ⓑ $I(g(X);Y)$ vs $I(X,Y)$

$H(g(X)|X)=0$ $\quad H(Y|X)=H(Y|X,g(X))$; $H(X|Y)\geq H(g(X)|Y)$

$$I(g(X);Y)=H(g(X))-H(g(X)|Y)\leq H(X)-\underbrace{H(g(X)|Y)}_{?}$$

$$I(X,g(X);Y)=I(X;Y)+\underbrace{I(g(X);Y|X)}_{=0}=$$

$$=I(g(X);Y)+I(X;Y|g(X))$$

$$I(g(X);Y|X)=\underbrace{H(g(X)|X)}_{0}-\underbrace{H(g(X)|X,Y)}_{0}=0$$

$$I(X;Y)=I(g(X);Y)+\underbrace{I(X;Y|g(X))}_{\geq 0}\Rightarrow$$

$$\boxed{I(X;Y)\geq I(g(X);Y)}$$

DOKAŽANO!!!

- MARKO E POKAŽANO OD DATA-PROCESSING INEQUALITY KADE $I(X;Y)\geq I(X;Z)$

$I(X;Z)\leq I(X;Y)$ i.e.

ⓒ $H(X_0|X_{-1})$ vs. $H(X_0|X_{-1},X_1)$

(X_{-1},X_0,X_1) $\quad H(X_{-1},X_0,X_1)=H(X_{-1})+H(X_0|X_{-1})+$
$+H(X_1|X_{-1}X_0)$

$H(X_0|X_{-1})\geq H(X_0|X_{-1},X_1)$ CONDITIONING REDUCES ENTROPY.

$D(p\|q)\geq 0$ $\quad \sum_{x\in X} p(x)\log\frac{p(x)}{q(x)}\geq 0$

$I(X;Y)=H(X)-H(X|Y)\geq 0$ $\quad \boxed{H(X)\geq H(X|Y)}$

$$I(X_0;X_1|X_{-1})=H(X_0|X_{-1})-H(X_0|X_{-1},X_1)\geq 0\Rightarrow$$

$$\boxed{H(X_0|X_{-1})\geq H(X_0|X_{-1},X_1)}$$

DOKAŽANO!!!

ⓓ $H(X,Y)/(H(X)+H(Y))$ vs. 1

$$H(X,Y)=H(X)+\underbrace{H(Y|X)}_{\text{CONDITIONING REDUCES ENTROPY}}\leq H(X)+H(Y)$$

$H(X,Y)\leq H(X)+H(Y)$ $\quad \boxed{\frac{H(X,Y)}{H(X)+H(Y)}\leq 1}$ DOKAŽANO!!!

• <u>HW2S SOLUTIONS:</u> (ISTITE REZULTATI NO MALKU POLAGANJE [illegible])

(a) $Y = f(X) = 5X$, f" IS BIJECTIVE FUNCTION i.e

$$X = f^{-1}(Y) = g(Y)$$

- ZNAEME DEKA : $H(X) \geq H(f(X)) = H(5X)$
- OD DRUGA STRANA $H(Y) \geq H(g(Y)) = H(X)$

$$H(5X) \geq H(X)$$

ZNAČI EDNAŠ DOBIVAŠ DEKA $H(X) \geq H(5X)$ A EDNAŠ DEKA $H(5X) \geq H(X) \Rightarrow \boxed{H(X) = H(5X)}$

(b) $I(g(X); Y)$ vs. $I(X; Y)$

MOŽE DA SE FORMIRA MARKOV CHAIN:

$$Y \rightarrow X \rightarrow \underbrace{g(X)}$$

→ OVA SEGMENTO NE ZAVISI OD Y T.E SE RABOTI ZA MARKOV LANEC

- DATA PROCESSING INEQUALITY:

$$I(Y; X) \geq I(Y; g(X)) \Rightarrow I(X; Y) \geq I(g(X); Y)$$

$$p(X, Z | Y) = \frac{p(X, Y, Z)}{p(Y)} = \frac{p(X,Y) \cdot p(Z|X,Y)}{p(Y)} = \underline{p(X|Y) \cdot p(Z|Y)}$$

$\boxed{2.43}$ MUTUAL INFORMATION OF HEADS AND TAILS

(a) CONSIDER A FAIR COIN FLIP. WHAT IS THE MUTUAL INFORMATION BETWEEN TOP AND BOTTOM SIDES OF THE COIN?

(b) A SIX-SIDE FAIR DIE IS ROLLED. WHAT IS THE MUTUAL INFORMATION BETWEEN THE TOP SIDE AND THE FRONT SIDE (THE SIDE MOST FACING YOU)?

(a) ~~$X \in \{PETKA, GLAVA\} = \{T, H\}$~~

~~$p(x) = \{\frac{1}{2}, \frac{1}{2}\}$~~

$X \in \{TAIL, NON\text{-}TAIL\}$ $Y \in \{HEAD, NON\text{-}HEAD\}$

p(x,y)

X\Y	H	NH	p(x)
T	0	$\frac{1}{2}$	$\frac{1}{2}$
NT	$\frac{1}{2}$	0	$\frac{1}{2}$
p(y)	$\frac{1}{2}$	$\frac{1}{2}$	

$$I(X;Y) = H(X) - H(X|Y)$$

P(X|Y)

X\Y	H	NH
T	0	1
NT	1	0
▨	▨	▨

$$P(X|Y) = \frac{P(X,Y)}{P(Y)}$$

$$H(X) = \frac{1}{2} ld 2 + \frac{1}{2} ld 2 = \frac{1}{2} + \frac{1}{2} = 1$$

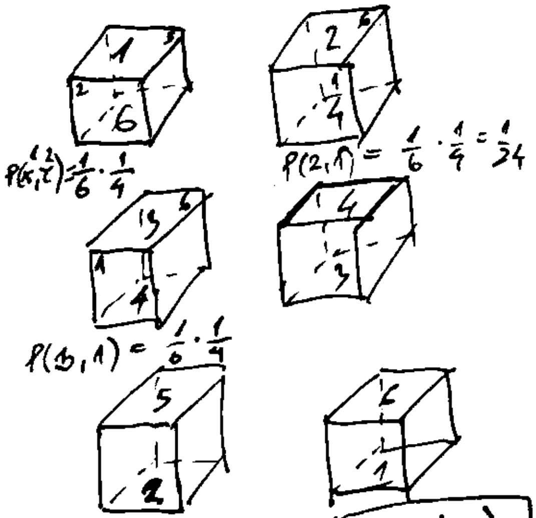
$$H(X|Y) = \sum_{x,y} p(x,y)\, ld \frac{1}{P(x|y)} = 0\, ld 0 + \frac{1}{2} \overset{0}{\cancel{ld 1}} + \frac{1}{2} \overset{0}{\cancel{ld 1}} = \underline{\underline{0}}$$

$$I(X;Y) = H(X) - \underbrace{H(X|Y)} = 1 - 0 = 1\ bit$$

LOGIČNO E! AKO GO ZNAEŠ "Y" SIGURNO GO ZNAEŠ "X"

⑥ X = GORNA STRANA ∈ {1,2,3,4,5,6}

Y = PREDNA STRANA ∈ {1,2,3,4,5,6}



P(X,Y)

X \ Y	1	2	3	4	5	6	P(X)
1	0	$\frac{1 \cdot 1}{6 \cdot 4}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	0	$\frac{1}{6}$
2	$\frac{1}{24}$	0	$\frac{1}{24}$	$\frac{1}{24}$	0	$\frac{1}{24}$	$\frac{1}{6}$
3	$\frac{1}{24}$	$\frac{1}{24}$	0	0	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{6}$
4	$\frac{1}{24}$	$\frac{1}{24}$	0	0	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{6}$
5	$\frac{1}{24}$	0	$\frac{1}{24}$	$\frac{1}{24}$	0	$\frac{1}{24}$	$\frac{1}{6}$
6	0	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	0	$\frac{1}{6}$
P(Y)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

P(Y|X)

X \ Y	1	2	3	4	5	6
1	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0
2	1/4	0	1/4	1/4	0	1/4
3	1/4	1/4	0	0	1/4	1/4
4	1/4	1/4	0	0	1/4	1/4
5	1/4	0	1/4	1/4	0	1/4
6	0	1/4	1/4	1/4	1/4	0

$$I(X,Y) = H(Y) - H(Y|X)$$

$$H(Y) = \cancel{6} \cdot \frac{1}{\cancel{6}} ld 6 = ld 6 = 2.58$$

$$H(Y|X) = -\sum_{x,y} \underline{\underline{P(x,y)}}\, ld P(Y|X)$$

$$= \cancel{24} \cdot \frac{1}{\cancel{24}} \cdot ld 4 = ld 4 = 2$$

$$\boxed{I(X,Y) = ld 6 - ld 4 = ld \frac{3}{2} = 0{,}585}$$

- POZNAVANJETO NA VREDNOSTA GORNATA STRANA JA NAMALUVA NEIZVESNOSTA NA VREDNOSTA NA PREDNATA STRANA ZA CELI 2 BITA!!!

- ISTO GO REČE VO $\boxed{HW2S}$ SO TOA ŠTO DIREKTNO ODAT SO FORMULATA ZA TRANSFORMACIJA

- $I(B;T) = H(B) - \underset{0}{\underline{H(B|T)}} =$ (AKO GO ZNAEŠ TOP GO ZNAEŠ BOTTOM) $= H(B) - 0 = 2 \cdot \frac{1}{2} ld 2 = 1$

- $I(T;F) = H(T) - H(F|T) = \cancel{6} \cdot \frac{1}{\cancel{6}} ld 6 - 4 \cdot \frac{1}{4} ld 4 = ld \frac{3}{2}$ } AKO RAZGLEDUVA DADENA GORNA STRANA ZA PREDNATA STRANA IMA 4 MOŽNOSTI SO EDNAKVI VEROJ

- PRIMEROT NA REŠENIETO VO HW2S TE OMRAZUVA DA NE ODIŠ SO CELATA FORMULA ZA $H(Y|X)$ TUKU SO SKLAPENATA T.E $Y|X$ GO TRETIRAŠ KAKO EDNA SLUČAJNA PROMENLIVA ZA DA NE ODIŠ SO FORMULATA $-\sum_{XY} P(XY) \operatorname{ld} P(Y|X)$.

- SEPAK MOETO REŠENIE E MATEMATIČKI POIZBRZANO A REŠENIETO VO HW2S E POPLASTIČNO I POINŽENERSKO.

$\boxed{2.44}$ PURE RANDOMNES. WE WISH TO USE THREE-SIDED COIN TO GENERATE FAIR COIN TOSS. LET THE COIN X HAVE PROBABILITY MASS FUNCTION

$$\boxed{p_C = 1 - p_A + p_B}$$

$$x = \begin{cases} A, & p_A \\ B, & p_B \\ C, & p_C \end{cases}$$

WHERE p_A, p_B, p_C ARE UNKNOWN.

(a) HOW WOULD YOU USE TWO INDEPENDENT FLIPS X_1, X_2 TO GENERATE (IF POSSIBLE) BERNOULLI$\left(\frac{1}{2}\right)$ RANDOM VARIABLE, Z?

(b) WHAT IS THE RESULTING MAXIMUM EXPECTED NUMBER OF FAIR BITS GENERATED!

(a) AA, AB, AC, , BB, BA, BC, CC, CA, CB

$$H(X) = -p_A \operatorname{ld} p_A - p_B \operatorname{ld} p_B - p_C \operatorname{ld} p_C$$

$$X \in \{A, B, C\} \qquad W \in \{W_1, W_2, \dots W_{2^n}\}$$

$$\boxed{W_i = \{W_{i1}, W_{i②}\}}_{\hookrightarrow n=2} \qquad \boxed{H(W) = n\,H(X)}$$

$$H(X) = -p_A \operatorname{ld} p_A - p_B \operatorname{ld} p_B - (1-p_A-p_B) \operatorname{ld} p_C =$$

$$= \underline{-p_A \operatorname{ld} p_A} - \underline{p_B \operatorname{ld} p_B} - \operatorname{ld} p_C + \underline{p_A \operatorname{ld} p_C} + \underline{p_B \operatorname{ld} p_C} =$$

$$= p_A \operatorname{ld} \frac{p_C}{p_A} + p_B \operatorname{ld} \frac{p_C}{p_B} - \operatorname{ld} p_C = \operatorname{ld}\left(\frac{p_C}{p_A}\right)^{p_A} + \operatorname{ld}\left(\frac{p_C}{p_B}\right)^{p_B} - \operatorname{ld} p_C$$

$$= \operatorname{ld}\left[\left(\frac{p_C}{p_A}\right)^{p_A} \cdot \left(\frac{p_C}{p_B}\right)^{p_B} \cdot \frac{1}{p_C}\right]$$

$$P(W) = \{p_A^2, p_A p_B, p_A p_C, p_B^2, p_B p_A, p_B p_C, p_C^2, p_C p_A, p_C p_B\}$$

$$\underset{\frac{1}{8}}{p_A} + \underset{\frac{3}{8}}{p_B} = \frac{1}{2} \qquad p_C = \frac{1}{2} \qquad \underbrace{\left\{\frac{1}{8^2}, \frac{1}{8}\cdot\frac{3}{8}, \frac{1}{8}\cdot\frac{1}{2}, \dots\right\}}_{\text{NEMA}}$$

$\{AA, AB, AC, BB, BA, BC, CC, CA\ CB\}$

$H(X) \geq H(f(X))$

$\{\underbrace{AA, AB, AC, BA, CA}_{\frac{1}{2}}, \underbrace{BB, BC, CC, CB}_{\frac{1}{2}}\}$

$$\left.\begin{aligned} p_A^2 + 2p_Ap_B + 2p_Ap_C &= \frac{1}{2} \\ p_B^2 + 2p_Bp_C + p_C^2 &= \frac{1}{2} \\ p_A + p_B + p_C &= 1 \end{aligned}\right\}$$

$x^2 + 2xy + 2xz = \frac{1}{2}$

$y^2 + 2yz + z^2 = \frac{1}{2}$

$x + y + z = 1$

$z = 1 - (x+y)$

$x^2 + \cancel{2xy} + 2x - 2x^2 - \cancel{2xy} = \frac{1}{2}$

$2x^2 + \cancel{4}x - \cancel{4}x^2 = 1$

$-2x^2 + 4x - 1 = 0$

$2x^2 - 4x + 1 = 0$

$x_{1,2} = \frac{4 \pm \sqrt{16 - 8}}{4}$

$x_{1,2} = 1 \pm \frac{1}{4}\sqrt{8} = 1 \pm \frac{1}{4}2\sqrt{2} = 1 - \frac{\sqrt{2}}{2} = \boxed{\frac{2-\sqrt{2}}{2}} = \underline{\underline{0.293}}$

$y^2 + 2y(1-x-y) + (1-(x+y))^2 = \frac{1}{2} =$

$= y^2 + 2y - \frac{2-\sqrt{2}}{\cancel{2}} \cdot \cancel{2}y - 2y^2 + \left(1 - \frac{2-\sqrt{2}}{2} - y\right)^2 = \frac{1}{2}$

$= y^2 + \cancel{2y} - \cancel{2y} + \sqrt{2}y \underline{-2y^2} + \left(\frac{\cancel{2}-\cancel{2}+\sqrt{2}}{2} - y\right)^2 = -y^2 + \sqrt{2}y + \left(\frac{\sqrt{2}}{2} - y\right)^2 =$

$= -y^2 + \sqrt{2}y + \left(\frac{2}{4} - 2\frac{\sqrt{2}}{\cancel{2}}y + y^2\right) = \cancel{-y^2} + \cancel{\sqrt{2}y} + \frac{1}{2} - \cancel{\sqrt{2}y} + \cancel{y^2} =$

$\Rightarrow \boxed{\frac{1}{2} = \frac{1}{2}}$

$z = 1 - x - y = 1 - \frac{2-\sqrt{2}}{2} - y$

$\boxed{z = \frac{\sqrt{2}}{2} - y}$

$z = \frac{2-2+\sqrt{2}}{2} - y = \frac{\sqrt{2}}{2} - y$

IF: $y = \frac{1}{4} \Rightarrow z = \frac{\sqrt{2}}{2} - \frac{1}{4} = \underline{0.707 - 0.25 = 0.457}$

$\boxed{0.457 + 0.293 + 0.25 = 0.750 + 0.250 = 1}$

$y^2 + 2y\left(\frac{\sqrt{2}}{2} - y\right) + \left(\frac{\sqrt{2}}{2} - y\right)^2 = \cancel{y^2} + \cancel{y\sqrt{2}} - \cancel{2y^2} + \frac{1}{2} - \cancel{\sqrt{2}y} + \cancel{y^2} = \frac{1}{2}$

$p(x) = \{p_A, p_B, p_Z\} = \{0.293;\ 0.25;\ 0.457\}$

$H(X) = -0.293\,\mathrm{ld}\,0.293 - 0.25\,\mathrm{ld}\,0.25 - 0.457\,\mathrm{ld}\,0.457 = 1.535$

$$H(W) = \sum_{i=1}^{9} p_i(W)\, ld\, p_i(W) = -p_A^2\, ld\, p_A^2 - p_A p_B\, ld\, p_A p_B - \dots - p_C p_D\, ld\, p_C p_D$$

$$\boxed{H(W) = 3.0704} \qquad \boxed{H(W) = 2 \cdot H(X) = 2 \cdot 1.535 = 3.070}$$

- THE MAXIMUM NUMBER OF PURE RANDOM BITS THAT CAN BE PRODUCED FROM THREE SIDE COIN IS:

$$\boxed{4\ H(X)}$$

- VO OBST SLUCAJ:

$$p(x) = \left[\frac{2-\sqrt{2}}{2}, y, \frac{\sqrt{2}}{2} - y\right]$$

$$\boxed{H(X) = -\frac{2-\sqrt{2}}{2}\, ld\, \frac{2-\sqrt{2}}{2} - y\, ld\, y - \left(\frac{\sqrt{2}}{2} - y\right) ld\left(\frac{\sqrt{2}}{2} - y\right)}$$

$$\frac{dH(X)}{dy} = 0 \qquad \left[y\, ld\, y + \left(\frac{\sqrt{2}}{2} - y\right) ld\left(\frac{\sqrt{2}}{2} - y\right)\right]' = 0$$

MAPLE:
MultihopMIMOCap.mw

$$\boxed{y = \frac{\sqrt{2}}{4}} \Rightarrow H(X)\ \text{MAXIMUM}$$

$$H_m(X) = 1.5795 \doteq 1.580$$

⇒ BEST DISTRIBUTION FOR X IS:

$$p(x) = \{p_A, p_B, p_C\} = \left\{\frac{2-\sqrt{2}}{2}, \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{4}\right\}$$

$$\boxed{p(x) = \left\{\frac{2-\sqrt{2}}{2}, \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}\right\}}$$

$$\boxed{H(W) = 2 \cdot 1.580 = 3.160} \Rightarrow \text{MAXIMUM NUMBER OF PURE RANDOM BITS.}$$

2.45 FINITE ENTROPY SHOW FOR A DISCRETE RANDOM VARIABLE $X \in \{1, 2, \dots\}$ IF $E[ld\, X] < \infty$ THEN $H(X) < \infty$.

JENSEN'S INEQUALITY: $E[f(x)] \geq f(E[x])$
IF $f(x)$ IS CONVEX FUNCTION

$$H(X) = -\sum_{x \in X} p(x)\, ld\, p(x) = -E[ld\, p(x)] \geq -ld\, E\{p(x)\}$$

$$E[ld\, p(x)] \leq ld\, E[p(x)]$$

$$E[p(x)] = \sum_{x \in X} p^2(x)$$

$$-E[ld\, p(x)] \geq -ld\, E[p(x)]$$

$ld(x)$ → CONCAVNA FUNKCIJA

$$2^{-H(x)} = 2^{+E[ld\, p(x)]} \leq E\left[2^{ld\, p(x)}\right]$$

$$= E[p(x)] = \sum_{x \in X} p^2(x)$$

$$2^{-H(x)} \leq \sum_{x \in X} p^2(x)$$

$H(x) \geqslant -ld\, E[p(x)] = -ld\left[\sum_{x\in X} p^2(x)\right] =$

$x \in \{1, 2, \ldots\}$ $\quad p(x) = \{p_1, p_2, \ldots\}$

$E[x] = \sum_{x\in X} x\cdot p(x) = \sum_{i=1}^{\infty} i\, p_i \qquad \sum_{i=1}^{\infty} i\cdot x^i = \frac{\otimes}{(1-\otimes)^2}$

$E[ld\, x] = \sum_{x\in X} p(x)\, ld(x) \leq ld\left[\sum_{x\in X} x\cdot p(x)\right] =$

$E[ld\, x] \leq ld\, E[x]$

$\otimes \Rightarrow -H(x)\, \cancel{ld}\, 2^{p'} \leq ld \sum_{x\in X} p^2(x) \qquad H(x) \geqslant ld \sum_{x\in X} p^2(x)$

- ТРЕБА ДА ПОКАЖАМ ДЕКА $H(x) \leq \underline{E[ld\, x]}$

$\sum_{x\in X} p(x)\, ld(x) = \sum_{x\in X} ld\, x^{p(x)} = ld\left\{\prod_{x\in X} x^{p(x)}\right\}$

$x = ld\, 2^x \qquad \sum_{x\in X} x\, p(x) = \sum_{x\in X} ld\, 2^x\, p(x) = E[ld\, 2^x]$

$\leqslant ld\, 2^{E[x]} = E[x]$

$H(x) = -E[ld\, p(x)] \geqslant -ld\, E[p(x)] = -ld \sum_{x\in X} p^2(x)$

$-H(x) \leqslant ld \sum_{x\in X} p^2(x) \qquad \boxed{2^{-H(x)} \leq \sum_{x\in X} p^2(x)}$

$D(p\|q) = \sum_{i=0}^{\infty} p_i\, ld \frac{p_i}{q_i} \geqslant 0 \qquad \sum_{i=0}^{\infty} p_i\, ld\, p_i - \sum_{i=0}^{\infty} p_i\, ld\, q_i \geqslant$

$\sum_{i=0}^{\infty} p_i\, ld\, p_i \geqslant \sum_{i=0}^{\infty} p_i\, ld\, q_i. \qquad -\sum_{i=0}^{\infty} p_i\, ld\, p_i \leq -\sum_{i=0}^{\infty} p_i\, ld\, q_i$

$\boxed{q_i = 2^{-i}} \qquad -\sum_{i=0}^{\infty} p_i\, ld\, p_i \leq -\sum_{i=0}^{\infty} p_i\, ld\, 2^{-i} = +\underbrace{\sum_{i=0}^{\infty} i\, p_i}_{E[X]}$

$\boxed{H(X) \leq +E[X]}$

$\boxed{\text{ИСТОТО Е КОРИСТЕНО ВО ПР. 2.13}}$

$\boxed{ld\, x \leq \frac{1}{\ln 2}(x-1)}$

$\boxed{\ln x \leq x-1}$

$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \ldots$

$\mathrm{ld}\, x \leq \frac{1}{\ln 2}(x-1)$ $\boxed{\ln(x) \leq x-1}$ TAYLOR

$E[\mathrm{ld}(x)] \leq \mathrm{ld}[E(x)] \leq \mathrm{ld}\, E(x)+1] \leq E[x]$

$E[\mathrm{ld}\, x] \leq \frac{1}{\ln 2} E[x-1] = \frac{1}{\ln 2}\left[E[x]-1\right] \geq \frac{1}{\ln 2}\left[H(x)-1\right]$

$\mathrm{ld}\, x \leq \mathrm{ld}(x+1) \leq x$

ZNAČI!

① $E[x] \geq E[\mathrm{ld}(x)]$

② $E[x] \geq H(x)$

$E[\mathrm{ld}(x)] < \infty$ NE IMPLICIRA DEKA $E(x) < \infty$

$H(x) = -\sum_{x\in\mathcal{X}} p(x)\, \mathrm{ld}\, p(x) \geq -\sum_{x\in\mathcal{X}} p(x)\, \mathrm{ld}(p(x)+1) \geq$

$\geq -\sum_{x\in\mathcal{X}} p(x)\cdot p(x) = -\sum_{x\in\mathcal{X}} p^2(x)$

$\mathrm{ld}\, x = \frac{\ln x}{\ln 2}$

$\ln x = \ln 2 \cdot \mathrm{ld}\, x$

$\mathrm{ld}\, Y \geq \frac{1}{\ln 2} - \frac{1}{\ln 2 (Y)}$

$-H(x) \leq \sum_{x\in\mathcal{X}} p^2(x)$

- ISTOTO SE DOBIVA ISO JENSEN'S INEQUALITY

$H(x) = -\sum_{x\in\mathcal{X}} p(x)\, \mathrm{ld}[p(x)] = -E[\underbrace{\mathrm{ld}(p(x))}_{\text{CONCAVNA}}] \geq -\mathrm{ld}\{E[p(x)]\} \geq$

$\geq -\mathrm{ld}\{E[p(x)]+1\} \geq -E[p(x)] = -\sum_{x\in\mathcal{X}} p^2(x)$

$E[\mathrm{ld}(x)] < \infty$ $E[\mathrm{ld}(x)] \leq \mathrm{ld}\{E[x]\}$

$\ln x \leq x-1$ $Y = \frac{1}{x}$ $x = \frac{1}{Y}$ $-\ln Y \leq \frac{1}{Y} - 1$

$\boxed{\ln Y \geq 1 - \frac{1}{Y}}$ $\boxed{Y > 0}$ $Y \in \{1,2,3 \ldots\}$

$p(Y) = \{p_1, p_2, \ldots\}$.

$E[\ln Y] \geq \sum_{Y\in\mathcal{Y}} \left(1-\frac{1}{Y}\right) p(Y) = 1 - \sum_{Y\in\mathcal{Y}} \frac{p(Y)}{Y} = 1 - E\left[\frac{1}{Y}\right]$

$H(\tau) = -\sum_{\tau\in\tau} p(\tau)\, ld\, p(\tau)$ $\qquad ld\, p(\tau) \le p(\tau) - 1$

$\boxed{-ld\, p(\tau) \ge 1 - p(\tau)}$ $\quad p \in [0,1]$

$H(\tau) = -\sum_{\tau\in\tau} p(\tau)\, ld\, p(\tau) \ge \sum_{\tau\in\tau} p(\tau) - p^2(\tau) = 1 - \sum_{\tau\in\tau} p^2(\tau)$

$E[\ln \tau] \ge 1 - E[\frac{1}{\tau}]$ $\qquad \boxed{H(\tau) \ge 1 - \sum_{\tau\in\tau} p^2(\tau) \ge 1 - 2^{-H(\tau)}}$

$H(\tau) + 2^{-H(\tau)} \ge 1 \;/\, ld$ $\qquad \boxed{ld\, H(\tau) - H(\tau) \ge 0}$?

$ld\, H(\tau) \ge H(\tau)$?

$H(\tau) \ge 1 - \sum_{\tau\in\tau} p^2(\tau)$ $\qquad E[\ln \tau] \ge \underline{1 - E[\frac{1}{\tau}]}$

$E[ld\, \tau] \le ld\, E[\tau]$

$H(\tau) = -p_1\, ld\, p_1 - p_2\, ld\, p_2 - p_3\, ld\, p_3 - \dots$

$E[ld\, \tau] = p_1\, ld\, \tau_1 + p_2\, ld\, \tau_2 + p_3\, ld\, \tau_3 \dots$

$\tau \in \{1, 2, 3, \dots\}$

$E[ld\, \tau] = p_2\, ld\, 2 + p_3\, ld\, 3 + p_4\, ld\, 4 + \dots$

$E[ld\, \tau] = \sum_{i=1}^{\infty} p_i\, ld\, i = \sum_{i=1}^{\infty} p_i\, ld\, \frac{1}{(\frac{1}{i})}$

$H(\tau) = \sum_{i=1}^{\infty} p_i\, ld\, \frac{1}{p_i} = \sum_{i=1}^{\infty} p_i \cdot 0 - \sum_{i=1}^{\infty} p_i\, ld\, p_i = \underline{-\sum_{i=1}^{\infty} p_i\, ld}$

$E[ld\, \tau] \ge \frac{1 - E[\frac{1}{\tau}]}{\ln 2} = \left[1 - \sum_{i=1}^{\infty} \frac{p(\tau)}{\tau}\right] \cdot \frac{1}{\ln 2}$

$ld\, p_i = \left| \begin{matrix} \text{IF} \\ p_i = 2^{-i} \end{matrix} \right| = ld\, 2^{-i} = -i$

$H(\tau) = -\sum_{i=1}^{\infty} p_i\, ld\, p_i$ $\qquad E[ld\, \tau] \ge \left[1 - \sum_{i=1}^{\infty} \frac{p(\tau)}{\tau}\right] \cdot \frac{1}{\ln 2}$

$H(\tau) \ge 1 - \sum_{i=1}^{\infty} p^2(\tau)$

ŠTO E POREDEMO ? $\quad \sum_{i=1}^{\infty} \frac{p(\tau)}{\tau} \cdot i < i \quad \sum_{i=1}^{\infty} p^2(\tau)$?

$$\sum_{\tau\in T}\frac{\gamma(\tau)}{\tau} \geqslant \sum_{\tau\in T}\gamma^2(\tau)$$ EDNAKVI SE ZA $\tau = 1$.

VIDI MAKE MultiHopMIMOCapacity.mw 3.27.15

$$\frac{1}{\ln 2}\Bigg[1 - \underbrace{\sum_{\tau\in T}\frac{\gamma(\tau)}{\tau}}_{[0,\dots 1]}\Bigg] \leqslant 1 - \underbrace{\sum_{\tau\in T}\gamma^2(\tau)}_{\leqslant 1}$$

$[0 \dots 1]$

FINAL !!!

$$\boxed{H(x) \geqslant \epsilon[ld(x)]}$$

JAS MISLAM DEKA E ODRAZO T.e. DEKA AKO $H(x) < \infty$ TOGAŠ $\epsilon[ld(X)] \leqslant \infty$ T.e.

$$\underbrace{\epsilon[ld(x)] < H(x) < \infty}_{\Rightarrow\ \epsilon[ld(x)] < \infty}$$

I REARMITE TEST VREDNOSTI ZA $H(X)$, $\epsilon[ld(X)]$ GO PONATO-VAAT OVA. MAKE.

$\boxed{\gamma = \alpha\cdot\beta^i} \Rightarrow$ VO GENERALEN SLUČAJ: $X \in \{i\}$ $i = 1, 2, \dots$

$$\frac{1}{\ln 2} - \frac{3\alpha\beta}{2\ln 2} \leqslant 1 - 2\alpha^2\beta^2$$

$$\frac{1}{\ln 2} - \frac{\alpha}{\ln 2} \leqslant 1 - \alpha^2$$

$$1 - \alpha \leqslant \ln 2 - \ln 2\,\alpha^2$$

$$\boxed{\ln 2\,\alpha^2 - \alpha + 1 - \ln 2 \leqslant 0}$$

OVA E SO POMOŠ NA TEOREM (2.19) Ed.2

$$2 - 3\alpha\beta \leqslant 2\ln 2 - 4\ln 2\,\alpha^2\beta^2$$

$$4\ln 2\,\alpha^2\beta^2 - 3\alpha\beta + 2 - 2\ln 2 \leqslant 0$$

$$\boxed{\gamma = \alpha\cdot\beta^i} \qquad H(x) = \sum_{i=1}^{\infty}\alpha\cdot\beta^i\, ld(\alpha\cdot\beta^i)^{-1} =$$

$$= \sum_{i=1}^{\infty}\alpha\cdot\beta^i\, ld\,\alpha^{-1} + \sum_{i=1}^{\infty}\alpha\cdot\beta^i\cdot i\, ld(\beta^{-1}) = ld(\alpha^{-1})\underbrace{\sum_{i=1}^{\infty}\alpha\cdot\beta^i}_{1} +$$

$$+ ld\,\beta^{-1}\underbrace{\sum_{i=1}^{\infty}(\alpha\cdot\beta^i)\cdot i}_{\epsilon[x]} = ld\,\alpha^{-1} + ld\,\beta^{-1}\cdot\epsilon[X]$$

$$\underset{\alpha,\beta\in[0..1]}{\epsilon[ld\,x]} \leqslant ld\,\epsilon[x] \leqslant \underset{x\in[1,2,\dots]}{\epsilon[x]}.$$

$$H(x) = ld\,\alpha^{-1} + ld\,\beta^{-1}\cdot E[x] \geq ld\,\alpha^{-1} + ld\,\beta^{-1} E[ld(x)]$$

$$E[ld(x)] \leq \frac{H(x) - ld\,\alpha}{ld\,\beta} \qquad \underbrace{H(x) \geq -ld\,\alpha \cdot ld\,\beta\, E[ld(x)]}_{\text{RAZLIČNO OD REZULTATOT VO 2.2.30}}$$

OVA KAZUVA DEKA VO GENERALEN SLUČAJ ~~ZA~~

$p = \alpha\cdot\beta^i$ ZA $\alpha, \beta \in [0..1]$ $\quad H(x) \geq E[ld(x)]$

• AKO REZULTATOT SE BAZIRA NA PROBLEM (2.30) TOGAŠ:

$$E[ld\,x] \leq ld\{\underline{E[x]}\} \underbrace{\leq}_{\text{ZOŠTO}} E[x] \qquad x \in [1, 2, \ldots]$$

$$E[x] = \sum_{i=1}^{\infty} i\,p(i)$$

AKO $E[ld\,x] < \infty$ NE MORA DA ZNAČI DEKA $E[x] < \infty$!!!

NO SEPAK AKO $E[ld\,x] = const \leq \infty$ TOGAŠ

$$D(p\|q) \geq 0 \qquad \sum_{i=0}^{\infty} p_i\, ld\frac{p_i}{q_i} \geq 0$$

$$+\sum_{i=0}^{\infty} p_i\, ld\,p_i - \sum_{i=0}^{\infty} p_i\, ld\,q_i \geq 0 \qquad -\sum_{i=0}^{\infty} p_i\, ld\,p_i \leq -\sum_{i=0}^{\infty} p_i\, ld\,q_i =$$

$$= \left| q_i = \alpha\cdot\beta^i \right| = -\sum_{i=0}^{\infty} p_i\, ld(\alpha\cdot\beta^i) = -\sum_{i=0}^{\infty} p_i\, ld\,\alpha - \sum_{i=0}^{\infty} p_i\, ld\,\beta^i =$$

$$= -ld\,\alpha - ld\,\beta \sum_{i=0}^{\infty} i\,p_i = \underline{-ld\,\alpha - ld\,\beta\, E[x]} \quad ⊛$$

$$\alpha, \beta \in [0..1] \Rightarrow H(x) \leq -ld\,\alpha - ld\,\beta\, E[x]$$

$$H(x) \leq ld\,\alpha^{-1} + ld\,\beta^{-1} E[x] = \left| \begin{matrix} \alpha = 1 \\ \beta = 1/2 \end{matrix} \right| = 0 + E[x]$$

$H(x) \leq E[x]$ PAK SE DOAĐA NA ISTOTO !!

$$\sum_{i=0}^{\infty} \alpha\cdot\beta^i = \alpha\frac{1}{1-\beta} = 1 \qquad \alpha\sum_{i=0}^{\infty} i\,\beta^i = \frac{\alpha\cdot\beta}{(1-\beta)^2} = A$$

$$\overset{=1}{\frac{\alpha}{1-\beta}}\cdot\frac{\beta}{1-\beta} = A \qquad \frac{\beta}{1-\beta} = A \qquad \boxed{\frac{\alpha}{1-\beta} = 1} \qquad \boxed{\alpha = 1-\beta}$$

$$\beta = 1-\alpha$$

$$\frac{1-\alpha}{1-1+\alpha} = A \qquad 1-\alpha = \alpha\cdot A \qquad \alpha(A+1) = 1 \qquad \alpha = \frac{1}{A+1}$$

$$\beta = 1 - \frac{1}{A+1} = \frac{A+1-1}{A+1} = \frac{A}{A+1} \quad //$$

$$\boxed{p = \frac{1}{A+1}\left(\frac{A}{A+1}\right)^i}$$

- AKO TEGNEJ SO ODLAJEN EKEONi

$$E[x] = \sum_{i=0}^{\infty} i\,p(i) = A \qquad \boxed{p(i) = \frac{1}{A+1}\left(\frac{A}{A+1}\right)^{i}}\ ①$$

$$\boxed{\left.\begin{array}{l} E[ld(x)] \leq ld\,\underbrace{E[x]}_{E[x] = A < \infty} \\ (A5) \end{array}\right\} \Rightarrow E[ld(x)] < \infty}$$

$$-H(x) = E\{ld[p(x)]\} \leq ld\,\underbrace{E[p(x)]}_{\in[0..1]} = ld \sum_{i=0}^{\infty} p^2(i) \underset{\text{MACLOREN}}{\leq} E[p(x)]-1$$

ILI:

$$-H(x) = E[ld(p(x))] \leq ld\{\underbrace{E[p(x)]}_{\in[0...1]}\} \leq ld\{E[p(x)]+1\} \leq \underline{\underline{E[p(x)]}}$$

$$N13.12 \Rightarrow +H(x) \leq -ld\alpha - ld\beta \cdot E[x] =$$

$$= -ld\frac{1}{1+A} - ld\frac{A}{1+A}\,E[x] = ld(A+1) + ld\left(\frac{A+1}{A}\right) E[x]$$

$$H(x) \leq ld(A+1) + ld(A+1) \cdot E[x] - ld(A)\,E[x] =$$

$$= ld(A+1)\,[1 + E[x]] - ld A\,E[x] = \underline{(1+A)\,ld(1+A) - A\,ld A}$$

$$\leq (1+A)\cdot\left(1-\frac{1}{A}\right) - A\left(1-\frac{1}{A}\right) = (1+A)\frac{A-1}{A} - \frac{A-1}{A}\cdot A$$

$$= \frac{A-1}{A} \qquad \boxed{H(x) \leq \frac{A-1}{A}}$$

- VO OVOJ SLUČAJ ZA BILO KAKVO A = p(x) SEKOGAŠ E KONVEKSNA FUNKCIJA (VIDI MAPLE MULTIPOP MIMOCapacity.mw (3.27.25))

$$H(x) = E[ld\,p(x)] \leq ld\,E[p(x)] \leq ld\,[E[p(x)]+1] \leq E[p(x)]$$

$$\leq p(E[x]) - 1$$

$$H(x) = \sum_{i=0}^{\infty} p\,ld\frac{1}{p} = -\sum_{i=0}^{\infty} \frac{1}{A+1}\left(\frac{A}{A+1}\right)^{i} ld\,\frac{1}{A+1}\left(\frac{A}{A+1}\right)^{i}$$

$$\text{FOR } A \to 1 \qquad \boxed{p(x) \doteq \left(\frac{1}{2}\right)^{x}}$$

$$H(x) = -E[ld\,p(x)] = -E\left[ld\left(\frac{1}{2}\right)^{x}\right] = E[ld\,2^{x}] = E[x]$$

$$H(X) \leq \underbrace{(1+A)\,ld(1+A) - A\,ld A}_{\text{MAKSIMALNA ENTROPIJA PR.2.30}} = ld(1+A) + \underbrace{\cancel{A\,ld(A+1) - A\,ld A}}_{\text{PRIBLIŽNO ZA GOLEMO A (VEĆE OD } A \geq 4)}$$

$$H(X) \leq \underbrace{ld(1+A) \leq A}$$

$$ld(1+x) \leq x \quad \Big\} \text{ MACLORENOV RED}$$

ZNAČI KONEČNO, AKO SE ZEME PDF-OT DA BIDE KAKO ① TOGAŠ:

$$\boxed{H(X) \leq A = E[X]} \qquad \underline{(\text{ZA } A \geq 4)}$$

BIDEJĆI: $E[ld(x)] \leq ld\,E[X] \leq E[X]$

USLOVOT $E[ld(X)] < \infty$ ZA DA $H(X) < \infty$ UŠTE POVEĆE E SIGURNO ISPOLNET ZOŠTO $E[X] < \infty$.

