y(t)$ $a \le t \le b$

$C: r(t) = x(t)\vec{i} + y(t)\vec{j}$ C – smooth curve

$r'(t) \neq 0$



$\Delta s_1, \Delta s_2, \Delta s_3, \ldots \Delta s_n$

$P_i^*(x_i^*, y_i^*)$

$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

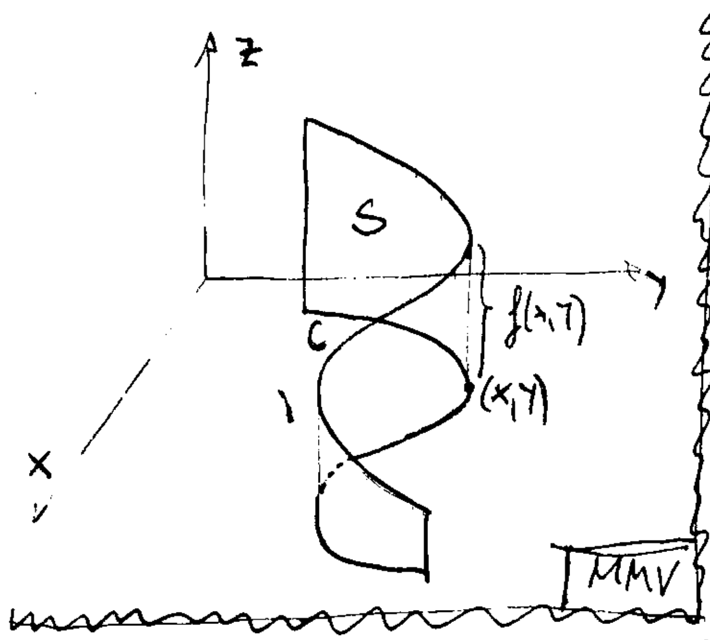
$$\boxed{L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt}$$

$$\boxed{\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt}$$

$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ $ds = \left(\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\right) dt$

$(a,0)$ $(b,0)$ $x = x$ $y = 0$

$$\int_C f(x,y)\,ds = \int_C f(x,0)\,dx$$

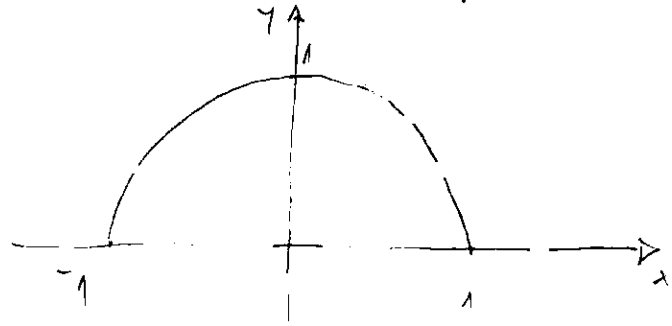


EXP 1

$$\int_C (2+x^2y)\,ds$$

NICO 6900. ENERGY SISTEM

$C: x^2+y^2=1 \Rightarrow$ UPPER HALF



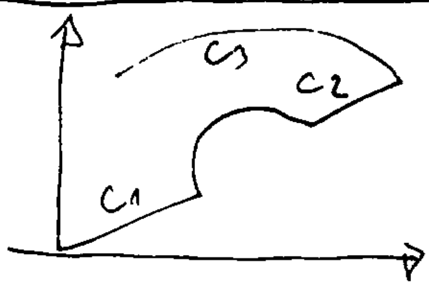
$$f(x,y) = 2+x^2y \qquad I = \int_C (2+x^2y)\,ds = \int_a^b \left(2+\cos^2 t(\sin t)\right)\sqrt{x'(t)^2+y'(t)^2}\,dt$$

$$\left.\begin{aligned} x &= \cos t \\ y &= \sin t \end{aligned}\right| \qquad \frac{dx}{dt} = -\sin t \qquad \frac{dy}{dt} = \cos t$$

$$I = \int_0^{\pi} (2+\cos^2 t\cdot\sin t)\sqrt{\sin^2 t+\cos^2 t}\,dt = \int_0^{\pi} (2+\cos^2 t\sin t)\,dt$$

$$= \int_0^{\pi} 2\,dt - \int_0^{\pi} \cos^2 t\,d\cos t = 2\pi - \left.\frac{\cos^3 t}{3}\right|_0^{\pi} = 2\pi - \left(-\frac{1}{3}-\frac{1}{3}\right)$$

$$\boxed{I = 2\pi + \frac{2}{3}}$$



$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds$$

EXP. 2

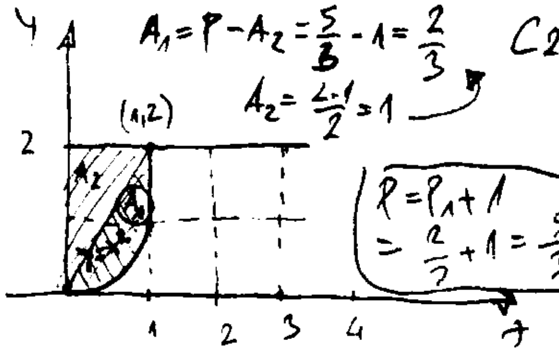
$$\int_C 2x\,ds$$

$C_1: y = x^2 \quad (0,0)$ TO $(1,1)$

C_2: VERTICAL LINE SEGMENT $(1,1)$ TO $(1,2)$

$A_1 = P - A_2 = \frac{5}{3} - 1 = \frac{2}{3}$

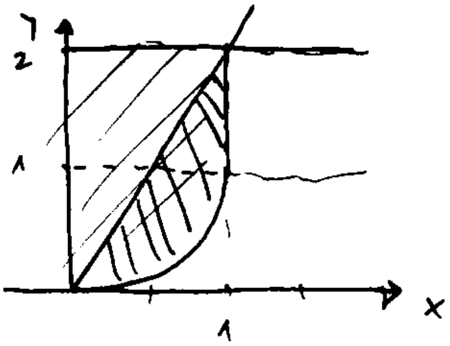
$A_2 = \frac{2\cdot 1}{2} = 1$



$$P = P_1 + 1 = \frac{2}{3}+1 = \frac{5}{3}$$

$$P_1 = \int_0^1 x(y)\,dy = \int_0^1 \sqrt{y}\,dy = \star$$

$$= \left.\frac{y^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_0^1 = \left.\frac{y^{3/2}}{3/2}\right|_0^1 = \frac{2}{3}\left(1^{3/2}-0\right) = \frac{2}{3}$$



$$\int_C 2x\,ds \qquad C_1: y = x^2 \quad x=\sqrt{y} \qquad C2: x = 1$$

$$I_1 = \int_{C_1} 2x\sqrt{x'^2+y'^2}\,dt = \int_{C_1} 2x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$I_1 = \int_0^1 2\sqrt{y}\sqrt{1+(2x)^2}\,dx = 2\int_0^1 x\sqrt{1+4x^2}\,dx = \frac{5}{6}\sqrt{5}-\frac{1}{6} = \frac{1}{6}(5\sqrt{5}-1)$$

$$I_2 = 2\int_1^2 1\sqrt{1+\underbrace{\left(\frac{dx}{dy}\right)^2}_{\to 0}}\,dy = 2\int_1^2 1\,dy = y\Big|_1^2 = 2$$

$$I = I_1 + I_2 = \frac{1}{6}(5\sqrt{5}-1)+2 = \frac{1}{6}(5\sqrt{5}-1)+\frac{12}{6} = \frac{1}{6}(5\sqrt{5}+11)$$

$$\int_C 2x\,ds = \int_{C_1} 2x\,ds + \int_{C_2} 2x\,ds = \frac{1}{6}(5\sqrt{5}-1)+2$$

LINISKIOT INTEGRAL VO OVOJ SLUČAJ NE ODGOVARA NA POVRŠINATA

⊙ PHYSICAL INTERPRETATION OF LINE INTEGRAL DEPENDS ON PHYSICAL REPRESENTATION OF THE FUNCTION $f(x,y)$.

• E.g. $\rho(x,y)$ LINEAR DENSITY AT POINT (x,y) OF THIN WIRE SHAPED LIKE „C"

MASS OF THE WIRE:

$$m = \lim_{n\to\infty}\sum_{i=1}^{n}\rho(x_i^*,y_i^*)\Delta s = \int_C \rho(x,y)\,ds$$

MMV

• E.g. IF $f(x,y) = 2+x^2y$ (Ex. 1) REPRESENTS DENSITY OF SEMICIRCULAR WIRE

$$\int_C (2+x^2y)\,ds \Rightarrow \text{MASS OF THE WIRE}$$

- CENTER OF MASS OF THE WIRE WITH DENSITY ρ

$$\bar{x} = \frac{1}{m}\int_C x\rho(x,y)\,ds \qquad \bar{y} = \frac{1}{m}\int_C y\rho(x,y)\,ds$$

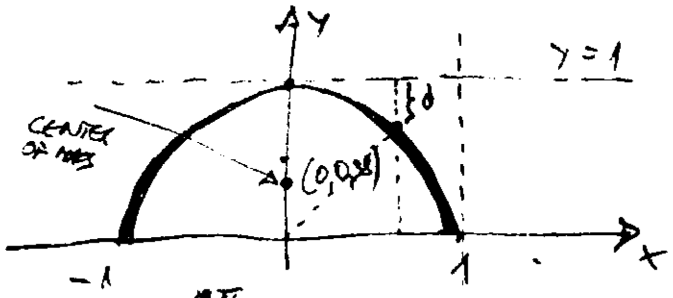
Ex. 3 WIRE TAKES SHAPE OF SEMICIRCLE

$x^2+y^2=1 \quad y \ge 0$

FIND THE CENTER OF MASS OF THE WIRE IF LINEAR DENSITY AT ANY POINT IS PROPORTIONAL TO DISTANCE FROM LINE $y=1$

SOLUTION: $x = \cos t$ $\quad x^2+y^2=1$ $\quad x'(t) = -\sin t$, $y'(t) = \cos t$

$y = \sin t$



$\rho(y) = c \cdot d; \quad d = 1 - \sin t = 1 - y$

$\rho(y) = c(1-\sin t) = c(1-y)$

$$m = \int_a^b c(1-\sin t)\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

$$m = \int_0^{\pi} c(1-\sin t)\,dt = c\cdot\left(\pi - \int_0^{\pi}\sin t\,dt\right) = c\left(\pi + \cos t\Big|_0^{\pi}\right)$$

$$\boxed{m = c(\pi - 2)}$$

$ds = \sqrt{x'^2(t) + y'^2(t)}\,dt = dt$

$$\bar{x} = \frac{1}{m}\int_C x\,\rho(x,y)\,ds = \frac{1}{m}\int_0^{\pi}\cos t\cdot c(1-\sin t)\,dt \Rightarrow$$

$$\bar{x} = \frac{c}{m}\int_0^{\pi}\cos t(1-\sin t)\,dt = 0$$

$$\bar{y} = \frac{c}{m}\int_0^{\pi}\sin t(1-\sin t)\,dt = \frac{c}{m}\left(2 - \frac{\pi}{2}\right)$$

$$\bar{y} = \frac{c}{c(\pi-2)}\left(\frac{4-\pi}{2}\right) = \cancel{\frac{4-\pi}{4\cdot 2\pi}} = \frac{4-\pi}{2\pi-4} = \frac{4-\pi}{2(\pi-2)}$$

CENTER OF MASS IS: $\left(0, \frac{4-\pi}{2(\pi-2)}\right) = (0, 0.28)$

- LINE INTEGRALS OF „f" ALONG „C" WITH RESPECT TO „x" & „y"

$$\int_C f(x,y)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i,y_i)\Delta x_i$$

$$\int_C f(x,y)\,dy = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i,y_i)\Delta y_i$$

$\int_C f(x,y)\,ds \Rightarrow$ LINE INTEGRAL WITH RESPECT TO ARC LENGTH.