OVA SEKOGAŠ VAŽI AKO ZEMEŠ $p(x) = 2^{-x}$ RELATIVNA ENTROPIJA $(D(p\|q) \geq 0)$ KAKO DOKAZOT E SO NA PP7.

SUŠTINSKIOT DOKAZ E (1.8).

$\boxed{2.46}$ AXIOMATIC DEFINITION OF ENTROPY. IF WE ASSUME CERTAIN AXIOMS FOR OUR MEASURE OF INFORMATION, WE WILL BE FORCED TO USE A LOGARITMIC MEASURE SUCH AS ENTROPY. SHANNON USED THIS TO JUSTIFY HIS INITIAL DEFINITION OF ENTROPY. IN THIS BOOK WE RELAY MORE ON THE OTHER PROPERTIES OF ENTROPY RATHER THEN ITS AXIOMATIC DERIVATION TO JUSTIFY ITS USE.

IF THE SEQUENCE OF SYMETRIC FUNCTIONS $H_m(p_1, p_2, \ldots p_m)$ SATISFIES THE FOLOWING PROPERTIES:

- NORMALIZATION: $H_2\left(\frac{1}{2}, \frac{1}{2}\right) = 1$
- CONTINUITY: $H_2(p, 1-p)$ IS CONTINUOUS FUNCTION OF "p"
- GROUPING: $H_m(p_1, p_2, \ldots, p_m) = H_{m-1}(p_1+p_2, p_3, \ldots, p_m) + (p_1+p_2)\,H_2\left(\frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2}\right)$

PROVE THAT H_m MUST BE OF THE FORM:

$$H_m(p_1, p_2, \ldots p_m) = -\sum_{i=1}^{m} p_i \log p_i \qquad m = 2, 3, \ldots$$

GROUPING PROPERTY i.e. AXIOM:

$$H_m(p_1, p_2, \dots p_m) = H_{m-1}(p_1+p_2, p_3, p_4, \dots p_m) + \underbrace{(p_1+p_2) H_2\left(\frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2}\right)}_{\circledast}$$

$$H_m = \sum_{i=1}^{m} p_i \, ld \frac{1}{p_i} = p_1 \, ld \, p_1^{-1} + p_2 \, ld \, p_2^{-1} + \dots + p_m \, ld \, p_m^{-1}$$

$$+ (p_1+p_2) \, ld \frac{1}{p_1+p_2} - (p_1+p_2) \, ld \frac{1}{p_1+p_2}$$

$$H_{m-1}(p_1+p_2, p_3, p_4, \dots p_m) = (p_1+p_2) \, ld \frac{1}{p_1+p_2} + p_3 \, ld \frac{1}{p_3} + \dots$$

$$\circledast = (p_1+p_2)\left[\frac{p_1}{p_1+p_2} \, ld \frac{p_1+p_2}{p_1} + \frac{p_2}{p_1+p_2} \, ld \frac{p_1+p_2}{p_2}\right] =$$

$$= p_1 \, ld\left(\frac{p_1+p_2}{p_2}\right) + p_2\left(ld \frac{p_2+p_2}{p_2}\right)$$

$$p_1 \, ld \frac{1}{p_1} + p_2 \, ld \frac{1}{p_2} - p_1 \, ld \frac{1}{p_1+p_2} - p_2 \, ld \frac{1}{p_1+p_2} =$$

$$= p_1 \, ld \frac{p_1+p_2}{p_1} + p_2 \, ld \frac{p_1+p_2}{p_2} = (p_1+p_2) H\left(\frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2}\right)$$

ZNAČI:

DOKAZANO!!

$$H_m = (p_1+p_2) H_2\left(\frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2}\right) + H_{m-1}(p_1+p_2, p_3, p_4, \dots p_m)$$

⊙ EDITION 1 SOLUTION:

- EXTEND THE GROUPING AXIOM BY INDUCTION AND PROVE THAT:

$$H_m(p_1, p_2, \dots, p_m) = H_{m-k}(p_1+p_2+\dots+p_k, p_{k+1}, \dots, p_m) +$$

$$+ (p_1+p_2+\dots+p_k) H_k\left(\frac{p_1}{p_1+p_2+\dots+p_k}, \dots, \frac{p_k}{p_1+p_2+\dots p_k}\right)$$

LET $f(m)$ BE ENTROPY OF UNIFORM DISTRIBUTION OF "m" SYMBOLS i.e.:

$$\boxed{f(m) = H_m\left(\frac{1}{m}, \frac{1}{m}, \dots, \frac{1}{m}\right)}$$

- FOR ANY TWO INTEGERS „r" AND „s":

$$f(r\cdot s) = f(r) + f(s)$$

- FOR RATIONAL: $q = \frac{r}{s}$

$$H_2(q, 1-q) = -q \,\mathrm{ld}\, q - (1-q)\,\mathrm{ld}(1-q)$$

$$f(n) = H_n\left(\frac{1}{n}, \frac{1}{n}, \dots \frac{1}{n}\right) = \underbrace{-\sum_{i=1}^{n} q_i \,\mathrm{ld}\, q_i}_{} =$$

DA IMA AKO DA ZNAEŠ DEFINICIJATA NA ENTROPIJATA, A ZADAČATA E [illegible] NE DA ZNAEŠ.

$$= \sum_{i=1}^{n} \frac{1}{n}\,\mathrm{ld}\, n = n \cdot \frac{1}{n}\,\mathrm{ld}\, n = \underline{\mathrm{ld}\, n}$$

$$\boxed{n = r\cdot s} \Rightarrow \underline{f(r\cdot s) = \mathrm{ld}(r\cdot s) = \mathrm{ld}\, r + \mathrm{ld}\, s}$$

$$H_2(q, 1-q) = \frac{r}{s}\,\mathrm{ld}\,\frac{s}{r} + \left(1-\frac{r}{s}\right)\mathrm{ld}\,\frac{1}{\frac{s-r}{s}} =$$

$$= \frac{r}{s}\,\mathrm{ld}\,\frac{s}{r} + \mathrm{ld}\,\frac{s}{s-r} - \frac{r}{s}\,\mathrm{ld}\,\frac{s}{s-r} =$$

$$= \frac{r}{s}\,\mathrm{ld}\left(\frac{s-r}{r}\right) + \mathrm{ld}\,\frac{s}{s-r}$$

$$= \frac{r}{s}\cdot \mathrm{ld}\,\frac{s}{r}\cdot\frac{s-r}{\cancel{s}} + \mathrm{ld}\,\frac{s}{s-r} =$$

$$= \left(1-\frac{r}{s}\right)\mathrm{ld}\,\frac{s}{s-r} = \frac{s-r}{s}\,\mathrm{ld}\,\frac{s}{s-r}$$

$$\boxed{h\left(\frac{q_2}{S_2}\right) = H\left(1-\frac{q_2}{S_2}, \frac{q_2}{S_2}\right) = H\left(\frac{q_1+q_2-q_2}{S_2}, \frac{q_2}{S_2}\right) = H\left(\frac{q_1}{S_2}, \frac{q_2}{S_2}\right)}$$

$$\boxed{S_3 = S_2 + q_3}$$

$$\boxed{S_k = \sum_{i=1}^{k} q_i} \qquad \boxed{h(q) = H_2(q, 1-q)}$$

$$H_n(q_1, \dots, q_n) = \underbrace{H_{n-1}(S_2, q_3, \dots, q_n)}_{\text{BY GROUPING AXIOM}} + S_2\, h\left(\frac{q_2}{S_2}\right)$$

$$= H_{n-2}(S_3, q_4, \dots, q_n) + S_3\, h\left(\frac{q_3}{S_3}\right) + S_2\, h\left(\frac{q_2}{S_2}\right)$$

$$\vdots$$

$$= H_{n-(k-1)}(S_k, q_{k+1}, \dots q_n) + \sum_{i=2}^{k} S_i\, h\left(\frac{q_i}{S_i}\right) \quad (\$)$$

- AKO SE UPOTREBI REPETITIVNO GROUPING AKSIOMATA NA: $H_k(q_1/S_k, q_2/S_k, \dots, q_k/S_k)$

$$H_k\left(\frac{q_1}{S_k}, \frac{q_2}{S_k}, \dots, \frac{q_k}{S_k}\right) = H_2\left(\frac{S_{k-1}}{S_k}, \frac{q_k}{S_k}\right) + \sum_{i=2}^{k-1} \frac{S_i}{S_k}\, h\left(\frac{q_i/S_k}{S_i/S_k}\right)$$

$$= H_2\left(\frac{S_k}{S_k}\right) + \sum_{i=2}^{k} \frac{S_i}{S_k}\, h\left(\frac{q_i}{S_i}\right) = \frac{1}{S_k}\sum_{i=2}^{k} h\left(\frac{q_i}{S_i}\right)$$

ZNAČI! $\boxed{H_K\left(\frac{p_1}{S_K}, \frac{p_2}{S_K}, \dots \frac{p_K}{S_K}\right) = \frac{1}{S_K}\sum_{i=2}^{K} S_i h\left(\frac{p_i}{S_i}\right)}$

AKO SE ZAMENI VO (4) SE DOBIVA

$$\boxed{H(p_1, p_2, \dots p_n) = H_{n-(k-1)}(S_k, p_{k+1}, \dots p_n) + \underline{\underline{S_k}}\, H_k\left(\frac{p_1}{S_k}, \frac{p_2}{S_k} \dots \frac{p_k}{S_k}\right)}$$

- SO OVA E DOKAZANA PROŠIRENATA FORMA NA AKSIOMATA NA GRUPIRANJE.

• CONSIDER: $f(m\cdot n) = H_{mn}\left(\frac{1}{mn}, \frac{1}{mn}, \dots, \frac{1}{mn}\right) =$

$$= H_{mn-n+1}\left(n\frac{1}{mn}, \frac{1}{mn}, \dots \frac{1}{mn}\right) + \underbrace{\frac{1}{m}}_{S_K} H_n\left(\frac{1}{n}, \frac{1}{n}, \dots \frac{1}{n}\right)$$

$$= H_{mn-n+1}\left(\frac{1}{m}, \frac{1}{mn}, \dots \frac{1}{mn}\right) + \frac{1}{m} H_n\left(\frac{1}{n}, \frac{1}{n}, \dots \frac{1}{n}\right)$$

$$= H_{mn-2n+1}\left(\frac{1}{m}, \frac{1}{m}, \frac{1}{mn} \dots \frac{1}{mn}\right) + \frac{2}{m} H_n\left(\frac{1}{n}, \frac{1}{n}, \dots \frac{1}{n}\right)$$

$\dots$

$$= H_m\left(\frac{1}{m}, \frac{1}{m}, \dots \frac{1}{m}\right) + H_n\left(\frac{1}{n}, \frac{1}{n}, \dots \frac{1}{n}\right)$$

$= f(m) + f(n)$ DOKAZANO !!!

- OTUKA AVTOMATSKI SLEDI: $\boxed{f(m^k) = k \cdot f(m)}$

UŠTE EDNO INTERESNO SVOJSTVO KOE ODMA SE GLEDA AKO ZNAČI DEKA ENTROPIJATA E DEFINIRANA LOGARITAMSKI:

$$H(m^k) = \sum_{i=1}^{m^k} \frac{1}{m^k}\, ld\, m^k = \cancel{m^k} \cdot \frac{1}{\cancel{m^k}}\, ld\, m^k = k\, \underbrace{ld\, m}_{H(m)}$$

• $H_2(1,0) = h(1) = 0$ $\boxed{p_1 + p_2 = 1}$

- PROOF:

$$H_3(p_1, p_2, 0) = H_2(p_1, p_2) + p_2 H_2(1,0) =$$

$$= H_2(p_1 + p_2, 0) + (p_1 + p_2) H_2\left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}\right) =$$

$$= H_2(1,0) + H_2(p_1, p_2)$$

$\Rightarrow p_2 H_2(1,0) = H_2(1,0)$ ZA $\forall p_2 \rightarrow H_2(1,0) = 0$

⊙ $f(n+1) - f(n) \to 0$ AS $n \to \infty$ (DA SE DOKAZE !!!)

$$f(n+1) = H_{n+1}\left(\frac{1}{n+1}, \frac{1}{n+1}, \cdots \frac{1}{n+1}\right) = H_{\underbrace{n+1-n+1}_{2}}\left(\frac{n}{n+1}, \frac{1}{n+1}\right)$$

$$+ \frac{n}{n+1} H_n\left(\cancel{\frac{\frac{1}{n+1}}{\frac{n}{n+1}}}, \frac{\frac{1}{n+1}}{\frac{n}{n+1}}, \cdots \frac{\frac{1}{n+1}}{\frac{n}{n+1}}\right) =$$

$$= H_2\left(\frac{n}{n+1}, \frac{1}{n+1}\right) = H_2\left(1-\frac{1}{n+1}, \frac{1}{n+1}\right) = h\left(\frac{1}{n+1}\right) =$$

$$= h\left(\frac{1}{n+1}\right) + \frac{n}{n+1} \underbrace{H_n\left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right)}_{f(n)}$$

$$\boxed{f(n+1) = h\left(\frac{1}{n+1}\right) + \frac{n}{n+1} f(n)}$$ ~~$f(n+1) - f(n) = h\left(\frac{1}{n+1}\right)$~~

$$\boxed{f(n+1) - \frac{n}{n+1} f(n) = h\left(\frac{1}{n+1}\right)}$$

$$\lim_{n\to\infty} \frac{(n+1)f(n+1) - nf(n)}{n+1} = \lim h\left(\frac{1}{n+1}\right)$$

$$\lim_{n\to\infty} f(n+1) - \frac{n}{n+1} f(n) = f(n) - f(n) = 0 \Rightarrow h\left(\frac{1}{n+1}\right) = 0$$

- LET US DEFINE:

$$\boxed{a_{n+1} = f(n+1) - f(n)} \qquad b_n = h\left(\frac{1}{n}\right)!$$

$$a_{n+1} = h\left(\frac{1}{n+1}\right) + \frac{n}{n+1} f(n) - f(n) = h\left(\frac{1}{n+1}\right) + \frac{nf(n) - }{n+1}$$ (n+1)f(✓

$$= h\left(\frac{1}{n+1}\right) + \frac{\cancel{nf(n)} - \cancel{nf(n)} - f(n)}{n+1} = h\left(\frac{1}{n+1}\right) - \frac{f(n)}{n+1}$$

$$\boxed{a_{n+1} = -\frac{1}{n+1} f(n) + b_{n+1}} \qquad a_{n+1} = -\frac{1}{n+1}\sum_{i=2}^{n} a_i + b_{n+1}$$

$$a_2 = f(2) - f(1) \qquad a_3 = f(3) - f(2) \qquad a_4 = f(4) - f(3)$$

$$a_n = f(n) - f(n-1) \qquad \boxed{a_2 + a_3 + a_4 + \dots a_n = f(2) - f(1) + \cancel{f(3)} - \cancel{f(2)} + f(4) - \cancel{f(3)} = -f(1) + f(n) = f(n)}$$

$(n+1)\cdot a_{n+1} = -\sum_{i=2}^{n} a_i + (b_{n+1})\cdot(n+1)$ $\quad \boxed{(n+1)\, b_{n+1} = (n+1)a_{n+1} + \sum_{i=2}^{n} a_i}$

• Suming over n

$$\sum_{n=2}^{N} n b_n = \sum_{n=2}^{N} n a_n + \sum_{n=2}^{N} \sum_{i=2}^{n} a_i =$$

$$= \sum_{n=2}^{N} (n a_n + a_{n-1} + a_{n-2} + \dots a_2)$$

$\boxed{n+1=m}$ $\qquad m b_m = m a_m + \sum_{i=2}^{m-1} a_i \quad \Big/ \sum_{m=2}^{N}$

$$\sum_{m=2}^{N} m b_m = \sum_{n=2}^{N} m a_m + \sum_{m=2}^{N} \sum_{i=2}^{m-1} a_i =$$

$$= \sum_{m=2}^{N} m a_m + a_{m-1} + a_{m-2} + \dots + a_2 \Rightarrow$$

$$\boxed{\sum_{n=2}^{N} n b_n = \sum_{n=2}^{N} (n a_n + a_{n-1} + a_{n-2} + \dots + a_2)}$$

$$\sum_{n=2}^{N} n a_n + \sum_{n=2}^{N} a_{n-1} + \sum_{n=2}^{N} a_{n-2} + \dots \underbrace{\sum_{n=2}^{N} a_2}_{(N-1)\, a_2}$$

$$\sum_{n=2}^{N} a_{n-2} = \left| \begin{matrix} m=n-2 & n=m-2 \\ n=2 & m=0 \\ n=N & m=N-2 \end{matrix} \right| = \sum_{n=0}^{N-2} a_n$$

$$\sum_{n=2}^{N} a_{n-1} = \left| \begin{matrix} m=n-1 & \\ n=2 & m=1 \\ n=N & m=N-1 \end{matrix} \right| = \sum_{n=1}^{N-1} a_n$$

$$\sum_{n=2}^{4} (n a_n + a_{n-1} + a_{n-2} + \dots a_2) = 2\underline{a_2} + \underline{\underline{3a_3}} + \underline{a_2} + 4a_4 + \underline{\underline{a_3}} + \underline{a_2} =$$

$$= 4a_2 + 4a_3 + 4a_4$$

$$\sum_{n=2}^{5} (n a_n + a_{n-1} + a_{n-2} + \dots a_2) = \underline{2a_2} + \underline{\underline{3a_3}} + \underline{a_2} + 4\underline{\underline{a_4}} + \underline{a_3} + \underline{a_2} +$$

$$+ 5a_5 + \underline{\underline{a_4}} + \underline{\underline{a_3}} + \underline{a_2} = 5a_2 + 5a_3 + 5a_4 + 5a_5$$

$$\sum_{n=2}^{N}\left(na_n + a_{n-1} + a_{n-2} + \cdots a_2\right) = N\cdot\sum_{i=2}^{N} a_i \qquad \Big/ \sum_{n=1}^{N} n = \frac{N(N+1)}{2}$$

$$\begin{array}{l|l} S = 1 + 2 + \cdots + (n-1) + n & \\ S = n + n-1 + \cdots + 2 + 1 & + \\ \hline 2S = (n+1) + (n+1) + \cdots + (n+1) & \end{array}$$

$$\boxed{S = \frac{n\cdot(n+1)}{2}}$$

$$\sum_{n=2}^{N} n b_n = N\sum_{i=2}^{N} a_i \Big/ \sum_{n=1}^{N} n = \frac{(N+1)N}{2}$$

$$\begin{array}{l|l} S = 2 + 3 + \cdots + n & \\ S = n + n-1 + \cdots + 2 & + \\ \hline 2S = (n+2) + (n+2) + \cdots + (n+2) & \end{array}$$

$$\boxed{S = \frac{(n-1)\cdot(n+2)}{2}}$$

$$\sum_{n=2}^{N} n = \frac{(N-1)(N+2)}{2}$$

$$\boxed{\frac{\sum_{n=2}^{N} n b_n}{\sum_{n=1}^{N} n} = \frac{2}{N+1}\sum_{i=2}^{N} a_i}$$

$$\boxed{b_n = h\left(\frac{1}{n}\right)}$$

$$h\left(\frac{1}{n}\right) = \left(1-\frac{1}{n}\right) ld\left(1-\frac{1}{n}\right) + \frac{1}{n} ld\left(\frac{1}{n}\right)^{-1} =$$

$$= \frac{n-1}{n} ld\left(\frac{n-1}{n}\right)^{-1} + \frac{1}{n} ld\, n = \frac{n-1}{n} ld \frac{n}{n-1} + \frac{1}{n} ld\, n$$

$$= \underline{\frac{n-1}{n} ld\, n} - \frac{n-1}{n} ld(n-1) + \underline{\frac{1}{n} ld\, n} = \left(\frac{n-\cancel{1}}{n} + \frac{\cancel{1}}{n}\right) ld\, n - \frac{n-1}{n} ld($$

$$= ld\, n - \frac{n-1}{n} ld(n-1) = 0$$

LOGIČNO $\in$ NO ! MAJKE VUKA TAKA!

$$\Rightarrow b_n \to 0 \quad \text{IF} \quad n \to \infty \Rightarrow \frac{\sum_{n=2}^{N} n b_n}{\sum_{n=1}^{N} n} \to 0$$

$$\Rightarrow a_{n+1} = \underbrace{-\frac{1}{n+1}\sum_{i=2}^{n} a_i}_{\to 0} + b_{n+1} \Rightarrow a_{n+1} \to 0 \Rightarrow$$

$$a_{n+1} = f(n+1) - f(n) = 0$$

DOKAZANO!!!

LEMA 2.0.1 LET THE FUNCTION $f(n)$ SATISFY THE FOLLOWING ASSUMPTIONS:

- $f(mn) = f(m) + f(n)$
- $\lim\limits_{n\to\infty}[f(n+1) - f(n)] = 0$
- $f(2) = 1$

THEN THE FUNCTION $f(n) = \log_2 n$.

PROOF: LET p BE ARBITRARY PRIME NUMBER AND LET:

$$\boxed{g(n) = f(n) - \frac{f(p)\cdot \mathrm{ld}\, n}{\mathrm{ld}\, p}}$$

$$g(m\cdot n) = f(m\cdot n) - \frac{f(p)\cdot \mathrm{ld}(m\cdot n)}{\mathrm{ld}\, p} =$$

$$= \underbrace{f(m\cdot n)} - \frac{f(p)\,\mathrm{ld}\, m}{\mathrm{ld}\, p} - \frac{f(p)\,\mathrm{ld}(n)}{\mathrm{ld}\, p} =$$

$$f(m\cdot n) = f(m) + f(n)$$

$$= f(m) - \frac{f(p)\,\mathrm{ld}\, m}{\mathrm{ld}\, p} + f(n) - \frac{f(p)\,\mathrm{ld}(n)}{\mathrm{ld}\, p} =$$

$$= g(m) + g(n)$$

ZNAČI $g(n)$ JA ISPOLNUVA PRVATA PRETPOSTAVKA OD LEMATA.

$$g(p) = f(p) - \frac{f(p)\cdot \cancel{\mathrm{ld}\, p}}{\cancel{\mathrm{ld}\, p}} = f(p) - f(p) = 0$$

$$a_n = g(n+1) - g(n) = f(n+1) - \frac{f(p)\cdot \mathrm{ld}(n+1)}{\mathrm{ld}\, p} - f(n) + \frac{f(p)\,\mathrm{ld}\, n}{\mathrm{ld}\, p}$$

$$= f(n+1) - f(n) - \frac{f(p)}{\mathrm{ld}\, p}\,\mathrm{ld}\,\frac{n+1}{n}$$

$$\underbrace{\lim_{n\to\infty}[f(n+1) - f(n)]}_{0} - \underbrace{\lim\left[\frac{f(p)}{\mathrm{ld}\, p}\,\mathrm{ld}\,\frac{n+1}{n}\right]}_{0} = 0$$

$$\Rightarrow \boxed{\lim_{n\to\infty}(a_n) = 0} \qquad \lim_{n\to\infty}[\ln x] = \ln \lim_{n\to\infty} \frac{\overset{1}{\cancel{n}}+1}{\cancel{n}} = 0$$

FOR AN INTEGER n, DEFINE

$$n^{(1)} = \left\lfloor \frac{n}{P} \right\rfloor \qquad n^{(1)} < \frac{n}{P} \qquad n = P\cdot n^{(1)} + l$$

$$\boxed{0 \leq l \leq P} \qquad g(P) = 0 \qquad g(Pn^{(1)}) = g(n^{(1)})$$

$$g(P\cdot n^{(1)}) = \underset{0}{\cancel{g(P)}} + g(n^{(1)}) = g(n^{(1)})$$

$$g(n) = \underline{\underline{g(n^{(1)})}} - g(\underline{P\cdot n^{(1)}}) + \underset{i-1}{\underline{g(n)}} = \underline{\underline{g(n^{(1)})}} + \underline{\sum_{i=Pn^{(1)}}^{n-1} a_i}$$

$$g(n) = g(n^k) + \sum_{i=1}^{k}\left(\sum_{i=P_n i}^{n} a_i\right)$$

$$n^{(k)} \leq \frac{n}{P^k} \qquad k = \left\lfloor \frac{\operatorname{ld} n}{\operatorname{ld} P} \right\rfloor + 1$$

$$a_i = g(i+1) - g(i)$$

$$\sum_{i=Pn^{(1)}}^{n-1} a_i = a_{Pn^{(1)}} + \dots a_{n-1} = g(Pn^{(1)}+1) - \cancel{g(P_n^{(1)})} + \cancel{g(P_n^{(1)})} - g(P_n^{(1)}-1) + \dots \cancel{g(n)} - g(n-1)$$

$$= g(P_n^{(1)}+1) - g(n-1)$$

- AKO DEFINICIJATA ~~ZA~~ a_i E: (KAKO NA P1.17)

$$a_i = g(i) - g(i-1)$$

$$\sum_{i=Pn^{(1)}}^{n-1} a_i = \cancel{g(P_n^{(1)})} - g(Pn^{(1)}-1) + \cancel{g(Pn^{(1)}+1)} - \cancel{g(Pn^{(1)})} + g(n-1) - \cancel{g(n-2)} = g(P_n^{(1)}-1) - g(n-1)$$

- SEKAK TREBA SO OVAA DEFINICIJA

$$\boxed{\begin{aligned}\sum_{i=Pn^{(1)}}^{n-1} a_i &= g(\cancel{P_n^{(1)}+1}) - \underline{g(Pn^{(1)})} + \cancel{g(P_n^{(1)}+2)} - g(P_n^{(1)}+1) + \\ &\quad + g(P_n^{(1)}+3) - \cancel{g(Pn^{(1)}+2)} + \dots + \underline{g(n)} - \cancel{g(n-1)} \\ \sum_{i=Pn^{(1)}}^{n-1} a_i &= g(n) - g(Pn^{(1)})\end{aligned}}$$

DOKAZANO!!!

- Kako što go definiravme $n^{(1)}$ od n se definira $n^{(2)}$ od $n^{(1)}$ i.e.

$$g(n^{(1)}) = g(n^{(2)}) + \sum_{i=pn^{(2)}}^{n^{(1)}-1} a_i$$

$$g(n) = g(n^{(1)}) + \sum_{i=pn^{(1)}}^{n-1} a_i = g(n^{2}) + \sum_{i=pn^{(2)}}^{n^{(1)}-1} a_i + \sum_{i=pn^{(1)}}^{n-1} a_i$$

$$g(n) = g(n^{k}) + \sum_{i=pn^{(k)}}^{n^{(k-1)}-1} a_i + \dots + \sum_{i=pn^{(2)}}^{n^{(1)}-1} a_i + \sum_{i=pn^{(1)}}^{n-1} a_i =$$

$$= g(n^{k}) + \sum_{j=1}^{k} \sum_{i=pn^{(j)}}^{n^{(j-1)}-1} a_i$$

$$n^{(k)} = \left\lfloor \frac{n}{p^k} \right\rfloor \qquad n^{(k)} \leq \frac{n}{p^k} \qquad n = p^k \cdot n^{(k)} + l \qquad 0 \leq l \leq p^k$$

$$k = \left\lfloor \frac{ld\, n}{ld\, p} \right\rfloor + 1 \qquad \left\lfloor \frac{ld\, n}{ld\, p} \right\rfloor = k-1$$

$$\underbrace{k-1 = \left\lfloor \frac{ld\, n}{ld\, p} \right\rfloor} \leq \frac{ld\, n}{ld\, p} \qquad \frac{ld\, n}{ld\, p} \geq k-1 \qquad ld\, n \geq ld\, p^{k-1} \qquad \boxed{n \geq p^{k-1}}$$

$$n^{(k)} = \left\lfloor \frac{p^{k-1}}{p^k} \right\rfloor = \left\lfloor \frac{1}{p} \right\rfloor = \left| p > 1 \right| = 0$$

$$n^{(k)} = 0 \Rightarrow g(n^{(k)}) = g(0) = 0$$

$$g(n) = \sum_{i=1}^{b_n} a_i \qquad b_n \leq p \cdot \left[\frac{ld\, n}{ld\, p} + 1 \right]$$

Since $a_n \to 0 \Rightarrow \frac{g(n)}{ld\, n} \to 0$ $g(n)$ HAS AT MOST $O(\log_2 n)$ TERMS a_i

$$\lim_{n \to \infty} \frac{f(n)}{ld\, n} = \frac{f(p)}{ld\, p} \qquad g(n) = f(n) - \frac{f(p)\, ld\, n}{ld\, p}$$

$$\frac{g(n)}{ld\, n} = \frac{f(n)}{ld\, n} - \frac{f(p)}{ld\, p} \Big/ \lim_{n \to \infty} \qquad \lim_{n \to \infty} \frac{f(n)}{ld\, n} = \frac{f(1)}{ld\, p}$$

∴ THIRD AXIOM $\frac{f(p)}{ld\, p} = c$ $\frac{f(2)}{ld\, 2} = 1 \Rightarrow \boxed{f(p) = ld\, p}$

$N = p_1 p_2 \cdots p_l \qquad f(N) = \sum f(p_i) = \sum \mathrm{ld}\, p_i = \mathrm{ld}\, N$

SAMA DA KAŽE DEKA ZNAČI PRIME NUMBERS MOŽEŠ DA OPIŠEŠ BILO KOJ INTEGER POZITIVEN BROJ. (FUNDAMENTAL THEOREM OF ARITHMETICS)

- SO OVA LEMATA E DOKAŽANA
- DOKAŽ AKO SE PRETPOSTAVI DEKA $f(n)$ E MONOTONA.
- ONLY FUNCTION $f(m)$ SUCH THAT $f(mn) = f(m) + f(n)$ FOR ALL INTEGERS m, n IS OF THE FORM $f(m) = \log_a(m)$ FOR SOME BASE „a".

LET $c = f(2)$. $\quad f(4) = f(2) + f(2) = f(2\cdot 2) = \underline{\underline{2c}}$

$$f(2^k) = k \cdot f(2) = k \cdot c = c \cdot \mathrm{ld}\, 2^k$$

- EXTENSION FOR INTEGERS THAT ARE NOT POWER OF 2.

FOR ANY INTEGER m, LET $r > 0$, BE ANOTHER INTEGER AND LET $2^k \leq m^r < 2^{k+1}$

$$kc \leq r f(m) < (k+1)c \qquad \underline{c\frac{k}{r} \leq f(m) < c\frac{k+1}{r}}$$

$$\mathrm{ld}\, 2^k \leq r\, \mathrm{ld}\, m \leq (k+1) \qquad \underline{\frac{k}{r} \leq \mathrm{ld}\, m < \frac{k+1}{r}}$$

$$\frac{ck}{r} \leq c\, \mathrm{ld}\, m \leq c\frac{k+1}{r} \qquad \frac{k}{r} \leq \frac{f(m)}{c} \leq \frac{k+1}{r}$$

$$\frac{k}{r \cdot c} \leq \frac{\mathrm{ld}\, m}{c} \leq \frac{k+1}{rc}$$

$$\boxed{|x| = \begin{cases} x & x > 0 \\ -x & x < 0 \end{cases}}$$

$$\left| f(m) - \frac{\mathrm{ld}\, m}{c} \right| = \begin{cases} f(m) - \frac{\mathrm{ld}\, m}{c} & f(m) - \frac{\mathrm{ld}\, m}{c} > 0 \\ \frac{\mathrm{ld}(m)}{c} - f(m) & f(m) - \frac{\mathrm{ld}\, m}{c} < 0 \end{cases}$$

$$\frac{k}{r} - \frac{k}{rc} \leq f(m) - \frac{\mathrm{ld}\, m}{c} \leq \frac{k+1}{r} - \frac{k+1}{rc}$$

$$L = \frac{kc - k}{rc} \qquad R = \frac{kc + c - k - 1}{rc} = \frac{c(k+1) - (k+1)}{rc} = \frac{(c-1)(k+1)}{rc}$$

$$\left| f(m) - \frac{\mathrm{ld}_2 m}{c} \right| < \frac{1}{r} \Rightarrow \text{MORA } f(m) = \frac{\mathrm{ld}_2 m}{c}$$

$$c = 1 \quad \text{ZOŠTO} \quad f(2) = \frac{\mathrm{ld}(2)}{c} = 1 \qquad c = \mathrm{ld}(2) = 1$$

- DOKAŽAVME DOKA:

$$f(n) = H_n\left(\frac{1}{n}, \frac{1}{n}, \cdots \frac{1}{n}\right) = \text{ld}\, n$$

- DA DOKAŽEME DEKA:

$$\boxed{H_2(q, 1-q) = -q\,\text{ld}\,q - (1-q)\,\text{ld}(1-q)} \quad (\#)$$

- BY ~~EXTENDED~~ GROUPING AXIOM!

$$\boxed{H(q_1, q_2, \cdots q_n) = H_{n-k+1}(s_k, q_{k+1}, \dots q_n) + s_k H_k\left(\frac{q_1}{s_k}, \frac{q_2}{s_k}, \frac{q_k}{s_k}\right)}$$

$$f(s) = H_s\left(\frac{1}{s}, \cdots \frac{1}{s}\right) = H\Big(\underbrace{\frac{1}{s}, \frac{1}{s}, \cdots \frac{1}{s}}_{r}, \frac{s-r}{s}\Big) + \frac{s-r}{s} f(s-r) = (*)$$

$$\boxed{s_k = (s-r)\cdot\frac{1}{s}}$$

$$f(s-r) = H\left(\frac{\frac{1}{\cancel{s}}}{\frac{s-r}{\cancel{s}}}, \frac{\frac{1}{\cancel{s}}}{\frac{s-r}{\cancel{s}}}, \cdots \frac{\frac{1}{\cancel{s}}}{\frac{s-r}{\cancel{s}}}\right)$$

$$= H_{s-r}\left(\frac{1}{s-r}, \frac{1}{s-r}, \cdots \frac{1}{s-r}\right)$$

$$(*) = H_2\left(\frac{r}{s}, \frac{s-r}{s}\right) + \frac{r}{s} H_r\left(\frac{\frac{1}{\cancel{s}}}{\frac{r}{\cancel{s}}}, \frac{\frac{1}{\cancel{s}}}{\frac{r}{\cancel{s}}}, \cdots, \frac{\frac{1}{\cancel{s}}}{\frac{r}{\cancel{s}}}\right) + \frac{s-r}{s} f(s-r)$$

$$= H_2\left(\frac{r}{s}, \frac{s-r}{s}\right) + \frac{r}{s} H_r\left(\frac{1}{r}, \frac{1}{r}, \cdots \frac{1}{r}\right) + \frac{s-r}{s} f(s-r)$$

$$= H_2\left(\frac{r}{s}, \frac{s-r}{s}\right) + \frac{r}{s} f(r) + \frac{s-r}{s} f(s-r) \quad (\$) \Rightarrow \text{P25}$$

$$f(s) = \text{ld}\, s \Rightarrow H_2\left(\frac{r}{s}, \frac{s-r}{s}\right) = \text{ld}\, s - \frac{r}{s} f(r) - \frac{s-r}{s} f(s-r) =$$

$$= \text{ld}\, s - \frac{r}{s}\text{ld}\, r - \frac{s-r}{s}\text{ld}(s-r) = \text{ld}\, s - \frac{r}{s}\text{ld}\, r - \text{ld}(s-r) + \frac{r}{s}\text{ld}(s-r)$$

~~$(*) = H_2\left(\frac{r}{s}, \frac{s-r}{s}\right) + \frac{s}{r} f(s) + \frac{s-r}{s} f(s-r)$~~

~~$H_2\left(\frac{r}{s}, \frac{s-r}{s}\right) = f(s) - \frac{s}{r} f(s) - \frac{s-r}{s} f(s-r) = \text{ld}\, s - \frac{s}{r}\text{ld}\, s$~~

~~$\left(1 + \text{ld}\frac{r}{s}\right) f(s-r) = \left(1 - \frac{s}{r}\right)\text{ld}\, s - \left(1 - \frac{r}{s}\right)\text{ld}(s-r)$~~

$-\frac{r}{s}\text{ld}\frac{1}{s}$ } TUKA E GREŠKATA!! NE MOŽE KOLIČNIKOT DA GO ROTIRAŠ!!!

$$= \text{ld}\, s - \frac{s}{r}\text{ld}\, s + \underline{\frac{s}{r}\text{ld}\, s} - \underline{\frac{r}{s}\text{ld}\, r} - \left(1 - \frac{r}{s}\right)\text{ld}(s-r) =$$

$$= \text{ld}\, s - \frac{s}{r}\text{ld}\, s - \frac{r}{s}\text{ld}\frac{r}{s} - \left(1 - \frac{r}{s}\right)\text{ld}(s-r) =$$

$$= -\frac{r}{s}\text{ld}\frac{r}{s} + \left(1 - \frac{s}{r}\right)\text{ld}\, s - \left(1 - \frac{r}{s}\right)\text{ld}(s-r) =$$

$$= -\frac{r}{s}\,\mathrm{ld}\frac{r}{s} - \left(1-\frac{s}{r}\right)\mathrm{ld}\frac{1}{s} - \left(1-\frac{r}{s}\right)\mathrm{ld}(s-r) = \bullet$$

$$-\left(1-\frac{s}{r}\right) = \left(\frac{s}{r}-1\right) = \frac{s-r}{r} \qquad -\left(1-\frac{r}{s}\right) = \left(\frac{r}{s}-1\right) = \frac{r-s}{s}$$

$$\bullet = -\frac{r}{s}\,\mathrm{ld}\frac{r}{s} + \frac{s-r}{r}\,\mathrm{ld}\frac{1}{s} + \frac{r-s}{s}\,\mathrm{ld}(s-r) =$$

$$= -\frac{r}{s}\,\mathrm{ld}\frac{r}{s} + \frac{r-s}{r}\,\mathrm{ld}\,s + \frac{r-s}{s}\,\mathrm{ld}(s-r) =$$

$$= -\frac{r}{s}\,\mathrm{ld}\frac{r}{s} + \left(1-\frac{s}{r}\right)\mathrm{ld}\,s + \left(\frac{r}{s}-1\right)\mathrm{ld}(s-r)$$

$$p = \frac{r}{s} \qquad \left(1-\frac{1}{r}\right)\mathrm{ld}\,s + (r-1)\,\mathrm{ld}(s-r)$$

IMA GREŠKA
VO IZVEDU-
VANJETO !!!
...