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$$

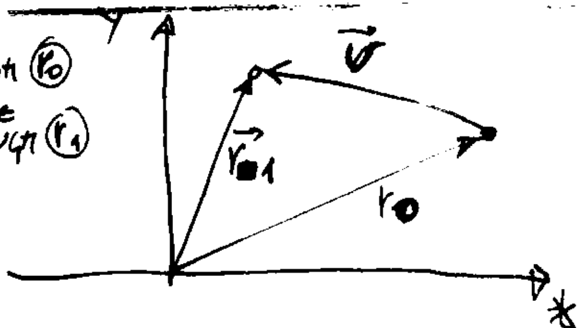
$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$$

$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy = \int_C P(x,y)\,dx + Q(x,y)\,dy$$

$\vec{r}(t) = \vec{r_0} + \vec{v}\,t$ LINE THROUGH r_0

$\vec{r}(t) = \vec{r_1} + \vec{v}\,t$ LINE THROUGH r_1

$\boxed{\text{MMV}}$



$\vec{v} + \vec{r_0} = \vec{r_1}$ $\quad \vec{v} = \vec{r_1} - \vec{r_0}$ $\quad \vec{r} = \vec{r_0} + v t = \vec{r_0} + (\vec{r_1} - \vec{r_0})t$

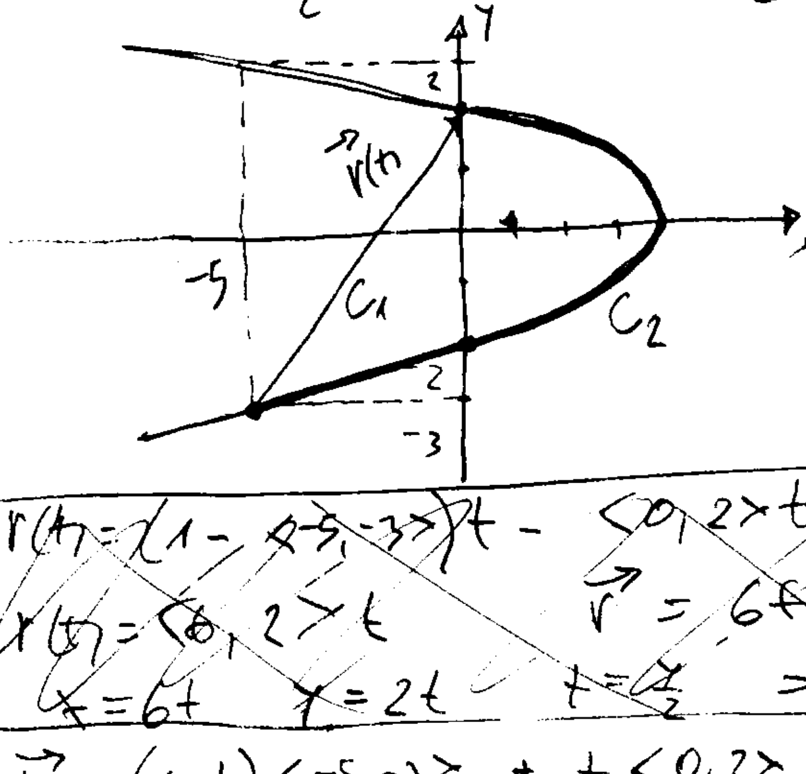
$$\boxed{r = (1-t)\vec{r_0} + t\cdot\vec{r_1}}$$

$$\boxed{r(t) = (1-t)\,\vec{r_0} + t\,\vec{r_1}}$$ → VECTOR REPRESENTATION OF LINE SEGMENT $\boxed{\text{MMV}}$

$\boxed{\text{Ex. 4}}$ $I = \int_C y^2\,dx + x\,dy$

(a) C_1: LINE SEGMENT $(-5,-3)$ TO $(0,2)$

(b) C_2: $x = 4 - y^2$ $(-5,-3)$ TO: $(0,2)$



$$\int_C P(x,y)\,dx = \int_a^b f(x,y)\,x'(t)\,dt$$

(a) $r(t) = (1-t)r_0 + r_1 t$

$r_0 = \langle -5, -3 \rangle$

$r_1 = \langle 0, 2 \rangle$

~~$r(t) = (1 - \langle -5,-3\rangle)t - \langle 0,2\rangle t \Rightarrow \langle 6,4\rangle t - \langle 0,2\rangle t$~~

~~$r(t) = \langle 6, 2\rangle t$ $\quad \vec{r} = 6t\,\vec{i} + 2t\,\vec{j}$~~

~~$x = 6t$ $\quad y = 2t$ $\quad t = \frac{y}{2} \Rightarrow x = 3$?~~

$\vec{r} = (1-t)\langle -5,-3\rangle + t\langle 0,2\rangle = \langle -5,-3\rangle + t\langle 5,3\rangle + t\langle 0,2\rangle$

$\vec{r} = \langle -5,-3\rangle + t\langle 5,5\rangle$ $\quad \underbrace{x = -5+5t \quad y = -3+5t}$

$\vec{r} = (-5+5t)\,\vec{i} + (-3+5t)\,\vec{j}$ $\quad t = \frac{y+3}{5}$

$x = -5 + y + 3 = y - 2$ $\quad \boxed{y = x+2}$ $\quad \boxed{x = y-2}$

$$\int_a^b y^2\,dx = \int_a^b y^2\,x'(t)\,dt = \int_{-5}^{0} (x+2)^2\,dx = \frac{35}{3}$$

$$I_1 = \int_C y^2 dx = \int_0^1 (-3+5t)^2 (-5+5t)' dt \quad \boxed{t = \frac{x+5}{5} \quad \begin{matrix} x=-5 & t=0 \\ x=0 & t=1 \end{matrix}}$$

$$I_1 = \int_0^1 (9 - 30t + 25t^2)(5)\,dt = 5\int_0^1 (5t-3)^2 dt = \frac{35}{3} \Big\}$$ ISTO SO PARAMETAR. PRISTAP !!!

$$I_2 = \int_C x\,dy = \int_{-3}^{2} (y-2)\,dy = -\frac{25}{2}$$

$$I = I_1 + I_2 = \frac{35}{3} - \frac{25}{2} = \frac{70-75}{6} = -\frac{5}{6}$$

(b) $$I = \int_{-5}^{0} (4-x)\,dx + \int_{-3}^{2} (4-y^2)\,dy = \frac{65}{2} + \frac{25}{3} = \frac{245}{6} = 40\frac{5}{6}$$

• ALTERNATIVELY TAKE „Y" AS PARAMETER

$x = 4 - y^2 \qquad y \in (-3, 2)$

$$I = \int_C y^2 dx + x\,dy = \int_{-3}^{2} y^2(-2y)\,dy + (4-y^2)\,dy =$$

$$= +\int_{-3}^{2} (-2y^3 - y^2 + 4)\,dy = \frac{245}{6} = 40\frac{5}{6}$$

• LINE SEGMENT FROM $(0,2)$ TO $(-5,-3)$

$$\vec{r} = (1-t)\vec{r_0} + t\vec{r_1} = (1-t)\langle 0,2\rangle + t\langle -5,-3\rangle =$$

$$= \langle 0,2\rangle - t\langle 0,2\rangle + t\langle -5,-3\rangle = \langle 0,2\rangle + t\langle -5,-5\rangle$$

$\boxed{x = -5t \quad y = 2-5t}$ $\qquad x = (0,-5) \quad y = (2,-3)$

$t = \frac{y-2}{-5} = \frac{2-y}{5} \quad \boxed{\begin{matrix} y=2 & t=0 \\ y=-3 & t=1 \end{matrix}}$

$x = (0,-5) \quad t = \frac{x}{-5} \qquad \begin{matrix} x=0 \\ x=-5 \end{matrix} \; \boxed{\begin{matrix} t=0 \\ t=1 \end{matrix}}$

$$I = \int_0^1 (2-5t)^2 \cdot (-5)\,dt + (-5t)\cdot(-5)\,dt = -5\int_0^1 \left[(2-5t)^2 - 5t\right] dt = \frac{5}{6}$$

DIFERENT DIRECTION – DIFERENT VALUE OF THE LINE INTEGRAL („–" VALUE)

$$\boxed{\int_{-C} f(x,y)\,dx = -\int_C f(x,y)\,dx \qquad \int_{-C} f(x,y)\,dy = -\int_C f(x,y)\,dy}$$

MMV

$$\boxed{\int_{-C} f(x,y)\,ds = \int_C f(x,y)\,ds}$$ NE MENUVA ZNAK AKO INTEGRIRAS PO „ARC LENGTH"

LINE INTEGRALS IN SPACE

$x = x(t) \quad y = y(t) \quad z = z(t) \quad a \le t \le b$

$$\int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta s_i$$

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

$$\int_C f(x,y,z)\,ds = \int_a^b f(\vec{r}(t))\,|r'(t)|\,dt$$

- $\boxed{f(x,y,z) = 1}$ $\quad \int_C ds = \int_a^b |r'(t)|\,dt = L$ — LENGTH OF CURVE „C"

- LINE INTEGRALS ALONG „C" WITH RESPECT TO „z"

$$\int_C f(x,y,z)\,dz = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta z_i$$
$$= \int_a^b f(x,y,z)\, z'(t)\,dt$$

$$\int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$$
$x = x(t) \quad y = y(t) \quad z = z(t) \quad dx = x'(t)dt \ldots$

Exp. 5 $\quad I = \int_C y \sin z\, ds \qquad \underbrace{C: x = \cos t,\ y = \sin t,\ z = t \quad 0 \le t \le 2\pi}_{\text{CIRCULAR HELIX}}$

$$I = \int_a^b \sin t \sin t \sqrt{(-\sin t)^2 + \cos^2 t + 1^2}\,dt = \int_0^{2\pi} \sin^2 t \sqrt{1+1^2}\,dt$$

$$I = \sqrt{2}\int_0^{2\pi} \frac{1}{2}(1 - \cos 2t)\,dt = \frac{\sqrt{2}}{2}\left[t\Big|_0^{2\pi} - \frac{1}{2}\int_0^{2\pi} \cos(2t)\,d2t\right] = \frac{\sqrt{2}}{2}\left[2\pi - \frac{\sin(2t)}{2}\Big|_0^{2\pi}\right] = \sqrt{2}\pi$$

$\cos(\alpha+\alpha) = \cos^2\alpha - \sin^2\alpha$

$\cos(\alpha-\alpha) = 1 = \cos^2\alpha + \sin^2\alpha$

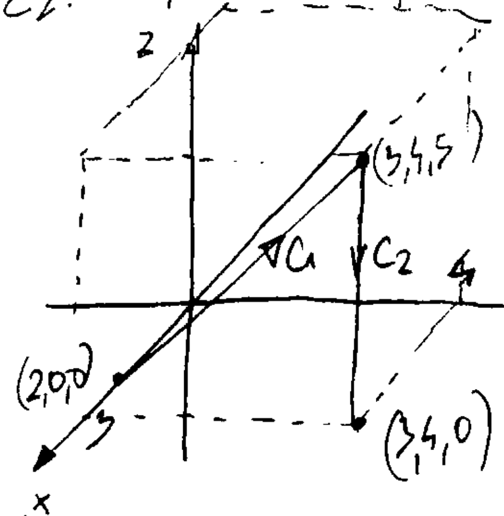
$\cos 2\alpha = 1 - \sin^2\alpha - \sin^2\alpha$

$\sin^2\alpha = \frac{1}{2}(1 - \cos 2\alpha)$

EXP6 $\int_C y\,dx + z\,dy + x\,dz$ $\quad C = C_1 \& C_2$

C_1: LINE SEGMENT FROM $(2,0,0)$ TO $(3,4,5)$

C_2: LINE SEGMENT FROM $(3,4,5)$ TO $(3,4,0)$



$\vec{r}_{C1} = \vec{r}_0(1-t) + \vec{r}_1(t)$

$= (1-t)\langle 2,0,0\rangle + t\langle 3,4,5\rangle$

$= \langle 2,0,0\rangle + t\langle 1,4,5\rangle$

C_1: $\boxed{x = 2+t \quad y = 4t \quad z = 5t}$

$\vec{r}_{C2} = (1-t)\langle 3,4,5\rangle + t\langle 3,4,0\rangle$

$= \langle 3,4,5\rangle - \langle 3,4,5\rangle t + t\langle 3,4,0\rangle$

$\vec{r}_{C_2} = \langle 3,4,5\rangle + \langle 0,0,-5\rangle t$ $\quad C_2$: $\boxed{x = 3 \quad y = 4 \quad z = 5-5t}$

- C_1 $\quad t = x-2 \quad x \in (2,3) \Rightarrow \underline{\underline{t \in (0,1)}}$

$I_1 = \int_{C_1} y\,dx + z\,dy + x\,dz = \int_0^1 4t\cdot dt + 5t\cdot 4dt + (2+t)5\cdot dt$

$I_1 = \int_0^1 (4t + 20t + 10 + 5t)\,dt = \int_0^1 (29t+10)\,dt = \left(29\frac{t^2}{2} + 10t\right)\Big|_0^1$

$\boxed{I_1 = \frac{29}{2} + 10 = \frac{29+20}{2} = \frac{49}{2}}$

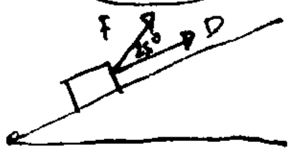
- C_2 $\quad I_2 = \int_{C_2} 4\cdot 0 + (5-5t)\cdot 0 + 3\cdot(-5)\,dt = \int_0^1 -15\,dt = \underline{\underline{-15}}$

$I = I_1 + I_2 = 24.5 - 15 = 9{,}5$

LINE INTEGRALS OF VECTOR FIELDS

$W = \int_a^b f(x)\,dx \qquad W = F\cdot D$

Ch12.3 EXP7 $\quad F = 200N \quad$ 8m UP THE RAMP



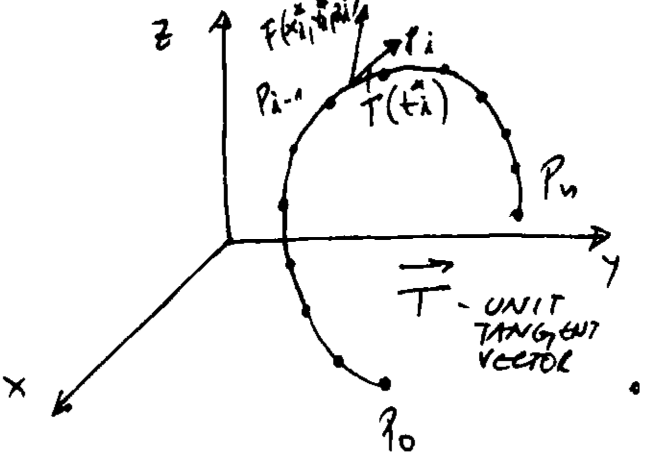
$W = F\cdot D = |F|\cdot|D|\cdot \cos 45 = 200\cdot 8\cdot \frac{\sqrt{2}}{2}$ ($\sqrt{2}/2$, 100)

$\boxed{W = 800\sqrt{2}\ Nm} = 1131{,}4\ Nm$

$= 1131{,}4\ J$

FORCE: $\vec F = P\vec i + Q\vec j + R\vec k$

- HOW TO COMPUTE WORK DONE BY THIS FORCE BY MOVING PARTICLE ALONG „C"



- WORK FOR MOVING PARTICLE FROM P_{i-1} TO P_i

$$\vec F(x_i^*, y_i^*, z_i^*)\cdot\left(\Delta s_i\, \vec T(t_i^*)\right) =$$
$$= \left[\vec F(x_i^*, y_i^*, z_i^*)\cdot \vec T(t_i^*)\right]\Delta s_i$$

- TOTAL WORK OF MOVING PARTICLE FROM P_{i-1} TO P_i IS

$$\sum_{i=1}^{n} \left[\vec F(x_i^*, y_i^*, z_i^*)\cdot \vec T(t_i)\right]\Delta s_i$$

$$\boxed{W = \int_C F(x,y,z)\cdot T(x,y,z)\,ds = \int_C \vec F \circ \vec T\,ds}$$
MMV

WORK IS LINE INTEGRAL WITH RESPECT TO ARC LENGHT OF THE TANGENTIAL COMPONENT OF FORCE!!!

$$\boxed{\int f(\vec r(t))\,|r'(t)|\cdot dt} \qquad \vec T = \frac{\vec{r'(t)}}{|r'(t)|}$$

$$W = \int_C \left[\vec F \circ \frac{r'(t)}{\cancel{|r'(t)|}}\right]\cdot \cancel{|r'(t)|}\,dt = \int_C F\cdot r'(t)\,dt$$

$$\boxed{W = \int_C \vec F(\vec r(t))\cdot r'(t)\,dt}$$
MMV

$$\boxed{W = \int_C \vec F\cdot d\vec r}$$
MMV

DEFINITION 13

$$\boxed{d\vec r = \vec{r'(t)}\,dt}$$

$$\boxed{\int_C \vec F\cdot d\vec r = \int_C \vec F(\vec r(t))\cdot r'(t)\,dt = \int_C \vec F\circ \vec T\,ds}$$
MMV

EX. 7 ~~...~~ $W = ?$ FOR $\vec F(x,y) = x^2\vec i - xy\vec j$ IN MOVING THE PARTICLE ALONG: $\vec r(t) = \cos t\,\vec i + \sin t\,\vec j \quad 0 \le t \le \frac{\pi}{2}$

$$W = \int_C \vec F\cdot d\vec r = \int_C \vec F(\vec r(t))\circ \vec r'(t)\,dt$$

$$W = \int_0^{\pi/2} (x^2 \vec{i} - xy \vec{j}) \cdot (\cos t \, \vec{i} + \sin t \, \vec{j})' \, dt =$$

$$= \int_0^{\pi/2} \left(\cos^2 t \, \vec{i} - \cos t \cdot \sin t \, \vec{j}\right) \cdot \left(-\sin t \, \vec{i} + \cos t \, \vec{j}\right) dt =$$

$$= \int_0^{\pi/2} \left(-\cos^2 t \cdot \sin t - \cos^2 t \cdot \sin t\right) dt = +2 \int_0^{\pi/2} \cos^2 t \, d(\cos t)$$

$$W = 2 \left. \frac{\cos^3 t}{3} \right|_0^{\pi/2} = \frac{2}{3}\left[0 - 1\right] = -\frac{2}{3}$$

$$\boxed{\int_{-C} \vec{F} \, d\vec{r} = -\int_C \vec{F} \, d\vec{r}}$$

$\boxed{\text{EXP 8}}$ $\quad \int_C \vec{F} \cdot d\vec{r} \qquad F(x,y,z) = xy \, \vec{i} + yz \, \vec{j} + zx \, \vec{k}$

$C: \; x = t \quad y = t^2 \quad z = t^3 \qquad 0 \leq t \leq 1$ (TWISTED CUBIC)

$$I = \int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(r(t)) \cdot \vec{r}'(t) \, dt = \int_0^1 \left(t \cdot t^2 \vec{i} + t^5 \vec{j} + t^4 \vec{k}\right) \cdot \left(\vec{i} + 2t \vec{j} + 3t^2 \vec{k}\right) dt$$

$$= \int_0^1 \left(t^3 + 2t^6 + 3t^6\right) dt = \int_0^1 t^3 + 5t^6 \, dt = \left[\frac{t^4}{4} + \frac{5t^7}{7}\right]_0^1$$

$$I = \frac{1}{4} + \frac{5}{7} = \frac{7 + 20}{28} = \frac{27}{28}$$

- CONNECTION BETWEEN LINE INTEGRALS OF VECTOR FIELDS AND LINE INTEGRALS OF SCALAR FIELDS.

$$F = P\vec{i} + Q\vec{j} + R\vec{k}$$

$$\int_C \vec{F} \, d\vec{r} = \int_a^b \left(P\vec{i} + Q\vec{j} + R\vec{k}\right) \cdot \left(x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k}\right) dt$$

$$= \int_a^b P(x,y,z) \cdot x'(t) \, dt + Q(x,y,z) \cdot y'(t) \, dt + R(x,y,z) \, z'(t) \, dt$$

$$\boxed{\int_C \vec{F} \, d\vec{r} = \int_C P \, dx + Q \, dy + R \, dz \quad \text{WHERE} \quad F = P\vec{i} + Q\vec{j} + R\vec{k}}$$

PMV

Ex6 $\int_C y\,dx + z\,dy + x\,dz$

$$\boxed{F = y\vec{i} + z\vec{j} + x\vec{k}}$$

$$\int \vec{F}\cdot d\vec{r} = \int_C y\,dx + z\,dy + x\,dz$$

Exc.3 $I = \int_C x y^4\,ds = ?$ $C:\ x^2+y^2=16$ $x=0 \div \infty$, $y = -\infty \div \infty$

$C:\ x = 4\cos t,\ y = 4\sin t$ $\boxed{t = -\frac{\pi}{2} \div \frac{\pi}{2}}$

$$I = 4^5 \int_a^b \cos t \cdot \sin^4 t \cdot \sqrt{x'^2(t) + y'^2(t)}\,dt = 4^6 \int_{a=-\pi/2}^{b=\pi/2} \sin^4 t \, d\sin t$$

$$I = 4^6 \frac{\sin^5 t}{5}\Big|_{-\pi/2}^{\pi/2} = 4^6 \frac{1}{5}\left(1^5 - (-1)^5\right) = \frac{2\cdot 4^6}{5} = 1638.4$$

Exc.7 $I = \int_C xy\,dx + (x-y)\,dy$ $C_1: (0,0) \div (2,0)$, $C_2: (2,0) \div (3,2)$

$$\vec{r}_{c_1} = (1-t)\vec{r}_0 + t\cdot\vec{r}_1 = (1-t)\langle 0,0\rangle + t\langle 2,0\rangle$$

$\boxed{\vec{r}_{c_1} = 2t\vec{i}}$ $\boxed{x = 2t \quad y = 0}$ C_1

$$\vec{r}_{c_2} = (1-t)\langle 2,0\rangle + t\langle 3,2\rangle = \langle 2,0\rangle + t\langle 3-2, 2-0\rangle$$

$\vec{r}_{c_2} = \langle 2,0\rangle + t\langle 1,2\rangle$ $\boxed{x = 2+t \quad y = 2t}$ C_2

$x\in(2\div 3)$ $t = x-2$ $t\in(0,1)$

$$I_1 = \int_{C_1} xy\,dx + (x-y)\,dy = 0$$

$$I_2 = \int_{C_2} xy\,dx + (x-y)\,dy = \int_0^1 (2+t)2t\cdot dt + (2+\underbrace{t-2t}_{-t})\cdot 2\,dt$$

$$I_2 = \int_0^1 \cancel{4t} + 2t^2 + 4 - \cancel{2t}\,dt = \int_0^1 2t^2 + 2t + 4\,dt =$$

$$= 2\int_0^1 t^2 + t + 2\,dt = 2\left[\frac{t^3}{3} + \frac{t^2}{2} + 2t\right]\Big|_0^1 = 2\left[\frac{1}{3} + \frac{1}{2} + 2\right]$$

$$I_2 = 2\left[\frac{2+3+12}{6}\right] = \frac{17}{3}$$ $I = I_1 + I_2 = \frac{17}{3}$

$t = \frac{y}{2}$ $x = 2 + \frac{y}{2}$ $\boxed{y = 2x - 4}$ $t = t$

$$I_2 = \int_2^3 x(2x-4) + (x - 2x + 4)\,2\,dx = \int_2^3 2x^2 - 4x + \cancel{4}2x + 8\,dx$$

$$I_2 = \int_2^3 2x^2 - 6x + 8\,dx = 2\int_2^3 x^2 - 3x + 4\,dx = 2\left[\frac{x^3}{3} - \frac{3x^2}{2} + 4x\right]\Big|_2^3$$

$= 17/3$

$\boxed{\text{Exc } 11}$ $I = \int_C x e^{yz}\,ds$ $C: (0,0,0) \div (1,2,3)$

$r = (1-t)\langle 0,0,0\rangle + t\langle 1,2,3\rangle$

$x = t$ $y = 2t$ $z = 3t$

$ds = \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}\,dt = \sqrt{1+4+9}\,(dt) = \sqrt{14}\,dt$

$$I = \sqrt{14}\int_0^1 t e^{2t\cdot 3t}\,dt = \sqrt{14}\int_0^1 t e^{6t^2}\,dt = \frac{\sqrt{14}}{12}\int_0^1 e^{6t^2}\,d(6t^2)$$

$$I = \frac{\sqrt{14}}{12} e^{6t^2}\Big|_0^1 = \frac{\sqrt{14}}{12}\left(e^6 - 1\right)$$

$$\underbrace{(a\vec{i} + b\vec{j})}_{\vec{A}} \circ \underbrace{(a\vec{i} + b\vec{j})}_{\vec{B}} = \vec{A}\cdot\vec{B} = A\cdot B\cdot\cos\theta$$

$\theta = \pi$ $\boxed{\cos\pi = -1}$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\circ\vec{T}\cdot ds = \int_C |\vec{F}|\cdot|\vec{T}|\cdot\cos\pi\,ds = -\int_C |\vec{F}||\vec{T}|\,ds$$

$17\cdot 500$ $17000/2 = 8.500/1000 = 8.5\ m$

$\boxed{\text{Exc. } 21}$ $F(x,y,z) = \sin x\,\vec{i} + \cos y\,\vec{j} + xz\,\vec{k}$ $\int_C \vec{F}\cdot d\vec{r} = ?$

$r(t) = t^3\vec{i} - t^2\vec{j} + t\vec{k}$ $d\vec{r} = r'(t)\,dt$ $t \in (0,1)$

$$\int_C \vec{F}\,d\vec{r} = \int_0^1 \left(\sin x\,\vec{i} + \cos y\,\vec{j} + xz\,\vec{k}\right)\cdot\left(3t^2\vec{i} - 2t\vec{j} + \vec{k}\right)dt$$

$x = t^3$ $y = -t^2$ $z = t$

$$I = \int_C \vec{F}\,d\vec{r} = \int_0^1 \left(\sin(t^3)\,\vec{i} + \cos t^2\,\vec{j} + \underbrace{t^3 \cdot t}\,\vec{k}\right) \cdot \left(3t^2\,\vec{i} + 2t\,\vec{j} + \vec{k}\right) dt$$

$$I = \int_0^1 \left(3t^2 \sin(t^3) + 2t \cdot \cos t^2 + t^4\right) dt = \frac{6}{5} - \cos(1) + \sin(1)$$

Exc 24

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\vec{T}\,ds = -K_1 \qquad x = -1 \ldots 0 \qquad |\vec{F}|\cdot|\vec{T}|\cos\theta \quad \pi > \theta > \frac{\pi}{2} \quad <0$$

$$\int_C \vec{F}\,d\vec{r} = 0 \qquad x = 0 \qquad |\vec{F}|\cdot|\vec{T}|\cdot\cos\frac{\pi}{2} \quad =0$$

$$\int_C \vec{F}\,d\vec{r} = +K_2 \qquad x = 0 \ldots 1 \qquad |\vec{F}|\cdot|\vec{T}|\cdot\cos\theta \quad 0 < \theta < \frac{\pi}{2} \quad >0$$

$$I = \int_C \vec{F}\cdot d\vec{r} = ? \qquad C:\ y = 1 + x^2$$

$$F(x,y) = \frac{x}{\sqrt{x^2+y^2}}\,\vec{i} + \frac{y}{\sqrt{x^2+y^2}}\,\vec{j}$$

$$\boxed{x = t \qquad y = 1+t^2 \qquad r(t) = t\,\vec{i} + (1+t^2)\,\vec{j}}$$

$$I = \int_{-1}^{1} \left(\frac{x}{\sqrt{x^2+(1+x^2)^2}}\,\vec{i} + \frac{1+x^2}{\sqrt{x^2+(1+x^2)^2}}\,\vec{j}\right) \cdot \underbrace{\left(x\,\vec{i} + (1+x^2)\,\vec{j}\right)'}_{\vec{i} + 2x\,\vec{j}}\,dx$$

$$I = \int_{-1}^{1} \frac{x}{\sqrt{x^2+(1+x^2)^2}} + \frac{(1+x^2)\cdot 2x}{\sqrt{x^2+(1+x^2)^2}}\,dx$$

$$\cancel{x^2 + 1 + 2x^2 + x^4 = x^4 + 3x^2 + 1}$$

$$\cancel{I = \int_{-1}^{1} \frac{}{\sqrt{x^4+3x^2+1}}\,dx =}$$

$$I = \int_{-1}^{1} \frac{x + 2x + 2x^3}{\sqrt{x^4+3x^2+1}}\,dx$$

$$I = \int_{-1}^{1} \frac{2x^3 + 3x}{\sqrt{x^4+3x^2+1}}\,dx$$

Exc. 28

$$\int_C \vec{F}\cdot d\vec{r} = ? \qquad F(x,y,z) = x^4 e^y\,\vec{i} + \ln z\,\vec{j} + \sqrt{y^2+z^2}\,\vec{k}$$

C: line segment $(1,2,1)$ $(6,4,5)$

$$r(t) = (1-t)\langle 1,2,1\rangle + t\langle 6,4,5\rangle = \langle 1,2,1\rangle + t\langle 5,2,4\rangle$$