~~$(r-1)\,\mathrm{ld}\,s + (r-1)\,\mathrm{ld}(s-r) =$~~

~~$(*) \Rightarrow f(s) = H_2\left(\frac{r}{s}, \frac{s-r}{s}\right) + \frac{s}{r} f(s) + \frac{s-r}{s} f(s-r) \Rightarrow$~~

~~$H_2\left(\frac{r}{s}, \frac{s-r}{s}\right) = \left(1-\frac{s}{r}\right) f(s) + \left(1-\frac{r}{s}\right) f(s-r) =$~~

~~$= \left(1-\frac{s}{r}\right)\mathrm{ld}\,s + \left(1-\frac{r}{s}\right)\mathrm{ld}(s-r)$~~ =

~~$= -\frac{r}{s}\,\mathrm{ld}\frac{r}{s} + \frac{r}{s}\,\mathrm{ld}\frac{r}{s} + \mathrm{ld}\,s - \frac{s}{r}\,\mathrm{ld}\,s + \left(1-\frac{r}{s}\right)\mathrm{ld}(s-r)$~~

~~$= -\frac{r}{s}\,\mathrm{ld}\frac{r}{s} + \frac{r}{s}\,\mathrm{ld}\,r - \frac{r}{s}\,\mathrm{ld}\,s + \mathrm{ld}\,s - \frac{s}{r}\,\mathrm{ld}\,s + (1-r)\,\mathrm{ld}(s-r)$~~

①

① ~~$= \mathrm{ld}\,s\left(-\frac{r}{s} + 1 - \frac{s}{r}\right) = \mathrm{ld}\,s\left(\frac{-r^2+rs-s^2}{sr}\right)$~~ ②

$$H_2\left(\frac{r}{s}, \frac{s-r}{s}\right) = \underbrace{\mathrm{ld}\,s - \frac{r}{s}\,\mathrm{ld}\,r}_{\text{PRAVILNO!!!}} - \frac{s-r}{s} f(s-r) = \boxed{\$}$$

$$= \mathrm{ld}\,s - \frac{r}{s}\,\mathrm{ld}\,r + \frac{r}{s}\,\mathrm{ld}\,s - \frac{r}{s}\,\mathrm{ld}\,s - \frac{s-r}{s}\,\mathrm{ld}(s-r) =$$

$$= \underline{\underline{\mathrm{ld}\,s}} - \underline{\frac{r}{s}\,\mathrm{ld}\,r} - \underline{\frac{r}{s}\,\mathrm{ld}\left(\frac{1}{s}\right)} - \underline{\underline{\frac{r}{s}\,\mathrm{ld}\,s}} - \frac{s-r}{s}\,\mathrm{ld}(s-r) =$$

$$= \mathrm{ld}\,s\left(1-\frac{r}{s}\right) - \frac{r}{s}\,\mathrm{ld}\frac{r}{s} - \frac{s-r}{s}\,\mathrm{ld}(s-r) =$$

$$= -\frac{r}{s}\,\mathrm{ld}\frac{r}{s} - \left(1-\frac{r}{s}\right)\mathrm{ld}\frac{1}{s} - \left(1-\frac{r}{s}\right)\mathrm{ld}(s-r) =$$

DOKAŽANO

$$= -\frac{r}{s}\,\mathrm{ld}\frac{r}{s} - \left(1-\frac{r}{s}\right)\mathrm{ld}\frac{s-r}{s} = -\frac{r}{s}\,\mathrm{ld}\frac{r}{s} - \left(1-\frac{r}{s}\right)\mathrm{ld}\left(1-\frac{r}{s}\right)$$

• ASSUME THAT N24.⊛ HOLDS FOR $n = n-1$

$$H_n(q_1, q_2, \dots, q_n) = \underline{H_{n-1}(q_1+q_2, q_3 \cdots q_n)} + (q_1+q_2) H_2\left(\frac{q_1}{q_1+q_2}, \frac{q_2}{q_1+q_2}\right)$$

$$= -\sum_{i=3}^{n} q_i \, ld\, q_i + (q_1+q_2)\, ld \frac{1}{(q_1+q_2)} + \cancel{(q_1+q_2)} \cdot \frac{q_1}{\cancel{q_1+q_2}}\, ld \frac{q_1+q_2}{q_1} +$$

$$+ \cancel{(q_1+q_2)} \frac{q_2}{\cancel{q_1+q_2}}\, ld \frac{q_1+q_2}{q_2} = -\sum_{i=2}^{n} q_i \, ld\, q_i + (q_1+q_2)\, ld \frac{1}{q_1+q_2}$$

$$+ \underline{q_1\, ld(q_1+q_2)} - \underline{q_1\, ld\, q_1} + \underline{q_2\, ld(q_1+q_2)} - \underset{2}{\underline{q_2\, ld\, q_2}}$$

$$= -\sum_{i=3}^{n} q_i \, ld\, q_i - \cancel{(q_1+q_2)\, ld(q_1+q_2)} + \cancel{(q_1+q_2)\, ld(q_1+q_2)} + -\sum_{i=1} q_i\, ld \frac{1}{q_i}$$

$$= \sum_{i=1}^{n} q_i \, ld\, q_i$$ DOKAZANO PO INDUKCIJI!!

INDUKCIJOM SE ISKORISTI

DVAJA ČLANA OD ... OVA VAŽI ZA [illegible]

1.) DOKAZANO ← $n = 2$
VRIJEDNOST

2.) AKO VAŽI ZA $n = n-1$ VAŽI I ZA $n = n$.

16.06.

$\boxed{17.06.12}$

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08.08.2012

$\boxed{2.47}$ ENTROPY OF MISSORTED FILE.

A DECK OF „n" CARDS IN ORDER $1, 2, \dots, n$ IS PROVIDED. ONE CARD IS REMOVED AT RANDOM, AND THEN REPLACED AT RANDOM. WHAT IS ENTROPY OF THE RESULTING DECK.

$$H(n) = \sum_{i=1}^{n} \frac{1}{n}\, ld\, n = n \frac{1}{n}\, ld\, n = \underline{\underline{ld\, n}}$$

$(n-1)$ - PAROVI

.1.2.3.4.

PRAVILNI ČETVERI NA N27

1 2 3 4	▨	▨	▨
2 1 3 4*	2 1 3 4*	3 2 1 4	4 2 3 1
3 2 1 4	1 3 2 4*	1 3 2 4*	1 4 3 2
4 2 3 1	1 4 3 2	1 2 4 3*	1 2 4 3*

VKUPNO KOMBINACIJ $n \cdot (n-1)$

$\boxed{1 2 3}$ 2 1 3* 3 2 1
2 1 3* 1 3 2* 1 3 2*
3 2 1

ADJESENT ④

VKUPNO IMA 6 MISSORTED KOMBINACIJ KOI SE PONAVLJAJU VEROJATNI

$4 \cdot 3 = 12$ MISSORTED KOMBINACII. ADJESENT $2(n-1) = 6$

$n \cdot (n-1)$ BILO KOJ OD N-TE BROEVI MOŽEŠ DA GO STAVIŠ NA $(n-1)$ MESTA

$2(n-1)$ $\Rightarrow$ OD NIVNOTE $n(n-1)$ IMA $(n-1)$ SOSEDNI PAROVI I SEKOJ SOSEDEN KESS IMA 2 DUPLIKATI.

$$P(x) = \begin{cases} \frac{1}{n^2} & \text{CASE I} \quad x(i)=j;\ x(j)=i;\ x(k)=k \quad |i-j|>1 \\ 2/n^2 & \text{CASE II} \quad x(i)=j;\ x(j)=i;\ x(k)=k \quad |i-j|=2 \\ \boxed{1/n} & \text{CASE III} \quad x(k)=k \text{ for ALL "k"} \end{cases}$$

- $n(n-1) - 2(n-1) = n^2 - n - 2n + 1 = n^2 - 3n + 1 =$
  $\hookrightarrow = \underbrace{(n-2)(n-1)}$ ISTANCES OF CASE „I"

DUPLIKATITE GI BROIŠ EDNAŠ

- $\overbrace{(n-1)}$ INSTANCES OF CASE II
- ① INSTANCE OF CASE III (ALL NUMBERS ARE AT ITS OWN RIGHT PLACE I.E NO MISSORTED FILES)
  „n" DUPLIKATI
  (VADIŠ „1" STAVAŠ NA 1, VADIŠ 2) STAVAŠ NA „2" ...

- OPCIJATA III E VOZMOŽNA ZOŠTO KARTATA MOŽE DA JA VRATIŠ VO NEJZINOTO INICIJALNO MESTO
- NE SE DOBNI PRIMERITE NA P.26 (TIE SE AKO IZVLEČEŠ 2 KARTI I GI ZAMENIŠ MESTATA)

1 2 3		
2 1 3*	2 1 3*	3 1 2
2 3 1	1 3 2*	1 3 2*

1 2 3 4	▨	▨	▨
*2 1 3 4	*2 1 3 4	3 1 2 4	4 1 2 3
2 3 1 4	1 3 2 4*	1 3 2 4*	1 4 2 3
2 3 4 1	1 3 4 2	1 2 4 3*	1 2 4 3*

(*) ADJESENT CASES

$$H(x) = \sum_{i=1}^{(n-2)(n-1)} \frac{1}{n^2} \operatorname{ld} n^2 + \sum_{i=1}^{(n-1)} \frac{2}{n^2} \operatorname{ld} \frac{n^2}{2} + \boxed{\frac{1}{n} \operatorname{ld} n}$$

$$= \frac{(n-2)(n-1)}{n^2} \operatorname{ld} n^2 + \frac{2(n-1)}{n^2} \operatorname{ld} \frac{n^2}{2}$$

~~$(n-1) + (n-2)(n-1) + 1 = n - 1 + 1 + n^2 - 3n + 1 = n^2 - 2n + 1 = (n-1)^2$~~

~~$P(x=x_1) = \frac{(n-2)\cancel{(n-1)}}{(n-1)^2} = \frac{n-2}{n-1}$~~ ~~$P(x=x_3) = \frac{n}{(n-1)^2}$~~

~~$P(x=x_2) = \frac{n-1}{(n-1)^2} = \frac{1}{n-1}$~~

$P(x=x_1) = \frac{n-2}{n-1}$ ~~VEROJATNOST DA SE SLUČI MISSORTING CASE I~~

$P(x=x_2) = \frac{1}{n-1}$ } ~~VEROJATNOST DA SE SLUČI MISSORTING CASE II~~

$P(x=x_3) = \frac{1}{(n-1)^2}$ } ~~VEROJATNOST DA SE SLUČI MISSORTING CASE III T.E. SORTING~~

- ~~SAMO ZA EDEN MISSORTING OD DADEN TIP~~

$$\gamma_1 = \frac{P(x=x_1)}{(n-2)(n-1)} = \frac{1}{(n-1)^2}$$?

$$\gamma_2 = \frac{P(x=x_2)}{}$$

VADIŠ 1 STAVAŠ NA 1, VADIŠ 2 STAVAŠ NA 2, ...!

VKUPNO NASTANI:

$$n(n-1) + \boxed{n} = \underline{n^2 - \cancel{n} + \cancel{n}} = n^2$$

$$P(x\in X_1) = \frac{(n-2)(n-1)}{n^2} \qquad P(x\in X_2) = \frac{2(n-1)}{n^2}$$

$$P(x\in X_3) = \frac{n}{n^2} \qquad (n-2)(n-1) + 2(n-1)$$

- SAMO ZA EDEN MISSORTING OD DADEN TIP

$$\gamma_1 = P(x=x_1) = \frac{P(x\in X_1)}{(n-2)(n-1)} = \frac{1}{n^2} \qquad \gamma_2 = P(x=x_2) = \frac{P(x\in X_2)}{\boxed{(n-1)}}$$

$$\gamma_3 = \frac{P(x\in X_3)}{\boxed{1}} = \frac{1}{n}$$

DUPLIKATITE ZA ZGOLEMUVAAT VEROJATNOSTA

$$\boxed{\gamma_1 = \frac{1}{n^2} \;;\; \gamma_2 = \frac{2}{n^2} \;;\; \gamma_3 = \frac{1}{n}}$$

$$H(X) = (n-1)(n-2)\cdot\frac{1}{n^2}\,ld\, n^2 + \underbrace{(n-1)}_{\text{BEZ DUPLIKATI}}\cdot\frac{2}{n^2}\cdot ld\,\frac{n^2}{2} + \frac{1}{\underbrace{n}_{\text{BEZ DUPLIKATI}}}\,ld\, n$$

$$= \frac{2(n-1)(n-2)}{n^2}\,ld\, n + \frac{4(n-1)\,ld\, n}{n^2} - \frac{2(n-1)\,ld\, 2}{n^2} + \frac{1}{n}\,ld\, n =$$

$$= ld(n)\left[\frac{2(n-1)(n-2) + 4(n-1) + n}{n^2}\right] - \frac{n-1}{n^2}\,ld\, 4$$

$$2(4-1)(4-2)+4(4-1)+4 = 2\left(4^2-2\,4-4+2\right)+4\,4-\cancel{4}+4 =$$

$$= 2\,4^2 - 6\,4 + \cancel{4} + 5\,4 - \cancel{4} = 2\,4^2 - 4\,\cancel{44} = 4(2\,4-1)$$

$$H(X) = \frac{4(2\,4-1)}{4^2}\, ld(4) - \frac{4-1}{4^2}\, ld\, 4$$

$$\boxed{H(X) = \frac{2\,4-1}{4}\, ld\, 4 - \frac{4-1}{4^2}\, ld\, 4}$$

МОЖЕБИ ЗАДАЧАТА ТРЕБА ДА СЕ ВИКА: ENTROPY OF TAKING OUT AND RETURNING BACK THE FILE FROM ARCHIVE.

2.48 SEQUENCE LENGTH; HOW MUCH INFORMATION

THE SEQUENCE GIVE

DOES THE LENGHT OF THE SEQUENCE? SUPPOSE

ABOUT THE CONTENT OF THE SEQUENCE?

WE CONSIDER A BERNULLI$\left(\frac{1}{2}\right)$ PROCESS $\{X_i\}$.

STOP THE PROCESS WHEN THE FIRST 1 APPEARS.

LET „N" DESIGNATE THIS STOPING TIME. THUS,

X^N IS AN ELEMENT OF THE SET OF ALL FINITE-LENGHT BINARY SEQUENCES $\{0,1\}^* = \{0,1,00,01,10,11,000,\ldots\}$.

(a) FIND $I(N;X^N)$. (b) FIND $H(X^N|N)$ (c) FIND $H(X^N)$

LET'S NOW CONSIDER A DIFERENT STOPING TIME. FOR THIS PART, AGAIN ASSUME THAT $X_i \sim$ BERNULLI$\left(\frac{1}{2}\right)$ BUT STOP AT TIME $N=6$, WITH PROBABILITY 1/3 AND STOP AT TIME $N=12$ WITH PROBABILITY 2/3. LET THIS STOPING TIME BE INDEPENDENT OF THE SEQUENCE $X_1, X_2, \ldots X_{12}$.

(d) FIND $I(N;X^N)$. (e) FIND $H(X^N|N)$. (f) FIND $H(X^N)$.

(a) $P(N) = \underbrace{\frac{1}{2}\cdot\frac{1}{2}\cdot\ldots\cdot\frac{1}{2}}\left(1-\frac{1}{2}\right) = \left(\frac{1}{2}\right)^N$

$$H(N) = \sum_{N=1}^{\infty}\left(\frac{1}{2}\right)^N ld\, 2^N = ld\, 2 \sum_{N=1}^{\infty} N\left(\frac{1}{2}\right)^N = ld\, 2\,\frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2}$$

$$H(N) = ld\, 2\, \frac{\frac{1}{2}}{\frac{1}{\cancel{4}\,2}} = (ld\, 2)\cdot 2 = \underline{\underline{2\, \cancel{ld}\, 2}} = 2$$

$$I(N;X^N) = H(N) - \underbrace{H(N|X^N)}_{0} = H(N) = 2\, ld\, 2 = 2$$

АКО ГО ЗНАЕШ X^N ГО ЗНАЕШ И „N"

⑥ $\underline{H(X^N|N)} = ?$ $\quad I(N;X^N) = H(N) - \underbrace{H(N|X^N)}_{0} = H(N)$

$$H(X^N|N) = H(X_1|N) + H(X_2|N,X_1) + \dots H(X_N|N,X_1,\dots X_{N-1})$$

$$X^N \in \{X_1, X_2, \dots X_N\}$$

$$H(X^N|N) = \sum_{i=1}^{N} P(i)\cdot H(X^N|N=i)$$

$$\left.\begin{aligned} I(N;X^N) &= \boxed{H(X^N) - H(X^N|N)} = N - H(X^N|N) \\ H(X^N) &= N\cdot H(X_i) = N\cdot\left[-q\,ld\,q - (1-q)\,ld\,(1-q)\right] = N \\ H(X^N/N) &= N - I(N;X^N) = N - 2\,\cancel{ld\,2}^{\,1} \end{aligned}\right\} ?$$

$$X^N \in \{0,1,00,01,10,11,000,001,010,011,100,101,110,111\}$$

$$N=1 \Rightarrow X^N \in \{1,10,11,100,101,110,111\}$$

$$X^N \in \{0,1\}$$

$\boxed{N=2}$ $\quad X^N \in \{0,1,00,0①,10,11\}$

$$H(X^N|N=2) = \varnothing$$

АКО ФАНАТА „1" е ТОКЕ N=2

А X^N е ОГРАНИЧЕНО НА ДОЛЖИНА

2 НЕМА НЕОДРЕДЕНОСТ.

$\boxed{N=3}$ $\quad X^N \in \{0,1,00,01,10,11,000,00①,010,011,100,101,110,111\}$

$$H(X^N|N=2) = P(00)\,ld\frac{1}{P(00)} + P(01)\,ld\frac{1}{P(01)} + P(10)\,ld\frac{1}{P(10)}$$

$$+ P(11)\,ld\frac{1}{P(11)} = \left(\frac{1}{4}\cdot ld\,4\right)\cdot 4 = \frac{4}{2}\cdot 4 = 2$$

$$H(X^N|N=n) = \left(\frac{1}{\cancel{2^n}}\,ld\,2^n\right)\cdot\cancel{2^n} = n\,ld\,2 = n$$

$$H(X^N|N) = \sum_{n=1}^{N} p(n)\,H(X^N|N=n) = \sum_{n=1}^{N}\frac{1}{2^n}\,n$$

$$S = \sum_{n=1}^{N} n x^n \qquad \int x^n dx = \frac{x^{n+1}}{n+1} \qquad \int x^{n-1}dx = \frac{x^n}{n}$$

$$\frac{S}{x} = \sum_{n=1}^{N} n x^{n-1} \Big/ \int \Rightarrow \int \frac{S}{x}\,dx = \sum_{n=1}^{N}\cancel{n}\,\frac{x^n}{\cancel{n}}$$

$$\int \frac{S\,dx}{x} = \sum_{k=1}^{N} x^k$$

$$S = x + x^2 + \dots x^N$$

$$\frac{S}{x} = 1 + x + \dots x^{N-1}$$

$$-$$

$$S - \frac{S}{x} = x^N - 1 \qquad S\left(1 - \frac{1}{x}\right) = x^N - 1$$

$$\boxed{S = \frac{x^N - 1}{\frac{x-1}{x}} = \frac{x^{N+1} - x}{x - 1}}$$

$$\int \frac{S\,dx}{x} = \frac{x^{N+1} - x}{x - 1} \qquad \frac{S}{x} = \left(\frac{x^{N+1} - x}{x - 1}\right)' = \frac{[(N+1)x^N - 1](x-1) - x^{N+1} + x}{(x-1)^2}$$

$$S = x \cdot \frac{(N+1)x^N(x-1) - \cancel{x} + 1 - x^{N+1} + \cancel{x}}{(x-1)^2} = x\,\frac{(N+1)x^{N+1} - (N+1)x^N + 1 - x}{(x-1)^2}$$

$$S = x\,\frac{Nx^{N+1} + \cancel{x^{N+1}} - Nx^N - x^N + 1 - \cancel{x^{N+1}}}{(x-1)^2} = x\,\frac{Nx^{N+1} - (N+1)x^N + 1}{(x-1)^2}$$

$$\boxed{S = x \cdot \frac{x^N(Nx - N - 1) + 1}{(x-1)^2}} \qquad \boxed{x = \frac{1}{2}}$$

$$S = \frac{1}{\cancel{2}}\,\frac{\frac{1}{2^N}\left(\frac{N}{2} - N - 1\right) + 1}{\frac{1}{\cancel{4}\,2}} = 2\left[\frac{1}{2^N}\left(-\frac{N}{2} - 1\right) + 1\right]$$

$$\boxed{S = 2 - \frac{1}{2^{N\,\text{[illegible]}\,1}}\left(1 + \frac{N}{2}\right)} \qquad \boxed{S = 2 - \frac{N+2}{2^N}}$$

$$\boxed{S = 2 - N \cdot 2^{-N} - 2^{1-N}}$$

$$\boxed{H(X^N|N) = 2 - \frac{N+2}{2^N}} \qquad N \to \infty \quad H(X^N|N) = 2$$

(c) $I(N; X^N) = H(X^N) - H(X^N|N) = 2 \Rightarrow$

$$H(X^N) = 2 + H(X^N|N) = 4 - \frac{N+2}{2^N}$$

$\boxed{N = 1}$ $\quad H(X^N|1) = 2 - \frac{3}{2} = 0.5$

$$H(X^N|1) = P(N=1) \cdot H(X^N|N=1) = \frac{1}{2} \cdot \Big[\overbrace{\tfrac{1}{2}\,\mathrm{ld}\,2 + \tfrac{1}{2}\,\mathrm{ld}\,2}^{1}\Big] = \frac{1}{2}$$

$$\boxed{H(X^N) = 4 - \frac{N+2}{2^N}}$$

(d) $P(N=6)=\frac{1}{3}$ $P(N=12)=\frac{2}{3}$

$$H(N)=P(N=6)\,ld\frac{1}{P(N=6)}+P(N=12)\cdot ld\frac{1}{P(N=12)}=$$

$$=\frac{1}{3}ld3+\frac{2}{3}ld\frac{3}{2}=\frac{1}{3}ld3+\frac{2}{3}ld3-\frac{2}{3}ld2$$

$$=ld3-\frac{2}{3}ld2 \qquad H(N)=ld3-\frac{2}{3}ld2=ld3-\frac{2}{3}$$

$$I(N;X^N)=H(N)-H(N|X^N)=H(N)=ld3-\frac{2}{3}=0{,}92$$

(e) $H(X^N|N)=?$ $I(N;X^N)=H(X^N)-H(X^N|N)$

$$H(X^N|N=6)=-P(000000)\,ld\,P(000000)-P(000001)\,ld\,P(000001)-$$

$$-P(000010)\,ld\,P(000010)-P(000011)\,ld\,P(000011)-\dots-P(111111)\,ld\,P(111111)$$

$$H(X^N|N=6)=\left(\frac{1}{\cancel{2^6}}ld\,2^6\right)\cdot\cancel{2^6}=ld\,2^6=6$$

$$H(X^N|N=12)=\left(\frac{1}{\cancel{2^{12}}}ld\,2^{12}\right)\cdot\cancel{2^{12}}=12$$

$$H(X^N|N)=P(N=6)\cdot H(X^N|N=6)+P(N=12)\,H(X^N|N=12)=$$

$$=\frac{1}{3}\cdot 6+\frac{2}{3}\cdot 12=2+8=\underline{\underline{10}}$$

$H(X^N|N)=10$

(d) $H(X^N)=?$ $H(X^N)=I(N;X^N)+H(X^N|N)\Rightarrow$

$$H(X^N)=ld3-\frac{2}{3}+10=ld3+\frac{30-2}{3}=ld3+\frac{28}{3}$$

$$\boxed{H(X^N)=10{,}92}$$

CHAPTER 3 ASYMPTOTIC EQUIPARTITION

$$\frac{1}{n}ld\frac{1}{p(x_1,x_2,\dots,x_n)}\rightarrow H \qquad x_1,x_2,\dots,x_n$$

$$p(x_1,x_2,\dots,x_n)$$

$$\left(p(x_1,x_2,\dots,x_n)\right)^{-\frac{1}{n}}\rightarrow 2^H \qquad \left[p(x_1,x_2,\dots,x_n)\right]^{\frac{1}{n}}=2^{-H} \qquad (\;)^n$$

$$\boxed{p(x_1,x_2,\dots,x_n)\rightarrow 2^{-nH}}$$

SE NANAŠAM NA PROBLEM 2.48

$H(X^N) = 4 - \frac{N+2}{2^N}$

$P(0{,}00000000001) = \frac{1}{2^{12}}$ $2^{-4H} \doteq 2^{4N + \frac{N(N+2)}{2^N}}$

- TYPICAL SET WHERE SAMPLE ENTROPY IS CLOSE TO THE TRUE ENTROPY.
- EXAMPLE. LET RANDOM VARIABLE $X \in \{0,1\}$ HAVE PROBABILITY MASS FUNCTION $p(1) = p$ $p(0) = q$

IF $X_1, X_2, \dots, X_n$ ARE i.i.d PROBABILITY OF $x_1, x_2, \dots, x_n$ IS $\prod_{i=1}^{n} p(x_i)$

e.g: $(1,0,1,1,0,1) \rightarrow p(X) = p^4 \cdot q^2 = p^{\sum x_i} q^{n-\sum x_i}$

NOT ALL SEQUENCES OF LENGHT „n" HAS SAME PROBABILITY.

$$\Pr\{X_1, X_2, \dots X_n : p(x_1, x_2, \dots x_n) = 2^{-n(H \pm \epsilon)}\} \doteq 1$$

$$p(x_1, x_2, \dots, x_n) = p^{\sum x_i} q^{n - \sum x_i}$$

NUMBER OF 1'S IN SEQUENCE IS CLOSE TO $n \cdot p$ AND ALL SUCH SEQUENCES HAVE RUGHLY THE SAME PROBABILITY $2^{-nH(p)}$

$x_1 x_2 x_3 x_4 x_5$ $p = 0.6$ $q = 0.4$

$n = 5$ $\boxed{n \cdot p = 0.6 \cdot 5 = 3}$ $\boxed{n \cdot q = 5 \cdot 0.4 = 2}$

$p^3 \cdot q^2 = (0.6)^3 \cdot (0.4)^2 = 0.03456$ $2^{-5 \cdot H(p)} = 0.03575$

DEFINITION: (CONVERGENCE OF RANDOM VARIABLES) GIVEN A SEQUENCE OF RANDOM VARIABLES $X_1, X_2, \dots$, WE SAY THAT SEQUENCE $X_1, X_2, \dots$ CONVERGES TO RANDOM VARIABLE X:

1. In probability IF EVERY $\epsilon > 0$, $\Pr\{|X_n - X| > \epsilon\} \to$
2. In mean square if $E[(X_n - X)^2] \to 0$
3. With probability 1 (also called almost surely) IF $\Pr\{\lim_{n \to \infty} X_n = X\} = 1$

3.1 ASYMPTOTIC EQUIPARTITION PROPERTY THEOREY

THEOREM 3.1.1 (AEP) If $x_1, x_2, \dots$ ARE i.i.d $\sim p(x)$ THEN: $\boxed{-\frac{1}{n} ld\, p(x_1, x_2, \dots, x_n) \xrightarrow{\text{I.P.}} H(X)}$ IN PROBABILITY.

OBNEZNO GLEDAJ GI IZVEDUVANJATA VO MATRE!!! [illegible]

$$-\frac{1}{n} ld\, p(x_1, x_2, \dots x_n) = -\frac{1}{n}\left[ld\, P(x_1) + ld\, P(x_2) + \dots + ld\, P(x_n)\right] =$$

$$= \underbrace{-\frac{1}{n}\sum_{i=1}^{n} ld\, P(x_i)}_{\text{SAMPLE AVERAGE}} \rightarrow -E\left[ld\, P(x_i)\right] \text{ IN PROBABILITY}$$

$= H(X)$

EXPECTED VALUE OF $ld(X_i)$

LAW OF LARGE NUMBERS

$$\Rightarrow ld\left[p(x_1, x_2, \dots, x_n)\right]^{-\frac{1}{n}} = H(x) \qquad p^{-\frac{1}{n}}(x_1, x_2, \dots x_n) = 2^{H(x)}$$

$$\boxed{p(x_1, x_2, \dots, x_n) = 2^{-n \cdot H(x)}} \quad (*)$$

EDNAKVOSTA VAZI ZA SEKVENCI SO $n \cdot p$ BROJ NA EDINICI

DEFINITION: THE TYPICAL SET $A_\varepsilon^{(n)}$ WITH RESPECT TO $p(x)$ IS THE SET OF SEQUENCES $(x_1, x_2 \dots x_n) \in \mathcal{X}^n$ WITH THE PROPERTY

$$\boxed{2^{-n(H(x)+\varepsilon)} \leq p(x_1, x_2, \dots x_n) \leq 2^{-n(H(x)-\varepsilon)}} \quad (\$)$$

ALTERNATIVNA FORMA: $\left|-\frac{1}{n} ld\, P(x_1, x_2, \dots x_n) - H(x)\right| \leq \varepsilon$

TYPICAL SET IS SET WHERE SAMPLE ENTROPY IS CLOSE TO REAL ENTROPY.

THEOREM 3.1.2 AS A CONSEQUENCE OF AEP, WE CAN SHOW THAT THE SET $A_\varepsilon^{(n)}$ HAS THE FOLLOWING PROPERTIES:

(1) If $(x_1, x_2, \dots, x_n) \in A_\varepsilon^{(n)}$, THEN $\boxed{H(X) - \varepsilon \leq -\frac{1}{n} ld\, p(x_1, x_2, \dots x_n) \leq H(X) + \varepsilon}$

(2.) $Pr\{A_\varepsilon^{(n)}\} \geq 1 - \varepsilon$ for „n" SUFFICIENTLY LARGE

(3.) $|A_\varepsilon^{(n)}| \leq 2^{n(H(x)+\varepsilon)}$ WHERE $|A|$ DENOTES THE NUMBER OF ELEMENTS IN THE SET A.

(4.) $|A_\varepsilon^{(n)}| \geq (1-\varepsilon)\, 2^{n(H(x)-\varepsilon)}$

THUS THE TYPICAL SET HAS PROBABILITY NEARLY 1, ALL ELEMENTS OF TYPICAL SET ARE NEARLY EQUIPROBABLE, AND NUMBER OF ELEMENT IS NEARLY 2^{nH}

PROOF: (1) $-n(H(X)+\epsilon) \leq \text{ld}\, p(x_1,x_2,\dots x_n) \leq -n(H(X)-\epsilon)$

$n(H(X)+\epsilon) \geq -\text{ld}\, p(x_1,x_2,\dots x_n) \geq n(H(X)-\epsilon) \Rightarrow$

$$\boxed{H(X)-\epsilon \leq -\frac{1}{n}\text{ld}\, p(x_1,x_2,\dots x_n) \leq H(X)+\epsilon}$$

(2) $\circledast \Rightarrow \quad p(x_1,x_2,\dots x_n) = 2^{-nH(X)}$

$$\lim_{n\to\infty} p(x_1,x_2,\dots x_n) = \lim_{n\to\infty} 2^{-nH(X)} = \frac{1}{2^{nH}} \to 0$$

- FOR ANY $\delta>0$ THERE EXIST n_0 SUCH THAT $n \geq n_0$ WE HAVE:

$$Pr\left\{\left|-\frac{1}{n}\text{ld}\, p(x_1,x_2,\dots,x_n) - H(X)\right|<\epsilon\right\} > 1-\delta.$$

$000,\ \underbrace{001}_{a},\ \underbrace{010}_{b},\ \underbrace{011}_{1},\ \underbrace{100}_{c},\ \underbrace{101}_{2},\ \underbrace{110}_{3},\ 111$ $\quad \boxed{\begin{matrix} p=0.7 \\ q=0.3 \end{matrix}}$

$(0.7)^2(0.3) = \underline{0.49\cdot 0.3}$ $\quad \textcircled{\$}\ (0.3)^2(0.7) = \underline{0.09\cdot 0.7 = 0.063}$

(3) $1 = \sum_{x\in X^n} p(x) \geq \sum_{x\in A_\epsilon^{(n)}} p(x) \overset{\downarrow}{\geq} \sum_{x\in A_\epsilon^{(n)}} 2^{-n(H(X)+\epsilon)} =$

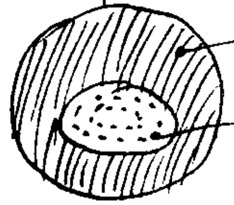
$= \underbrace{|A_\epsilon^{(n)}|\cdot 2^{-n(H(X)+\epsilon)}}_{\text{vo } \Sigma \text{ CLENOVITE SE EDNAKVI}}$ $\Rightarrow \left| \begin{matrix} 2^{n(H(X)+\epsilon)} \geq |A_\epsilon^{(n)}| \\ \text{i.e. } |A_\epsilon^{(n)}| \leq 2^{n(H(X)+\epsilon)} \end{matrix} \right.$

(4) $(1-\epsilon) \leq Pr\{A_\epsilon^{(n)}\} \underset{\uparrow \textcircled{\$}}{\leq} \sum_{x\in A_\epsilon^{(n)}} 2^{-n(H(X)-\epsilon)} = |A_\epsilon^{(n)}|\cdot 2^{-n(H(X)-\epsilon)}$

$\boxed{|A_\epsilon^{(n)}| \geq (1-\epsilon)\, 2^{+n(H(X)-\epsilon)}}$ DOKAZANO!!!

3.2 CONSEQUENCES OF THE AEP: DATA COMPRESSION

IN $A_\epsilon^{(n)}$ THERE ARE $\leq 2^{n(H+\epsilon)}$ SEQUENCES WHICH CAN BE INDEXED BY $n(H+\epsilon)+1$ BITS. PREFIX 0 FOR EACH SEQUENCE $\Rightarrow$ LENGHT OF INDEXING NUMBER $\leq n(H+\epsilon)+2$ bits



NON-TYPICAL SET: $n\,\text{ld}|X| + 2$ bits

TYPICAL SET: $n(H+\epsilon)+2$ bits

- x^n DENOTES A SEQUENCE $x_1, x_2, \dots, x_n$

$l(x^n)$ – LENGHT OF THE CODEWORD CORRESPONDING TO x^n

– IF n IS SUFFICIENTLY LARGE SO THAT $P_r\{A_\varepsilon^{(n)}\} \geq 1-\varepsilon$ THE EXPECTED LENGTH OF CODEWORD IS:

$$\boxed{E[l(X^n)] = \sum_{x^n} p(x^n)\cdot l(x^n)}$$

$$E[l(X^n)] = \sum_{x^n} p(x^n)\, l(x^n) = \sum_{x^n\in A_\varepsilon^{(n)}} p(x^n)\, l(x^n) + \sum_{x^n\in A_\varepsilon^{(n)c}} p(x^n)\, l(x^n)$$

$$\leq \sum_{x^n\in A_\varepsilon^{(n)}} p(x^n)\left(n(H+\varepsilon)+2\right) + \sum_{x^n\in A_\varepsilon^{(n)c}} p(x^n)\left(n\, ld|\mathcal{X}|+2\right) =$$

$$= P_r\{A_\varepsilon^{(n)}\}\left(n(H+\varepsilon)+2\right) + P_r\{A_\varepsilon^{(n)c}\}\left(n\, ld|\mathcal{X}|+2\right) \leq$$

$$n(H+\varepsilon) + \varepsilon n\left(ld|\mathcal{X}|\right) + 2 = n\left(H+\varepsilon'\right)$$

$$\varepsilon' = \varepsilon\cdot ld|\mathcal{X}| + \frac{2}{n} + \varepsilon .$$

THEOREM 3.2.1 LET X^n be i.i.d $\sim p(x)$. LET $\varepsilon>0$ THEN THERE EXISTS A CODE THAT MAPS x^n OF length n INTO BINARY STRINGS SUCH THAT THE MAPPING IS ONE-TO-ONE (AND THEREFORE INVERTIBLE) AND:

$$E\left[\frac{1}{n} l(X^n)\right] \leq H(X)+\varepsilon$$

HENCE WE CAN REPRESENT SEQUENCE $\mathcal{X}^n$ USING $nH(X)$ BITS ON AVERAGE.

3.3 HIGH PROBABILITY SETS AND TYPICAL SET

FROM THE DEFINITION OF $A_\varepsilon^{(n)}$ IT IS CLEAR THAT $A_\varepsilon^{(n)}$ IS FAIRLY SMALL SET THAT CONTAINS MOST OF THE PROBABILITY. WHETHER IT IS THE SMALLEST SET? WE WILL PROVE THAT THE TYPICAL SET HAS ESSENTIALLY THE SAME NUMBER OF ELEMENTS AS THE SMALLEST SET, TO FIRST ORDER IN EXPONENT.