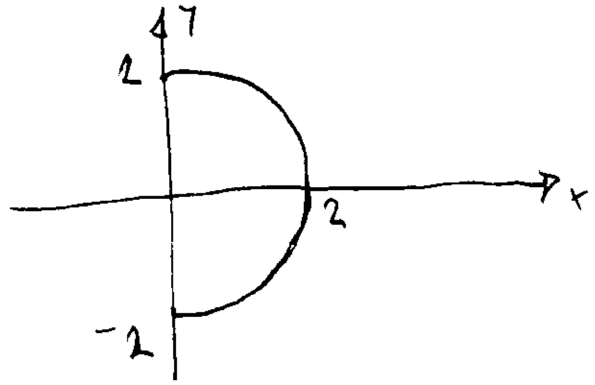
$$\boxed{x = 1 + 5t \qquad y = 2 + 2t \qquad z = 1 + 4t}$$

$r'(t) = 5\vec{i} + 2\vec{j} + 4k \qquad t = \frac{x-1}{5} \quad x \in (1,6) \quad t \in (0,1)$

$$I = \int_0^1 \vec{F}(r(t)) \cdot r'(t)\,dt = \int_0^1 \left[(1-5t)^4 e^{2+2t}\vec{i} + \ln(1+4t)\vec{j} + \sqrt{(2+2t)^2+(1+4t)^2}\,\vec{k}\right] \cdot (5\vec{i} + 2\vec{j} + 4\vec{k})$$

$$I = \int_0^1 5(1-5t)^4 e^{2+2t} + 2\ln(1+4t) + 4\sqrt{(2+2t)^2+(1+4t)^2}\,dt$$

Exc. 31 $x^2+y^2=4$ SEMICIRCLE $x \geq 0$
linear density is CONSTANT k



$\rho(x) = k \qquad x = 2\cos t, \quad y = 2\sin t$

$$m = \int_C \rho(x)\,ds = \int_{-\pi/2}^{\pi/2} k \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$\boxed{ds = 2\,dt}$$

$$m = \int_{-\pi/2}^{\pi/2} k \cdot 2\,dt = 2k\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \underline{\underline{2\pi k}}$$

$$\bar{x} = \frac{2}{m}\int_{-\pi/2}^{\pi/2} x\rho(x,y)\,dt = \frac{\cancel{2}}{\cancel{2}\pi \cancel{k}} \int_{-\pi/2}^{\pi/2} \cancel{k}\,\underbrace{x}_{2\cos t}\,dt$$

$$\bar{x} = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \cos t\,dt = \frac{2}{\pi}\sin t\Big|_{-\pi/2}^{\pi/2} = \frac{2}{\pi}(1+1) = \frac{4}{\pi}$$

$$\bar{y} = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \sin t\,dt = -\frac{2}{\pi}\cos t\Big|_{-\pi/2}^{\pi/2} = -\frac{2}{\pi}(0-0) = 0$$

CENTER OF MASS OF WIRE IS $\underline{\left(\frac{4}{\pi}, 0\right)}$

Ex. 37 Center of mass for helix

$x = 2\sin t \quad y = 2\cos t \quad z = 3t \quad 0 \le t \le 2\pi$

$\rho(x,y,z) = k$

$$m = \int_0^{2\pi} k \sqrt{4\sin^2 t + 4\cos^2 t + 9}\, dt = \underline{\underline{\sqrt{17}k \cdot 2\pi}}$$

$$\bar{x} = \frac{\sqrt{17}}{m} \int_0^{2\pi} x \cdot k\, dt = \frac{\cancel{\sqrt{17}k}}{\cancel{\sqrt{17}k} \cdot \cancel{2}\pi} \int_0^{2\pi} \cancel{2}\sin t\, dt = \frac{1}{\pi}(-\cos t)\Big|_0^{2\pi} = 0$$

$$\bar{y} = \frac{1}{\pi}\int_0^{2\pi} \cos t\, dt = \frac{1}{\pi} \sin t \Big|_0^{2\pi} = 0$$

$$\bar{z} = \frac{1}{2\pi}\int_0^{2\pi} 3 \cdot t\, dt = \frac{1}{2\pi} \cdot 3 \frac{t^2}{2}\Big|_0^{2\pi} = \frac{3 \cdot \cancel{4}\pi^{\cancel{2}}}{\cancel{4\pi}} = 3\pi$$

$$\boxed{\text{CENTER OF MASS} = (0, 0, 3\pi)}$$

Ex. 37 $F(x,y) = x\vec{i} + (y+2)\vec{j}$ FORCE MOVING PARTICLE ALONG $\vec{r}(t) = (t - \sin t)\vec{i} + (1 - \cos t)\vec{j}$ $0 \le t \le 2\pi$

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left(x\vec{i} + (y+2)\vec{j}\right) \cdot \left((t - \sin t)\vec{i} + (1 - \cos t)\vec{j}\right)' dt$$

$$= \int_0^{2\pi} \left(x\vec{i} + (y+2)\vec{j}\right) \cdot \left((1 - \cos t)\vec{i} + \sin t\, \vec{j}\right) dt =$$

$$= \int_0^{2\pi} (t - \sin t)(1 - \cos t) + (1 - \cos t + 2) \cdot \sin t\, dt =$$

$$= \int_0^{2\pi} t - \cancel{\sin t} - t\cos t + \cancel{\sin t \cos t} + 2 \cdot \sin t - \cancel{\cos t \sin t}\, dt$$

$$W = \int_0^{2\pi} \left(t - t\cos t + 2\sin t\right) dt = 2\pi^2$$

Exc. $\boxed{41}$ Man 160-lb = 72,6 kg $r = 20\,ft = 6{,}1\,m$

Can 25-lb = 11.34 kg $h = 90\,ft = 27{,}43\,m$

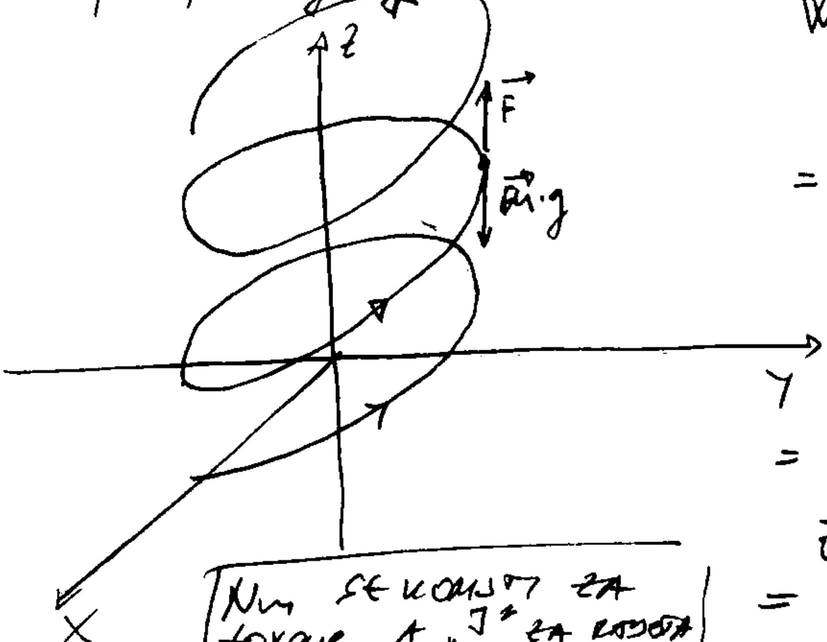
helix: $x = \underset{20}{\wedge}\cos t$; $y = \underset{20}{\wedge}\sin(t)$; $z = \underset{}{\wedge} t$ $0 \le t \le 2\pi$

$\frac{90}{2\pi} = k$

$90 = 2\pi \cdot k$ $k = \frac{90}{2\pi} = 4{,}775 \cdot 3 = 14{,}324$

helix: $x = 20\cos t$ $y = 20\sin t$ $z = \frac{90}{2\pi} \cdot t$ $t = 0 .. 2\pi$

$F = + \overset{185}{\cancel{mg}} \cdot \vec{k}$



$$W = \int_C \vec{F} \circ d\vec{r} =$$

$$= \int_0^{2\pi} \left(+\underbrace{m \cdot g}_{185}\,\vec{k}\right) \circ \left(20\cos t\,\vec{i} + 20\sin t\,\vec{j} + \frac{90}{2\pi} t\right)' \cdot dt$$

$$= \int_0^{2\pi} \left(+mg\,\vec{k}\right) \circ \left(-20\sin t\,\vec{i} + 20\cos t\,\vec{j} + \frac{90}{2\pi}\right) dt$$

Nu sekousty za torque, $\vec{i}$ u $\vec{j}$ za rosoja

$$= \int_0^{2\pi} +mg \cdot \frac{90}{2\pi}\, dt$$

$$W = +mg \frac{90}{2\pi} t \Big|_0^{2\pi} = -mg \frac{90}{2\pi}(2\pi) = +90 \, \overset{185\,lb}{\widetilde{m \cdot g}}$$

~~$W = -90\,ft \cdot 185\,lb \cdot 9.81 \frac{m}{s^2} = -90 \cdot 185\,lb \cdot 32{,}19 \frac{ft}{s^2}$~~

~~$W = 535.963{,}5$ lb-ft~~ ~~$W = -2\,90 \cdot 185\,lb$~~

~~SI: $W = -27{,}43\,m \cdot 83.91 \cdot 9.81 \frac{m}{s^2} = 22\,581{,}7$ Nm~~ J

$$W = +\underset{ft}{90} \cdot \underset{lb}{185} = 16.650{,}00 \text{ lb-ft}$$

$\boxed{22.477.5 \text{ J}}$

GOTOVA FORMULA OD INTERNETA

1 lb-ft $= 1.35582$ Nm

J (=) Nm

DIMENZIONALNO

~~$W = 22{,}43 \cdot 83.91 = \ldots$ Nm~~

1 lb + 1 ~~pound~~ feet / s^2 (=) $0.45\,kg \times 0.305\,m / s^2 = 0.13825$ N

$\boxed{1N = \frac{kg\,m}{s^2}}$ $\boxed{1 \text{ lb-ft}/s^2 = 0.13825 N}$

$$\boxed{1\,kp = 1kg \cdot 9{,}81 \frac{m}{s^2} = 9.81\,N}$$

$9.81 \frac{m}{s^2}$ $9{,}81$

$$16.650\ lb\text{-}ft = \underbrace{16.650 \cdot 0.45}_{kg} \cdot \underbrace{0.305}_{m} = 2285{,}212 \cdot 9{,}81 = 22417\ Nm$$

$\boxed{Exc\,46}$ AMPER LOW

$$\int_C \vec{B} \cdot d\vec{r} = \mu_0 I$$

B - MAGNETIC FIELD

μ_0 - PERMEABILITY OF FREE SPACE

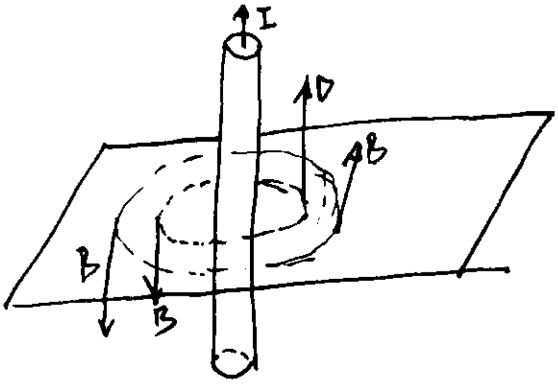
$C: \ x^2 + y^2 = r^2$

show that $|\vec{B}| = B = \dfrac{\mu_0 I}{2\pi r}$

$x = r\cos t \quad y = r\sin t$

$r(t) = r\cos t\,\vec{i} + r\sin t\,\vec{j}$

$t = 0..2\pi$



$$I = \int_C \vec{B} \cdot d\vec{r} = \int_0^{2\pi} \left(B_x\vec{i} + B_y\vec{j}\right)\left(r\cos t\,\vec{i} + r\sin t\,\vec{j}\right) dt$$

$$I = \int_0^{2\pi} -B_x \cdot r \cdot \sin t + B_y \cdot r\cos t \ dt = -B_x r \int_0^{2\pi} \sin t\,dt + B_y r \int_0^{2\pi} \cos t\,dt$$

$$I = +B_x r\cos t \Big|_0^{2\pi} + B_y r \cdot \sin t \Big|_0^{2\pi} = \emptyset \quad \text{LOGICNO}$$

$$\vec{a} \cdot \vec{a} = \left(x_1\vec{i} + y_1\vec{j} + z_1\vec{k}\right) \cdot \left(x_1\vec{i} + y_1\vec{j} + z_1\vec{k}\right) =$$

$$= x_1^2 + y_1^2 + z_1^2$$

$\vec{B}$ TANGENT TO A CIRCLE $\vec{r}$

$$\vec{B} = |\vec{B}| \cdot \vec{n} = |\vec{B}| \cdot \vec{T} \qquad |\vec{T}|^2 = 1$$

$$I = \int_C \vec{B} \cdot d\vec{r} = \int_C |\vec{B}| \cdot \underbrace{\vec{T} \cdot \vec{T}}_{dot}\, ds = \int_C |\vec{B}|\, ds$$

$$ds = \sqrt{x'^2(t) + y'^2(t)}\,dt = \underline{r \cdot dt}$$

$$\int_C |\vec{B}|\,ds = \int_0^{2\pi} |\vec{B}|\, r \cdot dt = |\vec{B}| \cdot r \int_0^{2\pi} dt = |\vec{B}| \cdot r \cdot 2\pi = \mu_0 I$$

$$\boxed{|\vec{B}| = \frac{\mu_0 I}{2\pi r}}$$ PROOFED!!

ALTERNATIVE: (SOLUTION Stewart)

$$\vec{B} = |\vec{B}| \cdot \vec{T} = |\vec{B}| \langle -\sin\theta, \cos\theta \rangle$$

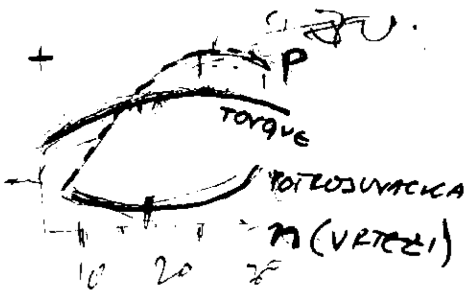
NEGATIVNA NASOKA VO ~~ODNOS~~ SO STRELKITE NA CASOVNIKOT

$$\int_C \vec{B} \cdot dr = \int_0^{2\pi} |\vec{B}| \langle -\sin\theta, \cos\theta \rangle \cdot \langle -r\sin\theta, r\cos\theta \rangle \, d\theta$$

$$= \int_0^{2\pi} |\vec{B}|\, r(\sin^2\theta + \cos^2\theta)\, d\theta = |\vec{B}| \cdot r \cdot 2\pi = I\mu_0$$

$$\boxed{|\vec{B}| = \frac{I \cdot \mu_0}{2\pi r}}$$ PROOFED!!!

THE FUNDAMENTAL THEOREM FOR LINE INTEGRALS



$$\frac{P}{n} = \frac{kg\,m}{s}$$

VRTEZEN MOMENT!!

105 KS

77 kW

250 Nm

1900 min^{-1} = 31,67 vrt/sec

$$\frac{P}{31{,}67} = 250\,Nm \qquad P = \underline{7917.5\,W}$$

$$\int_a^b F'(t)\,dt = F(b) - F(a)$$

$$g(x) = \int_b^x f(t)\,dt \qquad \boxed{g' = f}$$

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

$$\boxed{F(t) = \int f(t)\,dt}$$

$$\frac{d\,g(t)}{dt} = \int_b^x f(t)\,dt = f(t)$$

$$\int_a^b f(x)\,dx = F(b) - F(a) \qquad \underline{\underline{F'(x) = f(x)}} \Rightarrow$$

NET CHANGE THEOREM.

$$\boxed{\int_a^b F'(x)\,dx = F(b) - F(a)} \ ⊛$$

$$g(x) = \int_a^x f(t)\,dt \quad \text{THEN} \quad g'(x) = f(x)$$

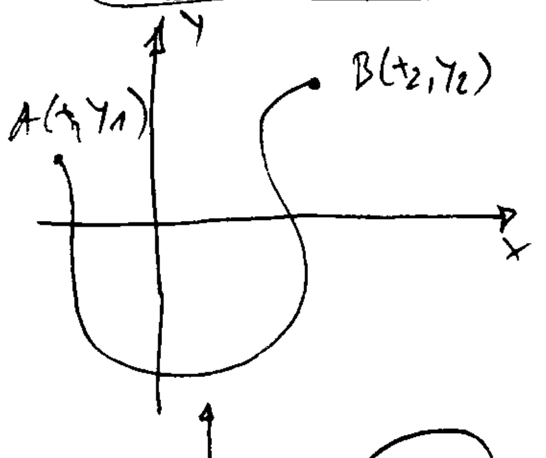
$$\boxed{g'(x) = f(x) = \frac{d}{dx}\int_a^x f(t)\,dt} = \underline{\underline{\int_a^x f'(t)\,dt}}$$

- IF WE THINK OF GRADIENT ∇f OF A FUNCTION $f(x,y,z)$ AS A SORT OF DERIVATE OF "f" THEN THE VERSION OF FUNDAMENTAL CALCULUS THEOREM IS:

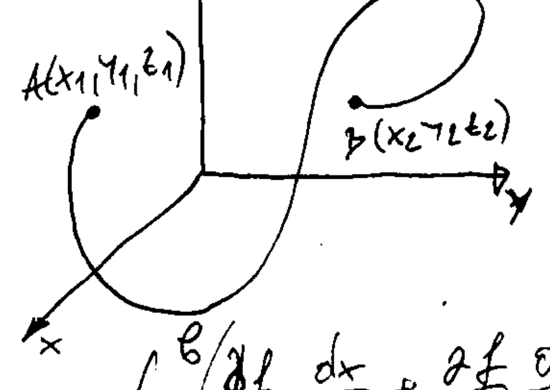
THEOREM 2 "C" smooth curve given by $\vec{r}(t)$, $a \le t \le b$; ∇f is <u>continuous</u> on "C". Then:

$$\boxed{\int_C \nabla f \cdot d\vec{r} = f(r(b)) - f(r(a))}$$

MMV



$$\underbrace{\int_C \nabla f \cdot d\vec{r}}_{\text{LINE INTEGRAL OF } \nabla f \text{ IS NET CHANGE IN "}f\text{"}} = f(x_2,y_2) - f(x_1,y_1)$$



$$\int_C \nabla f \cdot dr = f(x_2,y_2,z_2) - f(x_1,y_1,z_1)$$

<u>PROOF</u>:

$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f \cdot \vec{r}'(t)\,dt =$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right)dt = \int_a^b \frac{d}{dt} f(\vec{r}(t))\,dt = f(\vec{r}(b)) - f(\vec{r}(a))$$

⊛ →

$\boxed{\text{Exp 1}}$ $\vec{F}(\vec{x}) = \frac{m \cdot MG}{|x|^2} \cdot \vec{x}$ $(3,4,12)$ TO $(2,2,0)$

$W = \int_C \vec{F} \cdot d\vec{r}$ $r(t) = (1-t)\langle 3,4,12\rangle + t\langle 2,2,0\rangle$

$r(t) = \langle 3,4,12\rangle + t\langle -1,-2,-12\rangle = \langle 3,4,12\rangle - t\langle 1,2,12\rangle$

$x = 3 - t$ $\quad y = 4 - 2t$ $\quad z = 12 - 12t$ $\quad t = 3 - x$, $t \in (0,1)$

$W = -\int_0^1 \frac{m \cdot MG}{|x|^3} (\vec{i} + \vec{j} + k) \cdot \left((3-t)\vec{i} + (4-2t)\vec{j} + (12-12t)\vec{k}\right) dt$

$W = -m \cdot M \cdot G \int_0^1 \frac{(3-t) + (4-2t) + (12-12t)}{\left[(3-t)^2 + (4-2t)^2 + (12-12t)^2\right]^{3/2}} dt$

$\boxed{\vec{F}(\vec{x}) = \nabla f(x,y,z)}$ $\quad f(x,y,z) = \boxed{\frac{mMG}{\sqrt{x^2+y^2+z^2}}}$

$\nabla f(x,y,z) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$

$\frac{\partial f}{\partial x} = -\frac{1}{\cancel{2}} \frac{mMG \cdot \cancel{2}x}{\sqrt{(x^2+y^2+z)^3}} = -\frac{mMG\,x}{\sqrt{(x^2+y^2+z)^3}}$

$\frac{\partial f}{\partial y} = -\frac{m \cdot M \cdot G\,y}{\sqrt{(x^2+y^2+z^2)^3}}$; $\frac{\partial f}{\partial z} = -\frac{mMGz}{\sqrt{(x^2+y^2+z^2)^3}}$

$F(x) = -\frac{mMG}{\sqrt{(x^2+y^2+z^2)^3}} \cdot (x\vec{i} + y\vec{j} + zk)$

$\boxed{F(\vec{x}) = -\frac{mMG}{|\vec{x}|^2} \cdot \vec{x}}$

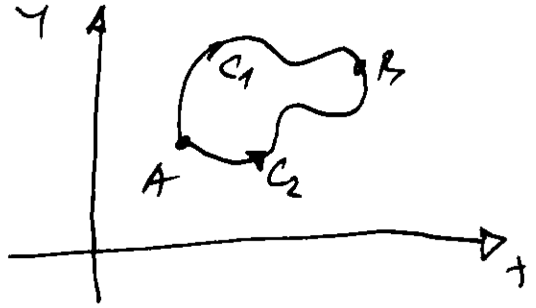
$\boxed{\text{MMV}}$

$\boxed{W = \int_C \vec{F} \cdot d\vec{r} = \int_0^1 \nabla f(\vec{r}(t))d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0))}$

$= -\frac{mMG}{\sqrt{9+16+144}} + \frac{mMG}{\sqrt{4+4+0}} = mMG\left(\frac{-1}{\sqrt{169}} + \frac{1}{\sqrt{8}}\right) = mMG \cdot \left(-\frac{1}{13} + \frac{\sqrt{2}}{2\sqrt{2}}\right)$

INDEPENDENCE OF PATH

$$\int_{C_1} \vec{F}\cdot d\vec{r} \neq \int_{C_2} \vec{F}\, d\vec{r}$$

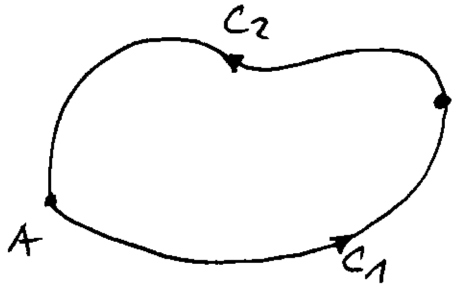


$$\boxed{\int_{C_1} \nabla f\cdot dr = \int_{C_2} \nabla f\, dr}$$

MMV

INTEGRAL OF CONSERVATIVE VECTOR FIELD DEPENDS ONLY ON THE INITIAL ~~[illegible]~~ AND TERMINAL POINT OF ~~[illegible]~~ A CURVE.
I.E INDEPENDENT OF PATH!!

has POTENTIAL FUNCTION $f(x,y,z)$



$$\int_{C} \vec{F}\, d\vec{r} = \int_{C_1} \vec{F}\, d\vec{r} + \int_{C_2} \vec{F}\, d\vec{r} =$$

$$= \int_{C_1} \vec{F}\, dr - \int_{-C_2} \vec{F}\, dr = 0$$

C_1 & $-C_2$ HAVE SAME INITIAL AND END POINTS

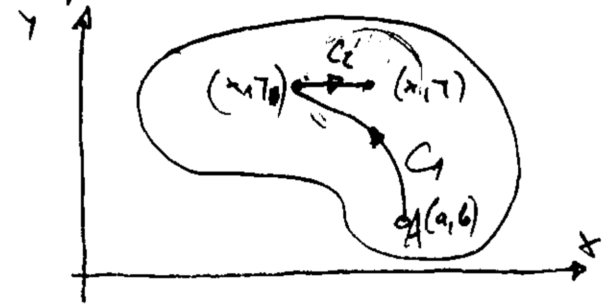
IF $\int_C \vec{F}\, d\vec{r} = 0$ FOR ANY C IN DOMAIN D

$$0 = \int_C \vec{F}\, d\vec{r} = \int_{C_1} \vec{F}\, d\vec{r} + \int_{-C_2} \vec{F}\, d\vec{r} = \int_{C_1} \vec{F}\, d\vec{r} - \int_{C_2} \vec{F}\, d\vec{r} \Rightarrow$$

$$\boxed{\int_{C_1} \vec{F}\, d\vec{r} = \int_{C_2} \vec{F}\, d\vec{r}}$$

THEOREM > $\int_C \vec{F}\, d\vec{r}$ IS INDEPENDENT OF PATH „D" IF AN ONLY IF $\int_C \vec{F}\, d\vec{r} = 0$ FOR EVERY CLOSED PATH „C" IN „D"!!

- IF „$\vec{F}$ IS CONSERVATIVE VECTOR FIELD THEN: $\int_C \vec{F}\, d\vec{r} = 0$



- „D" IS OPEN I.E. IT DOESN'T CONTAIN ANY OF IT'S BOUDARY POINTS.
- „D" IS CONNECTED I.E. ANY TWO POINTS IN D CAN BE JOINED BY PATH LEING IN „D".

Theorem 4 LET $\vec{F}$ IS A VECTOR FIELD THAT IS CONTINUOUS ON OPEN CONNECTED REGION D. IF $\int_C \vec{F} d\vec{r}$ IS INDEPENDENT OF PATH IN "D" THEN $\vec{F}$ IS CONSERVATIVE VECTOR FIELD ON "D"; i.e. THERE EXIST FUNCTION f SUCH THAT $\nabla f = \vec{F}$.