DEFINITION: FOR EACH $n = 1, 2, \dots$ LET $B_\delta^{(n)} \subset \mathcal{X}^n$ BE THE SMALLEST SET WITH:

$$P_r\{B_\delta^{(n)}\} \geq 1-\delta$$

WE ARGUE THAT $B_\delta^{(n)}$ MUST HAVE SIGNIFICANT INTERSECTION WITH $A_\varepsilon^{(n)}$ AND THEREFORE MUST HAVE ABOUT

AS MANY ELEMENTS. IN PROBLEM 3.11 WE OUTLINE THE PROOF OF FOLLOWING THEOREM:

$\boxed{\text{THEOREM 3.3.1.}}$ LET $x_1, x_2, \ldots, x_n$ BE I.I.D. ~ $p(x)$, FOR $\delta < \frac{1}{2}$ AND $\delta' > 0$, if $\Pr\{B_\delta^{(n)}\} > 1-\delta$ THEN

$$\frac{1}{n}\log|B_\delta^{(n)}| > H - \delta'$$ FOR "n" SUFFICIENTLY LARGE.

THUS, $B_\delta^{(n)}$ MUST HAVE AT LEAST $\boxed{2^{nH}}$ ELEMENTS TO FIRST ORDER IN THE EXPONENT. $A_\epsilon^{(n)}$ HAS $2^{n(H(X)\pm\epsilon)}$ ELEMENTS.

DEFINITION: THE NOTATION $\boxed{a_n \doteq b_n}$ MEANS

$$\lim_{n\to\infty}\left[\frac{1}{n}\log\frac{a_n}{b_n}\right] = 0$$

THIS IMPLIES THAT a_n & b_n ARE EQUAL TO THE FIRST ORDER IN THE EXPONENT.

- THEREFORE, if $\delta_n \to 0$ & $\epsilon_n \to 0$ THEN

$\boxed{\text{MMV}}$ $$\boxed{|B_{\delta_n}^{(n)}| \doteq |A_{\epsilon_n}^{(n)}| \doteq 2^{nH}}$$

$\boxed{071309700}$

$\boxed{3.1}$ MARKOV INEQUALITY & CHEBISHEV INEQUALITY.

(a) (Markov's inequality) FOR ANY NONNEGATIVE RANDOM VARIABLE X AND ANY $t>0$, SHOW THAT

$$\boxed{\Pr\{X \geq t\} \leq \frac{E[X]}{t}}$$

EXHIBIT RANDOM VARIABLE THAT ACHIEVES THIS INEQUALITY WITH EQUALITY.

(b) (Chebishev's inequality) LET Y BE A RANDOM VARIABLE WITH MEAN μ AND VARIANCE σ^2. BY LETTING $X = (Y-\mu)^2$, SHOW THAT FOR ANY $\epsilon > 0$

$$\Pr\{|Y-\mu| > \epsilon\} \leq \frac{\sigma^2}{\epsilon^2}$$

(a) $\Pr\{X \geq t\} = 1 - \Pr\{X \leq t\}$

$$E[X] = \sum_{i=1}^{n} p(x_i)\cdot x_i$$

$$x \in \{1, 2, \ldots, n\}$$

$$p(x) = \{p_1, p_2, \ldots, p_n\}$$

$$Pr(x > t) = 1 - Pr(x \leq t) = 1 - \sum_{i=1}^{t} \cdot p_i$$

$$\frac{E[X]}{t} = \frac{1}{t} \sum_{i=1}^{n} x_i p_i \geq 0 \qquad \boxed{\frac{E[X]}{t} \geq Pr(x > t)}$$

$$\frac{1}{t} \sum_{i=1}^{n} x_i p_i \geq 1 - \sum_{i=1}^{t} x_i p_i = \sum_{i=t+1}^{n} x_i p_i$$

$$\sum_{i=1}^{n} x_i p_i \geq t \sum_{i=t+1}^{n} x_i p_i \qquad \sum_{i=t+1}^{n} x_i p_i - \sum_{i=t+1}^{n} t x_i p_i + \sum_{i=1}^{t} x_i p_i \geq 0$$

$$\sum_{i=t+1}^{n} (1-t) x_i p_i + \sum_{i=1}^{t} x_i p_i \geq 0 \qquad ?$$

$$\sum_{i=1}^{n} x_i p_i \geq t - t \sum_{i=1}^{t} x_i p_i \qquad \sum_{i=1}^{t} x_i p_i + \sum_{i=t+1}^{n} x_i p_i + t \sum_{i=1}^{t} x_i p_i \geq t$$

$$(1+t) \sum_{i=1}^{t} x_i p_i + \sum_{i=t+1}^{n} x_i p_i \geq t$$

$$\frac{1}{t} \sum_{i=1}^{n} x_i p_i = \frac{1}{t} \left[\underbrace{\sum_{i=1}^{t} x_i p_i} + \sum_{i=t+1}^{n} x_i p_i \right]$$

$$x \in [1, 2, \ldots, t, \ldots n]$$

$$p(x) = \{p_1, p_2, \ldots p_t \ldots p_n\}$$

$$E[X] = \sum_{i=1}^{n} i \cdot p_i = p_1 + 2p_2 + 3p_3 + \ldots t p_t + \ldots + n p_n$$

$$\frac{E[X]}{t} = \frac{p_1}{t} + \frac{2p_2}{t} + \ldots p_t + \ldots + \frac{n}{t} p_n \geq$$

$$p_1 + p_2 + \ldots + p_t + \ldots + \frac{n}{t} p_n$$

$$1 - \sum_{i=1}^{t} p_i = 1 - p_1 - p_2 - p_3 - \ldots - p_n$$

$$Pr(x>t) = \sum_{i=t+1}^{n} p_i = p_{t+1} + p_{t+2} + \dots p_n$$

$$\frac{E[x]}{t} = \frac{p_1}{t} + \frac{2p_2}{t} + \dots + p_t + \frac{(t+1)p_{t+1}}{t} + \dots + \frac{n}{t} p_n.$$

$$\leq \frac{p_1}{t} + \frac{2p_2}{t} + \dots + p_t + \underbrace{(t+1)p_{t+1} + \dots + (n-1)p_{n-1}}_{Pr(x>t)}$$

$S = 1+2+ \dots +n$
$S = n+n-1+ \dots +1$
$2S = (n+1)\cdot n$
$\boxed{S = \frac{(n+1)\cdot n}{2}}$

$$x \in [t, 2t, 3t, \dots nt]$$

$$\boxed{\frac{E[x]}{t} = \sum_{i=1}^{n} i p_i} \qquad Pr(x>t] = \sum_{i=2}^{n} t p_i$$

$$\frac{E[x]}{t} = \sum_{i=1}^{n} i p_i \qquad Pr(x>t) = t \sum_{i=2}^{n} p_i$$

$$\frac{E[x]}{t} = p_1 + \sum_{i=2}^{n} i p_i$$ ~~$\geq \frac{Pr(x>t)}{t} \geq Pr(x>t)$~~

$$\frac{E[x]}{t} = \frac{p_1}{t} + \frac{2p_2}{t} + \dots + p_t + \frac{(t+1)p_{t+1}}{t} + \dots + \frac{n-1}{t} p_{n-1}$$

$$\geq \underbrace{p_1 + p_2 + + \dots p_t}_{1 - Pr(x>t)} + \frac{(t+1)p_{t+1}}{t} + \dots + \frac{n-1}{t} p_{n-}$$

$$\boxed{E[x] = \frac{1}{n} \sum_{i=1}^{n} i = \frac{1}{\cancel{n}} \frac{\cancel{n} \cdot n+1}{2} = \boxed{\frac{n+1}{2}}}$$

$$\frac{E[x]}{t} = \frac{1}{t} \cdot \frac{n+1}{2} \qquad x \in [1, 2, \dots t, \dots n]$$

$$Pr(x>t) = \sum_{i=t+1}^{n} p_i$$

$$Pr(x>t) = p_{t+1} + p_{t+2} + \dots p(n)$$

$$\frac{E[x]}{t} = \frac{p_1}{t} + \frac{2p_2}{t} + \dots + \frac{(t+1)p_{t+1}}{t} + \dots \frac{n p_n}{t} \leq \frac{p_1}{t} + \frac{2p_2}{t} + \dots + \overbrace{p_{t+1} + \dots + p_n}^{Pr(x>t)}$$

- PRIMER ZA UNIFORMNA RASPREDELBA:

$$x \in \{1, 2, 3, \dots, n\}$$

$$p(x) = \left\{\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right\}$$

$$\boxed{E[x] = \frac{1}{n}\sum_{i=1}^{n} i = \frac{1}{\cancel{n}}\frac{\cancel{n}(n+1)}{2} \qquad E[x] = \frac{n+1}{2}}$$

$$P_r\{x > t\} = (n-t)\cdot\frac{1}{n}$$

$$\frac{E[x]}{t} = \frac{n+1}{2t} \quad (\$)$$

$$\frac{n+1}{2t} \geqslant \frac{(n-t)}{n} \qquad n^2+n \geqslant 2nt - 2t^2$$

$$n^2+n+2t^2-2nt \geqslant 0$$

$$f(t) = 2t^2 - 2nt + n(n+1) \quad \boxed{\text{JPSSDP}}$$

VO OVOJ SLUČAJ NE VAŽI NERAVENSTVOTO.

- PRIMERI SO GEOMETRISKA RASPREDELBA

$$x \in \{1, 2, 3, \dots, n\}$$

$$p(x) = \left\{\frac{1}{2}, \frac{1}{2^1}, \dots \frac{1}{2^{n-1}}, \frac{1}{2^{n-1}}\right\}$$

$$S = q^1 + q^2 + \dots + q^n$$

$$qS = q^2 + q^3 + \quad + q^{n+1}$$

$$S(1-q) = q - q^{n+1}$$

$$\boxed{S = \frac{1-q^{n+1}}{1-q}}$$

$$\boxed{S = \frac{2-2^{n+1}}{1-2}}$$

$$\boxed{S = \frac{\frac{1}{2} - \frac{1}{2}^{n+1}}{1-\frac{1}{2}}}$$

$$\boxed{S = \frac{\cancel{\frac{1}{2}}\left(1-\frac{1}{2^n}\right)}{\cancel{\frac{1}{2}}}}$$

$$S = \left(1 - \frac{1}{2^n}\right)$$

$$S = \left(1 - \frac{1}{2^{n-1}}\right) + \frac{1}{2^{n-1}} = 1$$

$$\boxed{P_r(x > t) = 1 - \left(1 - \frac{1}{2^t}\right) = \frac{1}{2^t}}$$

MAKE!!!

$$E[x] = \sum_{i=1}^{n-1} i \cdot p(i) \underbrace{=}_{+n\cdot\frac{1}{2^{n-1}}} \sum_{i=1}^{n-1} i\cdot\frac{1}{2^i} + \frac{n}{2^{n-1}} = -\cancel{\frac{n}{2^{n-1}}} - \frac{1}{2^{n-1}} + 2 + \cancel{\frac{n}{2^{n-1}}}$$

$$\boxed{E[x] = 2 - \frac{1}{2^{n-1}}}$$

~~$n\frac{1}{2^{n-1}} = \frac{n}{2^n}(2^n-1) + \frac{1}{2^{n-1}}(2^n-1)$~~

~~$E[X] = (2^{n}-1) \dots = \frac{n+2}{2^n} = (n+2) \dots$~~

~~$n=10$ $\quad E[X] = \dots \frac{1}{2^{n}}$~~

$$\frac{E[X]}{t} = \frac{2}{t} - \frac{1}{t \cdot 2^{n-1}} \qquad Pr(X>t) = \frac{1}{2^t}$$

$$\frac{2}{t} - \frac{1}{t \cdot 2^{n-1}} \geqslant \frac{1}{2^t} \qquad \frac{2 \cdot 2^t}{t} - \frac{2^t}{t \cdot 2^{n-1}} \geqslant 1$$

$$2 \cdot 2^t - 2^{t-n+1} \geqslant t \qquad \boxed{2^{t-1} - 2^{t-n+1} - t \geqslant 0}$$

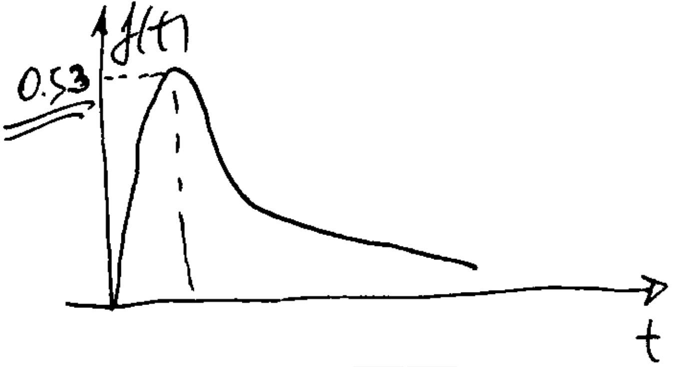
$$\frac{2^t}{t}\left[2 - \frac{1}{2^{n-1}}\right] \geqslant 1 \qquad \left(2 - \frac{1}{2^{n-1}}\right) \geqslant \boxed{\frac{t}{2^t}}$$

$$\boxed{\underbrace{\left(2 - \frac{1}{2^{n-1}}\right)}_{g(n)} \geqslant \underbrace{t \cdot 2^{-t}}_{f(t)}}$$

$$f(0) = 0$$

$$f(1) = \frac{1}{2}$$

$$f(2) = \frac{2}{2^2} = \frac{1}{4}$$



$$g(n): \; g(1) = 1$$

$$g(2) = 2 - \frac{1}{2} = \frac{3}{2}$$

$$g(3) = 2 - \frac{1}{4} = \frac{7}{4}$$

VAŽI NERAVENSTVOTO !!!

- SE NAVRAĆAM NA UNIFORMNATA ④ N13.40

$$Pr\{X \geqslant t\} = \frac{(n-t+1)}{n} \qquad \frac{n+1}{2t} \geqslant \frac{n-t+1}{n}$$

$$n^2+n \geqslant 2t \cdot n - 2t^2 + 2t \qquad 2t^2 - 2(n+1)t + n(n+1) \geqslant 0$$

ISTOTO SE DOBIVA KAKO I ZA $Pr\{X>t\}$

- SEGA DA GLEDAME ZA: $X = \left\{\frac{1}{2^{n-i}} : i = 1 \dots n\right\}$

$$P = \left[\left\{\frac{1}{2^i} : i = 1 \dots n-1\right\}, \frac{1}{2^{n-1}}\right]$$

$$E[X] = \sum_{i=1}^{n} x_i \, p(i) = \sum_{i=1}^{n-1} \frac{1}{2^{n-i}} \cdot \frac{1}{2^i} + \frac{1}{2^{n-1}} \cdot 1 = \sum_{i=1}^{n-1} \frac{1}{2^n} = \frac{n-1}{2^n} + \frac{1}{2^{n-1}} = \frac{n+1}{2^n}$$

$$\boxed{E[X] = \frac{n+1}{2^n}}$$

$\frac{E[X]}{t} = \frac{n+1}{t \cdot 2^n}$ $P_r[X > t] = ?$ $P_r[X > t] = \frac{1}{2^n}$

$x = \left[\frac{1}{2^{n-1}}, \frac{1}{2^{n-2}}, \dots, t, \dots, \frac{1}{2^2}, \frac{1}{2^1}, 1 \right]$

$p(x) = \left[\frac{1}{2}, \frac{1}{2^2}, \dots, \frac{1}{2^n} \dots \frac{1}{2^{n-2}}, \frac{1}{2^{n-1}}, \frac{1}{2^{n-1}} \right]$

$t = \frac{1}{2^{n-m}}$ $t = \frac{1}{2^n} \cdot 2^n$ $\boxed{2^n = t \cdot 2^n}$

$2^n = \frac{1}{256} \cdot 2^{10} = 4$

$n = ld\, 4 = 2$

$$\boxed{P_r[X > t] = \frac{1}{t \cdot 2^n}}$$

$\cancel{\frac{n+1}{t \cdot 2^n}} \geq \cancel{\frac{1}{t \cdot 2^n}}$ $n+1 \geq 1$ $\boxed{n \geq 0}$

/ VO OVOJ SLUČAJ VAŽI RAVENSTVOTO.

$X = [1, 2, \dots, t, \dots n]$ $\frac{E[X]}{t} = \frac{1}{t} \frac{\cancel{x}(n+1)}{2\cancel{x}}$

$$\boxed{\frac{E[X]}{t} = \frac{n+1}{2t}}$$

$\left.\frac{1}{t}, \frac{1}{t}, \dots \frac{1}{t}\right\}$ OPT.1

$p = [p_1, p_2, \dots p_t, \dots p_n]$

$$\boxed{\sum_{i=1}^{n} p_i = \underbrace{p_t + p_{t+1} \cdots p_n}_{\frac{n+1}{2}\}\text{ OPT 1}} = \frac{n+1}{2 \cdot t}}.$$

$p = \left[\frac{1}{n}, \frac{1}{n-1}, \dots, \frac{1}{t}, \dots \frac{1}{2} \right]$

$\boxed{P_r[X \geq t] \leq \frac{E[X]}{t}}$ $\sum_{i=t}^{n} p_i \leq \frac{1}{t} \sum_{i=1}^{n} p_i \cdot x_i$

$$t\cdot\sum_{i=t}^{n}\gamma_i \le \sum_{i=1}^{t-1}\gamma_i x_i + \sum_{i=t}^{n}\gamma_i x_i$$

$$\sum_{i=t}^{n}(\textcircled{t}-\gamma_i)x_i \le \boxed{\sum_{i=1}^{t-1}\gamma_i x_i}$$

$$\boxed{\gamma_1 x_1+\gamma_2 x_2+\dots\gamma_{t-1}x_{t-1}+(t-\gamma_t)X_t+(t-\gamma_{t+1})X_{t+1}+\dots+(t-\gamma_n)X \ge 0}$$

$$X\in[x_1,x_2,\dots,x_t,\dots x_n]=[x_1,x_2,\dots,\textcircled{\gamma_t},\dots\gamma_t]$$

$$\gamma\in[\gamma_1,\gamma_2,\dots\gamma_t,\dots\gamma_n]=[\gamma_1,\gamma_2,\dots\gamma_t,\dots\gamma_n]$$

$$\frac{E[x]}{t}=\sum_{i=1}^{t-1}\gamma_i x_i+\sum_{i=t}^{n}\underbrace{\gamma_i}_{\gamma_t} x_i=\sum_{i=1}^{t-1}\gamma_i x_i+\gamma_t\sum_{i=t}^{n}x_i$$

EDITION 1 SOLUTION

(a) IF X HAS DISTRIBUTION $F(x)$

$$E[X]=\int_0^\infty x\,dF=\int_0^\delta x\,dF+\int_\delta^\infty x\,dF\ge\int_\delta^\infty x\,dF$$

$$\ge\int_\delta^\infty\delta\,dF=\delta\cdot Pr[x\ge\delta] \qquad \boxed{Pr[x\ge\delta]\le\frac{E[x]}{\delta}}$$

PROOFED!!!

$$E[X]=E[X\,|X\le\delta]\,Pr(x\le\delta)+E[X\,|x>\delta]\cdot Pr[x>\delta]$$

$$\circledast\ E[x]=E[x|x\ge\delta]\cdot Pr\{x\ge\delta\}+E[x|x<\delta]\,Pr\{x<\delta\}$$

$$E[x\,|Y=y]=\sum_{x\in X}x\cdot P(X=x|Y=y)=\sum_{x\in X}x\,\frac{P(X=x,Y=y)}{P(Y=y)}$$

$$E[x|x\le\delta]=\sum_{x\in X}x\cdot p(x|x\le\delta)=\sum_{x\in X}x\cdot\frac{p(x,x\le\delta)}{p(x\le\delta)}$$

$$E[x]=\sum_{x\in X}x\cdot p(x)=\sum_{x\in X}x\cdot\sum_{n}p(x|n)$$

$$E[x]\ge E[x|x\ge\delta]\cdot Pr\{x\ge\delta\}\ge\delta\cdot Pr\{x\ge\delta\}$$

JAKO!!!

- CONDITIONAL EXPECTATION AND TOTAL EXPECTATION

- IF R IS RANDOM VARIABLE AND "E" IS EVENT THEN CONDITIONAL EXPECTATION IS DEFINED BY:

$$Ex(R|E) = \sum_{\omega \in S} R(\omega)\cdot Pr(\omega|E) \quad .$$

- FOR EXAMPLE, LET R BE THE NUMBER THAT COMES UP ON A ROLL OF A FAIR DIE, AND LET "E" BE THE EVENT THAT NUMBER IS EVEN. LET'S COMPUTE $Ex(R|E)$, THE EXPECTED VALUE OF A DIE ROLL, GIVEN THAT THE RESULT IS EVEN.

$$Ex(R|E) = \sum_{\omega \in E} R(\omega)\cdot Pr(\omega|E) = 1\cdot 0 + 2\cdot\frac{1}{3} + 3\cdot 0 + 4\cdot\frac{1}{3} + 5\cdot 0 + 6\cdot\frac{1}{3} =$$

$$= \frac{2}{3} + \frac{4}{3} + \frac{6}{3} = \frac{12}{3} = 4$$

THEOREM (TOTAL EXPECTATION)

$$Ex(R) = Ex(R|E_1)\cdot Pr(E_1) + Ex(R|E_2)Pr(E_2) + \dots + Ex(R|E_n)\cdot Pr(E_n)$$

ANALOGUS TO TOTAL PROBABILITY THEOREM

$$P(B) = P(B|A_1)P(A_1) + P(B|A_2)\cdot P(A_2) + \dots + P(B/A_n)\cdot P(A_n)$$

- FOR EXAMPLE, LET R BE THE NUMBER: THAT COMES UP ON FAIR DIE, AND "E = THE EVENT THAT THE RESULT IS EVEN. THEN THE E IS THE EVENT THAT THE RESULT IS ODD. SO TOTAL EXPECTATION THEOREM SAYS:

$$Ex(R) = \underbrace{Ex(R|E)}_{4}\cdot\underbrace{Pr(E)}_{1/2} + \underbrace{Ex(R|\overline{E})}_{?}\cdot\underbrace{Pr(\overline{E})}_{1/2}$$

$= 7/2$

171
341
$\boxed{512}$

$$Ex(R|\overline{E}) = 1\cdot\frac{1}{3} + 2\cdot 0 + 3\cdot\frac{1}{3} + 4\cdot 0 + 5\cdot\frac{1}{3} + 6\cdot 0$$

$$Ex(R|\overline{E}) = \frac{1+3+5}{3} = \frac{9}{3} = \textcircled{3}$$

$$\boxed{Ex(R) = 4\cdot\frac{1}{2} + 3\cdot\frac{1}{2} = 2 + \frac{3}{2} = \frac{4+3}{2} = \frac{7}{2}}$$

P(R,E)

E\R	1	2	3	4	5	6	7	8	9	10	P(E)
E	0	1/4	0	1/16	0	1/64	0	1/256	0	1/512	171/512
$\overline{E}$	1/2	0	1/8	0	1/32	0	1/128	0	1/512	0	341/512
P(R)	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256	1/512	1/512	1

$P(R|E)$

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$E \backslash R$	1	2	3	4	5	6	7	8	9	10
E	0	$\frac{512}{171}\cdot\frac{1}{4}$	0	$\frac{512}{171}\frac{1}{16}$	0	$\frac{512}{171}\frac{1}{64}$	0	$\frac{512}{171}\frac{1}{256}$	0	$\frac{512}{171}\frac{1}{512}$
$\bar{E}$	$\frac{512}{341}\frac{1}{2}$	0	$\frac{512}{341}\frac{1}{8}$	0	$\frac{512}{341}\frac{1}{32}$	0	$\frac{512}{341}\frac{1}{128}$	0	$\frac{512}{341}\frac{1}{512}$	0

$$E_x(R) = \underbrace{E_x(R|E)}_{?}\cdot\underbrace{P_R(E)}_{\frac{171}{512}} + \underbrace{E_x(R|\bar{E})}_{?}\cdot\underbrace{P_R(\bar{E})}_{\frac{341}{512}}$$

CONTINUE FROM PP.43

- FOR GIVEN δ, THE DISTRIBUTION ACHIEVING: $P_r\{X \geq \delta\} = \frac{E[X]}{\delta}$ IS: $X = \begin{cases} \delta & \text{WITH PROB. } \mu/\delta \\ 0 & \text{WITH PROB. } 1-\mu/\delta \end{cases}$

WHERE $\mu \leq \delta$

$$E[X] = \cancel{\delta}\cdot\frac{\mu}{\cancel{\delta}} + \cancel{0}\cdot\left(1-\frac{\mu}{\delta}\right) \qquad \boxed{E[X]=\mu}$$

$$P_r\{X\geq\delta\} = \mu/\delta \qquad \boxed{\mu/\delta = \mu/\delta} \ ✓$$

- IZGLEDA VAZI ZA. OSTE. POEDNOSTAVNA DISTRO:

G: $X = \begin{cases} \delta & \text{WITH PROBABILITY } \mu \\ 0 & \text{WITH PROBABILITY } 1-\mu \end{cases}$

$$E[X] = \delta\cdot\mu + \cancel{0}\cdot(1-\mu) = \delta\mu$$

$$P_r\{X\geq\delta\} = \mu \Rightarrow \mu = \frac{\cancel{\delta}\mu}{\cancel{\delta}} \qquad \boxed{\mu=\mu} \ ✓$$

(b) Chebyshev INEQUALITY: LET Y BE A RANDOM VARIABLE WITH MEAN μ AND VARIANCE δ^2. BY CREATING $Z=(Y-\mu)^2$, SHOW THAT FOR ANY $\varepsilon>0$

$$P_r\{|Y-\mu| > \varepsilon\} \leq \frac{\delta^2}{\varepsilon^2}$$

$$\boxed{P_r\{X\geq\delta\} \leq \frac{E[X]}{\delta}}$$

MARKOV INEQUALITY

$$\rightarrow P_r\{[Y-\mu]^2 > \varepsilon^2\} \leq \underbrace{P_R\{(Y-\mu)^2 \geq \varepsilon^2\} \leq \frac{E[(Y-\mu)^2]}{\varepsilon^2}}_{\text{USE MARKOV INEQUALITY}}$$

$$E[(Y-\mu)^2] = \delta^2 \rightarrow P_r\{[Y-\mu]^2 > \varepsilon^2\} \leq \frac{\delta^2}{\varepsilon^2}$$

$$P_r\{[Y-\mu]^2 > \varepsilon^2\} = P_r\{(Y-\mu) > \varepsilon\} \Rightarrow P_r\{(Y-\mu)>\varepsilon\} \leq \frac{\delta^2}{\varepsilon^2}$$

PROVED !!!

(c) (SPECIAL IN EDITION 1) PROOVE THE WEEK LAW OF LARGE NUMBERS. LET $Z_1, Z_2, \dots, Z_n$ BE A SEQUENCE OF i.i.d RANDOM VARIABLES WITH MEAN μ AND VARIANCE σ^2. LET $\bar{Z}_n = \frac{1}{n}\sum_{i=1}^{n} Z_i$ BE A SAMPLE MEAN.

SHOW THAT:

$$P_r \cdot \{|\bar{Z}_n - \mu| \geq \varepsilon\} \leq \frac{\sigma^2}{n\varepsilon^2}$$

THUS $P_r\{|\bar{Z}_n - \mu| \geq \varepsilon\} \to 0$ AS $n \to \infty$. THIS IS KNOW AS THE WEAK LAW OF LARGE NUMBERS.

$$E[\bar{Z}_n] = \frac{1}{n} E\left[\sum_{i=1}^{n} Z_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[Z_i] = \frac{1}{\cancel{n}} \cdot \cancel{n} \cdot \mu = \mu$$

$$Var[\bar{Z}_n] = E[(\bar{Z}_n - \mu)^2] = E[\bar{Z}_n^2 - 2\mu \bar{Z}_n + \mu^2] =$$

$$= E[\bar{Z}_n^2] - 2\mu \underbrace{E[\bar{Z}_n]}_{\mu} + \mu^2 = E[\bar{Z}_n^2] - \mu^2$$

$$E[\bar{Z}^2] = \frac{1}{n^2} E[(Z_1 + Z_2)\cdot(Z_1 + Z_2)] = \frac{1}{n^2} E[Z_1^2 + 2Z_1Z_2 + Z_2^2]$$

$$\overset{n=2}{=} \frac{1}{n^2}\left\{E[Z_1^2] + 2\underbrace{E[Z_1Z_2]} + E[Z_2^2]\right\} = \frac{2}{\cancel{n^2}_{4}} \cdot \underbrace{E[Z_1^2]}_{\sigma^2 + \mu^2} = \frac{1}{2}(\sigma^2 + \mu^2)$$

FOR n: $E[\bar{Z}^2]^{0} = \frac{n}{n^2}(\sigma^2 + \mu^2) = \frac{1}{n}(\sigma^2 + \mu^2)$

$\boxed{\text{NE E DOBRO VAKA. GO ADJUSTAM } \cdot \frac{1}{n}}$

$$Var[\bar{Z}] = \frac{1}{n}(\sigma^2 + \mu^2) - \mu^2$$

$$\sigma^2 = E[(Z_1 - \mu)^2] = E[Z_1^2] - 2\mu E[Z_1] + \mu^2 = E[Z_1^2] - \mu^2$$

$$\bullet\ Var[\bar{Z}_n] = E\left[\left(\frac{1}{n}\sum_{i=1}^{n} Z_i - \mu\right)^2\right] = \frac{1}{n^2} E\left[\sum Z_i^2\right] - 2\mu \underbrace{E\left(\frac{1}{n}\sum_{i=1}^{n} Z_i\right)}_{\mu} + \mu^2$$

$$\boxed{Var[\bar{Z}_n] = \frac{1}{n^2} E\left[\sum Z_i^2\right] - \mu^2}$$

VAKA E DOBRO!!!

$$E\left[\left(\sum_{i=1}^{2} Z_i\right)^2\right] = E[(Z_1 + Z_2)(Z_1 + Z_2)] = E[Z_1^2] + 2\underbrace{E[Z_1Z_2]}_{0 \text{ STATISTIČKI NEZAVISNI!}} + E[Z_2^2]$$

$$E\left[\left(\sum_{i=1}^{2} Z_i\right)^2\right] = 2\cdot(\sigma^2 + \mu^2)$$

ANALOGNO: $E\left[\left(\sum_{i=1}^{n} Z_i\right)^2\right] = n\cdot(\sigma^2 + \mu^2)$

$$Var[\bar{Z}_n] = \frac{1}{n^2} n \cdot (\sigma^2 + \mu^2) - \mu^2 = \frac{\sigma^2}{n} + \frac{\mu^2}{n} - \mu^2$$

$\bar{Z}_n = \frac{1}{n}\sum_{i=1}^{n} z_i$ $\boxed{\epsilon[z_i] = \mu \quad \epsilon[(z_i-\mu)^2] = \sigma^2}$ ⊛

$\epsilon[\bar{Z}_n] = \frac{1}{n}\sum_{i=1}^{n}\epsilon[z_i] = \frac{1}{n}\cdot n\cdot\mu = \mu$

~~$\epsilon[(\bar{Z}_n-\mu)^2] = \epsilon[\bar{Z}_n^2] - 2\mu\,\epsilon[\bar{Z}_n] + \mu^2 = \epsilon[\bar{Z}_n^2] - \mu^2$~~

$\epsilon[(\bar{Z}_n-\mu)^2] =$ ~~$\epsilon\left[\frac{1}{n^2}\left(\sum_{i=1}^{n} z_i\right)^2\right] = \frac{1}{n^2}\cdot n\cdot\epsilon[z_1^2]$~~

$= \epsilon\left[\left(\frac{1}{n}\sum_{i=1}^{n} z_i - \mu\right)^2\right] = \epsilon\left[\left(\frac{1}{n}\sum_{i=1}^{n} z_i - \frac{1}{n}\sum_{i=1}^{n}\mu\right)^2\right] =$

$= \epsilon\left[\frac{1}{n}\sum_{i=1}^{n}(z_i-\mu)\right]^2 \underset{\text{STATISTI. INDEPEND.}}{=} \frac{1}{n^2}\sum_{i=1}^{n}\epsilon[\underbrace{(z_i-\mu)^2}_{\sigma^2}] = \frac{1}{n^{\cancel{2}}}\cdot\cancel{n}\cdot\sigma^2 = \frac{\sigma^2}{n}$

OVA E DOBRO IZVEDUVANJE!!! ⊛ ↗ σ^2 $\boxed{\text{MMV}}$

$[(z_1-\mu)+(z_2-\mu)][(z_1-\mu)+(z_2-\mu)] =$

$(z_1-\mu)^2 + 2(z_1-\mu)\cdot(z_2-\mu) + (z_2-\mu)^2$

$\epsilon[(z_1-\mu)\cdot(z_2-\mu)] = \underbrace{\epsilon[z_1 z_2]}_{} - \mu\cdot\underbrace{\epsilon[z_1]}_{\mu} - \mu\,\underbrace{\epsilon[z_2]}_{\mu} + \mu^2 = \underline{\underline{-\mu^2}}$

- OVA JOKE KAŽUVA NEŠTO DRUGO NO MISLAM DEKA
POJAKO E TOA ŠTO AKO NA STATISTIČKI NEZAVISNI
ZVKŠIŠ POJAVANJE ILI ODZEMANJE NA KOMP-
LOMENCIVI STATISTIČKI NEZAVISNI TA
KATA ĆE OTPAGUVAAT $\epsilon[(z_1-\mu)\cdot(z_2-\mu)] = 0$.
ZATOA:

$Z = X + Y$: $\epsilon[X] = \mu_X$ $\epsilon[Y] = \mu_Y$ $\epsilon[(X-\mu_X)^2] = \sigma_X^2$

$\sigma_X^2 = \epsilon[X^2] - \mu_X^2$

$\epsilon[(Y-\mu_Y)^2] = \sigma_Y^2$

$\epsilon[Z] = \epsilon[X+Y] = \epsilon[X] + \epsilon[Y] = \mu_X + \mu_Y$

$\epsilon[(Z-\mu_Z)^2] = \epsilon[Z^2] - 2\mu_Z\cdot\epsilon[Z] + \mu_Z^2 = \epsilon[Z^2] - \mu_Z^2$

$\epsilon[Z^2] = \epsilon[X^2 + 2XY + Y^2] = \epsilon[X^2] + \epsilon[Y^2] = \sigma_X^2 + \mu_X^2 + \sigma_Y^2 + \mu_Y^2$

$\epsilon[(Z-\mu_Z)^2] = \sigma_X^2 + \mu_X^2 + \sigma_Y^2 + \mu_Y^2 - (\mu_X + \mu_Y)^2 =$ ↗ I VO TEORIJA NA INFORM.

$\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2 - 2\mu_X\mu_Y$

ZA NORMALNA DISTRI-BUCIJA IMA DOKAZ DOKAZ DEKA $\sigma_Z^2 = \sigma_Y^2 + \sigma_X^2$ SO KARAKTERISTIČNI FUNKCII (VIDI WIKIPED.)

CHEBISHEV INEQUALITY:

$$P_r\{|Y-\mu| > \varepsilon\} \le \frac{\sigma^2}{\varepsilon^2} \qquad \bar{Z}_n = \frac{1}{n}\sum_{i=1}^{n} z_i$$

$$E[\bar{Z}_n] = \mu \qquad Var\{\bar{Z}_n\} = \frac{\sigma^2}{n}$$

$$P_r\{|\bar{Z}_n - \mu| > \varepsilon\} \le \frac{\sigma^2}{n \cdot \varepsilon^2}$$

BY LETTING $n \to \infty$ ⇒ $P_r\{|\bar{Z}_n - \mu| > \varepsilon\} \to 0$

PROVED!!!

$\boxed{3.2}$ AEP AND MUTUAL INFORMATION. LET (X_i, Y_i) BE i.i.d ~ $p(x,y)$. WE FORM THE LOG LIKELIHOOD RATIO OF THE HYPOTHESIS THAT X AND Y ARE INDEPENDENT VS. HYPOTHESIS THAT X AN Y ARE DEPENDENT. WHAT IS THE LIMIT OF:

$$I = \frac{1}{n}\, ld \frac{p(X^n)\, p(Y^n)}{p(X^n, Y^n)}$$

$$-\frac{1}{n}\, ld\, p(x_1, x_2, \dots x_n) = -\frac{1}{n} \underbrace{\sum_{i=1}^{n} ld\, p(x_i)}_{\text{SAMPLE AVERAGE}} \to -E[ld(X_i)] = H(X)$$

$$\boxed{p(x_1, x_2, \dots x_n) \to 2^{-nH(X)}}$$

$$I = \frac{1}{n}\, ld\, p(X^n) + \frac{1}{n}\, ld\, p(Y^n) - \frac{1}{n}\, ld\, p(X^n, Y^n) \xrightarrow[n\to\infty]{}$$

$$\to -H(X) - H(Y) + H(X,Y) = -H(X) - H(Y) + (H(Y) + H(X|Y))$$

$$= -H(X) \pm H(X|Y) = -\underline{I(X,Y)}$$

$$\lim_{n\to\infty} \frac{1}{n}\, ld \frac{p(X^n)\cdot p(Y^n)}{p(X^n, Y^n)} = I(X,Y)$$

CONCORDIA UNIVERSITY SOLUTION

$$\frac{1}{n}\, ld \frac{p(X^n)\cdot p(Y^n)}{p(X^n, Y^n)} = \frac{1}{n}\, ld \prod_{i=1}^{n} \frac{p(x_i)\cdot p(y_i)}{p(x_i, y_i)} =$$

$$= \frac{1}{n} \sum_{i=1}^{n} ld \frac{p(x_i)\cdot q(y_i)}{p(x_i, y_i)} \bullet \lim_{n\to\infty} \frac{1}{n} ld \frac{p(x^n)\cdot q(y^n)}{p(x^n, y^n)} =$$

$$= \sum_{i=1}^{n} \underbrace{\lim_{n\to\infty} \frac{1}{n} ld \frac{p(x_i)\cdot q(y_i)}{p(x_i, y_i)}}_{I(x_i, y_i)} = \sum_{i=1}^{n} I(x_i, y_i)$$

VLATKO
075318796

$$\frac{1}{n} \sum_{i=1}^{n} ld \frac{p(x_i)\cdot q(y_i)}{p(x_i, y_i)} \to E\left[ld \frac{p(x_i) q(y_i)}{p(x_i, y_i)} \right] =$$

$$= - I(X, Y)$$

PROVED!!