PROOF: $A(a,b)$ FIXED POINT IN „D"

$$f(x,y) = \int_{(a,b)}^{x,y} \vec{F} \cdot d\vec{r} \qquad \text{FOR ANY } (x,y) \text{ IN } D.$$

(x_1, y) $\quad x_1 < x$

NE ZAVISI OD „x"

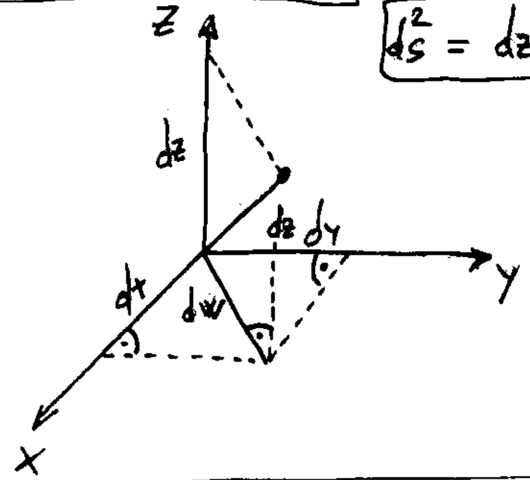
$$f(x,y) = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} = \int_{(a,b)}^{(x_1 y)} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

$$\frac{\partial f(x,y)}{\partial x} = 0 + \frac{\partial}{\partial x} \int_{C_2} \vec{F} \cdot d\vec{r} \qquad \boxed{\vec{F} = P\vec{i} + Q\vec{j}}$$

$$\int_{C_2} \vec{F} \circ d\vec{r} = \int_{C2} P\,dx + Q\,dy$$

ON „C_2" Y IS CONSTANT $\Rightarrow dy = 0$

$$\boxed{ds^2 = dz^2 + dw^2 = dx^2 + dy^2 + dz^2}$$



$$\int_C \vec{F} \circ \vec{T} \cdot ds = \int_C \vec{F} \circ \vec{T} \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$x \triangleq t \qquad x_1 \leq t \leq x$

$$\frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x} \int_{C_2} P\,dx + Q\,dy = \frac{\partial}{\partial x} \int_{x_1}^{x} P(t,y)\,dt = \underline{\underline{P(x,y)}}$$

$$\boxed{g(x) = \int_a^x f(t)\,dt \qquad g'(x) = f(x)}$$

SIMILARLY: $$\frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y} \int_{C_2} P\,dx + Q\,dy = \frac{\partial}{\partial y} \int_{y_1}^{y} Q(x,t)\,dt = Q(x,y)$$

$$\vec{F} = P(x,y)\vec{i} + Q(x,y)\vec{j} = \frac{\partial f(x,y)}{\partial x}\vec{i} + \frac{\partial f(x,y)}{\partial y}\vec{j} = \nabla f$$

- HOW IS IT POSSIBLE TO DETERMINE WHETER OR NOT A VECTOR FIELD $\vec{F}$ IS CONSERVATIVE?

$$\boxed{\vec{F} = P\vec{i} + Q\vec{j} \qquad \vec{F} = \nabla f \qquad P = \frac{\partial f}{\partial x} \ \& \ Q = \frac{\partial f}{\partial y}}$$

BY CLAIRAUT'S THEOREM

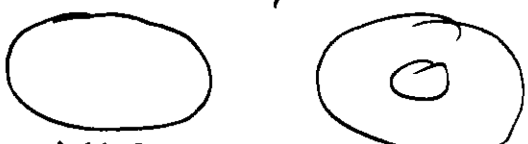
$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}}$$

MMV

THEOREM 5 IF $\vec{F}(x,y)\vec{i} + Q(x,y)\vec{j}$ IS CONSERVATIVE VECTOR FIELD WHERE $P \& Q$ HAVE CONTINOUS FIRST ORDER DERIVATES ON DOMAIN D, THEN THROUGHOUT "D" WE HAVE:

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$$

- SIMPLE CURVE (DOESN'T INTERSECT ITSELF)
- SIMPLY-CONNECTED REGION



YES NO

THEOREM 6 $\boxed{\vec{F} = P\vec{i} + Q\vec{j}}$ VECTOR FIELD ON AN OPEN SIMPLY-CONNECTED REGION D. IF "P" & "Q" HAVE CONTINOUS FIRST DERIVATES AND

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$ THROUGH D

THEN F IS CONSERVATIVE.

EXP 2 $\vec{F}(x,y) = (x-y)\vec{i} + (x-2)\vec{j}$ WHETHER $F(x,y)$ IS CONSERVATIVE?

$P = (x-y) \qquad Q = x-2 \qquad \boxed{\frac{\partial P}{\partial y} = -1 \qquad \frac{\partial Q}{\partial x} = 1}$

$\frac{\partial P}{\partial x} \neq \frac{\partial Q}{\partial y} \Rightarrow \vec{F}(x,y)$ IS NOT CONSERVATIVE!

EXP 3 $F(x,y) = (3+2xy)\vec{i} + (x^2-3y^2)\vec{j}$ WHETHER IS CONSERVATIVE?

$\frac{\partial P}{\partial y} = 2x \qquad \frac{\partial Q}{\partial x} = 2x \qquad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \to$ CONSERVATIVE!!

DOMAIN IS ENTIRE PLAN $D = R^2$

$\boxed{\text{EXP. 4}}$ ⓐ $F(x,y) = (3+2xy)\vec{i} + (x^2-3y^2)\vec{j}$ $f = ?$ SUCH $\vec{F} = \nabla f$

ⓑ EVALUATE $\int_C \vec{F}\cdot dr$ WHERE $C: r(t) = e^t \sin t\,\vec{i} + e^t \cos t\,\vec{j}$ $0 \le t \le \pi$

MY IDEA!

ⓐ $f(x,y) = \int_{(a,b)}^{(x,y)} \vec{F}\, d\vec{r}$ $\quad \dfrac{\partial f(x,y)}{\partial x} = \int_{C_2} P(x,y)\,dx + Q(x,y)\,dy$

$$\frac{\partial f(x,y)}{\partial x} = \int_{x_1}^{x} P(t,y)\,dt = P(x,y)$$

$$f(x,y)\Big|_x = \int P(x,y)\,dx = 3x + \cancel{2}\frac{x^2 y}{\cancel{2}} = \boxed{3x + x^2 y}$$

$$f(x,y)\Big|_y = \int Q(x,y)\,dy = x^2 y - \cancel{3}\frac{y^3}{\cancel{3}} = x^2 y - y^3$$

$$f(x,y)\Big|_x + f(x,y)\Big|_y = 3x + 2x^2 y - y^3 = f(x,y)$$

$$\frac{\partial f(x,y)}{\partial x} = 3 + 4x \qquad \frac{\partial f(x,y)}{\partial y} = 2x^2 - 3y^2$$

IF: $f(x,y) = 3x + x^2 y - y^3$

$$\boxed{\frac{\partial f}{\partial x} = 3 + 2xy \qquad \frac{\partial f}{\partial y} = x^2 - 3y^2}$$

$f_x(x,y) = 3 + 2xy \Big/ \int dx$ $\qquad f_y(x,y) = x^2 - 3y^2$

$f(x,y) = 3x + x^2 y + g(y) \Big/ \dfrac{\partial}{\partial y}$

$f_y(x,y) = x^2 + g'(y)$ $\Rightarrow g'(y) = -3y^2$

$g(y) = -\int 3y^2\,dy = -\cancel{3}\dfrac{y^3}{\cancel{3}} + K = \underline{-y^3 + K}$

$$\boxed{f(x,y) = 3x + x^2 y - y^3 + K}$$

ⓑ $\int_C \vec{F}\cdot d\vec{r} = f(r(b)) - f(r(a))$

$f(x,y) = 3x + x^2y - y^3 + K$

$\vec{r}(t) = e^t \sin t\,\vec{i} + e^t \cos t\,\vec{j} \qquad \underbrace{0}_{a} \le t \le \underbrace{\pi}_{b}$

$x = e^t \sin t \qquad y = e^t \cos t$

$f(x,y) = 3e^t \sin t + e^{2t}\sin^2 t \cdot e^t \cos t - e^{3t}\cos^3 t + K$

$f(r(\pi)) = 3e^{\pi}\underbrace{\sin\pi}_{0} + e^{2\pi}\underbrace{\sin^2\pi}_{0}\cdot e^{\pi}\cos\pi - e^{3\pi}\cos^3\pi + K$

$f(r(\pi)) = -e^{3\pi}\cos^3\pi = -e^{3\pi}(-1)^3 = e^{3\pi} + K$ 1

$f(r(0)) = 3e^0\underbrace{\sin 0}_{0} + e^{20}\underbrace{\sin^2 0}_{0}\cdot e^0\cos 0 - e^{30}\underbrace{\cos^3 0}_{1} + K = -1 + K$

$\int \vec{F}\,d\vec{r} = f(r(\pi)) - f(r(0)) = e^{3\pi} + \cancel{K} + 1 - \cancel{K} = e^{3\pi} + 1$

---

or: $r(0) = \langle 0, 1\rangle \qquad r(\pi) = (0, -e^{\pi})$

$\int_C \vec{F}\,d\vec{r} = f(0, -e^{\pi}) - f(0,1) = +e^{3\pi} - (-1) = e^{3\pi} + 1$

$\boxed{\text{EXP. 5}}$ $F(x,y,z) = y^2\vec{i} + (2xy + e^{3z})\vec{j} + 3ye^{3z}\vec{k} \quad f = ?$

SUCH AS: $\boxed{\nabla f = F(x,y,z)}$

$f_x(x,y,z) = y^2 \qquad f_z(x,y,z) = 3ye^{3z}$ (**)

$f_y(x,y,z) = 2xy + e^{3z}$ (*)

$\boxed{g(y,z) \text{ CONSTANT WITH RESPECT TO } x}$

$f(x,y,z) = y^2x + g(y,z)$

$f(x,y,z) = xy^2 + e^{3z}y + k(x,z)$

$f(x,y,z) = 3y\frac{1}{3}e^{3z} = y\cdot e^{3z} + m(x,y)$

• $\frac{\partial f}{\partial y} = 2yx + g_y'(y,z)$; (*) $\Rightarrow g_y'(y,z) = e^{3z}$

$\boxed{g(y,z) = e^{3z}\cdot y + C} \qquad \boxed{g_y(y,z) = \frac{\partial g(y,z)}{\partial y}}$ (A)

• $\frac{\partial f}{\partial z} = 3e^{3z}y + \underbrace{k'(x,z)}_{0 \leftarrow (**)} \qquad k'(x,z) = 0 \qquad \boxed{k = C}$

$f(x,y,z) = y^2x + e^{3z}\cdot y + C$

 $f_x = y^2$; $\quad f_y = 2yx + e^{3z} \quad f_z = 3e^{3z}\cdot y$

$$\frac{\partial f}{\partial z} = 3y \cdot e^{3z} + m'(x,y) \Leftarrow (**) \Rightarrow m'(x,y) = \emptyset$$

$$\boxed{m(x,y) = y^2x + C}$$

ALTERNAT: (1) $\Rightarrow$ $g(y,z) = e^{3z} \cdot y + h(z)$

$$f(x,y,z) = y^2x + e^{3z} \cdot y + h(z)$$

$$\frac{\partial f}{\partial z} = 3e^{3z} \cdot y + h'(z) \; ; \; (**) \Rightarrow h'(z) = \emptyset \quad \underline{h(z) = K}$$

$$\boxed{f(x,y,z) = y^2x + e^{3z}y + K} \quad \underline{\nabla f = \vec{F}}$$

(6) CONSERVATION OF ENERGY

- CONTINOUS FORCE $\vec{F}$ MOVING AN OBJECT ALONG A PATH "C" GIVEN BY:

$r(t) \quad a \le t \le b \qquad \vec{r}(a) = A \qquad \vec{r}(b) = B$

- NEWTON SECOND LAW:

$$\boxed{\vec{F}(\vec{r}(t)) = m \cdot \vec{r}''(t)}$$

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt = \int_a^b m \cdot \vec{r}''(t) \circ \vec{r}'(t)\, dt$$

$$\frac{d}{dt}[r'(t) \circ r'(t)] = r''(t) \circ r'(t) + r'(t) \circ r''(t) = \underline{2 \cdot r''(t) \circ r'(t)}$$

$$W = \frac{m}{2} \int_a^b \frac{d}{dt}[r'(t) \circ r'(t)]\, dt = \frac{m}{2} \int_a^b \frac{d}{dt} |r'(t)|^2\, dt$$

$$\left| \frac{d}{dx} \int_a^x F(t)\, dt = F(x) \qquad \int_a^b F'(t)\, dt = F(b) - F(a) \right|$$

$$W = \frac{m}{2} |r'(t)|^2 \Big|_a^b = \frac{m}{2} \left(|r'(b)|^2 - |r'(a)|^2 \right)$$

$$W = \frac{1}{2} m |r'(b)|^2 - \frac{1}{2} m |r'(a)|^2 = \underbrace{\frac{1}{2} m |v(b)|^2}_{\text{KINETIC ENERGY OF OBJECT}} - \frac{1}{2} m |v(a)|^2$$

$$\boxed{W = K(B) - K(A)}$$

- ASSUME $\vec{F}$ IS CONSERVATIVE FORCE VECTOR FIELD

$$\vec{F} = \nabla f = -\nabla P$$

$$W = \int_C \vec{F}\, d\vec{r} = \int_C \nabla f\, d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) = -\int_C \nabla P\, d\vec{r} = +P(\vec{r}(a)) - P(\vec{r}(b))$$

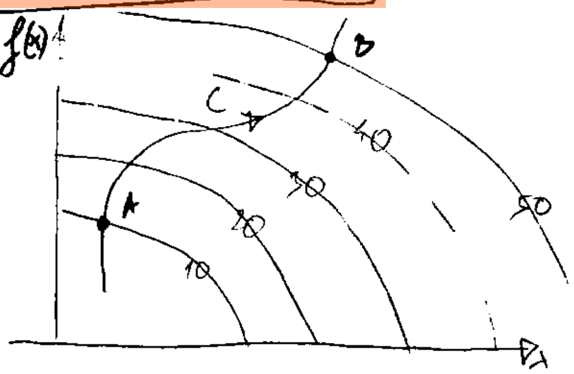
$$W = P(A) - P(B) \qquad W = K(B) - K(A)$$

$$P(A) - P(B) = K(B) - K(A) \Rightarrow \boxed{P(A) + K(A) = K(B) + P(B)}$$

- LAW OF CONSERVATION OF ENERGY (REASON WHY $\vec{F}$ IS CALLED CONSERVATIVE)

16.3 EXERCISES

Ex.1 $f(x)$



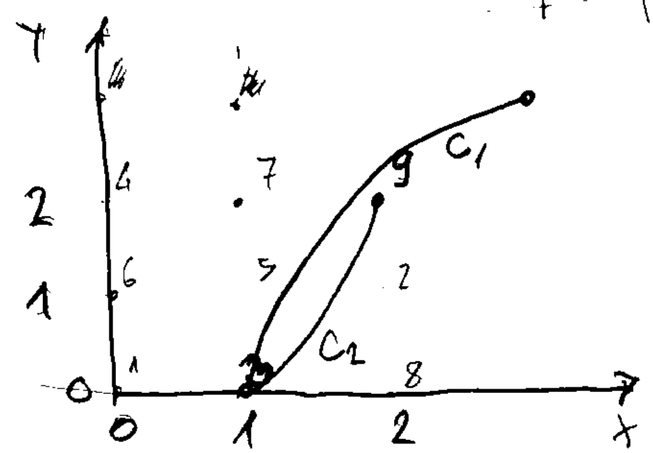
$$\int_C \nabla f\, dr = f(B) - f(A) = 50 - 10 = 40$$