3.3 PIECE OF CAKE. A CAKE IS SLICED ROUGHLY IN HALF, THE OTHER PIECES DISCARDED. WE WILL ASSUME THAT A RANDOM CUT CREATES PIECES OF PROPORTIO

$$P = \begin{cases} \left(\frac{2}{3}, \frac{1}{3}\right) & \text{WITH PROBABILITY } \left(\frac{3}{4}\right) = p \\ \left(\frac{2}{5}, \frac{3}{5}\right) & \text{WITH PROBABILITY } \left(\frac{1}{4}\right) = q \end{cases}$$

THUS FOR EXAMPLE, THE FIRST CUT (AND CHOICE OF LARGEST PIECE) MAY RESULT IN PIECE SIZE $\frac{3}{5}$. CUTTING AND CHOOSING FROM THIS PIECE MIGHT REDUCE IT TO SIZE $\left(\frac{3}{5}\right)\cdot\left(\frac{2}{3}\right)$ AT TIME 2, AND SO ON. HOW LARGE TO FIRST ORDER (IN EXPONENT) IS THE PIECE OF CAKE AFTER "n" CUTS.

1) $\frac{1}{4}\cdot\frac{1}{2} = P\left(\frac{3}{5}\right) = \frac{1}{8}$ $\quad P\left(\frac{2}{5}\right) = \frac{1}{4}\cdot\frac{1}{2} = \frac{1}{8}$

2) $P\left(\frac{3}{5}\cdot\frac{2}{3}\right) = P\left(\frac{3}{5}\right)\cdot P\left(\frac{2}{3}\right) = \frac{1}{8}\cdot\frac{3}{4}\cdot\frac{1}{2} = \frac{1}{8}\frac{3}{8} = \frac{3}{64}$

$$P(2) = p\left[\frac{1}{2} P\left(\frac{2}{3}\right) + \frac{1}{2} P\left(\frac{1}{3}\right)\right]$$

$$P\left(\frac{3}{15}\right) = P\left(\frac{3}{5}\cdot\frac{1}{3}\right) = P\left(\frac{3}{5}\right)\cdot P\left(\frac{1}{3}\right) = \frac{1}{8}\cdot\frac{3}{4}\cdot\frac{1}{2} = \frac{3}{64} = P\left(\frac{3}{5}\cdot\frac{2}{3}\right) = P\left(\frac{6}{15}\right.$$

$$P\left(\frac{4}{15}\right) = P\left(\frac{2}{5}\cdot\frac{2}{3}\right) = P\left(\frac{2}{5}\right)\cdot P\left(\frac{2}{3}\right) = \frac{1}{8}\cdot\frac{3}{4}\cdot\frac{1}{2} = \frac{3}{64} = P\left(\frac{2}{5}\cdot\frac{1}{3}\right) = P\left(\right.$$

- AND SE POČNE OD „1"

$P\left(\frac{2}{3}\right) = \frac{3}{4}\cdot\frac{1}{2} = \frac{3}{8}$ $\quad P\left(\frac{1}{3}\right) = \frac{3}{4}\cdot\frac{1}{2} = \frac{3}{8}$

$$P\left(\frac{4}{15}\right) = P\left(\frac{2}{3}\cdot\frac{2}{5}\right) = P\left(\frac{2}{3}\right)\cdot P\left(\frac{2}{5}\right) = \frac{3}{8}\cdot\frac{1}{8} = \frac{3}{64} = P\left(\frac{1}{3}\cdot\frac{2}{5}\right) = P\left(\frac{2}{15}\right)$$

$$P\left(\frac{6}{15}\right) = P\left(\frac{2}{3}\cdot\frac{3}{5}\right) = P\left(\frac{2}{3}\right)\cdot P\left(\frac{3}{5}\right) = \frac{3}{64} = P\left(\frac{1}{3}\cdot\frac{3}{5}\right) = P\left(\frac{3}{15}\right)$$

AKO SEKOGAŠ GO IZBIRAŠ POGOLEMOTO PARČE PROCESOT SE POEDNOSTAVUVA :

$$P = \begin{cases} \left(\frac{2}{3} \middle| \frac{1}{3}\right) & \text{WITH PROBABILITY } \left(\frac{3}{4}\right), \text{"}p\text{"} \\ \left(\frac{2}{5} \middle| \frac{3}{5}\right) & \text{WITH } -\text{"}- \ \left(\frac{1}{4}\right), \text{"}q\text{"} \end{cases}$$

• AKO SE TRGNE OD $\left(2/3\right)$

① $P\left(\frac{2}{3}\right) = P\left(\frac{3}{4}\right)\cdot\frac{1}{2} = \frac{3}{4}\,\frac{1}{2} = \frac{3}{8} = \frac{p}{2}$

$P\left(\frac{1}{3}\right) = \left(\frac{3}{4}\right)\cdot\frac{1}{2} = \frac{3}{8} = \frac{p}{2}$

} SEKOL GO BIRAŠ POGOLEMOTO T.E 2/3

② $P\left(\frac{2}{3}\cdot\frac{3}{5}\right) = P\left(\frac{2}{3}\right)\cdot P\left(\frac{3}{5}\right) = \frac{3}{8}\cdot\frac{1}{4}\cdot\frac{1}{2} = \frac{3}{8}\cdot\frac{1}{8} = \frac{3}{64}$

$P\left(\frac{2}{3}\cdot\frac{2}{5}\right) = P\left(\frac{2}{3}\right)\cdot P\left(\frac{2}{5}\right) = \frac{3}{8}\cdot\frac{1}{8} = \frac{3}{64}$

} GO BIRAŠ POGOLEMOTO 3/5

$P\left(\frac{2}{3}\cdot\frac{2}{3}\right) = P\left(\frac{2}{3}\right)\cdot P\left(\frac{2}{3}\right) = \frac{3}{8}\cdot\frac{3}{8} = \frac{9}{64}$

$P\left(\frac{2}{3}\cdot\frac{1}{3}\right) = P\left(\frac{2}{3}\right)\cdot P\left(\frac{1}{3}\right) = \frac{3}{8}\cdot\frac{3}{8} = \frac{9}{64}$

} GO BIRAŠ POGOLEMOTO 2/3

$P\left(\frac{2}{3}\cdot\frac{3}{5}\right) = P\left(\frac{6}{15}\right) =$

$P\left(\frac{6}{15}\right) = P\left(\frac{2}{3}\cdot\frac{3}{5}\right) = \frac{p}{2}\cdot\frac{q}{2} = \frac{pq}{2^2}$

$P\left(\frac{4}{9}\right) = P\left(\frac{2}{3}\cdot\frac{2}{3}\right) = \frac{p}{2}\cdot\frac{p}{2} = \frac{p^2}{2^2}$

} VO VTORIOT ČEKOR GO IMAŠ ILI $P\left(\frac{6}{15}\right)$ ILI $P\left(\frac{4}{9}\right)$

- AKO SE TRGNE OD $(3/5)$ VO VTOR ČEKOR OČITE SE :

$P\left(\frac{3}{5}\cdot\frac{2}{3}\right) = \frac{pq}{2^2}$ $\quad P\left(\frac{3}{5}\cdot\frac{3}{5}\right) = \frac{q^2}{2^2}$

} MOŽEBI NE ~~TREBA~~ DA MOŽIŠ SO 1/2 ZOŠTO PLOVICO ← OD GO ZEMAŠ POGOLEMOTO PARČE !!!

ENTROPIJATA ZA VTORIOT ČEKOR E:

$$H = -\sum_{k=0}^{2}\binom{n}{k}p^n\cdot q^{n-k}\ ld\binom{n}{k}p^n\cdot q^{n-k} =$$

$\binom{2}{1} = \frac{2!}{1!\cdot1!} = 2$

ZATOA OVDE NEMA „2"

$$= -\left[\binom{n}{0}p^0\cdot q^2\, ld\binom{n}{0}p^0\cdot q^2 + \cancel{\binom{n}{1}}p^1\cdot q^1\, ld\binom{n}{1}pq + \binom{n}{2}p^2\cdot q^0\, ld\binom{n}{2}p^2\right]$$

$$= -\left[q^2\, ld\, q^2 + 2pq\, ld(pq) + p^2\, ld\, p^2\right]$$

VO GENERALEN SLUČAJ ENTROPIJATA ĆE BIDE :

$$\boxed{H(S) = -\sum_{k=0}^{n}\binom{n}{k}p^n q^{n-k}\, ld\binom{n}{k}p^n q^{n-k}}$$

$S = \{x_1, x_2 \dots x_n\}$ ENTROPIJA NA NIZATA OD SLUČAJNI PROM.

(30) $P\left(\frac{2}{3}\cdot\frac{3}{5}\cdot\frac{2}{3}\right) = \Big|$ ODAM BEZ REDENJE PO 2 ZARADI IZBOR NA POGOLEMITE VELIČINA $\Big| = p\cdot q\cdot p = \underline{p^2 q}$

$P\left(\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}\right) = p^3$; $P\left(\frac{2}{3}\cdot\frac{3}{5}\cdot\frac{3}{5}\right) = \underline{p\cdot q^2}$;

$P\left(\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{3}{5}\right) = \underline{p^2\cdot q}$; $P\left(\frac{3}{5},\frac{2}{3},\frac{2}{3}\right) = \underline{q\cdot p^2}$;

$P\left(\frac{3}{5}\cdot\frac{2}{3}\cdot\frac{3}{5}\right) = \underline{\underline{q^2\cdot p}}$; $P\left(\frac{3}{5}\cdot\frac{3}{5}\cdot\frac{2}{3}\right) = \underline{q^2\cdot p}$; $P\left(\frac{3}{5}\frac{3}{5}\frac{3}{5}\right) = q^3$

IMAŠ VKUPNO: $1\times p^3$, $3\times p^2 q$, $3\times p q^2$, $1\times q^3$

MOŽNI ISHODI:

$P\left(\frac{9}{64}\right)$; $3\cdot P\left(\frac{9}{16}\cdot\frac{1}{4}\right) = 3\cdot P\left(\frac{9}{64}\right)$; $3\,P\left(\frac{3}{4}\cdot\frac{1}{16}\right) = 3\cdot P\left(\frac{3}{64}\right)$

; $P\left(\left(\frac{1}{4}\right)^3\right) = P\left(\frac{1}{64}\right)$

$X_i \in \begin{cases} \frac{2}{3} & \text{PROBABILITY } p \\ \frac{3}{5} & \text{PROBABILITY } q = 1-p \end{cases}$

$P(x_1, x_2, \dots, x_i, \dots x_n) \to 2^{-nH(X)} = \underline{2^{-n\cdot\left(p\,ld\frac{1}{p} + q\,ld\frac{1}{q}\right)}}$

$$\boxed{H(X) = p\,ld\frac{1}{p} + q\,ld\frac{1}{q} = \frac{3}{4}ld\frac{4}{3} + \frac{1}{4}ld\,4 = \underline{\underline{0.8113}}}$$

- TIPIČNIOT SET ĆE IMA
  - $n\cdot\frac{3}{4}$ 2/3-KI OTSEČOCI (DVOTRETINSKI)
  - $n\cdot\frac{1}{4}$ 3/5-KI (TRIPETINSKI) ISEČOCI

$$\boxed{\begin{aligned} e^{\ln x} &= x \\ \ln x &= \ln x \end{aligned}}$$

ZA DOVOLNO GOLEMO „n" $\quad \underline{P_r\{A_\varepsilon^{(n)}\} \geq 1-\varepsilon}$

PRIMER: $n = 4$ $\Rightarrow$ NAJVEROJATNATA GOLEMINA NA IZLEZOT ĆE BIDE

$\boxed{\left(\frac{2}{3}\right)^{\frac{3n}{4}}\cdot\left(\frac{3}{5}\right)^{\frac{n}{4}}} = \left(\frac{2}{3}\right)^3\cdot\left(\frac{3}{5}\right)^1 = \frac{8}{27}\cdot\frac{3}{5} = \frac{24}{135} = \frac{8}{45}$

$p\left(\left(\frac{2}{3}\right)^3\cdot\left(\frac{3}{5}\right)^1\right) \doteq 2^{-nH(X)} = 2^{\underline{-4\cdot 0.8113}}$

$$\boxed{\text{DOKAZ DELA} \quad p^{n\cdot p}\cdot q^{n\cdot q} = 2^{-n\cdot H(X)}}$$

$$\boxed{\begin{aligned} &2^{-4\left(\frac{3}{4}ld\frac{4}{3} + \frac{1}{4}ld\,4\right)} = 2^{-4\left(\frac{3}{4}ld\frac{4}{3} + \frac{1}{2}\right)} = 2^{-3\,ld\frac{4}{3} - 2} = 2^{-2}\cdot 2^{-3\,ld\frac{4}{3}} \\ &= \frac{1}{4}\cdot 2^{ld\left(\frac{4}{3}\right)^3} = \frac{1}{4}\cdot\left(\frac{4}{3}\right)^3 \end{aligned}}$$

• Според мене најверојатната големина на типичното тогаш кога „n" = четири ќе биде:

$$L(n)=\left(\frac{2}{3}\right)^{\frac{3}{4}n}\cdot\left(\frac{3}{5}\right)^{\frac{n}{4}} \qquad L(4)=\frac{8}{45}=0,178 \qquad L(10)=0.0133$$

$$\boxed{2^{n\cdot H} = \Big/ n=4 \Big/ = 2^{4\cdot 0.8} = 9.48}$$

$H(X)=0.8113$

максимален број на елементи во типичното множество.

$$\binom{4}{1}=\frac{4!}{3!}=\frac{4\cdot\cancel{3\cdot2\cdot1}}{\cancel{3\cdot2\cdot1}}=4$$

$\boxed{3.4}$ AEP. Let X_i be iid $\sim p(x)$, $x\in\{1,2,\dots,m\}$

Let $\mu = E[X]$ and $H=-\sum p(x)\log p(x)$.

Let $A^n=\{x^n\in X^n : |-\frac{1}{n}\log p(x^n)-H|\le\varepsilon\}$.

Let $B^n=\{x^n\in X^n : |\frac{1}{n}\sum_{i=1}^{n}x_i-\mu|\le\varepsilon\}$. LAW OF LARGE NUMBERS WEEK LAW

(a) Does $Pr\{X^n\in A^n\}\to 1$. Yes

(b) Does $Pr\{X^n\in A^n\cap B^n\}\to 1$. Yes

(a.) A^n е типично множество / ~~за~~ него важи $Pr\{A^{(n)}\}\le 1-\varepsilon$ Theorem 3.1.2.2

(b) B^n - SMOLEST SET $P_r\{B_\delta^{(n)}\}\ge 1-\delta$

(c) Show that: $|A^n\cap B^n|\le 2^{n(H+\varepsilon)}$ for $\forall n$

(d) Show that: $|A^n\cap B^n|\ge\left(\frac{1}{2}\right)\cdot 2^{n(H-\varepsilon)}$ for n sufficiently large

(c) $|A_\varepsilon^{(n)}|\le 2^{n(H+\varepsilon)}$ A_n is typical set

$\Rightarrow |A^{(n)}\cap B^{(n)}|\le|A^{(n)}|$ $\qquad |A^{(n)}|\ge|A^{(n)}\cap B^{(n)}|$

$2^{n(H+\varepsilon)}\ge|A^{(n)}|\ge|A^{(n)}\cap B^{(n)}|$ $\qquad \underline{|A^{(n)}\cap B^{(n)}|\le 2^{n(H+\varepsilon)}}$

(d) $|A^n|\ge(1-\varepsilon)2^{n(H-\varepsilon)}$

$$1=\sum_{x^n\in X^n}p(x^n)\ge\sum_{x^n\in A_\varepsilon^n}p(x^n)=\Big|\ p(x^n)\ge 2^{n(H(x)-\varepsilon)}\Big/=ⓐ$$

$$2^{-n(H(x)+\varepsilon)}\le p(x^n)\le 2^{-n(H(x)-\varepsilon)}$$

① $= \sum_{x^n \in A^n} p(x^n) \geq \sum_{x^n \in A^n} 2^{-n(H(X)+\epsilon)} = |A_n|\, 2^{-n(H(X)+\epsilon)}$

$|A_n| \leq 2^{n(H(X)+\epsilon)}$

$1 = \sum_{x^n \in X^n} p(x^n) \leq \sum_{x^n \in X^n} 2^{-n(H(X)-\epsilon)} \leq |X^n| \cdot 2^{-n(H(X)-\epsilon)}$

$|X^n| \geq 2^{n(H(X)-\epsilon)}$ $\qquad |A^n \cap B^n| \leq |A_n| \qquad p(x^n)$

$|A^n| \geq \boxed{2^{n(H-\epsilon)}} - \epsilon 2^{n(H-\epsilon)}$

$2^{n(H-\epsilon)} \leq |A^n| + \epsilon \cdot 2^{n(H-\epsilon)}$ $\qquad \boxed{\frac{1}{2^{n(H-\epsilon)}}} \geq \frac{1}{|A^n| + \epsilon 2^{n(H-\epsilon}}$

$p(x^n) \geq \frac{1}{|A^n| - \epsilon 2^{(H-\epsilon)}}$ $\qquad 2^{-n(H(X)-\epsilon)} \geq \frac{1}{|A^n| + \epsilon 2^{n(H-\epsilon)}}$

$|A^n|\, 2^{-n(H(X)-\epsilon)} + \epsilon \geq 1$ $\qquad |A^n| \geq (1-\epsilon)\, 2^{n(H(X)-\epsilon)}$

ИСТОТ РЕЗУЛТАТ..!!?

$(1+\epsilon) \geq P_r\{X^{(n)}\} \geq P_r\{A^{(n)}\} = \sum_{x^n \in A^n} p(x^n) \geq \sum_{x^n \in A^n} 2^{-n(H+\epsilon)}$

$= |A^n|\, 2^{-n(H+\epsilon)}$ $\qquad (1+\epsilon) \geq |A^n|\, 2^{-n(H+\epsilon)}$

$$\boxed{|A^n| \leq (1+\epsilon)\, 2^{+n(H+\epsilon)}}$$

$|A^{(n)} \cap B^n| \leq |A^n| \leq (1+\epsilon)\, 2^{+n(H+\epsilon)}$

$(1+\epsilon)\, 2^{n(H+\epsilon)} \geq (1-\epsilon)\, 2^{n(H-\epsilon)}$ $\qquad (1+\epsilon)\, 2^{n\epsilon} \geq (1-\epsilon)\, 2^{-n\epsilon}$

$(1+\epsilon)\, 2^{2n\epsilon} \geq 1-\epsilon$

$\boxed{2^{2n\epsilon} \geq \frac{1-\epsilon}{1+\epsilon}}$ $\qquad 2^{n\epsilon} \geq \sqrt{\left(\frac{1-\epsilon}{1+\epsilon}\right)}$ $\qquad \boxed{2^{-n\epsilon} \leq \sqrt{\frac{1+\epsilon}{1-\epsilon}}}$

$|A^n| \geq \underline{(1-\epsilon)}\, 2^{nH} \cdot 2^{-n\epsilon}$ $\qquad (1-\epsilon) \leq (1+\epsilon)\, 2^{2n\epsilon}$

$\longrightarrow$ SLEDI OD TEOREMA 3.1.2.2

$(1+\epsilon) \geq -(1+\epsilon)\, 2^{2n\epsilon}$

⊡ UNIVERSITY OF TORRONTO SOLUTIONS:

$P_r\{X^n \in A^n \cap B^n\} = P_r\{X^n \in A^n\} + P_r\{X^n \in B^n\} - P_r\{X^n \in A^n \cup B^n\}$

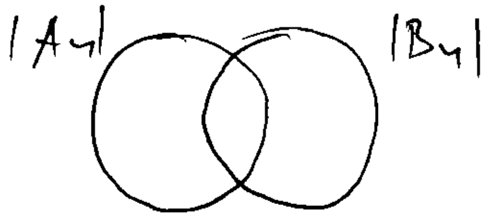
$\geq (1-\epsilon) + (1-\epsilon_1) - P_r\{X^n \in A^n \cup B^n\} \geq 1-\epsilon-\epsilon_1 \Rightarrow P_r\{X^n \in A^n \cap B^n\} \to 1$

(c) (самостојно откако го видов решението на „б“)

$|A^n \cap B^n| \le 2^{n(H+\varepsilon)}$ заш ~~секако~~ претпоставив

$|A^n| \ge |A^n \cap B^n|$

$|A^n \cap B^n| = |A^n| + |B^n| - |A^n \cup B^n|$



$|A^n| + |B^n| = |A^n \cup B^n| + |A^n \cap B^n|$

~~$|B^n| \le |A^n| + |A \cup B|$~~

$|A^n \cap B^n| = |A^n| + |B^n| - |A^n \cup B^n|$

$|A^n| + |B^n| \ge |A^n \cap B^n|$ $|A^n \cap B^n| \le |A^n| + |B^n| \le$

$|A^n \cap B^n| = |A^n| + |B^n| - |A^n \cup B^n| =$ / $|A^n \cup B^n| \ge |B^n|$ / $|B^n| \le |A^n \cup B^n|$ /

$\le |A^n| + \cancel{|A^n \cup B^n|} - \cancel{|A^n \cup B^n|} = |A^n|$

$$\boxed{|A^n \cap B^n| \le |A^n|}$$

доказ на ~~мојата~~ претпоставка !!!

Сега важи претходниот доказ. Задача 3.1.2.3

$|A^n| \le 2^{n(H+\varepsilon)} \Rightarrow |A^n \cap B^n| \le |A^n| \le 2^{n(H+\varepsilon)}$

$|A^n \cap B^n| \le 2^{n(H+\varepsilon)}$

(d) $|A^n \cap B^n| \ge \frac{1}{2} 2^{n(H-\varepsilon)}$ $|A^n| \ge (1-\varepsilon) 2^{n(H-\varepsilon)}$

$|A^n \cap B^n| = |A^n| + |B^n| - |A^n \cup B^n| \ge$ / $|A^n \cup B^n| \le 2|B^n|$ / $|A^n| \ge |B^n|$ /

$\ge |A^n| + |B^n| - 2|B^n| = |A^n| - |B^n|$

$|A^n| \ge (1-\varepsilon) 2^{+n(H-\varepsilon)}$ / $\Pr\{A^n\} \ge (1-\varepsilon)$ /

$|A^n| \ge$

(c) UNIVERSITY OF TORONTO SOLUTION

$$1 \geqslant \sum_{x^n \in A^n \cap B^n} p(x^n) \geqslant \sum_{x^n \in A^n \cap B^n} 2^{-n(H+\epsilon)} = |A^n \cap B^n| \cdot 2^{-n(H+\epsilon)}$$

$$\Rightarrow |A^n \cap B^n| \leqslant 2^{n(H+\epsilon)}$$ PROVED!!!

(d) UNIVERSITY OF TORONTO SOLUTIONS

OD (C) SLEDUVA DEKA AKO $|\underbrace{A^n \cap B^n}_{C^n}| \leq 2^{n(H+\epsilon)}$

$|C^n| \leq 2^{n(H+\epsilon)} \Rightarrow C^n \in$ TIPICNO MNOZESNO

$\Rightarrow \Pr\{C^n\} \to 1 \Rightarrow \Pr\{A^n \cap B^n\} \to 1$

$$\frac{1}{2} \leqslant \Pr\{A^n \cap B^n\} = \sum_{x^n \in A^n \cap B^n} p(x^n) \leqslant \sum_{x^n \in A^n \cap B^n} 2^{-n(H-\epsilon)} =$$

$$= |A^n \cap B^n| \, 2^{-n(H-\epsilon)}$$

$$\boxed{|A^n \cap B^n| \geqslant \frac{1}{2} 2^{+n(H-\epsilon)}}$$ PROVED!!!

3.5 SETS DEFINED BY PROBABILITIES. LET $X_1, X_2, \ldots$ BE AN i.i.d SEQUENCE OF DISCRETE RANDOM VARIABLES WITH ENTROPY $H(X)$. LET

$$C_n(t) = \{x^n \in \mathcal{X}^n : p(x^n) \geqslant 2^{-nt}\}$$

DENOTE THE SUBSET OF n-SEQUENCES WITH PROBABILITIES $\geqslant 2^{-nt}$.

(a) SHOW THAT $|C_n(t)| \leq 2^{nt}$

(b) FOR WHAT VALUES OF „t" DOES $P\{X^n \in C_n(t)\} \to 1$

(a) $2^{-n(H(X)+\epsilon)} \leqslant p(x^n) \leqslant 2^{-n(H(X)-\epsilon)}$ / ld

$$1 = \sum_{x^n \in X^n} p(x^n) \geqslant \sum_{x^n \in X^n} 2^{-nt} \geqslant |C_n(t)| \, 2^{-nt} \Rightarrow$$

$$\boxed{|C_n(t)| \leq 2^{nt}}$$

(b) $t = H(X) + \epsilon \Rightarrow |C_n(t)| \leq 2^{n(H(X)+\epsilon)}$

$-n(H(X)+\epsilon) \leq \mathrm{ld}\, p(x^n) \leq -n(H(X)-\epsilon)$ /(-1)·

$n(H(X)+\epsilon) \geqslant -\mathrm{ld}\, p(x^n) \geqslant n[H-\epsilon] \Rightarrow$

$$\boxed{n\left(H(X)-\varepsilon\right) \le -\frac{1}{n} ld\left(p(x^n)\right) \le n\left(H(X)+\varepsilon\right)}$$

$$1-\varepsilon = \sum_{x^n\in\mathcal{X}^n} p(x^n) \ge \sum_{x^n\in X^n} 2^{-nt} = |C_n(t)|\cdot 2^{-nt}$$

$$2^{-nt} \le \frac{1-\varepsilon}{|C_n(t)|} \; ; \quad -nt \le ld\,\frac{1-\varepsilon}{|C_n(t)|} \; ;$$

$$\boxed{t \ge \frac{1}{n}\, ld\, \frac{|C_n(t)|}{1-\varepsilon}}$$

- UNIVERSITY OF TORONTO SOLUTIONS

$$\frac{1}{n}\, ld\, |B_\delta^{(n)}| > H-\delta \qquad |B_\delta^{(n)}|^{\frac{1}{n}} > 2^{(H-\delta)}$$

$$\boxed{|B_\delta^{(n)}| > 2^{n(H-\delta)}} \Rightarrow |C_n(t)| > 2^{n(H-\varepsilon)}$$

- FROM PART (a) $|C_n(t)| \le 2^{nt} \Rightarrow$

$\boxed{t > H-\varepsilon}$ IS NECESSARY CONDITION SUCH $P_r\{x^n\in C_n(t)\}\to 1$

$\left\{\begin{array}{l} t \ge H+\varepsilon \text{ IS SUFFICIENT CONDITION } (\varepsilon>0) \\ \text{SUCH } A_\varepsilon^{(n)} \subseteq C_n(t) \text{ WHERE } P_r\{X^n\in C_n(t)\}\to 1 \end{array}\right.$

⊡ WHETHER $t=H$ IS SUFFICIENT CONDITION
(IT DEPENDS TO THE DISTRIBUTION.

⊟ E.G. $P_r\{x=0\}=P_r\{x=1\}=\frac{1}{2}$ $\quad t=H=1$

$$H=\frac{1}{2} ld 2+\frac{1}{2} ld 2 = 1$$

ALL SEQUENCES HAVE PROBABILITY 2^{-n}

$\left.\begin{array}{ll} x_1 & 00 \\ x_2 & 01 \\ x_3 & 10 \\ x_4 & 11 \end{array}\right\}$ $\quad P(x_i)=\frac{1}{2^n}=\frac{1}{4} \quad i=1,2,3,4$

NAVISTINA E TAKA!!!

⊟ AND $P_r\{X=0\}=p$ $\quad P_r\{X=1\}=1-p \quad 0<p<1$

$t=H$ IS NOT SUFFICIENT !!!

- NA PRIMER ZA $n = 2$

$$\left.\begin{array}{l} x_1 \doteq 00 \\ x_2 = 01 \\ x_3 = 10 \\ x_4 = 11 \end{array}\right\} \quad p(x_1) = p^2 \cdot \quad p(x_2) = p(1-p) \quad p(x_3) = p(1-p) \quad p(x_4) = (1-p)^2 .$$

E.G. $p = 0.6$ $q = 0.4$ $p(x_1) = (0.6)^2 = 0.36$

$p(x_2) = 0.6 \cdot 0.4 = 0.24$ $p(x_3) = 0.24$ $p(x_4) = 0.16$

$H = -0.6 \, ld\, 0.6 - 0.4 \, ld\, 0.4 = 0.971$ $\boxed{p(x_1) > 0.26}$

$2^{-nH} = 2^{-2H} = 2^{-1.942} = \underline{\underline{0.26}}$ $\boxed{p(x_2..x_4) < 0.26}$

- ZNAČI, NEOPHODEN USLOV $Pr\{X_n \in C_n(t)\} \rightarrow 1 \Leftarrow$ $\boxed{t > H - \varepsilon}$

$\boxed{36}$ AEP-like limit. Let $X_1, X_2, \dots,$ be i.i.d DRAWN ACORDING TO PROBABILITY MASS FUNCTION $p(x)$. FIND: $\lim_{n\to\infty} \left[p(X_1, X_2, \dots, X_n)\right]^{\frac{1}{n}}$.

$$2^{-n(H+\varepsilon)} \leq p(x_1, x_2, \dots, x_n) \leq 2^{-n(H(X)-\varepsilon)}$$

$$-\frac{1}{n} ld\, p(x_1, x_2, \dots x_n) \xrightarrow[n\to\infty]{} H(X)$$

$$\boxed{2^{-(H+\varepsilon)} \leq p^{\frac{1}{n}}(x_1 x_2 \dots x_n) \leq 2^{-H(X)-\varepsilon}}$$

$$\frac{1}{n} ld\, p(x_1, x_2, \dots x_n) \rightarrow -H(X)$$

$$\boxed{\lim_{n\to\infty} p^{\frac{1}{n}}(x_1 x_2 \dots x_n) = 2^{-H(X)}}$$

- TRINITY College DUBLIN SOLUTION:

$$\lim_{n\to\infty} p^{\frac{1}{n}}(x_1, x_2, \dots x_n) = \lim_{n\to\infty} 2^{\frac{1}{n} ld\, p(x_1, x_2, \dots, x_n)} =$$

$$= \lim_{n\to\infty} 2^{\frac{1}{n}\sum_{i=1}^{n} ld\, p(x_i)} = \left| \begin{array}{c} \text{STRONG LAW OF LARGE NUMBERS} \\ \Downarrow \\ \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} p(x_i) = E[X] \end{array} \right| =$$

$$= \lim_{n\to\infty} 2^{E[ld\, p(X)]} = 2^{-H(X)}$$

STRONG LAW: ALMOST SURELY

$$\frac{1}{n}\sum_{i=1}^{n} p(x_i) \xrightarrow{A.S.} E[X] .$$

$\boxed{3.7}$ AEP AND SOURCE CODING: A DISCRETE MEMORYLESS SOURCE EMITS A SEQUENCE OF STATISTICALLY INDEPENDENT BINARY DIGITS WITH PROBABILITIES $p(1) = 0.005$ AND $p(0) = 0.995$. THE DIGITS ARE TAKEN 100 AT TIME AND BINARY CODEWORD IS PROVIDED FOR EVERY SEQUENCE OF 100 DIGITS CONTAINING THREE OR FEWER 1'S.

(a) ASSUMING THAT ALL CODEWORDS ARE THE SAME LENGHT, FIND THE MINIMUM LENGHT REQUIRED TO PROVIDE CODEWORDS FOR ALL SEQUENCES WITH THREE OR FEWER 1'S.

(b) CALCULATE THE PROBABILITY OF OBSERVING A SOURCE SEQUENCE FOR WHICH NO CODEWORD HAS BEEN ASSIGNED.

(c) USE CHEBYSHEV'S INEQUALITY TO BOUND THE PROBABILITY OF OBSERVING A SOURCE SEQUENCE FOR WHICH NO CODEWORD HAS BEEN ASIGNED. COMPARE THE BOUND WITH ACTUAL PROBABILITY COMPUTED IN PART (b).

(b)
- 2 SLUČAI VO 1000 SEKVENCII
- 9 SLUČAI VO 5000 SEKVENCII
- 18 SLUČAI VO 10000 SEKVENCII
- 85 SLUČAI VO 50000 SEKVENCII

MAKE EXPERIMENT

MultihopMIMOCapa-city1.m

- VEROJATNOST DA SE GENERIRA SEKVENCA SO >3 EDINICI e:

$$\boxed{\begin{array}{r}5000\\4991\\\hline 9\end{array}}$$

$$p = 1 - P(S_3 + S_2 + S_1 + S_0)$$

$S_i \to$ SEQUENCE WITH i = ONES.

$$P(S_0+S_1+S_2+S_3) = \binom{100}{3} p^{97} \cdot q^{3} + \binom{100}{2} p^{98} q^{2} + \binom{100}{1} p^{99} q + \binom{100}{0} p^{100}$$

000
001
010
011
100
101
110
111

$$\underbrace{\binom{3}{2} = \frac{6}{2 \cdot 1!} = 3}_{①} \qquad \underbrace{\binom{3}{1} = \frac{6}{2!\,1!} = 3}_{①}$$

$p(0)=0.995$ $\qquad$ $p(1)$

$$H(p) = p \operatorname{ld} \frac{1}{p} + q \operatorname{ld} \frac{1}{q} = \underline{0.0454}$$

$$\boxed{\begin{aligned} n \cdot H(X) &= 100 \cdot 0.0454 \\ &= 4{,}545 \approx 5 \text{ BITS} \end{aligned}}$$

- VEROJATNOSTA DA ~~SE~~ GENERIRA SEKVENCATA SO POVEĆE OD 3 EDINICI E ISTA SO VEROJATNOSTA DA SE GENERIRA IZVORNA SEKVENCA ZA KOJA NE E GENERIRAN KODEN ZBOR

$$p = 1 - \binom{100}{3} p^{97} q^{3} - \binom{100}{2} p^{98} q^{2} - \binom{100}{1} p^{99} q - \binom{100}{0} p^{100} = \underline{0.0017}$$

(a) MINIMALNATA DOLŽINA NA KODNITE ZBOROVI E

$$n\cdot H(q) = 100\cdot\left(q\, ld\frac{1}{q} + (1-q)\, ld\frac{1}{(1-q)}\right) \qquad \underline{q = 0.995}$$

$n\cdot H(q) = 100\cdot 0.0454 = 4,5415 \doteq 5$ BITS

(c) $\boxed{\mu = q\cdot 0 + z\cdot 1 = 0.005}$

$$\sigma^2 = \sum_{x=0}^{1}(x-\mu)^2 q(x) = (1-0.005)^2\cdot 0.005 + (0-0.005)^2\cdot 0.995$$

$$\boxed{\sigma^2 = (0.995)^2\cdot 0.005 + (0.005)^2\cdot 0.995 = 0.005} = (0.995+0.005)\cdot 0.005\cdot 0.995 = 0.005\cdot 0.995$$

$P\{|X-\mu|>\varepsilon\} \le \frac{\sigma^2}{\varepsilon^2}\ \underline{\forall\varepsilon>0}$ SO DRUGI ZBOROVI: $\boxed{\cancel{X \xrightarrow{a.s.} \mu}}$

$$P\{|X-\mu|>\varepsilon\} \le \frac{0.005}{\varepsilon^2}$$

$$\underbrace{P\{|x-0.005|>\varepsilon\}} \le \frac{0.005}{\varepsilon^2}$$

OVA E EKVIVALENTNO NA $\underline{P_r\{x=1\} = 0.005}$

NAJGOLEMA VREDNOST ZA „ε" E $\varepsilon = 0.995$

$$P\{|x-0.005|>\varepsilon\} \le \frac{0.005}{0.995^2} = \underline{0.0051}$$

SE SLUČUVA KOGA $\boxed{x=1}$

ZNAČI UPPER BOUND E: 0.005.

☐ SOLUTION 1st EDITION

(a) $\binom{100}{0} + \binom{100}{1} + \binom{100}{2} + \binom{100}{3} = 166751$

$ld(166751) = 17.34 \doteq 18$ BITI — REAL USED LENGHT

$n\cdot H(q) = 4.5 \doteq 5$ BITI ⇒ MINIMUM LEGHT

18 IS QUITE LARGER THEN $nH(q)$ ([illegible])

(b) $$P = 1 - \sum_{i=0}^{3}\binom{100}{i} \underbrace{p^{100-i}}_{„0"} \cdot \underbrace{z^{i}}_{„1"} = 0.00167$$

(c) S_n – RANDOM VARIABLE THAT IS SUM OF „n" i.i.d.
CHEBISHEV INEQUALITY IS: $P_r\{|S_n - n\mu|>\varepsilon\} \le \frac{n\cdot\sigma^2}{\varepsilon^2}$

$n=100 \quad \mu = 1\cdot 0.005 + 0\cdot 0.995 = 0.005$

$\sigma^2 = |\,pp.53\,| = 0.005\cdot 0.995$

$\boxed{S_{100} \geq 4}$ IF AND ONLY IF: $|S_{100} - 100.0.005| \geq 3.5$

$|S_{100} - 0.5| \geq 3.5 \Rightarrow \boxed{\varepsilon = 3.5} \Rightarrow$ NAJGOLEMA VREDNOST ~~ZA~~ ε KOJA IMAME 4 BITNA SEKVENCA. 5-BITNA SEKVENCA SEKAKO ĆE GO NADMINE OVA $\varepsilon =$ USLOVOT

$$\Pr\{|S_n - n\cdot\mu| \geq \varepsilon\} \leq \frac{n\sigma^2}{\varepsilon^2}$$

OVAA DEFINICIJA GO VGRADUVA ~~ZA~~ KOI ĆE E DELA ZBORUVAME ~~ZA~~ SEKVENCI DEFINIRAN KODEN ZBOR.

ZO TOA OVA ĆE BIDE VEROJATNOST. ~~ZA~~ SEKVENCIJATA IMA 4 ILI POVEĆE BITA.

$$\Pr\left\{\begin{matrix}\text{NEMA}\\ \text{KODEN ZBOR}\end{matrix}\right\} \leq \frac{100\cdot(0.005\cdot 0.995)}{3.5^2} = \frac{0.5\cdot 0.995}{3.5^2} = 0.04061$$

$0.04061 >> 0.0167$ } LOŠ BOUND ~~DAVA~~ CHEBISHEVATA NEEDNAKVOST.