Ex.2 $\int_C \nabla f \cdot d\vec{r} = ?$ $\quad C: x = t^2+1,\ y = t^3+t,\ 0 \le t \le 1$

$$t = \sqrt{x-1} \qquad y = \sqrt{(x-1)^3} + \sqrt{x-1} = \sqrt{x-1}\,(x-1+1)$$

$$\boxed{y = x\sqrt{x-1}}$$

y \ x	0	1	2
0	1	6	4
1	3	5	7
2	8	2	9



$$\boxed{\int_C \nabla f \cdot dr = f(2,2) - f(1,0) = 9 - 3 = 6}$$

$$r'(t) = 2t\vec{i} + (3t^2+1)\vec{j}$$

$$3t^2+1 \neq 0 \Rightarrow r'(t) \neq 0$$

$\Rightarrow r(t)$ IS smooth CURVE $\Rightarrow \nabla f$ IS CONTINUOUS

VO MAPLE ZA CRTAM C^4

Exc. 9 WHETHER $\vec{F}$ IS CONSERVATIVE VECTOR FIELD?
FIND f SUCH $\nabla f = \vec{F}$

$$\vec{F}(x,y) = \underbrace{(ye^x + \sin y)}_{P(x,y)}\vec{\imath} + \underbrace{(e^x + x\cos y)}_{Q(x,y)}\vec{\jmath}$$

$$\frac{\partial P}{\partial y} = e^x + \cos y \qquad \frac{\partial Q}{\partial x} = e^x + \cos y$$

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}} \Rightarrow \vec{F} \text{ IS CONSERVATIVE}$$

$$f_x(x,y) = ye^x + \sin y$$
$$f_y(x,y) = e^x + x\cos y$$

$$f(x,y) = \int f_x(x,y)\,dx = ye^x + x\sin y + g(y)$$

$$\frac{\partial f(x,y)}{\partial y} = e^x + x\cos y + \underbrace{g'(y)}_{=0} \qquad g(y) = \underline{\underline{C}}$$

$$\boxed{f(x,y) = e^x + x\cos y + C}$$

Exc. 11 $F(x,y) = \langle 2xy, x^2\rangle = \underbrace{2xy}_{P(x,y)}\vec{\imath} + \underbrace{x^2}_{Q(x,y)}\vec{\jmath}$ A(1,2) B(3,2)

$$\frac{\partial P}{\partial y} = 2x \qquad \frac{\partial Q}{\partial x} = 2x \qquad \underline{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 2x}$$ CONSERVATIVE

$$F(x,y) = \nabla f \qquad \int_C \vec{F}\,d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$$

$$f_x(x,y) = 2xy \qquad f_y(x,y) = x^2$$
$$f(x,y) = \int 2xy\,dx = x^2y + g(y)$$
$$\frac{\partial f}{\partial y} = x^2 + g'(y) \qquad g'(y) = 0 \quad g(y) = C$$
$$f(x,y) = x^2y + C \qquad \int_C \vec{F}\,dr = f(3,2) - f(1,2) = 9\cdot 2 - 2 = \underline{\underline{16}}$$

$\boxed{\text{Exc. 15}}$ $F(x,y,z) = yz\vec{i} + xz\vec{j} + (xy+2z)\vec{k}$

$C: \ (1,0,-2)$ to $(4,6,3)$ $\quad \int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = ?$

$\vec{r}(t) = (1-t)\langle 1,0,-2\rangle + t\langle 4,6,3\rangle =$

$= \langle 1,0,-2\rangle + t\langle 3,6,5\rangle$

$\boxed{x = 1+3t \quad y = 6t \quad z = -2+5t}$ $\quad x \in (1,4)$

$t = \frac{x-1}{3} \Rightarrow t \in (0,1)$

$f_x(x,y,z) = yz \quad f_y(x,y,z) = xz \quad f_z(x,y,z) = xy+2z$

$f(x,y,z) = \int yz\,dx = \underline{xyz} + g(y,z)$

$f_y(x,y,z) = xz + g_y'(y,z) \quad g_y'(y,z) = 0 \quad g(y,z) = \underline{\underline{h(z)}}$

$f_y(x,y,z) = xz \quad \boxed{f(x,y,z) = xyz + h(z)}$

$\frac{\partial f}{\partial z} = xy + h'(z) \Rightarrow h'(z) = 2z \quad \underline{h(z) = z^2 + C}$

$\boxed{f(x,y,z) = xyz + z^2 + C}$

$\frac{\partial P}{\partial y} = z \quad \frac{\partial Q}{\partial x} = z$

$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A) = f(4,6,3) - f(1,0,-2) =$

$= 4\cdot 6\cdot 3 + 3^2 + C - \left(1\cdot 0\cdot(-2) + 4 + C\right) = 72+9+\cancel{C}-4-\cancel{C}$

$= 81-4 = 77$

$\boxed{\text{Exc. 21}}$ $W = ?$ $\quad F(x,y) = 2y^{3/2}\vec{i} + 3x\sqrt{y}\,\vec{j} \quad P(1,1)\ Q(2,4)$

$W = \int_C \vec{F}\,d\vec{r} = \int_C \nabla f \cdot d\vec{r} \quad \frac{\partial P}{\partial y} = \underline{\underline{3y^{1/2}}} = \frac{\partial Q}{\partial x} = \underline{3\sqrt{y}}$

$f_x(x,y) = 2y^{3/2} \quad f_y = 3x\sqrt{y}$

$f(x,y) = \int f_x(x,y)\,dx = 2y^{3/2}x + g(y) \quad f_y(x,y) = \underline{\underline{3\sqrt{y}}}\,x + g'(y)$

$g'(y) = 0 \quad g(y) = C \quad f(x,y) = 2xy^{3/2} + C$

$W = f(2,4) - f(1,1) = 2\cdot 2\cdot\sqrt{4^3} - 2\cdot 1\cdot 1 = 4\cdot 8 - 2 = 32-2 = 30$

Exc. 24 $F(x,y) = (2xy + \sin y)\vec{i} + (x^2 + x\cos y)\vec{j}$

$\frac{\partial P}{\partial y} = 2x + \cos y \qquad \frac{\partial Q}{\partial x} = 2x + \cos y$ } CONSERVATIVE

Exc. 27 $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$

(*) $\boxed{\begin{matrix} P & Q & R \\ x & y & z \end{matrix}} = \boxed{\begin{matrix} Q & R & P \\ z & x & y \end{matrix}} \qquad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} \qquad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$

$F = \nabla f$ i.e. $\boxed{P = f_x \quad Q = f_y \quad R = f_z}$

CLAIRAUT'S THEOREM

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial Q}{\partial x}$$

$$f_{xy} = f_{yx}$$

$$\frac{\partial P}{\partial z} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial z \partial x} = \frac{\partial^2 f}{\partial x \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial R}{\partial x}$$

$$f_{xz} = f_{zx}$$

$$\frac{\partial Q}{\partial z} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial z \partial y} = \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial R}{\partial y}$$

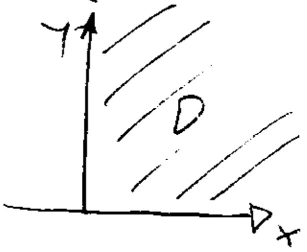
$$f_{yz} = f_{zy}$$

MMV

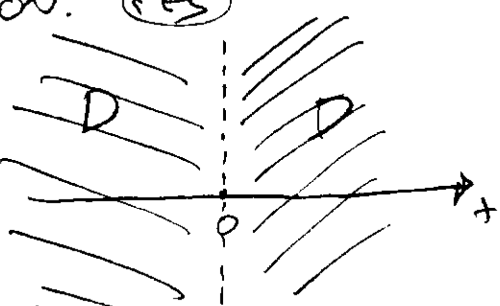
SUŠTINATA E DA SE KORISTI CLAIRAUT TEOREMATA A NE DA PAMTAM NA PAMET KAKO POKAŽANO (*)

Exc. 29 WHETHER OR NOT GIVEN SET IS (a) OPEN, (b) CONNECTED, AND (c) SIMPLY-CONNECTED

$\{(x,y) \mid x > 0, y > 0\}$



(a) OPEN YES
(b) CONNECTED YES
(c) SIMPLY CON. YES

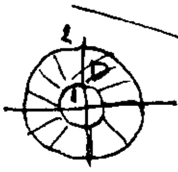


Exc. 30 $\{(x,y) \mid x \neq 0\}$

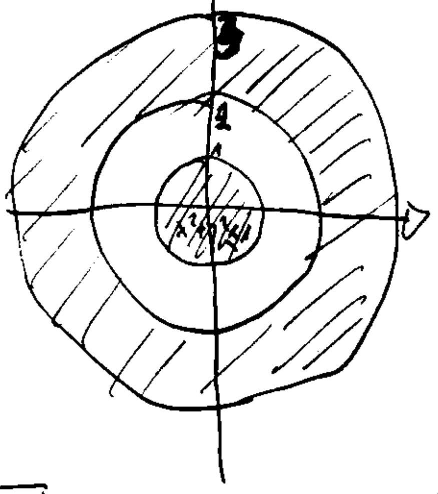
(a) YES (b) NO (c) NO

Exc. 31 $\{(x,y) \mid 1 < x^2 + y^2 < 4\}$

(a) YES (b) YES (c) NO



$\boxed{32}$ $\{(x,y) \mid x^2+y^2 \le 1 \text{ OR } 4 \le x^2+y^2 \le 9\}$



$x^2+y^2 \le 1$ (a) YES, (b) YES (c) YES

$4 \le x^2+y^2 \le 9$ (a) NO, (b) YES (c) NO

OR

(a) NO (b) NO (c) NO

$\boxed{\text{Ex. 33}}$ $F(x,y) = \dfrac{-y\vec{i} + x\vec{j}}{x^2+y^2}$

(a) $$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right) = -\frac{x^2+y^2-2y\cdot y}{(x^2+y^2)^2} = -\frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial Q}{\partial x} = \frac{x^2+y^2-x\cdot 2x}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2} = -\frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\left.\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}\right\} \Rightarrow \vec{F}(x,y) \text{ IS CONSERVATIVE}$$

(b) $\int_C \vec{F}\cdot d\vec{r}$ IS NOT INDEPENDENT OF PATH

C: $\boxed{\begin{array}{l} x = \cos t \\ y = \sin t \end{array}}$

$r = \cos t\,\vec{i} + \sin t\,\vec{j}$

$x^2+y^2 = \cos^2 t + \sin^2 t = 1$

$\boxed{F(x,y) = -\sin t\,\vec{i} + \cos t\,\vec{j}}$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot dr + \int_{C_2} \vec{F}\cdot d\vec{r}$$

C_1: $y \ge 0$ $\quad t = 0..\pi$

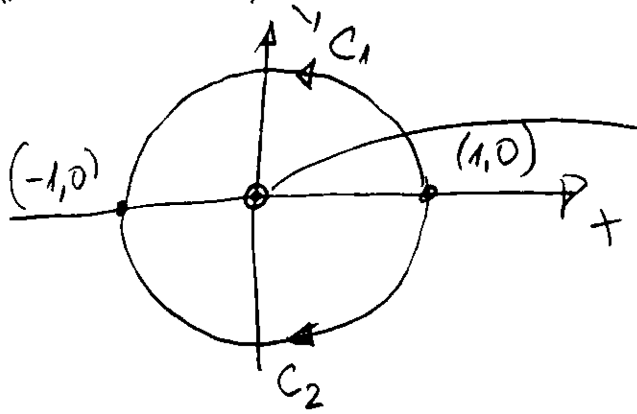
C_2: $y \le 0$ $\quad t = \pi..2\pi$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_0^{\pi} (-\sin t\,\vec{i} + \cos t\,\vec{j})(-\sin t\,\vec{i} + \cos t\,\vec{j})\,dt$$

$$= \int_0^{\pi} (\sin^2 t + \cos^2 t)\,dt = \pi - 0 = \pi$$

$$-\int_{\pi}^{2\pi}\vec F\cdot dr = -\int_{\pi}^{2\pi}\left(-\sin t\,\vec i+\cos t\,\vec j\right)\left(-\sin t\,\vec i+\cos t\,\vec j\right)dt = -(2\pi-\pi)=$$

$$\int_{C_2}\vec F\,d\vec r=\int_{2\pi}^{\pi}\vec F\,\vec r\,'(t)\,dt$$



DOMAIN OF $\vec F$ IS $\mathbb{R}^2$ EXCEPT THE ORIGIN $\Rightarrow$ HENCE THE DOMAIN IS NOT SIMPLY CONECTED !!