3.8 PRODUCTS

LET

$$x = \begin{cases} 1, & \text{WITH PROBABILITY } 1/2 \\ 2, & \text{WITH PROBABILITY } 1/4 \\ 3, & \text{WITH PROBABILITY } 1/4 \end{cases}$$

MMV IMA [illegible] UDAVA SIMULACIJA VO MAPLE. MultiLingMinoCapd. mw

LET $x_1, x_2 \ldots x_n$ BE DRAWN I.I.D ACCORDING TO THIS DISTRIBUTION. FIND THE LIMITTING BEHAVIOR OF THE PRODUCT

$$(X_1\cdot X_2\cdot \ldots X_n)^{1/n}.$$

$$(X_1\cdot X_2\cdots X_n)^{\frac{1}{n}} = 2^{\frac{1}{n}\,ld(X_1\cdot X_2\cdots X_n)} = 2^{\overbrace{\frac{1}{n}\sum_{i=1}^{n} ld(X_i)}^{\text{BY LAW OF LARGE NUMB.}}}$$

$$\xrightarrow{a.s.} 2^{E[ld(X)]} = 2^{\frac{1}{4}(1+ld3)} = \underline{1.56508}$$

$$H(X) = \frac{1}{2}ld2 + \frac{1}{4}ld4 + \frac{1}{4}ld4 = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$$

$$E[ld(X_i)] = \overset{0}{\cancel{ld(1)}}\cdot\frac{1}{2} + \overset{1}{\cancel{ld2}}\cdot\frac{1}{4} + ld3\cdot\frac{1}{4} = \frac{1}{4}(1+ld3) = \frac{1}{4}ld6$$

$\boxed{3.9}$ **AEP.** LET $x_1, x_2, \dots$ BE INDEPENDENT, IDENTICALLY DISTRIBUTED RANDOM VARIABLES DRAWN ACCORDING TO THE PROBABILITY MASS FUNCTION $p(x)$, $x \in \{1,2,\dots,m\}$. THUS, $p(x_1,x_2,\dots,x_n) = \prod_{i=1}^{n} p(x_i)$. WE KNOW THAT $-\frac{1}{n} \operatorname{ld} p(x_1,x_2,\dots,x_n) \to H(X)$ IN PROBABILITY.

LET $q(x_1,x_2,\dots,x_n) = \prod_{i=1}^{n} q(x_i)$, WHERE „q" IS ANOTHER PROBABILITY MASS FUNCTION ON $\{1,2,\dots,m\}$

(a) EVALUATE $\lim\left(-\frac{1}{n}\operatorname{ld} q(x_1,x_2,\dots,x_n)\right)$ WHERE $x_1, x_2 \dots x_n$ ARE i.i.d $\sim p(x)$

(b) NOW EVALUATE THE limit OF THE „LOG" LIKELIHOOD RATIO $\frac{1}{n}\operatorname{ld}\frac{q(x_1,x_2\dots x_n)}{p(x_1,x_2,\dots,x_n)}$ WHEN $x_1, x_2, \dots$ ARE i.i.d $\sim p(x)$. THUS, THE ODDS FAVORING „q" ARE EXPONENTIALLY SMALL WHEN „p" IS TRUE.

(a) $$-\frac{1}{n}\operatorname{ld} q(x_1,x_2,\dots,x_n) = -\frac{1}{n}\sum_{i=1}^{n}\operatorname{ld} q(X_i) \to -E_q[\operatorname{ld} q]$$

$$= -\sum_{i=1}^{n} q(x_i)\operatorname{ld} q(x_i) = \underline{H_q(X)}$$

$$-\frac{1}{n}\operatorname{ld} q(x_1,x_2,\dots,x_n) = -\frac{1}{n}\left[\sum_{i=1}^{n}\operatorname{ld} q(x_i) - \sum_{i=1}^{n}\operatorname{ld} p(x_i) + \sum_{i=1}^{n}\operatorname{ld}(p(x_i))\right]$$

$$= -\frac{1}{n}\sum_{i=1}^{n}\operatorname{ld}\frac{q(x_i)}{p(x_i)} - \frac{1}{n}\sum_{i=1}^{n}\operatorname{ld} p(x_i) \to -E_p\left[\operatorname{ld}\frac{q(x_i)}{p(x_i)}\right] - E_p[\operatorname{ld} p(x_i)]$$

$$= E\left[\operatorname{ld}\frac{p(x_i)}{q(x_i)}\right] + H(X) = \underline{D(p\|q) + H(X)}$$

(b) $$\frac{1}{n}\operatorname{ld}\frac{q(x_1,x_2,\dots x_n)}{p(x_1,x_2,\dots x_n)} \to -E_p\left[\operatorname{ld}\frac{p(x_1,\dots x_n)}{q(x_1 \dots x_n)}\right] = -D(p\|q)$$

ISTO GO REŠILE VO EDITION 1 SO MALA VARIJACIJA

(a) $$-\lim_{n\to\infty}\frac{1}{n}\operatorname{ld} q(x_1,x_2,\dots x_n) = -\lim_{n\to\infty}\sum_{i=1}^{n}\operatorname{ld} q(x_i) \text{ w.p. } 1 = -\sum p(x)\operatorname{ld} q(x)$$

$$= +\sum p(x)\operatorname{ld}\frac{p(x)}{q(x)} - \sum p(x)\operatorname{ld} p(x) = \underline{D(p\|q) + H(p)}$$

(b) $\lim_{n\to\infty} \frac{1}{n} \operatorname{ld} \frac{q(x_1,x_2,\dots x_n)}{p(x_1 x_2,\dots x_n)} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \operatorname{ld}\frac{q(x_i)}{p(x_i)} \overset{\text{W.P.1}}{=} E_p\left[\operatorname{ld}\frac{q(x_i)}{p(x_i)}\right]$

$\boxed{090700865}$

$= -D(p\|q)$

$\boxed{3.10}$ RANDOM BOX SIZE AN n-DIMENSIONAL RECTANGULAR BOX WITH SIDES $X_1, X_2, \dots, X_n$ IS TO BE CONSTRUCTED. THE VOLUME IS $V_n = \prod_{i=1}^{n} X_i$. THE EDGE LENGHT l OF A n-CUBE WITH THE SAME VOLUME AS A RANDOM BOX IS $l = V_n^{1/n}$. LET $X_1, X_2, \dots X_n$ BE UNIFORM RANDOM VARIABLES OVER UNIT INTERVAL $[0,1]$. FIND $\lim_{n\to\infty} V_n^{1/n}$ AND COMPARE TO $(E[V_n])^{1/n}$.

CLEARLY THE EXPECTED EDGE LENGHT DOES NOT CAPTURE THE IDEA OF THE VOLUME OF THE BOX. THE GEOMETRIC MEAN, RATHER THAN THE ARITHMETIC MEAN, CHARACTERIZES THE BEHAVIOR OF THE PRODUCTS.

$$H(x) = -\int_0^1 p(x)\operatorname{ld} p(x)\,dx = \int_0^1 p(x)\operatorname{ld}\frac{1}{p(x)}\,dx = 0$$

$$p(x) = \left[\frac{1}{2}, \frac{1}{2}\right] \qquad \boxed{H(x) = \frac{1}{2}\operatorname{ld}2 + \frac{1}{2}\operatorname{ld}2 = 1}$$

$$p(x) = \left[\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right] \qquad H(x) = \left(\frac{1}{4}\cdot\operatorname{ld}4\right)4 = \left(\frac{1}{2}\right)\cdot 4 = 2$$

$$p(x) = \left[\frac{1}{8}, \frac{1}{8}, \dots, \frac{1}{8}\right] \qquad H(x) = \left(\frac{1}{8}\cdot 3\right)8 = \frac{24}{8} = 3$$

$$\int\frac{dx}{x} = \ln x \qquad (\ln x)' = \frac{1}{x}$$

$$\left(\ln\left(\frac{1}{x}\right)\right)' = \frac{1}{\frac{1}{x}}\cdot\left(\frac{1}{x}\right)' = x\cdot x^{-1-1}\cdot(-1) = -x\cdot\frac{1}{x^2} = \underline{\underline{-1/x}}$$

(*) NE GLEDO!

- X_i ZAMISLI GI KAKO DA SE PDF ZOŠTO SE SO VREDNOST VO RANG $[0..1]$ (NE MORA DA ODI SO OVAA APSTRAKCIJA)

① $\lim_{n\to\infty}(x_1\cdot x_2\cdots x_n)^{\frac{1}{n}} = e^{\lim\left[\frac{1}{n}\ln(x_1\cdot x_2\cdots x_n)\right]} = e^{\lim\left[\frac{1}{n}\sum_{i=0}\ln x_i\right]}$

$\xrightarrow{\text{A.S.}} e^{E[\ln x_i]} = \underbrace{e^{-1}}_{\text{PP.63}} = \frac{1}{e}$

e^x E KONTINUIRANA FUNKCIJA NA ZATOA lim ODI VO EXPONENTOT. (STEWART CONTINUITY THEOR.)

$$E[V_n]^{1/n} = E[x_1 \cdot x_2 \cdots x_n]^{1/n}$$

- IST REZON IMAŠE VO EDITION 1 SOLUTIONS SAMO (*) MI E GREŠNO T.E NEMA POREDA NA PROSMETKA NA HA

$$E[\ln(x)] = \int_0^1 \underbrace{①}_{\psi(x)} \ln(x)\,dx = \left|\begin{array}{l} u = \ln x \\ du = \frac{1}{x}\cdot dx \\ v = \int 1 \cdot dx = x \end{array}\right| = x\cdot\ln x - \int \cancel{x}\cdot\frac{dx}{\cancel{x}} = x\ln x - x$$

$$E[\ln(x)] = (x\ln x - x)\Big|_0^1 = -1$$

(-1)

TEOREMA ZA SREDNA VREDNOST ILI $\int \psi(x)\cdot\ln(x)\,dx$ OD VEROJATN

$$\lim_{n\to\infty} V_n^{\frac{1}{n}} = e^{\lim_{n\to\infty}\left[\frac{1}{n}\ln V_n\right]} = \frac{1}{e} < \frac{1}{2}$$

ALT. ①: $\lim_{n\to\infty} \ln(V_n)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{1}{n}\ln(x_1 x_2 \cdots x_n) = \lim_{n\to\infty}\left[\frac{1}{n}\sum_{i=1}^{n}\ln(x_i)\right]$

$$\xrightarrow[n\to\infty]{A.S} E[\ln(x)] = -1$$

OD OVAA SLEDI DEKA EFEKTIVNATA DOLŽINA NA STRANA NA OVA TELO E: e^{-1}

$$E[x_1 \cdot x_2 \cdots x_n] = \prod_{i=1}^{n} E[x_i] = \left(\frac{1}{2}\right)^n$$

$$\int_{x_1}\int_{x_2}\cdots\int_{x_n} x_1 \cdot x_2 \cdots x_n \underbrace{f(x_1, \ldots x_n)}_{= f(x_1)\cdot f(x_2)\cdots f(x_n)} dx_1 dx_2 \ldots dx_n$$

$E[x_i] = \frac{1}{2} = \frac{0+1}{2}$ } ARITHMETIC MEAN

$$① \Rightarrow \underbrace{(x_1 \cdot x_2 \cdots x_n)^{\frac{1}{n}}}_{} = 1/e < \left(\frac{1}{2}\right).$$

DEFINITION FOR GEOMETRIC MEAN

$\left(\frac{1}{e}\right)^n < \left(\frac{1}{2}\right)^n \Rightarrow$ SREDEN VOLUMEN NA VAKVOTO GEOMETRISKO TELO

- SAMO DA KAŽE DEKA EFEKTIVNATA (SREDNATA) DOLŽINA DOBIENA KAKO GEOMETRISKA SREDINA E POMALA OD ONAA DOBIENA KAKO ARITMETIČKA SREDINA.

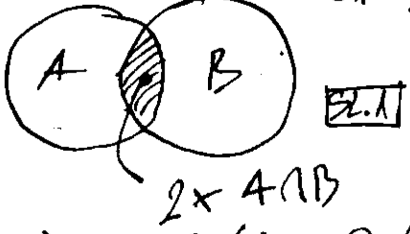
$\boxed{3.11}$ PROOF OF THEOREM 3.3.1. 0075426898

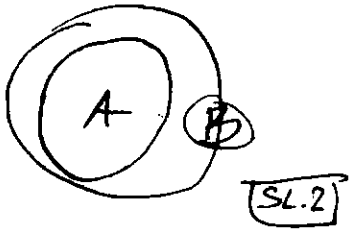
IGOR

THIS PROBLEM SHOWS THAT THE SIZE OF THE SMALLEST "PROBABLE" SET IS ABOUT 2^{nH}. LET $x_1, x_2, \ldots, x_n$ BE i.i.D $\sim p(x)$. LET $B_\delta^{(n)} \subset X^n$ SUCH THAT $Pr(B_\delta^{(n)}) > 1-\delta$. FIX $\varepsilon < \frac{1}{2}$.

(a) GIVEN ANY TWO SETS A, B SUCH AS $Pr(B_\delta^{(n)}) > 1-\varepsilon_1$ AND $Pr(B) > 1-\varepsilon_2$, SHOW THAT $Pr(A \cap B) > 1-\varepsilon_1-\varepsilon_2$. HENCE $Pr(A_\varepsilon^{(n)} \cap B_\delta^{(n)}) \geq 1-\varepsilon-\delta$

$$Pr(A+B) = \underbrace{Pr(A)}_{>1-\varepsilon_1} + \underbrace{Pr(B)}_{>1-\varepsilon_2} - Pr(A\cap B) > 1-\varepsilon_1+1-\varepsilon_2-Pr(A\cap B)$$





$$Pr(A\cap B) = Pr(A)+Pr(B) - \underbrace{Pr(A+B)}_{\leq 1} \geq 1-\varepsilon_1+1-\varepsilon_2-Pr(A+B)$$

$$\geq 1-\varepsilon_1+\cancel{1}-\varepsilon_2-\cancel{1} \geq 1-\varepsilon_1-\varepsilon_2$$ PROVED!!!

(b) JUSTIFY THE STEPS IN THE CHAIN OF INEQUALITIES:

$$1-\varepsilon-\delta \underset{(a)}{\leq} Pr(A_\varepsilon^{n} \cap B_\delta^{(n)}) = \sum_{A_\varepsilon^{(n)} \cap B_\varepsilon^{(n)}} p(x^n) \leq \sum_{A_\varepsilon^{(n)} \cap B_\varepsilon^{(n)}} e^{-n(H-\varepsilon)}$$

$$\boxed{2^{-n(H+\varepsilon)} \leq p(x^n) \leq 2^{-n(H-\varepsilon)}}$$ ASYMPTOTIC EQUIPARTITION THEOREM

$$= |A_\varepsilon^{(n)} \cap B_\delta^{(n)}| \, e^{-n(H-\varepsilon)} \leq |A_\varepsilon^{(n)}| \, e^{-n(H-\varepsilon)} \overset{*}{\leq} |B_\delta^{n}| \, e^{-n(H-\varepsilon)}$$

↑ PRESEKOT E POMAL $\vee$ EDNAKOV OD SAMOTO MNOŽESTVO $\boxed{SL.2}$

$$|B_\delta^{(n)}| \geq (1-\varepsilon_1-\varepsilon_2) \cdot e^{n(H-\varepsilon)}$$

(c) COMPLETE THE PROOF OF THE TEOREM

$$(1-\varepsilon-\delta) \leq |B_\delta^{(n)}| \, e^{-n(H-\varepsilon)} \quad / \ln$$

$$\ln(1-\varepsilon-\delta) \leq \ln|B_\delta^{(n)}| - n(H-\varepsilon)$$

$$\ln|B_\delta^{(n)}| \geq \ln(1-\varepsilon-\delta) + n(H-\varepsilon) \geq \ln\left(1-\frac{1}{2}-\delta\right) + n(H-\varepsilon$$

$$= nH - n\varepsilon + \ln\left(\frac{1}{2}-\delta\right) \geq nH - \frac{n}{2} + \ln\left(\frac{1}{2}-\delta\right)$$

$$\frac{1}{n}\ln|B_\delta^{(n)}| \geq H + \frac{1}{n}\underbrace{\ln\left(\frac{1}{2}-\delta\right)}_{\leq 0} - \frac{1}{2} = H - \delta'$$

$$\boxed{\delta' = \frac{1}{2} + \frac{1}{n}\left|\ln\left(\frac{1}{2}-\delta\right)\right|} \qquad \delta' > 0$$

- EDITION 1 SOLUTION (WITH COMPLEMENTS)

- A^C KOMPLEMENT OF A (ČLENOVI OD $\mathcal{X}^n$ KOI НЕ ПРИПАЃААТ НА А.

$$P(A^C \cup B^C) = P(A^C) + P(B^C) - P(A^C \cap B^C)$$

$$\leq P(A^C) + P(B^C)$$

$$P(A) \geq 1 - \varepsilon_1 \Rightarrow P(A^C) = 1 - P(A) \leq \cancel{1} - \cancel{1} + \varepsilon_1 \Rightarrow \boxed{P(A^C) \leq \varepsilon_1}$$

$$P(B) \geq 1 - \varepsilon_2 \Rightarrow P(B^C) = 1 - P(B) \leq \cancel{1} - \cancel{1} + \varepsilon_2 = \varepsilon_2$$

$$\boxed{P(B^C) \leq \varepsilon_2}$$

$$P(A \cap B) = 1 - P(A^C \cup B^C) \geq 1 - P(A^C) - P(B^C) \geq \underline{\underline{1 - \varepsilon_1 - \varepsilon_2}}$$

$$X_n \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

$$A \in \{1, 2, 3, 4, 5, 6\} \qquad A^C \in \{7, 8, 9, 10\}$$

$$B \in \{1, 2, 3, 4, 5, 6, 7\} \qquad B^C \in \{8, 9, 10\}$$

$$A^C \cup B^C = A^C + B^C - A^C \cap B^C = \{7, 8, 9, 10\}$$

$$A \cap B = \{1, 2, 3, 4, 5, 6\}$$

$$\boxed{P(A \cap B) = 1 - P(A^C \cup B^C)}$$

ЕМПИРИСКИ ДОКАЖАНО !!!

$\boxed{N.4.4.(*)}$

$$|A_\varepsilon^{(n)} \cap B_\delta^{(n)}| e^{-n[H-\varepsilon]} \leq |B_\delta^{(n)}| e^{-n(H-\varepsilon)}$$

SLEDI OD FAKTOT ŠTO: $\boxed{A_{\varepsilon}^{(n)} \cap B_\delta^{(n)} \subseteq B_\delta^{(n)}}$

$\boxed{3.12}$ MONOTONIC CONVERGENCE OF EMPIRICAL DISTRIBUTION. LET $\hat{p}_n$ DENOTE EMPIRICAL PROBABILITY MASS FUNCTION CORRESPONDING TO $X_1, X_2, \dots, X_n$ i.i.d $\sim p(x)$, $x \in \mathcal{X}$, SPECIFICALLY,

$$\hat{p}_n(x) = \frac{1}{n} \sum_{i=1}^{n} I(X_i = x)$$

IS THE PROPORTION OF TIMES $X_i = X$ IN THE FIRST „n" SAMPLES, WHERE I IS INDICATOR FUNCTION.

(a) SHOW FOR $\mathcal{X}$ BINARY THAT

$$E[D(\hat{p}_{2n} \| p)] \le E[D(\hat{p}_n \| p)] \longrightarrow$$

EMPIRISKI SE DOKAZUVA VO MATLAB: MultiColMMCap1.m

THUS EXPECTED RELATIVE ENTROPY "DISTANCE" FROM THE EMPIRICAL DISTRIBUTION TO THE TRUE DISTRIBUTION DECREASES WITH SAMPLE SIZE.

(Hint: WRITE $\hat{p}_{2n} = \frac{1}{2}\hat{p}_n + \frac{1}{2}\hat{p}_n'$ AND USE THE CONVEXITY OF D.)

(b) SHOW FOR AN ARBITRARY DISCRETE $\mathcal{X}$ THAT

$$\boxed{E[D(\hat{p}_n \| p)] \le E[D(\hat{p}_{n-1} \| p)]}$$

(Hint: WRITE $\hat{p}_n$ AS THE AVERAGE OF „n" EMPIRICAL MASS FUNCTIONS WITH EACH OF THE „n" SAMPLES DELETED IN TURN.)

$$E[D(\hat{p}_{2n} \| p)] \rightarrow \frac{1}{m} D(\hat{p}_{2n} \| p)$$

m → BROJ NA MERENJA

$$D(\hat{p}_{2n} \| p) = \sum_{i=1}^{n} p_{2n} \, \mathrm{ld} \frac{p_{2n}(i)}{p(i)}$$

$$\hat{p}_{2n} = [p_n, p_n']$$

- ZA KONVEKSNI FUNKCII VAZI JENSEN'S INEQUALITY.

$$E[D(\hat{p}_{2n} \| p)] \ge D(E[\hat{p}_{2n} \| p])$$

$$D(\hat{p} \| \hat{p}) = \hat{p}_1 \cdot \mathrm{ld} \frac{\hat{p}_1}{p_1} + \hat{p}_2 \cdot \mathrm{ld} \frac{p_2}{p_2}$$

$$= E_{\hat{p}}\left[\mathrm{ld} \frac{\hat{p}}{p}\right]$$

$$D(\hat{p}_n\|p) = \frac{1}{n}\sum_{i=1}^{n} I(x_i=1)\cdot ld\left(\frac{\frac{1}{n}\sum_{i=1}^{n} I(x_i=1)}{p_1}\right) +$$

$$+ \frac{1}{n}\sum_{i=1}^{n} I(x_i=2)\, ld\left(\frac{\frac{1}{n}\sum_{i=1}^{n} I(x_i=2)}{p_2}\right)$$

$$D(\hat{p}_{2n}\|p) = \frac{1}{2n}\sum_{i=1}^{2n} I(x_i=1)\cdot ld\left(\frac{\frac{1}{2n}\sum_{i=1}^{2n} I(x_i=1)}{p_1}\right) +$$

$$+ \frac{1}{2n}\sum_{i=1}^{2n} I(x_i=2)\, ld\left(\frac{\frac{1}{2n}\sum_{i=1}^{2n} I(x_i=2)}{p_2}\right) =$$

$$= \frac{1}{2}\left[\frac{1}{n}\sum_{i=1}^{n} I(x_i=1)\, ld\, \frac{\frac{1}{2n}\sum_{i=1}^{2n} I(x_i=1)}{p_1} +\right.$$

$$\left.+ \frac{1}{n}\sum_{i=1}^{n} I(x_i=2)\, ld\, \frac{\frac{1}{2n}\sum_{i=1}^{2n} I(x_i=2)}{p_{2\,2n}}\right] +$$

$$+ \frac{1}{2n}\sum_{i=n+1}^{2n} I(x_i=1)\, ld\, \frac{\frac{1}{2n}\sum_{i=1}^{2n} I(x_i=1)}{p_1} +$$

$$+ \frac{1}{2n}\sum_{i=n+1}^{2n} I(x_i=2)\, ld\, \frac{\frac{1}{2n}\sum_{i=1}^{2n} I(x_i=2)}{p_2}$$

$$\hat{p}_{2n} = \frac{1}{2n}\sum_{i=1}^{2n} I(x_i=x) = \frac{1}{2}\underbrace{\frac{1}{n}\sum_{i=1}^{n} I(x_i=x)}_{p_n} + \frac{1}{2}\underbrace{\frac{1}{n}\sum_{i=n+1}^{2n} I(x_i=x)}_{p_n'}$$

$$\boxed{\hat{p}_{2n} = \frac{1}{2}p_n + \frac{1}{2}p_n'} \qquad \hat{p}_{2n} \to E[p_n] \quad \circledast$$

$$D(p_{2n}\|p_n) = \underline{p_{2n}(1)}\cdot ld\, \frac{p_{2n}(1)}{p(1)} + p_{2n}(0)\cdot ld\, \frac{p_{2n}(0)}{p(0)} \to$$

$$= \left(\frac{1}{2}p_n(1) + \frac{1}{2}p_n'(1)\right) ld\left[\frac{\frac{1}{2}p_n(1) + \frac{1}{2}p_n'(1)}{p(1)}\right] + p_{2n}(0)\, ld\, \frac{p_{2n}(0)}{p(0)}$$

$$D(\gamma_{2n}\|\gamma) = \frac{1}{2}\left(\gamma_n(1) + \gamma_n'(1)\right) \mathrm{ld} \frac{\gamma_n(1)+\gamma_n'(1)}{2\gamma(1)} + \frac{1}{2}\left(\gamma_n(0)+\gamma_n'(0)\right) \mathrm{ld} \frac{\gamma_n(0)+\gamma_n'(0)}{2\gamma(0)}$$

$$= \frac{1}{2}\left(\gamma_{n1}+\gamma_{n1}'\right) \mathrm{ld} \frac{\gamma_{n1}+\gamma_{n1}'}{2\gamma_1} + \frac{1}{2}\left(1-\gamma_{n1}+1-\gamma_{n1}'\right) \cdot \mathrm{ld}\left[\frac{2-\gamma_{n1}-\gamma_{n1}'}{2(1-\gamma_1)}\right]$$

$$= \frac12\left(\gamma_1+\gamma_1'\right) \mathrm{ld} \frac{\gamma_1+\gamma_1'}{2\gamma} + \frac12\left(2-\gamma_1-\gamma_1'\right) \mathrm{ld} \frac{2-\gamma_1-\gamma_1'}{2(1-\gamma)} =$$

$$= \frac12\left(\gamma_1+\gamma_1'\right) \mathrm{ld} \frac{\gamma_1+\gamma_1'}{2\gamma} - \frac12\left(\gamma_1+\gamma_1'\right) \mathrm{ld} \frac{2-\gamma_1-\gamma_1'}{2(1-\gamma)} + \mathrm{ld} \frac{2-\gamma_1-\gamma_1'}{2(1-\gamma)}$$

$$= \frac12\left(\gamma_1+\gamma_1'\right) \mathrm{ld} \frac{\left(\gamma_1+\gamma_1'\right) 2(1-\gamma)}{2\gamma\left(2-\gamma_1-\enclose{circle}{\gamma_1'}\right)} + \mathrm{ld} \frac{2-\gamma_1-\gamma_1'}{2(1-\gamma)}$$

$$D(\gamma_n\|\gamma) = \gamma_n(1)\, \mathrm{ld} \frac{\gamma_n(1)}{\gamma} + \gamma_n(0)\, \mathrm{ld} \frac{\gamma_n(0)}{1-\gamma}$$

$$\boxed{D(\gamma_n\|\gamma) = \gamma_1\, \mathrm{ld} \frac{\gamma_1}{\gamma} + (1-\gamma_1)\, \mathrm{ld} \frac{1-\gamma_1}{1-\gamma}}$$

ZA $n \to \infty$ $\boxed{\gamma_1 \doteq \gamma_1'}$

$$D(\gamma_{2n}\|\gamma) = \frac12\left(\gamma_1+\gamma_1'\right) \mathrm{ld} \frac{\left(\gamma_1+\gamma_1'\right)(1-\gamma)}{\gamma\left(2-\gamma_1-\gamma_1'\right)} + \mathrm{ld} \frac{2-\gamma_1-\gamma_1'}{2(1-\gamma)}$$

IF: $\gamma_1 = \gamma_1'$

$$D(\gamma_{2n}\|\gamma) = \gamma_1\, \mathrm{ld} \frac{2\gamma_1(1-\gamma)}{2\gamma(1-\gamma_1)} + \mathrm{ld} \frac{1-\gamma_1}{1-\gamma}$$

$$D(\gamma_{2n}\|\gamma) = \gamma_1\, \mathrm{ld} \frac{\gamma_1}{\gamma} + (1-\gamma_1)\, \mathrm{ld} \frac{1-\gamma_1}{1-\gamma} = D(\gamma_n\|\gamma)$$

$$D(\hat{\gamma}_{nn}\|\gamma) = \frac1n\left[\hat\gamma_{n1}(1)+\hat\gamma_{n2}(1)+\dots\hat\gamma_{nn}(1)\right] \mathrm{ld} \frac{\frac1n\left[\hat\gamma_{n1}(1)+\dots+\hat\gamma_{nn}(1)\right]}{\gamma(1)}$$

$$+ \frac1n\left[\hat\gamma_{n1}(0)+\hat\gamma_{n2}(0)+\dots+\hat\gamma_{nn}(0)\right] \mathrm{ld} \frac{\frac1n\left[\hat\gamma_{n1}(0)+\dots+\hat\gamma_{nn}(0)\right]}{\gamma(0)}$$

$$\to \underset{(??)}{\epsilon}\left[\gamma_{ni}(1)\right] \mathrm{ld} \frac{\epsilon\left[\gamma_{ni}(1)\right]}{\gamma(1)} + \epsilon\left[\gamma_{ni}(0)\right] \mathrm{ld} \frac{\epsilon\left[\gamma_{ni}(0)\right]}{\gamma(0)}$$

(b) ARBITRARY DISCRETE „$\mathcal{X}$" , $x \in \{x_1, x_2, \dots x_n\}$

$$E[D(\hat{p}_n \| p)] \leq E[D(\hat{p}_{n-1} \| p)]$$

$$D(\hat{p}_n \| p) = \hat{p}_n(x_1) \operatorname{ld} \frac{\hat{p}_n(x_1)}{p(x_1)} + \hat{p}_n(x_2) \operatorname{ld} \frac{\hat{p}_n(x_2)}{p(x_2)} + \dots$$

$$+ \hat{p}_n(x_n) \operatorname{ld} \frac{\hat{p}_n(x_n)}{p(x_n)} = E_{\hat{p}_n}\left[\operatorname{ld} \left(\frac{\hat{p}_n(x)}{p(x)} \right) \right]$$

- RETURN TO PG.67 ⊛

$$D(\hat{p}_{2n} \| p) = \hat{p}_{2n}(1) \operatorname{ld} \frac{\hat{p}_{2n}(1)}{p(1)} + \hat{p}_{2n}(0) \operatorname{ld} \frac{\hat{p}_{2n}(0)}{p(0)}$$

$$\hat{p}_{2n}(1) = \frac{1}{2} p_n(1) + \frac{1}{2} p_n'(1) \xrightarrow{a.s.} E[p_n(1)] = p(1)$$

$$p_n(1) = \frac{1}{n} \sum_{i=1}^{n} I(X_i = 1)$$ } KAKO RASTE „n" TAKA E LOGIČNO DA RASTE I $p_n(1)$ ZOŠTO MOŽE POVEĆE EDINICI.

SOGLASNO ZAKONOT ZA GOLEMI BROEVI PRICE NEGOE GOLEMO „n" BROJOT NA EDINICI PODELEN SO DOLŽINATA NA PRIMEROKOT ĆE BIDE EDNAKVA NA VEROJATNOSTA NA POJAVUVANJE NA EDINICA.

$$D(\hat{p}_n \| p) \xrightarrow[n \nearrow]{} \emptyset \Rightarrow \underline{D(\hat{p}_{2n} \| p) < D(\hat{p}_n \| p)}$$

$$\boxed{0 \leq \lambda \leq 1}$$

$$\boxed{D(\lambda p_1 + (1-\lambda) p_2 \| \lambda q_1 + (1-\lambda) q_2) \leq \lambda D(p_1 \| q_1) + (1-\lambda) D(p_2 \| q_2)}$$

→ CONVEXITY OF RELATIVE ENTROPY

☐ sol_2_229 SOLUTION

$$p_n'(x) = \frac{1}{n} \sum_{i=1}^{n} 1(X_{n+i} = x) \quad \text{FOR ALL } x \in \mathcal{X}$$

- N.67 ⊛ $\Rightarrow \hat{p}_{2n} = (\hat{p}_n + \hat{p}_n') \cdot \frac{1}{2}$

BY CONVEXITY OF RELATIVE ENTROPY

$$D(p_{2n} \| p) = D\left(\frac{1}{2}\hat{p}_n + \frac{1}{2}\hat{p}_n' \,\Big\|\, \frac{1}{2}p + \frac{1}{2}p\right) \leq \frac{1}{2} D(\hat{p}_n \| p) +$$

$$+ \frac{1}{2} D(\hat{p}_n' \| p) \quad / E[\dots]$$

$$\mathbb{E}[D(p_{2n}\|q)] = \frac{1}{2}\Big[\mathbb{E}[D(\hat p_n\|q)]\Big] + \frac{1}{2}\mathbb{E}[D(\hat p_n'\|q)] = \mathbb{E}[D(\hat p_n\|q)]$$

$$\boxed{\mathbb{E}[D(\hat p_n'\|q)] = \mathbb{E}[D(\hat p_n\|q)]}$$

$\hat p_{2n}, \hat p_n, \hat p_n'$ ARE RANDOM VARIABLES TAKING VALUES IN THE SPACE OF PROBABILITY DISTRIBUTIONS OF „$\mathcal{X}$"

(6) FOR $1 \le i \le n$, let $p_n^{(i)}$ DENOTE THE EMPIRICAL DISTRIBUTION OF $(n-1)$ RANDOM VARIABLES X_j $1 \le j \le n$ $j \ne i$, i.e.

$$\hat p_n^i(x) = \frac{1}{n-1}\sum_{j=1, j\ne i}^{n} 1(X_j = x) \quad \text{FOR AL } x \in \mathcal{X}$$

$$\hat p_n = \frac{1}{n}\sum_{i=1}^{n} \hat p_n^i$$

- BY USING CONVEXITY OF RELATIVE ENTROPY FUNCTION

$$\boxed{D(\hat p_n\|q) \le \frac{1}{n}\sum_{i=1}^{n} D(p_n^{(i)}\|q)} \Longrightarrow$$

ISTA LOGIKA KAKO ZA REZ- SOT $\hat p_n^i$

$$\mathbb{E}[D(\hat p_n\|q)] \le \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[D(p_n^{(i)}\|q)] = \mathbb{E}[D(\hat p_{n-1}\|q)]$$

$$\mathbb{E}[D(p_n^{(i)}\|q)] = \mathbb{E}[D(\hat p_{n-1}\|q)] \quad \text{FOR } \forall i$$

- RECALL: LOG-SUM INEQUALITY

$$\sum_{i=1}^{n} a_i \,\mathrm{ld}\frac{a_i}{b_i} \ge \left(\sum_{i=1}^{n} a_i\right)\mathrm{ld}\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}$$

- THEOREM 2.7.2 (CONVEXITY OF RELATIVE ENTROPY)

$$\underline{D(\lambda p_1 + (1-\lambda)p_2 \,\|\, \lambda q_1 + (1-\lambda)q_2) \le \lambda D(p_1\|q_1) + (1-\lambda)D(p_2\|q_2)}$$

$$(\lambda p_1 + (1-\lambda)p_2)\,\mathrm{ld}\frac{\lambda p_1 + (1-\lambda)p_2}{\lambda q_1 + (1-\lambda)q_2} \le \lambda p_1 \,\mathrm{ld}\frac{p_1}{q_1} + (1-\lambda)p_2\,\mathrm{ld}\frac{p_2}{q_2}$$

(p_1, q_1) & (p_2, q_2) ARE TWO PAIRS OF PROBABILITY MASS FUNCTIONS.

$\boxed{3.13}$ CALCULATION OF TYPICAL SET

: TO CLARIFY THE NOTION OF TYPICAL SET $A_\varepsilon^{(n)}$ AND THE SMALLEST SET OF HIGH PROBABILITY $B_\delta^{(n)}$, WE WILL CALCULATE THE SET FOR A SIMPLE ~~EXAMPLE~~. CONSIDER A SEQUENCE OF I.I.D BINARY RANDOM VARIABLES, $X_1, X_2, \dots X_n$ WHERE THE PROBABILITY THAT $X_i = 1$ IS 0.6 ($p=0.6$ $q=0.4$)

(a) CALCULATE $H(X)$

(b) WITH $n = 25$ AND $\varepsilon = 0.1$, WHICH SEQUENCES FALL IN THE TYPICAL SET $A_\varepsilon^{(n)}$? WHAT IS THE PROBABILITY OF THE TYPICAL SET? HOW MANY ELEMENTS ARE THERE IN THE TYPICAL SET? (THIS INVOLVES COMPUTATION OF THE TABLE OF PROBABILITIES FOR SEQUENCES WITH k 1'S, $0 \leq k \leq 25$, AND FINDING THOSE SEQUENCES THAT ARE IN THE TYPICAL SET.)

(c) HOW MANY ELEMENTS ARE THERE IN THE SMALLEST SET THAT HAS PROBABILITY 0.9?

(d) HOW MANY ELEMENTS ARE THERE IN THE INTERSECTION OF THE SETS IN (b) AND (c)? WHAT IS THE PROBABILITY OF THIS INTERSECTION.

$$2^{-n(H+\varepsilon)} \leq p(x^n) \leq 2^{-n(H-\varepsilon)}$$

$$Pr\{A_\varepsilon^{(n)}\} \geq 1-\varepsilon \qquad |A_\varepsilon^{(n)}| \leq \cdot 2^{n(H+\varepsilon)} \qquad |A_\varepsilon^{(n)}| \geq (1-\varepsilon) 2^{n(H-}$$

$$\boxed{(H-\varepsilon) \leq -\frac{1}{n} \, ld \, p(x^n) \leq (H+\varepsilon)}$$

ALL SEQUENCES FOR WHICH THIS IS TRUE, ε-TYPICAL SET

(a) $H(X) = p \, ld \frac{1}{p} + q \, ld \frac{1}{q} = 0.6 \, ld \frac{1}{0.6} + 0.4 \, ld \frac{1}{0.4} = \underline{0.970951}$

(b) $n = 25 \qquad 1 \leq k \leq 25$

TOTAL NUMBER OF SEQUENCES WITH GIVEN "K" IS

$\binom{n}{k}$ e.g. $\binom{n}{1} = \frac{25!}{24!} = 25 \nearrow$ THERE ARE 25 SEQUENCES WIT 1 ONE

PROBABILITY OF THOSE SEQUENCES IS

$\binom{n}{k} \cdot p^k \cdot q^{n-k}$ e.g. $25 \cdot (0.6)^1 \cdot (0.4)^{24} = 4.22212 \cdot 10^{-9}$

$-\frac{1}{n} \, ld \left((0.6)^1 \cdot (0.4)^{24} \right) = \underline{1.29853}$

- $A_\varepsilon^{(n)}$ VO OVA MNOZESTVO MISLAM DEKA PADAAT SITE SEKVENCI SO $25 \cdot 0.6 = 15{,}0$ EDINICI NO

$$\begin{matrix}0.970951-0.1\\=0.870951\end{matrix} \leq -\frac{1}{25} \, ld \, (0.6)^{15} \cdot (0.4)^{10} = 0.970951 \leq \begin{matrix}0.970951+0.1\\1.070951\end{matrix}$$

$H-\varepsilon \leqslant -\frac{1}{n}\, ld\, p(x^n) \le H+\varepsilon \qquad -\varepsilon \leqslant -\frac{1}{n}\, ld\, p(x^n) - H \le \varepsilon.$

$$\boxed{\left|-\frac{1}{n}\, ld\, p(x^n) - H\right| \le \varepsilon}$$

SKRATENA FORMA NA ISKAZOT ZA TIPIČNO MNOŽESTVO

$$\boxed{Pr\{A_\varepsilon^{(n)}\} \geqslant 1-\varepsilon \geqslant 0.9}$$

OVA ŠO ISPOLNUVAAT SITE SEKVENCI ŠO: 11, 12, ... 19 EDINICI

MAN

OVA E GLAVNATA MOVEKA!!! ISTATA JA PRAVAT I VO UNIVERSITY OF TORONTO.

- PROBABILITY OF TYPICAL SET IS:

$$P_A = \sum_{i=11}^{19}\binom{25}{i} p^i q^{25-i} = \underline{0.936246}$$

$$\boxed{P_A = 0.936246 \geqslant 0.9 = 1-\varepsilon}$$

- THE NUMBER OF ELEMENTS IN TYPICAL SET IS:

$$N_A = \sum_{i=11}^{19}\binom{25}{i} = \underline{\underline{26.366.510}} \text{ ELEMENTS}.$$

$$|A_\varepsilon^{(n)}| \le 2^{n(H+\varepsilon)} = 2^{25(1.070951)} = 114.737.700$$

$$\underline{26.366.510 \leqslant 114.737.700}$$

$\underline{8+7}+\underline{6+4}+\underline{3+2} = 15+10+5 = 30$

- MINIMAL SET $[12, 19]$ $\qquad P_B = 0.89284 \doteq \underline{\underline{0.9}}$

$$N_B = \sum_{i=12}^{19}\binom{25}{i} = 21.909.110$$

$$\boxed{P_B \doteq 0.9}$$

? #

SUM ZEMAL DA BIDE = A TREBA ≥

- NUMBER OF ELEMENTS IN THE INTERSECTION

$$|A_\varepsilon^{(n)} \cap B_\varepsilon^{(n)}| = |B_\varepsilon^{(n)}| = 21.909.110 \text{ ELEMENTS}$$

$$\frac{1}{ⓝ} \ln |B_\varepsilon^{(n)}| > H-\delta' \qquad \boxed{H = 0.970951}$$

$$\frac{1}{25} \ln (21.909.110) = \underline{0.97540}$$

$$\delta' = \frac{1}{2} + \frac{1}{n}\left|\ln\left(\frac{1}{2}-\delta\right)\right| = \frac{1}{2}+\frac{1}{25}\left|\ln(0.4)\right| = 0.53665$$

$$H-\delta' = 0.970951 - 0.53665 = 0.43430$$

?

$$25+24+23+\cdots 2 = \frac{n\cdot(n+1)}{2} - 1$$

RASPREDELBA P_i

$$s = 1+2+3+\cdots n$$
$$s = n+n-1+n-2+\cdots+1$$
$$\overline{2s = n+1+n+1+n+1+\cdots+(n+1)}$$

$$2s = (n+1)\cdot n$$
$$s = \frac{(n+1)\cdot n}{2}$$

- VRZ OSNOV NA SIMULACIITE VO MAKE DOBIV DEKA MINIMALEN SET E GO SOČINUVAAT SEKVENCIITE ŠTO IMAAT VKUPNO:

12, 13, 14 ... 20 EDINICI

SUMATA NA VEROJATNOSTITE NA OVIE SEKVENCI E ≥ 0.9, VOEDNO TOA E MNOŽESTVOTO SO NAJMAL BROJ NA ELEMENTI ŠTO GO ISPOLNUVA OVOJ USLOV.

$$\sum_{i=12}^{20}\binom{25}{i}p^i q^{25-i} = 0.9127 > \qquad \sum_{i=12}^{20}\binom{25}{i} = \underline{21.962.240}$$

$$\left|A_\varepsilon^{(n)} \cap B_\varepsilon^{(n)}\right| = \sum_{i=12}^{19}\binom{25}{i} = \underline{21.909.110} \quad ①$$

NAVISTINA PRESEKOT E OGRANEN

— V_0 (#) SLUČAJOT SUM ZA ~~DOBIL~~ ISTATA VEROST

A NE SUM BARAL MNOŽESTVOTO DA IMA MINIMALEN BROJ NA ELEMENTI.

WRIGHT COLLEGE SOLUTION

(c) $N\cdot p^{12}\cdot q^{13} \leq 0.9 - \sum_{j=13}^{25}\binom{25}{j}p^j\cdot q^{25-j}$ } ZEMAŠ ONOLKU SEKVENCI SO 12 EDINICI ZA DA VKUPNATA VEROJATNOST BIDE 0.9

$$\boxed{N \leq \frac{0.9 - \sum_{j=13}^{25}\binom{25}{j}p^j q^{25-j}}{p^{12}\cdot q^{13}}}$$

$$N = 3.68067\cdot 10^6$$

— ZNAČI VKUPNIOT BROJ NA ELEMENTI VO MINIMALNOTO MNOŽESTVO E:

$$P = N\cdot p^{12}\cdot q^{13} + \sum_{j=13}^{25}\binom{25}{j}p^j q^{25-j}$$

$$\left|B_\delta^{(n)}\right| = N + \sum_{j=13}^{25}\binom{25}{j} = \underline{2.04579\,10^7} \quad ②$$

$$|B_\delta^{(n)}| \doteq |A_\epsilon^{(n)}| = 2^{n\cdot H} = 2^{25\cdot 0.970951} = \underline{\underline{2.0283\cdot 10^7}}$$

- AKO SE SPOREDI NAŠATA PRESMETKA (BAZIRANA NA MARKE) I PRESMETKATA OD WRIGHT COLLEAGE, SE GLEDA RAZLIKA (① vs ② N.73):

$① = 21,90911\cdot 10^6$

$② = 20,4579\cdot 10^6$

$$\boxed{2^{nH} = 20,283\cdot 10^6}$$

$2^{n(H+\epsilon)} = 114,7377\cdot 10^6$

$n\cdot H = 25\cdot 0,97 \doteq$

REŠENIE KOE ZAŠ
DEFINITIVNO ② E PODOBRO MARKE ZOŠTO RAZGLEDUVAV CELI
NE ŠO VODIV VO SEKVENCI.
GRUPI NA SEKVENCI, A NE DEL OD
JE RABOTI ZA OPTIMIZACIONEN PROBLEM.

(b) $|A_\epsilon^{(n)} \cap B_\delta^{(n)}| = |A_{0.1}^{(n)} \cap B_{0.1}^{(n)}| =$

$A_\epsilon^{(n)} \quad k \in \{11, 12, \dots 19\}$

$B_\delta^{n} \quad k \in \{\text{part of } 12,\ 13 \dots 25\}$

$|A_{0.1}^{(n)} \cap B_{0.1}^{(n)}| \Rightarrow k \in \{\text{part of } 12, 13, \dots 19\}$

$$|A_{0.1}^{(n)} \cap B_{0.1}^{(n)}| \doteq N + \sum_{\lambda=13}^{19} \binom{25}{\lambda} = 3,68067\cdot 10^6 + 16.708810$$

$$= 2.03895\ 10^7$$

$$P_{AND} = N\cdot p^{12} q^{13} + \sum_{j=13}^{19} \binom{25}{j} p^j q^{25-j} = \underline{0.87064}$$

- SUŠTINATA ZOŠTO $B_{0.1}^{(n)}$: $k \in \{\text{part of } 12, 13 \dots 25\}$ E ZATOA ŠTO $p > q$ PA ~~ZATOA~~ ~~ZAPOČ~~NUVA OD $k=25$ PA SUMIRA NANOLU. TAKA PO REDU (ZA POMAL BROJ NA ČLENOVI) SE STIGA DO VKUPNA VEROJATNOST 0.9. MMV

CHAPTER 4 ENTROPY RATES OF ERGODIC PROCESSES

AEP ESTABLISH THAT $n\cdot H(X)$ BITS SUFFICE ON AVERAGE TO DESCRIBE „n" INDEPENDENT AND IDENTICALLY DISTRIBUTED RANDOM VARIABLES.

- IF $X_1, X_2, \ldots, X_n$ ARE **DEPENDENT** WE WILL SHOW THAT THE ENTROPY $H(X_1, X_2, \ldots X_n)$ GROWS (ASYMPTOTICALLY) LINEARLY WITH „n" A RATE $H(X)$, WHICH WE WILL CALL THE **ENTROPY RATE** OF THE PROCESS.

4.1 MARKOV CHAINS

A STOCHASTIC PROCESS $\{X_i\}$ IS AN INDEXED SEQUENCE OF RANDOM VARIABLES. IN GENERAL THERE CAN BE ARBITRARY DEPENDENCE AMONG THE RANDOM VARIABLES. PROCESS IS CHARACTERIZED BY THE JOINT PROBABILITY MASS FUNCTIONS:

$$\Pr\{(X_1, X_2, \ldots X_n) = (x_1, x_2, \ldots, x_n)\} = p(x_1, x_2, \ldots, x_n),$$

$$(x_1, x_2, \ldots x_n) \in \mathcal{X}^n \quad \text{for} \quad n = 1, 2, \ldots .$$

MMV

DEFINITION A STOCHASTIC PROCESS IS SAID TO BE **STATIONARY** IF THE JOINT DISTRIBUTION OF ANY SUBSET OF THE SEQUENCE OF RANDOM VARIABLES IS INVARIANT WITH RESPECT TO SHIFTS IN TIME INDEX, THAT IS:

$$\Pr\{X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n\} = \Pr\{X_{1+l} = x_1, X_{2+l} = x_2, \ldots X_{n+l} = x_n\}$$ ✓

FOR EVERY n AND EVERY SHIFT „l", AND FOR ALL $x_1, x_2, \ldots x_n \in \mathcal{X}$

DEFINITION A DISCRETE STOCHASTIC PROCESS $X_1, X_2, \ldots$ IS SAID TO BE A MARKOV CHAIN OR MARKOV PROCESS IF FOR $n = 1, 2, \ldots,$

$$\Pr(X_{n+1} = x_{n+1} | X_n = x_n, X_{n-1} = x_{n-1}, \ldots X_1 = x_1) = \Pr(X_{n+1} = x_{n+1} | X_n = x_n)$$

FOR ALL $x_1, x_2, \ldots, x_n, x_{n+1} \in \mathcal{X}$

$$p(x_1, x_2, \ldots x_n) = p(x_1)\, p(x_2|x_1) \cdot p(x_3|x_2) \cdots p(x_n|x_{n-1})$$

$$p(x_1, x_2, \ldots x_n) = p(x_1) \cdot p(x_2, x_3, \ldots x_n | x_1) = p(x_1) \cdot p(x_2|x_1) \cdot p(x_3, \ldots x_n | x_1 x_2)$$

$$= p(x_1) \cdot p(x_2|x_1) \cdot \underbrace{p(x_3|x_1 x_2)}_{= p(x_3|x_1)} \cdot p(x_4, \ldots x_n | x_1 x_2 x_3) = p(x_1) \cdot p(x_2|x_1)$$

$$\cdot\, p(x_3|x_2) \cdot \overbrace{p(x_4|x_1 x_2 x_3)}^{p(x_4|x_3)} \cdot p(x_5, \ldots x_n | x_1 x_2 x_3 x_4) \ldots$$

- FOR TIME INVARIANT MARKOV CHAIN:

$P_r(X_{n+1}=b|X_n=a)=P_r\{X_2=b|X_1=a\}$ for all $a,b\in X$

- If $\{X_i\}$ is a Markov chain, X_n is called the state at time "n". A time invariant Markov chain is characterized by its initial state and probability transition matrix

$P=[P_{ij}]$, $i,j\in\{1,2,\dots m\}$ $\quad P_{ij}=P_r\{X_{n+1}=j|X_n=i\}$

If it is possible to go with positive probability from any state of Markov chain to any other state in the finite number of steps, the Markov chain is said to be IRREDUCIBLE.

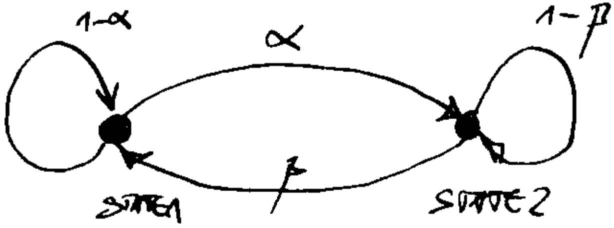
- If the probability mass function of random variable at time "n" is $p(x_n)$, the probability mass function at time $(n+1)$ is:

$$p(x_{n+1})=\sum_{x_n} p(x_n)\cdot P_{x_n x_{n+1}}$$

A distribution of states such that distribution at time $n+1$ is the same as the distribution at time "n" is called STATIONARY DISTRIBUTION.

$\boxed{\text{EXAMPLE 4.1.1}}$ Consider a two state Markov chain with probability transition matrix

✓ $$P=\begin{bmatrix}1-\alpha, & \alpha\\ \beta, & 1-\beta\end{bmatrix}$$



μ - vector representing stationary probabilities of states 1&2, respectively.

$\boxed{\mu\cdot P=\mu}$ } find stationary probabilities by solving of this equation.

$$[\mu_1\ \mu_2]\begin{bmatrix}1-\alpha, & \alpha\\ \beta, & 1-\beta\end{bmatrix}=[\mu_1(1-\alpha)+\mu_2\cdot\beta,\ \alpha\mu_1+(1-\beta)\mu_2]=[\mu_1,\mu_2]$$

$\mu_1(1-\alpha)+\mu_2\beta=\mu_1 \qquad -\alpha\mu_1+\mu_2\beta=0 \qquad \alpha\mu_1=\mu_2\beta$

$\alpha\mu_1+(1-\beta)\mu_2=\mu_2 \qquad \alpha\mu_1-\beta\mu_2=0 \qquad \boxed{\alpha\mu_1=\beta\mu_2}$

$\boxed{\mu_1+\mu_2=1} \qquad \mu_1=1-\mu_2 \qquad \alpha(1-\mu_2)=\beta\mu_2$

$\alpha - \alpha\mu_2 = \beta\mu_2$ $\quad (\alpha+\beta)\mu_2 = \alpha$ $\quad \boxed{\mu_2 = \frac{\alpha}{\alpha+\beta}}$

$$\boxed{\mu_1 = \frac{\beta}{\alpha}\cdot\mu_2 = \frac{\beta}{\alpha+\beta}}$$

- IF PROCESS IS STATIONARY THE ENTROPY OF THE STATE X_n AT TIME n IS:

$$H(X_n) = H\left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right) = \frac{\beta}{\alpha+\beta}\, ld \frac{\alpha+\beta}{\beta} + \frac{\alpha}{\alpha+\beta}\, ld \frac{\alpha+\beta}{\alpha}$$

$$H(Y|X) = \sum_{x\in X} p(x) H(Y|X=x) \qquad \boxed{H(Y|X) = \sum_{x,y} p(x,y)\, ld \frac{1}{p(y|x)}}$$

4.2 ENTROPY RATE

IF WE HAVE A SEQUENCE OF n RANDOM VARIABLES, A NATURAL QUESTION TO ASK IS: HOW DOES THE ENTROPY OF THE SEQUENCE GROW WITH n? WE DEFINE THE ENTROPY RATE AS THIS RATE OF GROWTH AS FOLLOWS.

DEFINITION: THE ENTROPY OF STOCHASTIC PROCESS $\{X_i\}$ IS DEFINED BY

$$\boxed{H(X) = \lim_{n\to\infty} \frac{1}{n} H(X_1, X_2, \dots, X_n)}$$

WHEN THE LIMIT EXISTS.

EXAMPLES:

1.) TYPEWRITER: m - EQUALLY LIKELY OUTPUT LETTERS
m^n - SEQUENCES OF LENGHT n, ALL EQUALLY LIKELY.

$$H(X_1, X_2, \dots X_n) = \sum_{i=1}^{m^n} \frac{1}{m^n} \cdot ld\, m^n = \cancel{m^n} \cdot \frac{1}{\cancel{m^n}} \cdot ld\, m^n$$

$H(X_1, X_2, \dots X_n) = ld\, m^n$ THE ENTROPY RATE IS:

$$H(X) = \lim_{n\to\infty} \frac{1}{n} H(X_1, X_2, \dots X_n) = \frac{1}{n}\, ld\, m^n = \boxed{ld\, m}$$

2.) $X_1, X_2, \dots X_n$ ARE i.i.D RANDOM VARIABLES. THEN

$$H(X) = \lim \frac{H(X_1, X_2 \dots X_n)}{n} = \lim \frac{\cancel{n} H(X_1)}{\cancel{n}} = H(X_1)$$

3. SEQUENCE OF INDEPENDENT BUT NOT IDENTICALLY DISTRIBUTED RANDOM VARIABLES

$$H(X_1, X_2, \dots X_n) = \sum_{i=1}^{n} H(X_i)$$

E.G. SEQUENCE OF DISTRIBUTION SUCH AS:

$\lim \frac{1}{n} \sum_{i=1}^{n} H(X_i)$ DOESN'T EXIST

$p_i = P(X_i = 1)$ IS CONSTANT BUT FUNCTION OF „i“

$$p_i = \begin{cases} 0.5 & \text{IF } 2k \le \log\log i \le 2k+1 \\ 0 & \text{IF } 2k+1 \le \log\log i \le 2k+2 \end{cases}$$

FOR $k = 0, 1, 2, \dots$

$H(X_i)$ WILL OSCILATE BETWEEN „0“ & „1“, AND WILL NOT HAVE THE LIMIT.

- QUANTITY OF ENTROPY RATE: (Alternative Deffinition)

$$\boxed{H'(X) = \lim_{n \to \infty} H(X_n | X_{n-1}, X_{n-2}, \dots X_1)}$$

$H(X)$ ~ PER SYMBOL ENTROPY OF „n“ RANDOM VARIABLES.

$H'(X)$ ~ CONDITIONAL ENTROPY OF LAST RANDOM VARIABLE GIVEN THE PAST.

THEOREM 4.2.1 FOR STATIONARY STOHASTIC PROCESS $H(X)$ AND $H'(X)$ EXIST AND ARE THE LIMITS EQUAL:

$$\boxed{H(X) = H'(X)}$$

$$H(X_1, X_2, \dots X_n) = H(X_1) + H(X_2|X_1) + H(X_3|X_2, X_1) \dots$$

$$H(X_1, X_2, \dots X_n) = \sum_{i=1}^{n} H(X_i | X_{i-1}, \dots X_1)$$

$$\lim_{n \to \infty} \frac{1}{n} H(X_1, X_2, \dots X_n) = \sum_{i=1}^{n} \underbrace{\lim \frac{1}{n} H(X_i | X_{i-1}, \dots X_1)}_{\text{IF EXISTS}}$$

THIS LIMIT EXISTS ALSO

THEOREM 4.2.2 FOR STATIONARY STOHASTIC PROCESS $H(X_n | X_{n-1}, \dots X_1)$ IS NONINCREASING IN „n“ AND HAS LIMIT $H'(X)$.

CONDITIONING REDUCES ENTROPY!!!

PROOF: $H(X_{n+1} | X_1, X_2, \dots X_n) \le H(X_{n+1} | X_n, \dots X_2) =$

$= H(X_n | X_{n-1}, \dots X_1)$ DUE TO DEFINITION OF STATIONARY STOHAST. PROCESS.

THEOREM 4.2.3 (Cesaro mean) IF $\boxed{a_n \to a}$ AND $b_n = \frac{1}{n} \sum_{i=1}^{n} a_i$ THEN $b_n \to a$

PROOF: LET $\varepsilon > 0$. SINCE $a_n \to a$ THERE EXIST NUMBER SUCH THAT $|a_n - a| < \varepsilon$ FOR ALL $n \geq N(\varepsilon)$
HENCE,

$$|b_n - a| = \left|\frac{1}{n}\sum_{i=1}^{n}(a_i - a)\right| \leq \frac{1}{n}\sum_{i=1}^{n}|(a_i - a)|$$

$$\leq \frac{1}{n}\sum_{i=1}^{N(\varepsilon)}|a_i - a| + \frac{n - N(\varepsilon)}{n}\cdot\varepsilon \leq \frac{1}{n}\sum_{i=1}^{N(\varepsilon)}|a_i - a| + \varepsilon$$

FIRST TERM GOES TO „0" AS $n \to \infty$, WE CAN MAKE $|b_n - a| \leq 2\varepsilon$ BY TAKING „n" large enough
HENCE, $b_n \to a$ AS $n \to \infty$.

- PROOF OF THEOREM 4.2.1 BY CHAIN RULE

$$H(X_1, X_2, \dots X_n) = \sum_{i=1}^{n} H(X_i | X_{i-1}, \dots X_1)$$

$$\frac{H(X_1, X_2, \dots X_n)}{n} = \frac{1}{n}\sum_{i=1}^{n} H(X_i | X_{i-1}, \dots X_1)$$

ENTROPY RATE IS $\boxed{\text{TIME}}$ AVERAGE OF CONDITIONAL ENTROPIES. (MMV)

$$\boxed{\frac{1}{n}\sum_{i=1}^{n} H(X_i | X_{i-1}, \dots X_1) =}$$ H(Xn|Xn-1,...X1)

$$H(X) \overset{\boxed{T4.2.1}}{=} \lim_{n \to \infty} \frac{1}{n} H(X_1, X_2, \dots X_n) \overset{\boxed{T4.2.3}}{=} \lim_{n \to \infty} H(X_n | X_{n-1} \dots X_1) = H'(X)$$

(PROOF IN 168)

- AEP FOR STATIONARY ERGODIC PROCESS

FOR ANY STATIONARY ERGODIC PROCESS:

$$\left. -\frac{1}{n}\log p(X_1, X_2, \dots X_n)\right] \to H(\mathcal{X})$$

MARKOV CHAINS

FOR STATIONARY MARKOV CHAIN, THE ENTROPY RATE IS GIVEN BY:

$$H(X) = H'(X) = \lim H(X_n | X_{n-1}, \dots 1) = \lim H(X_n | X_{n-1})$$
$$= \lim H(X_2 | X_1)$$

CONDITIONAL ENTROPY IS CALCULATED USING THE GIVEN STATIONARY DISTRIBUTION. STATIONARY DISTRIBUTION μ IS SOLUTION OF THE EQUATIONS

$$\mu_j = \sum_i \mu_i P_{ij} \quad \text{for all „}j\text{"}$$

$$[\mu_1\ \mu_2] = [\mu_1\ \mu_2]\ \overset{i\downarrow}{\begin{bmatrix} \overset{P_{11}}{1-\alpha} & \overset{P_{12}}{\beta} \\ \overset{P_{21}}{\beta} & \overset{2_{22}}{1-\alpha} \end{bmatrix}} = [\mu_i]\cdot[P_{ij}]$$

$$\boxed{\mu_j = \sum_i \mu_i P_{ij}}$$

(MOJA INTERPRETACIJA SUMA NA VEROJATNOSTITE-TRANZICIONI KOI OD BILO KOJA SOSTOJBA „i" VODAT VO SOSTOJBA „j")
2

$\boxed{j=1}$ $$\mu_1 = \mu_1(1-\alpha) + \mu_2\cdot\beta = \sum_{i=1} \mu_i\cdot P_{i1}$$

MMV

THEOREM 4.2.4 LET $\{X_i\}$ BE A STATIONARY MARKOV CHAIN WITH STATIONARY DISTRIBUTION „μ" AND TRANSITION MATRIX P. LET $\boxed{X_1 \sim \mu}$ THEN THE ENTROPY RATE IS!

MMV $$\boxed{H(\mathcal{X}) = -\sum_{ij} \mu_i P_{ij} \operatorname{ld} P_{ij}}$$

OVA E KLASIKA: PROCESOT TREBA DA E VO SOSTOJBA „i" $i\in\{1,2,\dots n\}$ I PREOGJA VO SOSTOJBA „j" SO VEROJATNOST P_{ij} AKO VEĆE E VO SOSTOJBA „i"

VKUPNA VEROJATNOST DA E VO SOSTOJBA „i" $\mu_i\cdot P_{ij}$

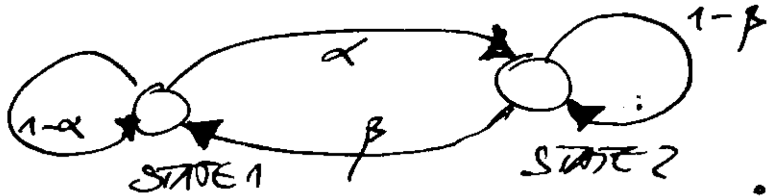
PROOF: $$H(\mathcal{X}) = H(X_2|X_1) = \sum_i \mu_i \underbrace{\sum_j P_{ij} \operatorname{ld}\frac{1}{P_{ij}}}$$

$$H(Y|X=x) = -\sum_{y\in Y} P(y|X=x) \operatorname{ld} P(y|X=x)$$

$$H(Y|X) = \overline{H(Y|X=x)} = -\sum_{\substack{x\in X\\ y\in Y}} P(x)P(y|x)\operatorname{ld}P(y|x)$$

$$= \sum_x\sum_y P(x,y)\operatorname{ld}P(y|x) = \sum_{x\in X} P(x)\,H(Y|X=x)$$

EXAMPLE 4.2.1 (TWO STATE MARKOV CHAIN) THE ENTROPY RATE OF TWO STATE MARKOV CHAIN IS:



$$H(\mathcal{X}) = -\sum_{ij} \mu_i P_{ij}\operatorname{ld}P_{ij} \qquad P_{ij} = \begin{array}{c|c} & j\rightarrow \\ \hline i\downarrow & \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix} \end{array}$$

$$= \sum_i \mu_i \sum_j P_{ij}\operatorname{ld}\frac{1}{P_{ij}} = \mu_1 \sum_j P_{1j}\operatorname{ld}\frac{1}{P_{1j}} +$$

$$\mu_2 \sum_j P_{2j}\operatorname{ld}\frac{1}{P_{2j}} = \mu_1\left[\alpha \operatorname{ld}\frac{1}{\alpha} + (1-\alpha)\operatorname{ld}\frac{1}{1-\alpha}\right] + \mu_2\left[\beta\operatorname{ld}\frac{1}{\beta} + 1-\beta\operatorname{ld}\frac{1}{1-\beta}\right.$$

$$\boxed{H(X) = \underbrace{\frac{\beta}{\alpha+\beta}}_{\mu_1} H(\alpha) + \underbrace{\frac{\alpha}{\beta+\alpha}}_{\mu_2} H(\beta)}$$

- **MOŽ IZMISLEN PRIMER ZA MARKOV PROCES SO DVE SOSTOJBI**

$\{x_1\ x_2\}$

$x_1 \in (0, 1)$ $\quad P(x_1) = \begin{bmatrix} \overset{0}{\frac{1}{2}}, & \overset{1}{\frac{1}{2}} \end{bmatrix}$

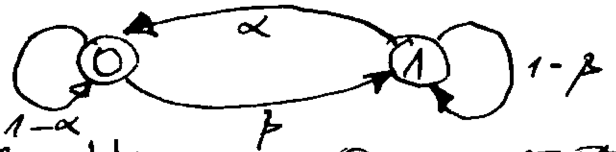
$x_2 \in (0, 1)$ $\quad P(x_2) = \begin{bmatrix} \overset{0}{\frac{1}{3}}, & \overset{1}{\frac{2}{3}} \end{bmatrix}$

- PROCESOT E : $\xi = \underline{x_1} \oplus x_2$ mod 2

$\xi \in [0, 1]$

x_1	x_2	ξ
0	0	0
0	1	1
1	0	1
1	1	0

ξ :



x_1	x_2	ξ	n
0	0	0	1

Ⓐ ĆE OSTANE VO SOSTOJBAT „0" AKO:

1) NOVOTO IZVLEČENO $x_1 = 0$
NOVOTO IZVLEČENO $x_2 = 0$

2) NOVOTO IZVLEČENO $x_1 = 1$
NOVOTO " $x_2 = 1$

VEROJATNOSTA DA PROCESOT ξ NE JA SMENI SOSTOJBATA VO „0" E :

$$P_0 = \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{2}{3} = \frac{1}{6} + \frac{3}{6} = \frac{3}{6} = \frac{1}{2} = \mu_1$$

Ⓑ ξ ĆE BIDE VO SOSTOJBA „1" AKO:

1) $\underline{x_1 = 0, \ x_2 = 1}$ ILI $x_1 = 1, \ x_2 = 0$

$$P_1 = \frac{1}{2}\cdot\frac{2}{3} + \frac{1}{2}\cdot\frac{1}{3} = \frac{2}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2} = \mu_2$$

Ⓒ ξ ĆE PREMINE VO SOSTOJBA 1 OD SOSTOJBA „0" AKO

	n	$n+1$
$x_1=0$ $x_2=0$	0	1
$x_1=1$ $x_2=1$	0	1

$$P(1|0) = P(00)\left[\underline{P(x_1=0) + P(x_2=1)}\right] + P(00)\left[\underline{P(x_1=1) + P(x_2=0)}\right] + P(11)\left[\underline{P(x_1=0) + P(x_2=1)}\right] + P(11)\left[\underline{P(x_1=1) + P(x_2 \neq 0)}\right]$$

$$= (P(11)+P(00))\left[P(x_1=0) + P(x_2=1)\right] + (P(11)+P(00))\left[P(x_1=1) + P(x_2=0)\right]$$

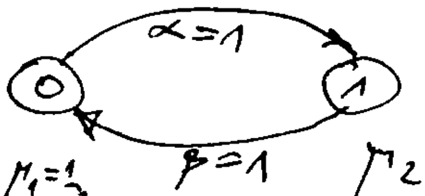
$$= \left(\frac{2}{6} + \frac{1}{6}\right)\left[\frac{1}{2} + \frac{2}{3}\right] + \left(\frac{2}{6} + \frac{1}{6}\right)\left[\frac{1}{2} + \frac{1}{3}\right] = \frac{1}{2}\cdot\frac{3+4}{6} + \frac{1}{2}\ \frac{3+2}{6} = \frac{12}{12}$$

	n	$n+1$
$x_1=0$ $x_2=1$	1	0
$x_1=1$ $x_2=0$	1	0

$$P(1|0) = P(01)\left[\underline{P(x_1=1)+P(x_2=1)}\right] + P(00)\left[\underline{P(x_1=0)+1(x_2=0)}\right] + P(10)\cdot\left[\underline{P(x_1=0)} \underline{+P(x_2=0)}\right] + P(10)\cdot\left[\underline{P(x_1=1)+P(x_2=1)}\right]$$

$$P(1|0) = \boxed{P(01)+P(10)}\cdot\left[P(x_1=1)+P(x_2=1)\right]+\left[P(01)+P(10)\right]\cdot\left[P(x_1=0)+P(x_2=0)\right] = \left(\frac{2}{6}+\frac{1}{6}\right)\left(\frac{1}{2}+\frac{2}{3}\right)+\left(\frac{2}{6}+\frac{1}{6}\right)\cdot\left(\frac{1}{2}+\frac{1}{3}\right) = \frac{1}{2}\,\frac{3+4}{6}+\frac{1}{2}\,\frac{3+2}{6} = \frac{12}{12}$$

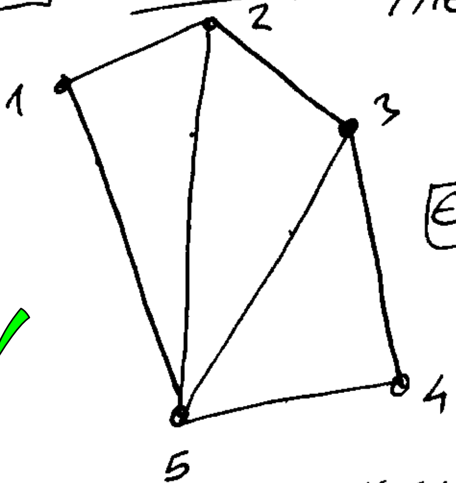
$$\boxed{\alpha = 1 \quad , \beta = 1}$$



$\mu_1=\frac{1}{2}$ $\quad$ $\mu_2=\frac{1}{2}$

AKO GO MODELIRAM VO MATRICE BI TREBAZO DA PODIJAM NAIZMENICNA SEKVENCA

$\boxed{4.3}$ EXAMPLE: ENTROPY RATE OF RANDOM WALK ON THE WEIGHTED GRAPH.



$\boxed{E=7}$

$w=\frac{1}{7}$

CONSIDER GRAPH WITH „m" NODES LABELED $\{1,2,\ldots m\}$, WITH WEIGHT $W_{ij}\geq 0$ ON THE EDGE JOINING i TO NODE j

$W_{ij} = W_{ji}$ $\}$ UNDIRECTED GRAPH

$W_{ij}=0$ IF THERE IS NO EDGE CONNECTING i & j

A PARTICLE WALKS RANDOMLY FROM NODE TO NODE. THE RANDOM WALK $\{X_n\}$, $X_n\in\{1,2,\ldots m\}$ GIVEN $X_n=i$, THE NEXT VERTEX j IS CHOSEN FROM THE NODES CONNECTED TO NODE i WITH PROBABILITY PROPORTIONAL TO THE WEIGHT OF THE EDGE CONNECTING i TO j

$$P_{ij} = \frac{W_{ij}}{\sum_k W_{ik}}$$

THE STATIONARY DISTRIBUTION FOR THIS MARKOV CHAIN ASSIGNS PROBABILITY TO NODE i. LET $W_i = \sum_j W_{ij}$ BE TOTAL WEIGHT OF EDGES EMANATING FROM NODE i, AND LET $W = \sum_{i,j:j>i} W_{ij}$ BE THE SUM OF WEIGHTS OF ALL EDGES. THEN: $\sum W_i = 2W$

- NOW WE GESS THAT STATIONARY DISTRIBUTION IS:

$$\boxed{\mu_i = \frac{W_i}{2W}}$$

WE CHECK THIS IS STATIONARY DISTRIBUTION BY CHECKING $\boxed{\mu P = \mu}$

$$\boxed{\sum_i \mu_i P_{ij} = \sum_i \frac{\cancel{W_i}}{2W} \frac{W_{ij}}{\cancel{W_i}} = \frac{W_j}{2W} = \mu_j}$$

THUS, THE STATIONARY PROBABILITY OF STATE „i" IS PROPORTIONAL TO THE WEIGHT OF THE EDGES EMANATING FROM NODE „i".

- ENTROPY RATE IS:

$$H(\mathcal{X}) = H(X_2|X_1) = -\sum_i \mu_i \sum_j P_{ij} \, ld P_{ij} =$$

$$= -\sum_i \frac{W_i}{2W} \sum_j \frac{W_{ij}}{W_i} \, ld \frac{W_{ij}}{W_i} = \frac{-1}{2W} \sum_{i,j} W_{ij} \, ld \frac{W_{ij}}{W_i}$$

$$= -\sum_{i,j} \frac{W_{ij}}{\boxed{2W}} \, ld \frac{W_{ij}}{2W} + \sum_{i,j} \frac{W_{ij}}{2W} \, ld \frac{W_i}{2W} =$$

$$= H\left(\ldots, \frac{W_{ij}}{2W}, \ldots\right) - H\left(\ldots, \frac{W_i}{2W}, \ldots\right)$$

- IF ALL EDGES HAVE EQUAL WEIGHT, STATIONARY DISTRIBUTION PUTS WEIGHT $E_i/2E$ ON NODE „i", WHERE E_i IS NUMBER OF EDGES EMANATING FROM NODE „i" AND E IS TOTAL NUMBER OF EDGES IN THE GRAPH.

$$-\sum_i \sum_j \frac{W_{ij}}{2W} \, ld \frac{W_{ij}}{2W} = \sum_i \sum_j \frac{W_{ij}}{2E} \, ld \frac{W_{ij}}{2E} =$$

$$/W_{ij} = w = const/ = -\frac{w}{2E} \sum_i \sum_j \, ld \frac{w}{2E} =$$

$$= /\boxed{w = \frac{1}{E}}/ = -\frac{1}{\cancel{2}E^{\cancel{2}}} \cdot \cancel{2E} \cdot ld \frac{1}{2E^2} = \frac{1}{E} \, ld \, 2E^2 = \frac{2}{E} \, ld (E\sqrt{2})$$

$\boxed{\text{FOR } w = \frac{1}{E}}$

DO KALGAVA ZEMAT $w=1$ AKO

OPIS 10 ~~EXAMPLE~~ 4.2 : (ZA $w=1$)

$$✓ \quad \sum_{i=1}^{7} \sum_{j=1}^{7} \frac{w}{2E} \, ld \frac{w}{2E} = \frac{w_{12}}{2E} \, ld \frac{w_{12}}{2E} + \frac{w_{15}}{2E} \, ld \frac{w_{15}}{2E} +$$

$$+ \frac{w_{21}}{2E} \, ld \frac{w_{21}}{2E} + \frac{w_{25}}{2E} \, ld \frac{w_{25}}{2E} + \frac{w_{23}}{2E} \, ld \frac{w_{23}}{2E} + \frac{w_{32}}{2E} \, ld \frac{w_{32}}{2E} +$$

$$\frac{w_{35}}{2E} \, ld \frac{w_{35}}{2E} + \frac{w_{34}}{2E} \, ld \frac{w_{34}}{2E} + \frac{w_{43}}{2E} \, ld \frac{w_{43}}{2E} + \frac{w_{45}}{2E} \, ld \frac{w_{45}}{2E} +$$

$w=1$

$$+ \frac{w_{51}}{2E} \, ld \frac{w_{51}}{2E} + \frac{w_{52}}{2E} \, ld \frac{w_{52}}{2E} + \frac{w_{53}}{2E} \, ld \frac{w_{53}}{2E} + \frac{w_{54}}{2E} \, ld \frac{w_{54}}{2E} = \cancel{2E} \cdot \frac{①}{\cancel{2E}} \, ld \frac{1}{2E}$$

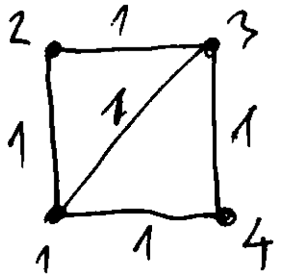
$$\text{FOR } W = 1 \rightarrow -\sum_i \sum_j \frac{w_{ij}}{2W} \, ld \frac{W_{ij}}{2W} = ld \, 2E$$

⇛ FOR EQUAL EDGES :

$$\boxed{H(X) = \underbrace{ld(2E)}_{\text{PROOF PP.83}} - \underbrace{H\left(\frac{E_1}{2E}, \frac{E_2}{2E}, \dots \frac{E_4}{2E}\right)}_{\text{PROOF PP.84}}}$$

$$-H\left(\frac{E_1}{2E}, \frac{E_2}{2E}, \dots \frac{E_4}{2E}\right) = \frac{E_1}{2E}\, ld \frac{2E}{E_1} + \frac{E_2}{2E}\, ld \frac{2E}{E_2} + \dots + \frac{E_4}{2E}\, ld \frac{2E}{E_4}$$

$$\sum_i \sum_j \frac{W_{ij}}{2W}\, ld \frac{W_i}{2W} = \sum_{i=1}^{4} \sum_{j=1}^{4} \frac{1}{2E}\, ld \frac{W_i}{2E} =$$



$$= \frac{w_{12}}{2E}\, ld \frac{W_1}{2E} + \frac{w_{14}}{2E}\, ld \frac{w_1}{2E} +$$

$$+ \frac{w_{13}}{2E}\, ld \frac{W_1}{2E} + \frac{w_{21}}{2E}\, ld \frac{w_2}{2E} +$$

$$+ \frac{W_{23}}{2E}\, ld \frac{w_2}{2E} + \frac{w_{31}}{2E} \cdot ld \frac{w_3}{2E} + \underset{=w_1}{\frac{w_{32}}{2E}}\, ld \frac{w_3}{2E} + \frac{w_{34}}{2E}\, ld \underset{=w_2}{\frac{w_3}{2E}}$$

$$\boxed{w_{ij}=1}$$

$$+ \underset{=w_3}{\frac{w_{41}}{2E}}\, ld \frac{w_4}{2E} + \underset{=w_4}{\frac{w_{42}}{2E}}\, ld \frac{w_4}{2E} = ③ \frac{1}{2E}\, ld \frac{\textcircled{w_1}}{2E} + ② \frac{1}{2E}\, ld \frac{w_2}{2E}$$

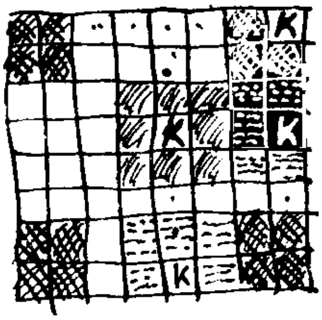
$$+ ③ \frac{1}{2E}\, ld \frac{w_3}{2E} + ② \frac{1}{2E}\, ld \frac{w_4}{2E} = \frac{w_1}{2E}\, ld \frac{w_1}{2E} + \frac{w_2}{2E}\, ld \frac{w_2}{2E} +$$

$$+ \frac{w_3}{2E}\, ld \frac{w_3}{2E} + \frac{w_4}{2E}\, ld \frac{w_4}{2E} = \sum_{i=1}^{4} \frac{w_i}{2E}\, ld \frac{w_i}{2E} =$$

$$= -H\left(\frac{w_1}{2E}, \frac{w_2}{2E}, \frac{w_3}{2E}, \frac{w_4}{2E}\right)$$

DOKAZANO!!!