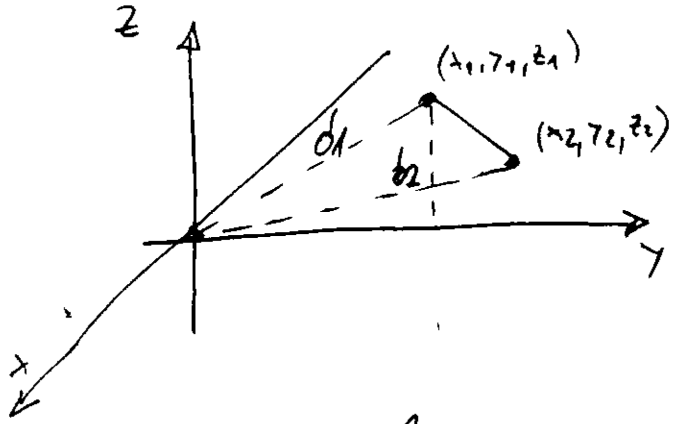
$\boxed{\text{Ex. 34}}$ $\quad \vec F(r)=\dfrac{c\,\vec r}{|\vec r|^3} \qquad \vec r = x\vec i+y\vec j+z\,k$

(1) $W=?$ $\quad \int_C \vec F\,d\vec r \qquad \vec F(r)=\dfrac{c}{(x^2+y^2+z^2)^{3/2}}\left(x\vec i+y\vec j+z\,k\right)$

$$f_x=\frac{c\cdot x}{(x^2+y^2+z^2)^{3/2}}$$

$$f_y=\frac{c\cdot y}{(x^2+y^2+z^2)^{3/2}}$$

$$f_z=\frac{c\,z}{(x^2+y^2+z^2)^{3/2}}$$



$$f(x,y,z)=\int f_x(x,y,z)\,dx=\int\frac{c\,x\,dx}{(x^2+y^2+z^2)^{3/2}}=-\frac{c}{\sqrt{x^2+y^2+z^2}}+g(y,z)$$

$$f_y(x,y,z)=-\left(-\frac12\right)\frac{c\cdot 2y}{\sqrt{(x^2+y^2+z^2)}^{3}}+g'(y,z)=\frac{y\cdot c}{\sqrt{(x^2+y^2+z^2)}^{3}}+g'(y,z)$$

$g'(y,z)=0 \qquad g(y,z)=h(z) \qquad \boxed{f(x,y,z)=-\dfrac{c}{\sqrt{x^2+y^2+z^2}}+h(z)}$

$$f_z(x,y,z)=\frac{z\cdot c}{\sqrt{(x^2+y^2+z^2)}^{3}}+h'(z) \qquad h'(z)=0 \qquad \underline{h(z)=K}$$

$$\boxed{f(x,y,z)=-\frac{c}{\sqrt{x^2+y^2+z^2}}+K}$$

$$\int_C\vec F\,dr=\int_C\nabla f\,d\vec r=f(d_2)-f(d_1)=-\frac{c}{d_2}+\frac{c}{d_1}=\frac{d_2-d_1}{d_1 d_2}\cdot c$$

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b) $\vec{F} = -\frac{mMG}{|\vec{r}|^3}\vec{r}$ $\int_C \vec{F}\,dr = \frac{c}{d_1} - \frac{c}{d_2}$

• W = ? DONE BY GRAVITATION FIELD WHEN ~~EARTH~~ MOVE FROM APHELION (MAX DISTANCE FROM SUN $1.52\cdot10^{8}$ km) TO PERIHELION (MIN DISTANCE $1.47\cdot10^{8}$ km)

$m = 5.97\cdot10^{24}$ kg $M = 1.99\cdot10^{30}$ kg $G = 6.67\cdot10^{-11}\ \frac{Nm^2}{kg^2}$

$C = -m\cdot MG = -5.97\cdot10^{24}\cdot1.99\cdot10^{30}\cdot6.67\cdot10^{-11}\ Nm^2$

$C = -79.241601\cdot10^{54-11} = -79{,}241601\cdot10^{43}$

$W = C\left(\frac{1}{d_1} - \frac{1}{d_2}\right) = -79{,}242\cdot10^{43}\left(\frac{1}{1{,}52\cdot10^{11}} - \frac{1}{1{,}47\cdot10^{11}}\right)$

$W = 79{,}241601\left(\frac{1}{1{,}47} - \frac{1}{1{,}52}\right)\cdot10^{32} = 1{,}7732188\cdot10^{32}\ Nm\ [J]$

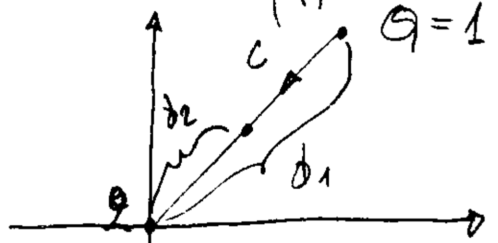
$\left(\frac{1}{1.52\cdot10^{11}} - \frac{1}{1{,}47\cdot10^{11}}\right) = -0{,}022377\cdot10^{-11}$

$\left(\frac{1}{1.52\cdot10^{8}} - \frac{1}{1.47\cdot10^{8}}\right) = -0{,}022377\cdot10^{-8} = -2{,}2377\cdot10^{-10}$ $\frac{Nm^2}{km}$

$W = 79{,}241601\cdot10^{43}\cdot0{,}022377\cdot10^{-8} = 1.7678\cdot10^{35}$

c) $\vec{E} = \frac{\varepsilon q Q\vec{r}}{|\vec{r}|^3}$ $q = -1.6\cdot10^{-19}$ C

$\overline{qQ} = 10^{-12}$ m $= d_1$ $d_2 = 0.5\cdot10^{-12}$



$C = \varepsilon\cdot q\cdot Q = 8{,}985\cdot10^{9}(-1.6)\cdot10^{-19}\cdot1$

$C = -14.3760\cdot10^{-9}$

$W = -14{,}376\cdot10^{3}\left(\frac{1}{10^{-12}} - \frac{2}{10^{-12}}\right)$

$\boxed{W = 14{,}376\cdot10^{3}\ J}$ $\boxed{W \doteq 1.4\cdot10^{4}\ J}$

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ch 16.4 GREEN'S THEOREM

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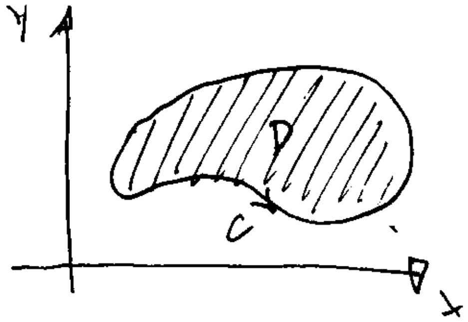

$r(t)\quad a \le t \le b$

POSITIVELY ORIENTED IF DIRECTION IS COUNTERCLOCKWISE !!

GREEN'S THEOREM:

HAV

$$\boxed{\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$

NOTATION:

$$\oint_C P\,dx + Q\,dy = \oint_C P\,dx + Q\,dy =$$

IF C IS CLOSED AND HAVE POSITIVE ORIENTATION

$$= \int_{\partial D} P\,dx + Q\,dy$$

$$\boxed{\int_C P\,dx + Q\,dy = \int_C \vec{F} \cdot d\vec{r}} \; *$$

$$\boxed{\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\partial D} P\,dx + Q\,dy}$$

$$* \quad \int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{r}'(t)\,dt = \int_C \left(P\vec{i} + Q\vec{j} + R\vec{k}\right)\left(x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k}\right) dt$$

$$= \int_C P\,x'(t)\,dt + Q\,y'(t)\,dt + R\,z'(t)\,dt = \int_C P\,dx + Q\,dy + R\,dz$$

~~[illegible]~~

~~[illegible]~~ [illegible]

$$\int_a^b F'(x)\,dx = F(b) - F(a) \qquad \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\partial D} P\,dx + Q\,dy$$

- Proof of GREEN'S THEOREM (WHERE D IS SIMPLE REGION)

~~[illegible]~~ ?

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$$\int_C P\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA \qquad *$$

$$\int_C Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA \qquad **$$

Proof of ⊛: We chose D TO BE TYPE I REGION

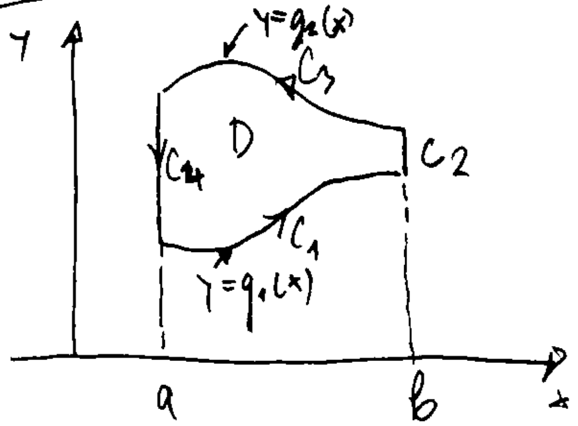
$$\oint_C P\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA \qquad \oint_C Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA$$

$$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

$$\iint_D \frac{\partial P}{\partial y}\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P(x,y)}{\partial y}\,dy\,dx = \int_a^b \underbrace{\Big[P(x, g_2(x)) - P(x, g_1(x))\Big]}_{\Delta\Delta}\,dx$$

$$\int_a^b f(x)\,dx = F(b) - F(a) \Leftarrow \boxed{f = F'}$$ F ANTIDERIVATIVE OF "f"

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$ ⇒ NET CHANGE THEOREM



$$\int_{C_1} P\,dx = \int_a^b P(x, g_1(x))\,dx$$

C_1: $\underbrace{x = x \quad y = g_1(x) \quad a \le x \le b}_{\text{PARAMETRIC EQUATION}}$

x - IS TAKEN AS PARAMETER

$$\int_{C_2} P\,dx = \int_{C_4} P\,dx = 0 \qquad dx = 0 \Rightarrow x \text{ IS CONSTANT}$$

$-C_3$: $x = x \quad y = g_2(x) \quad a \le x \le b$

$$\int_{-C_3} P(x,y)\,dx = \int_a^b P(x, g_2(x))\,dx = -\int_{C_3} P(x,y)\,dx$$

$$\int_{C_3} P(x,y)\,dx = -\int_a^b P(x, g_2(x))\,dx$$

$$\int_C P(x,y)\,dx = \int_{C_1} P\,dx + \underbrace{\int_{C_2} P\,dx}_{\to 0} + \int_{C_3} P\,dx + \underbrace{\int_{C_4} P\,dx}_{\to 0} =$$

$$= \int_a^b P(x, g_1(x))\,dx - \int_a^b P(x, g_2(x))\,dx = \int_a^b \Big[P(x, g_1(x)) - P(x, g_2(x))\Big]\,dx$$ $\Delta\Delta$

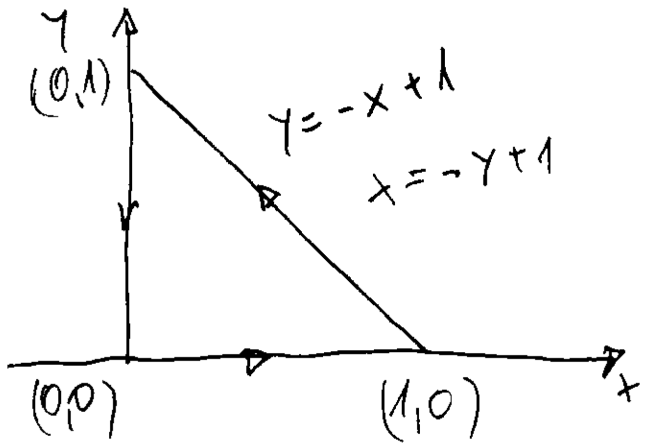
$$\boxed{\int_C P\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA} \quad (\Delta \text{II})$$

- ANALOGNO SO KORISTENJE NA REGION D OD TIP NA D SE DOBIVA:

$$\boxed{\int_C Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA} \quad (*\Delta)$$

$$(*\Delta) + (\Delta \text{II}) \Rightarrow \boxed{\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial y}\right) dA}$$

$\boxed{\text{Exp. 1}}$ $\int_C \overset{P}{x^4}\,dx + \overset{Q}{xy}\,dy$



$$I = \int_C x^4\,dx + xy\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy =$$

$$= \left/ \frac{\partial Q}{\partial x} = y \quad \frac{\partial P}{\partial y} = 0 \right/ =$$

$$= \iint_D y\,dx\,dy = \iint_D y\,dy\,dx$$

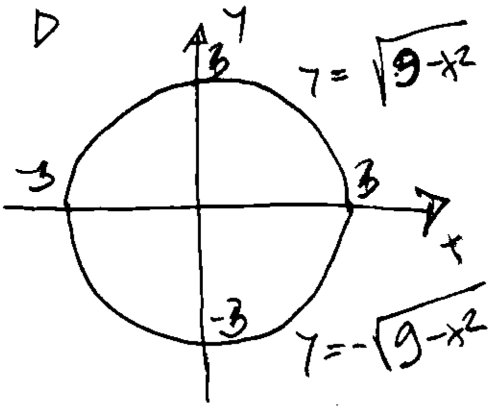
$$I = \int_0^1 \left(\int_0^{-x+1} y\,dy \right) dx = \int_0^1 \frac{y^2}{2}\Big|_0^{-x+1} dx = \int_0^1 \frac{(-x+1)^2}{2}\,dx$$

$$= \frac{1}{2}\int_0^1 (1-x)^2\,dx = -\frac{1}{2}\int_0^1 (1-x)^2\,d(1-x) = -\frac{(1-x)^3}{2\cdot 3}\Big|_0^1 = -\frac{1}{2}\left(0 - \frac{1}{3}\right) = \frac{1}{6}$$

$\boxed{\text{Exp. 2}}$ $I = \oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy \quad \boxed{C: x^2 + y^2 = 9}$

$$\frac{\partial P}{\partial y} = 3 \quad \frac{\partial Q}{\partial x} = 7 \qquad I = \iint_D (7-3)\,dx\,dy$$

$\iint_D (7-3)\,dx\,dy$ $\qquad C: x^2+y^2=9$



$I = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 4\cdot dy\,dx =$

$= \int_{-3}^{3} 4\cdot 2\sqrt{9-x^2}\,dx = 8\underbrace{\int_{-3}^{3}\sqrt{3^2-x^2}\,dx}_{\frac{9}{2}\pi}$

$I = 8\cdot\frac{9}{2}\pi = 4\cdot 9\pi = 36\pi$

$I = \int_{-3}^{3}\int_{9}^{6} 4\cdot dx\,dy = \int_{0}^{2\pi}\int_{0}^{3} 4\cdot r\,dr\,d\theta = \int_{0}^{2\pi} 4\cdot\frac{r^2}{2}\Big|_0^3 d\theta$

$= \int_{0}^{2\pi} 2\cdot 9\cdot d\theta = 18\cdot\theta\Big|_0^{2\pi} = \underline{\underline{36\cdot\pi}}$

$\iint_D 4\cdot dx\,dy = 4\cdot(r^2\pi) = 4\cdot 9\pi = \underline{\underline{36\pi}}$