EXAMPLE 4.3.1 (RANDOM WALK ON A CHESSBOARD) LET A KING MOVE AT RANDOM ON 8×8 CHESSBOARD.



$$H(X) = ld\, 2E - H\left(\frac{E_1}{2E}, \frac{E_2}{2E}, \dots, \frac{E_4}{2E}\right)$$

i = ▨ — INTERIOR 4 KOMBINACII VNATRE

e = ▤ — EDGE 2 KOMBINACII PO SEKCIJA VKUPNO 2·4 = 8 · KOMBINACII

c = ▩ — CORNER 4×1 KOMBINACIJA = 4

$\underbrace{\qquad}$ PATEKI ≜ EDGE W_{ij}

VKUPNIOT BROJ NA EDGES (W_{ij}) : ⓘ 4×8 = 32 EDGES

ⓔ 8×5 = 40 EDGES ⓒ 4×3 = 12 EDGES

W = E = 32 + 40 + 12 = 72 + 12 = 84 EDGES

Ⓑ ?? 11.85.

$H(x) = ld\,168 + \underline{4\left(\frac{8}{168}\, ld \frac{8}{168}\right)} + 8\left(\frac{5}{168}\, ld \frac{5}{168}\right) + 4\left(\frac{3}{168}\, ld \frac{3}{168}\right)$

$\mu_i = \frac{W_i}{2W}$ $\mu_i = \frac{8}{168}$ $\mu_e = \frac{5}{168}$ $\mu_c = \frac{3}{168}$

---

① 36 POZICII (ZADACI) × 8 LINKA = 288 LINKA

$\frac{36 \cdot 8}{288}$

ⓔ 24 POZICII (ZADACI) × 5 LINKA = 120 LINKA

ⓒ 4 POZICII × 3 LINKA = 12 LINKA

$2E$ = TOTAL = 420 LINKA

OVA E PRAVILO A NE ($)

076105

ZNAČI:

$\underbrace{\mu_i = \frac{8}{420} \quad \mu_e = \frac{5}{420} \quad \mu_c = \frac{3}{420}}_{\text{STACIONARNITE VEROJATNOSTI}}$

$H(x) = ld\,420 + 36\,\frac{8}{420}\, ld \frac{8}{420} + 24\,\frac{5}{420}\, ld \frac{5}{420} + 4\,\frac{3}{420}\, ld \frac{3}{420}$

$= \frac{1}{35}\left(72\, ld\,2 + ld\,3 + 10\, ld\,5\right) = \frac{1}{35}\, ld\left(2^{72} \cdot 3 \cdot 5^{10}\right) =$

$= \cancel{ld\,420} + \frac{288}{420}\, ld\,8 - \cancel{\frac{288}{420}\, ld\,420} + \frac{120}{420}\, ld\,5 - \cancel{\frac{120}{420}\, ld\,420} +$

$\frac{12}{420}\, ld\,3 - \cancel{\frac{12}{420}\, ld\,3} = \frac{288}{420}\, ld\,8 + \frac{120}{420}\, ld\,5 + \frac{12}{420}\, ld\,3$

$= \frac{288}{420} \cdot 3 + \frac{2}{7}\, ld\,5 + \frac{2}{70}\, ld\,3 = \frac{144}{210} \cdot 3 + \frac{2}{7}\, ld\,5 + \frac{1}{35}\, ld\,3$

$= \frac{72}{35} + \frac{2}{7}\, ld\,5 + \frac{1}{35}\, ld\,3 = \frac{72}{35} + ld\left(5^{\frac{2}{7}} \cdot 3^{\frac{1}{35}}\right) = 2.76584$

$= \frac{70}{35} + \frac{2}{35}\, ld\,2 + \frac{10}{35}\, ld\,5 + \frac{1}{35}\, ld\,3 = 2 + \frac{1}{35}\, ld\left(4 \cdot 5^{10} \cdot 3\right) =$

$= 2 + \frac{1}{35}\, ld\,12 + \frac{10}{35}\, ld\,5 = 2 + \frac{1}{35}\, ld\,12 + \frac{2}{7}\, ld\,5 =$

$= 2 + \frac{1}{35}\, ld\left(12 \cdot 5^{10}\right) = \boxed{0.32 \cdot ld\,8} \Rightarrow$ VO KNIGATA GO DOVEDUVAAT DO OVAA FORMA.

DADOT NA ZADACI

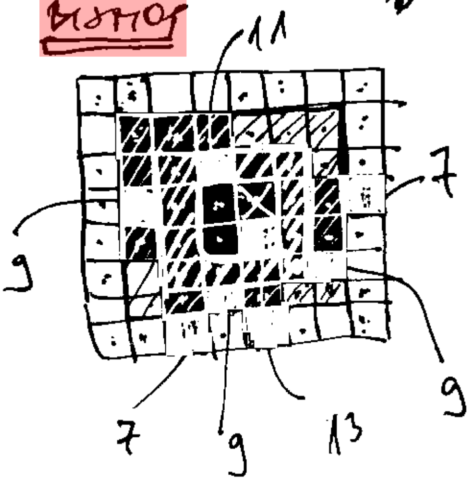
- ENTROPY RATE OF THE ROOKS

$\mu = \frac{14}{2W}$ $2W = 14 \cdot 8^2 = 14 \cdot 64 = 896$

$\mu = \frac{14}{896}$ $H(x) = ld\,896 + 64 \cdot \frac{14}{896} \cdot ld \frac{14}{896}$

$= ld\,896 \cdot \frac{14}{896} = ld \frac{896}{14} = 3.8$

- **BISHOP**



I: 7 - MOVES, 28 NODES (I-V KONCENTRIČEN KVADRAT)

II: 9 - MOVES, 12 + 8 = 20 NODES

III: 11 - MOVES, 8 + 4 = 12 NODES

IV: 13 - MOVES, 4 NODES

$$\begin{array}{r} 11 \cdot 12 \\ \hline 22 \\ 11 \\ \hline 132 \end{array}$$

$$2E = 7 \cdot 28 + 9 \cdot 20 + 11 \cdot 12 + 13 \cdot 4 = 196 + 180 + 132 + 52$$

$$= 376 + 184 = \underline{\underline{560}} \text{ LINKS}$$

$$\mu_I = \frac{7}{560} \quad \mu_{II} = \frac{9}{560} \quad \mu_{III} = \frac{11}{560} \quad \mu_{IV} = \frac{13}{560}$$

$$H(x) = ld\, \overset{560}{\cancel{2E}} + 28 \cdot \frac{7}{560} ld \frac{7}{560} + 20 \cdot \frac{9}{560} ld \frac{9}{360} +$$

$$+ 12 \cdot \frac{11}{360} ld \frac{11}{560} + 4 \cdot \frac{13}{560} ld \frac{13}{560}$$

$$= \frac{196}{560} ld\, 7 + \frac{180}{560} ld\, 9 + \frac{132}{560} ld\, 11 + \frac{52}{560} ld\, 13$$

$$= \frac{1}{560} \left(196\, ld\, 7 + 180\, ld\, 9 + 132\, ld\, 11 + 52\, ld\, 13 \right)$$

$$= 3.16053$$

- **QUEEN**

I: 7 + 14 = 21 MOVES, 28 NODES

II: 9 + 14 = 23 MOVES ; 20 NODES

III: 11 + ~~14~~ 14 = 25 MOVES ; 12 NODES

IV: 13 + 14 = 27 MOVES ; 4 NODES

$$\begin{array}{r} 21 \cdot 28 \\ \hline 168 \\ 42 \\ \hline 588 \end{array} \qquad \begin{array}{r} 25 \cdot 12 \\ \hline 50 \\ 25 \\ \hline 300 \end{array}$$

$$2E = 21 \cdot 28 + 23 \cdot 20 + 25 \cdot 12 + 27 \cdot 4 = 588 + 460 + 300 + 108$$

$$= 1048 + 408 = 1456 \qquad \mu_I = \frac{21}{1456} \quad \mu_{II} = \frac{23}{1456} \quad \mu_{III} = \frac{25}{1456}$$

$$\mu_{IV} = \frac{27}{1456} \qquad H(x) = ld\, 1436 + \boxed{28 \cdot \frac{21}{1456}} ld \frac{21}{1456} + 20 \frac{23}{1456} \cdot$$

$$\cdot\, ld \frac{27}{1456} + 12 \cdot \frac{25}{1456} ld \frac{25}{1456} + 4 \cdot \frac{27}{1456} ld \frac{27}{1456} =$$

$$= \frac{1}{1456} \left(588\, ld\, 21 + 460\, ld\, 23 + 300\, ld\, 25 + 108\, ld\, 27 \right) = \underline{4.5125}$$

KRALICATA IMA NAJGOLEMA ENTROPY RATE. MOE TOLKUVANJE E DEKA NAJNEIZVESNO E KADE ĆE SE NAJDE A SO TOA IMA NAJGOLEMA SLOBODA NA DVIŽENJE.

$\boxed{4.4.}$ SECOND LAW OF THERMODYNAMICS

ENTROPY OF AN ISOLATED SYSTEM IS NONDECREASING.

- IN STATISTICAL DYNAMICS ENTROPY IS „log" OF NUMBER OF MICROSTATES IN THE SYSTEM

① RELATIVE ENTROPY $D(\mu_n \| \mu_n')$ DECREASES WITH n.

LET μ_n & μ_n' BE TWO PROBABILITY DISTRIBUTIONS ON THE STATE SPACE OF MARKOV CHAIN AT TIME „n", AND μ_{n+1} AND μ'_{n+1} BE THE CORRESPONDING DISTRIBUTIONS AT TIME n+1. LET CORRESPONDING MASS FUNCTIONS BE DENOTED AS p & q. THUS:

$$p(x_n, x_{n+1}) = p(x_n)\, r(x_{n+1}|x_n) \qquad q(x_n, x_{n+1}) = q(x_n)\, r(x_{n+1}|x_n)$$

WHERE $r(\circ|\circ)$ IS PROBABILITY TRANSITION FUNCTION FOR MARKOV CHAIN.

$$D\big(p(x_n, x_{n+1}) \| q(x_n, x_{n+1})\big) = D\big(p(x_n) \| q(x_n)\big) +$$

$$+ \cancelto{0}{D\big(p(x_{n+1}|x_n) \| q(x_{n+1}|x_n)\big)} = \underline{D\big(p(x_{n+1}) \| q(x_{n+1})\big)} +$$

$$+ D\big(p(x_n|x_{n+1}) \| q(x_n|x_{n+1})\big)$$

p & q ARE DERIVED FROM SAME CHAIN ⇒

$$\boxed{p(x_{n+1}|x_n) = q(x_{n+1}|x_n) = r(x_{n+1}|x_n)}$$

$$D\big(p(x_{n+1}|x_n) \| q(x_{n+1}|x_n)\big) = 0$$

$$D\big(p(x_n) \| q(x_n)\big) \geq D\big(p(x_{n+1}) \| q(x_{n+1})\big) \quad \text{i.e.}$$

$$D(\mu_n \| \mu_n') \geq D(\mu_{n+1} \| \mu'_{n+1})$$

CONSEQUENTLY THE DISTANCE BETWEEN THE PROBABILITY MASS FUNCTIONS IS DECREASING WITH TIME „n" FOR ANY MARKOV CHAIN. **PRIMER:** AKO SE PRETPOSTAVI DEKA DANOCNIOT SISTEM ZA REDISTRIBUCIJA NA SOPSTVENOST E IST VO UK I CANADA, SO TEK NA VREME DISTRIBUCIJATA NA BOGATSTVOTO VO TIE DVE ZEMJI ĆE STANE SLIČNA.

② RELATIVE ENTROPY $D(\mu_n \| \mu)$ BETWEEN A DISTRIBUTION μ_n ON THE STATES AT TIME „n" AND A STATIONARY DISTRIBUTION μ DECREASES WITH „n"

e.g $\mu_n' = \mu$ $\qquad D(\mu_n \| \mu) \geq D(\mu_{n+1} \| \mu)$

WHICH MAKES THAT ANY STATE DISTRIBUTION CONVERGES TOWARDS STATIONARY DISTRIBUTION AS TIME PASSES.

③ ENTROPY INCREASES IF STATIONARY DISTRIBUTION IS UNIFORM.

IN GENERAL, THE FACT THAT RELATIVE ENTROPY DECREASES DOESN'T IMPLY THAT ENTROPY INCREASES.

- IF WE START THE MARKOV CHAIN FROM UNIFORM DISTRIBUTION, THE DISTRIBUTION WILL TEND TO THE STATIONARY DISTRIBUTION, HENCE ENTROPY DECREASES WITH TIME.
- IF STATIONARY DISTRIBUTION IS UNIFORM

$$D(\gamma\|q) = \sum_{i=1}^{n} \gamma \,\mathrm{ld}\frac{\gamma}{q} = \sum_{i=1}^{n} \gamma \,\mathrm{ld}\frac{\gamma}{\frac{1}{|2|}} = \mathrm{ld}|2| + \sum_{i=1}^{n} \gamma \,\mathrm{ld}\,\gamma$$

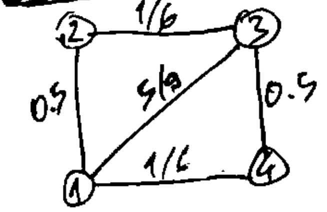
$$\boxed{D(\gamma\|2) = \mathrm{ld}|2| - H(\gamma)}$$

$$D(\mu_n\|\mu) = \mathrm{ld}|X| - H(\mu_n) = \underbrace{\mathrm{ld}|X| - H(\gamma_n)}$$

↙

SINCE FOR MARKOV CHAIN WE HAVE MONOTONIC DECREASE OF RELATIVE ENTROPY IT IMPLIES MONOTONIC INCREASE OF ENTROPY. THIS EXPLANATION TIES MOST CLOSELY TO STATISTICAL TERMODYNAMIC WHERE ALL MICROSTATES ARE EQUALLY LIKELY.

- POTTORNO ODIO DA DIZOZIRAM MARKOV PROCESS (Ex.2)



$\epsilon = 5$ $\quad 2\epsilon = 10$

$\mu_i = ?$ $\quad \mu_1 = \frac{3}{10}$ $\quad \mu_2 = \frac{2}{10}$ $\quad \mu_3 = \frac{3}{10}$ $\quad \mu_4 = \frac{2}{10}$

$$\boxed{P_{ij} = 0.2}$$

$$\mu_j = \sum_i \mu_i \cdot P_{ij}$$

$j = 2$

$$\mu_2 = \mu_1 \cdot P_{12} + \mu_3 \cdot P_{32} = \frac{3}{10}\cdot 0.2 + \frac{3}{10}\cdot 0.2 = 0.2\cdot 2\cdot\frac{3}{10} = 0.4\cdot\frac{3}{10} = \frac{1.2}{10} \quad ?$$

$$\mu_2 = \mu_1 \cdot P + \mu_3 \cdot P = \frac{3}{10}\cdot P \qquad P = \frac{10}{3}\cdot\mu_2 = \frac{\cancel{10}}{3}\cdot\frac{2}{\cancel{10}} = \frac{2}{3}$$

$$\mu_3 = \mu_1 \cdot P + \mu_2 \cdot P + \mu_4 \cdot P = \left(\frac{3}{10} + \frac{4}{10}\right)\cdot P = \frac{7}{10}\cdot\frac{2}{3} = \frac{14}{30}$$

$$[\mu_1\ \mu_2\ \mu_3\ \mu_4] = [\mu_1\ \mu_2\ \mu_3\ \mu_4] \begin{array}{c|cccc} 1 & 0 & 0.2 & 0.2 & 0.2 \\ 2 & 0.2 & 0 & 0.2 & 0 \\ 3 & 0.2 & 0.2 & 0 & 0.2 \\ 4 & 0.2 & 0 & 0.2 & 0 \end{array}$$

$\mu_1 = 0.2(\mu_2 + \mu_3 + \mu_4)$ $\mu_3 = 0.2(\mu_1 + \mu_2 + \mu_4)$

$\mu_2 = 0.2(\mu_1 + \mu_3)$ $\mu_4 = 0.2(\mu_1 + \mu_2)$

$$5 \cdot \mu_1 = 0.2(\mu_1 + \mu_3) + \mu_3 + 0.2(\mu_1 + 0.2(\mu_1 + \mu_3)) =$$

$$= \underline{0.2\mu_1} + \underline{0.2\mu_3} + \underline{\mu_3} + \underline{0.2\mu_1} + \underline{0.04\mu_1} + \underline{0.04\mu_3} =$$

$$= (2 \cdot 0.2 + 0.04)\mu_1 + (1.2\mu_3 + 0.04\mu_3) = 0.44\mu_1 + 1.24\mu_3$$

$$\boxed{4{,}56\mu_1 = 1.24\mu_3}$$

$$[0.3, 0.2, 0.3, 0.2] = [0.3, 0.2, 0.3, 0.2] \begin{bmatrix} P_{11}, P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \\ P_{41} & P_{42} & P_{43} & P_{44} \end{bmatrix}$$

$$= [0.3, 0.2, 0.3, 0.2] \begin{bmatrix} 0 & \underline{P_{12}} & \underline{P_{13}} & \underline{P_{14}} \\ P_{12} & 0 & \underline{P_{23}} & 0 \\ P_{13} & P_{23} & 0 & \underline{P_{34}} \\ P_{14} & 0 & P_{34} & 0 \end{bmatrix} =$$

\\\

$$\frac{18}{10} = \frac{9}{5}$$

$$= [0.3, 0.2, 0.3, 0.2] \begin{bmatrix} 0 & P_1 & P_2 & P_3 \\ P_1 & 0 & P_3 & 0 \\ P_2 & P_3 & 0 & P_4 \\ P_3 & 0 & P_4 & 0 \end{bmatrix}$$

$0.3 = 0.2P_1 + 0.3P_2 + 0.2P_3$ $0.3 = 0.3P_2 + 0.2P_3 + 0.2P_4$

$0.2 = \underline{0.3P_1 + 0.3P_3}$ $0.2 = 0.3P_3 + 0.3P_4$

$P_1 = P_4$, $P_2 = 0.556$, $P_3 = 0.67 - P_4$, } MAKE

$$\boxed{P_4 = 0.5 \quad P_1 = 0.5;\ P_2 = 0.556 \quad P_3 = 0.17}$$

$$\begin{array}{r} 0.15 \\ 0.051 \\ \hline 0.02 \\ \hline \end{array}$$

$0.3 \cdot P_1 + 0.3P_3 = 0.5 \cdot 0.3 + 0.3 \cdot 0.17 = 0.15 + 0.051$

$$\boxed{P_1 = 0.5;\ P_2 = \frac{5}{9};\ P_3 = \frac{1}{6};\ P_4 = 0.5}$$

$x^4 = \{x_1, x_2, x_3, x_4\}$

$P(x_2|x_1) = 0.5$; $P(x_3|x_2) = 1/6$; $P(x_3|x_1) = 5/9$;

$P(x_4|x_1) = 1.6$; $P(x_3|x_4) = 0.5$

$$P_{ij} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0.5 & 5/9 & 1/6 \\ 2 & 0.5 & 0 & 1/6 & 0 \\ 3 & 5/9 & 1/6 & 0 & 0.5 \\ 4 & 1/6 & 0 & 0.5 & 0 \end{array}$$

$\mu = [0.3, 0.2, 0.3, 0.2]$

$$H(x) = -\sum_i \mu_i \sum_j P_{ij} \operatorname{ld} P_{ij}$$

$$\boxed{H(x) = 1.2135}$$ NAPRE / MATLAB

x^4 : $X_1 = \{1, 0, 0, 0\}$ $\quad x_3 = \{0, 0, 1, 0\}$

$X_2 = \{0, 1, 0, 0\}$ $\quad x_4 = \{0, 0, 0, 1\}$

$$p(x_1) = \mu_1 (1-\mu_2)(1-\mu_3)(1-\mu_4) = 0.3 \cdot 0.8 \cdot 0.7 \cdot 0.8 = 0.1344$$

$$p(x_2) = \mu_2 (1-\mu_1)(1-\mu_3)(1-\mu_4) = 0.2 \cdot 0.7 \cdot 0.7 \cdot 0.8 = 0.0784$$

$$p(x_3) = \mu_3 (1-\mu_1)(1-\mu_2) \cdot (1-\mu_4) = 0.3 \cdot 0.7 \cdot 0.8 \cdot 0.8 = 0.1344$$

$$p(x_4) = \mu_4 (1-\mu_1)(1-\mu_2)(1-\mu_3) = 0.2 \cdot 0.7 \cdot 0.8 \cdot 0.7 = 0.0784$$

$$H(x_1, x_2, x_3, x_4) = -\sum_{i=1}^{4} p(x_i) \operatorname{ld} p(x_i) = \underline{\underline{0.93867}}$$

$$H(x) = \frac{1}{4} H(x_1, x_2, x_3, x_4) = \frac{1.35421}{4} = 0.23467$$

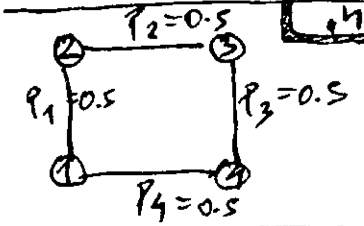
VAŽI NE DAVA PONE BROJKA. VIDI PP72 KADE OBAN SO u=4 u=1

NAPRAVI

• AKO SE ZEME $p(x_i) = \mu_i$ TOGAŠ

$$-\sum_{i=1}^{4} \mu_i \operatorname{ld} \mu_i = \underline{\underline{1.97095}} \qquad \frac{H(x_1, x_2, x_3, x_4)}{4} = \frac{1.97095}{4} = \underline{\underline{0.5}}$$

~~μ^4 = MODES & μ^4 LINKS CHANGE~~



$\mu_1 = 0.25$ $\quad \mu_2 = 0.25$ $\quad \mu_3 = 0.25$ $\quad \mu_4 = 0.25$

$$[0.25\ \ 0.25\ \ 0.25\ \ 0.25] \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \\ P_{41} & P_{42} & P_{43} & P_{44} \end{bmatrix} =$$

MOJOV PRIMER E IZGLEDA KEJS KADE PROCESOT E POVEĆE STOHASTIČ !!

$$= [0.25\ \ 0.25\ \ 0.25\ \ 0.25] \begin{bmatrix} 0 & P_1 & 0 & P_4 \\ P_1 & 0 & P_2 & 0 \\ 0 & P_2 & 0 & P_3 \\ P_4 & 0 & P_3 & 0 \end{bmatrix} = [0.25, 0.25, 0.25, 0.25]$$

$0.25 P_1 + 0.25 P_4 = 0.25$ $\quad 0.25 P_2 + 0.25 P_3 = 0.25$

$0.25 P_1 + 0.25 P_2 = 0.25$ $\quad 0.25 P_3 + 0.25 P_4 = 0.25$

$P_1 + P_4 = 1$ $\quad P_2 + P_3 = 1$

$P_1 + P_2 = 1$ $\quad P_3 + P_4 = 1$

- KVADRAT: $P_1 = P_3$ $P_2 = 1 - P_3$, $\underline{P_4 = 1 - P_3}$
- PETOAGOLNIK: $P_1 = 1 - P_3$; $P_2 = P_3$; $P_3 = 1 - P_3$; $P_4 = 1 - P_3$; $\underline{\underline{P_5 = P_3}}$

FOR $\boxed{P_3 = 0.5}$ $\boxed{P_1 = 0.5 \quad P_2 = 0.5 \quad P_3 = 0.5 \quad P_4 = 0.5}$

$\to P_3 = 0.5 \Rightarrow P_1 = 0.5 \quad P_2 = 0.5 \quad P_3 = 0.5, \; P_4 = 0.5, \; P_5 = 0.$

- ŠESTOAGOLNIK:

$P_1 = P_3$; $P_2 = 1 - P_3$; $P_3 = P_3$; $P_4 = 1 - P_3$; $P_5 = P_3$; $P_6 = 1 - P_3$

- PETOAGOLNIK

$$[\mu_1 \; \mu_2 \; \mu_3 \; \mu_4 \; \mu_5] \cdot \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 0.5 & 0 & 0 & 0.5 \\ 2 & 0.5 & 0 & 0.5 & 0 & 0 \\ 3 & 0 & 0.5 & 0 & 0.5 & 0 \\ 4 & 0 & 0 & 0.5 & 0 & 0.5 \\ 5 & 0.5 & 0 & 0 & 0.5 & 0 \end{array} = \underline{\underline{[0.2, 0.2, 0.2, 0.2, 0.2]}}$$

- TRIAGOLNIK

$$[\mu_1 \; \mu_2 \; \mu_3] \quad \begin{array}{ccc} 1 & 2 & 3 \\ 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{array} \quad \Rightarrow \quad \boxed{H(X) = 1}$$

$$-[\mu_1 \mu_2 \mu_3] * (P_{ij} .* \operatorname{ld} P_{ij}) = \textcircled{1}$$

SLUČAJ ZA NLO KORKAVA MATRICA

$$-\sum_{i=1}^{N} \frac{1}{N} \cdot \operatorname{ld} \frac{1}{N} = + \frac{\cancel{N}}{\cancel{N}} \cdot \operatorname{ld} N = \underline{\operatorname{ld} N} = H(x_1, x_2 \ldots x_N)$$

$$\frac{H(x_1 x_2, \ldots x_n)}{n} = \frac{\operatorname{ld} n}{n} \to 0$$

$$p(x_1) = \mu_1 (1 - \mu_2)(1 - \mu_3) \ldots (1 - \mu_n) = \underline{\underline{\mu (1 - \mu)^{n-1}}} = p(x_2) = \ldots = p(x_n)$$

$$-\sum_{i=1}^{N} p(x_i) \operatorname{ld} p(x_i) = -N \cdot p(x_1) \operatorname{ld} p(x_1) = -N \cdot \mu (1-\mu)^{N-1} \operatorname{ld} \mu (1-\mu)^{N-1}$$

NE TREBA DA RAZLIŽO N = NEO PROBLEM 4 TRETIRAJ KAKO SEKVENCU [illegible]

$$\lim_{N \to \infty} \frac{-N \mu (1-\mu)^{N-1} \operatorname{ld} \mu (1-\mu)^{N-1}}{\textcircled{N}} = \underline{\underline{-\lim_{N \to \infty} \mu (1-\mu)^{N-1} \operatorname{ld} \mu (1-\mu)^{N-1}}}$$

$$-\lim_{n \to \infty} \mu (1-\mu)^{n-1} \operatorname{ld} \mu \to 0 \quad 0 \le \mu \le 1$$

$$- \lim_{n \to \infty} \mu (1-\mu)^{n-1} \operatorname{ld} (1-\mu)^{n-1} = \left[\lim (n-1)(1-\mu)^{n-1}\right] \cdot \mu \cdot \operatorname{ld}(1-\mu)$$

$$\lim_{n\to\infty} p(1-p)^{n-1}\,\mathrm{ld}\,(1-p)^{n-1} = \lim_{n\to\infty} \frac{1}{n}\left(1-\frac{1}{n}\right)^{n-1}\mathrm{ld}\left(1-\frac{1}{n}\right)^{n-1} \to 0$$

$$= \lim_{n\to\infty} \frac{1}{n}\left(\frac{n-1}{n}\right)^{n-1}\mathrm{ld}\left(\frac{n-1}{n}\right)^{n-1} = \lim_{n\to\infty}\left(\frac{n-1}{n}\right)^{n}\cdot \cancel{\mathrm{ld}\left(\frac{n-1}{n}\right)}$$

$$\lim_{n\to\infty} \frac{1}{n}\left(1-\frac{1}{n}\right)^{n-1}\cdot \mathrm{ld}\,\frac{1}{n} = \lim_{n\to\infty}\frac{1}{n}\left(\frac{n-1}{n}\right)^{n-1}\mathrm{ld}\,\frac{1}{n} \to 0$$

- AKO SE ODI PO FORMULATA

$$H(X) = \mathrm{ld}\,2E - H\left(\frac{E_1}{2E}, \frac{E_2}{2E}, \dots \frac{E_n}{2E}\right) \qquad 4.667.501.082$$

$$= \underbrace{0.5\,\mathrm{ld}\,\frac{2E}{0.5} = 0.5\,\mathrm{ld}\,4E}_{\text{ZATOA ŠTO VO MODOT SLUČAJ}} \qquad w_j = 0.5 \quad E_1 = 2\cdot 0.5 = 1$$

- TRIAGOLNIK

$$H(X) = 0.5\,\mathrm{ld}\,4\cdot 3 + \cancel{3}\cdot\left(\frac{1}{\cancel{6}_2}\,\mathrm{ld}\,\frac{1}{6}\right) =$$

$$= 0.5\,\mathrm{ld}\,12 + 0.5\,\mathrm{ld}\,\frac{1}{6} = 0.5\,\mathrm{ld}\,4 + \cancel{0.5\,\mathrm{ld}\,6} - \cancel{0.5\,\mathrm{ld}\,6} \quad (?)$$

$$= 1 \;\; 0.5\,\mathrm{ld}\,3$$

- CETIRIAGOLNIK

$$H(X) = 0.5\,\mathrm{ld}(4\cdot 4) + \cancel{4}\left(\frac{\cancel{1}}{\cancel{8}_2}\,\mathrm{ld}\,\frac{1}{8}\right) = 0.5\,\cancel{\mathrm{ld}\,2}^{1} + \cancel{0.5\,\mathrm{ld}\,8} - \cancel{0.5\,\mathrm{ld}\,8}$$

$$= 0.5$$

NEMAV ZEMENO VO PREDVID DEKA $2E = 2\cdot N_E \cdot 0.5 = N_E$ NUMBER OF EDGES

$$H(X) = -2\cdot\cancel{N_E}\,\frac{0.5}{\cancel{N_E}}\,\mathrm{ld}\,\frac{0.5}{N_E} - H\left(\frac{E_1}{N_E}, \frac{E_2}{N_E}, \dots \frac{E_n}{N_E}\right) =$$

SUM OF WEIGHTS

$$= \mathrm{ld}\,2N_E - H\left(\frac{E_1}{N_E}, \frac{E_2}{N_E}, \dots \frac{E_n}{N_E}\right)$$

MAV

OK!!!

- TRIAGOLNIK

$$H(X) = \mathrm{ld}\,2\cdot 3 + \cancel{3}\,\frac{1}{\cancel{3}}\cdot\mathrm{ld}\,\frac{1}{3} = \mathrm{ld}\,2 + \cancel{\mathrm{ld}\,3} - \cancel{\mathrm{ld}\,3} = 1$$

- KVADRAT

$$H(X) = \mathrm{ld}\,2\cdot 4 + \cancel{4}\,\frac{1}{\cancel{4}}\,\mathrm{ld}\,\frac{1}{4} = \mathrm{ld}\,2 + \mathrm{ld}\,4 - \mathrm{ld}\,4 = 1$$

- PETOAGOLNIK

$$H(X) = \mathrm{ld}\,2\cdot 5 + 5\,\frac{1}{5}\,\mathrm{ld}\,\frac{1}{5} = \mathrm{ld}\,2 + \cancel{\mathrm{ld}\,5} - \mathrm{ld}\,5 = 1$$

PR. 90 ★ $x \in \{1, 2, 3, 4\}$ $\boxed{n=4}$ → DOLŽINA NA KODNATA AZBUKA.

- IZVOROT GENERIRA PO EDEN ZNAK SO DOLŽINA $\boxed{n=1}$

$$p(x) = \{p_1, p_2, p_3, p_4\} = \{0.1344, 0.0784, 0.1344, 0.0784\}$$

ENTROPIJATA NA SEKVENCATA E:

$$H(\mathcal{X}) = \frac{1}{4}H(\mathcal{X}_1) = \frac{1}{1}H(\mathcal{X}) = -\sum_{i=1}^{4} p_i \,\mathrm{ld}\, q_i = \underbrace{1.35421}_{\text{pp. 71}}$$ ★

$$\boxed{1.35421 \sim 1{:}2135}$$

ZNAČI ZA PRIMEROT 5 LINKA 4 NAZI DOBIVAM TOČKA NA TEOREMATA 4.24 T.E REZULTATOT DOBIEN TAKA ODGOVARA NA REZULTATOT DOBIEN SOGLASNO DEFINICIJATA ZA ENTROPY RATE. SEPAK ZA PRIMERITE „4" LINKA & „4" NODES NE USPEVAM DA GO DOKAŽAM TOA.

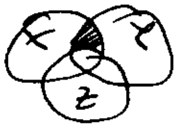
DEFINITION A PROBABILITY TRANSITION MATRIX $[P_{ij}]$, $P_{ij} = \Pr\{X_{n+1}=j \mid X_n = i\}$, IS CALLED DOUBLY STOCHASTIC IF

$$\sum_i P_{ij} = 1 \quad j=1,2,3,\ldots \quad \text{AND} \quad \sum_j P_{ij} = 1 \quad i=1,2,3\ldots$$

REMARK: THE UNIFORM DISTRIBUTION IS STATIONARY DISTRIBUTION OF P IF AND ONLY IF THE PROBABILITY TRANSITION MATRIX IS DOUBLY STOCHASTIC (PROBLEM 4.1)

④ THE CONDITIONAL ENTROPY $H(X_n|X_1)$ INCREASES WITH „n" FOR A STATIONARY MARKOV PROCESS. IF THE MARKOV PROCESS IS STATIONARY, $H(X_n)$ IS CONSTANT, SO THE ENTROPY IS NONINCREASING, HOWEVER $H(X_n|X_1)$ IS INCREASING WITH „n". THUS THE CONDITIONAL UNCERTAINTY OF THE FUTURE INCREASES.

$$H(X_n|X_1) \geq \underline{H(X_n|X_1X_2)} \text{ (CONDITIONING REDUCES ENTROPY)} =$$



$$= \underbrace{H(X_n|X_2)}_{\text{BY MARKOVITY}} = \underbrace{H(X_{n-1}|X_1)}_{\text{BY STATIONARITY}}$$

$$\boxed{H(X_n|X_1) \geq H(X_{n-1}|X_1)}$$

LOGIČNO MI E!! DO TEK NA VREME NEIZVESNOSTA RASTE.

- ALTERNATIVELY BY DATA-PROCESSING INEQUALITY

$$I(X;Y|Z) = H(X|Z) - \underbrace{H(X|Z,Y)}_{H(X|Y)} = H(Y|Z) - H(Y|X,Z)$$

$$I(X;Y|Z) = -I(X;\cancel{Z}) + I(Z,Z;X)$$

$$I(X,Z;Y) = I(X;Y) + I(Z;Y|X) = \underline{I(Z;Y) + I(X;Y|Z)}$$

$$I(X,Y;Z) = \underline{I(X;Z)} + I(Y;Z|X) = \underline{I(Y;Z)} + \underbrace{I(X;Z|Y)}_{0}$$

$$\boxed{I(Y;Z) \geq I(X;Z)}$$

$$I(X;YZ) = I(X;Y) + \underbrace{I(X;Z|Y)}_{0} = I(X;Z) + \underbrace{I(X;Y|Z)}_{\geq 0}$$

$$\boxed{I(X;Y) \geq I(X;Z)} \longrightarrow \text{DATAPROCESSING INEQUALITY}$$

$$X_1 \rightarrow X_{n-1} \rightarrow X_n$$

$$I(X_1; X_{n-1}) \geq I(X_1; X_n)$$

$$\boxed{H(X_1|X_{n-1}) \leq H(X_1|X_n)}$$

$$\cancel{H(X_1)} - H(X_1|X_{n-1}) \geq$$
$$\cancel{H(X_1)} - H(X_1|X_n)$$
$$\boxed{H(X_1|X_n) \geq H(X_1|X_{n-1})}$$

-1 OBRATNO:

$$H(X_{n-1}) - H(X_{n-1}|X_1) \geq H(X_n) - H(X_n|X_1).$$

$$\binom{3}{2} = \frac{6}{2} = 3$$

$$\boxed{\begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{matrix}}$$

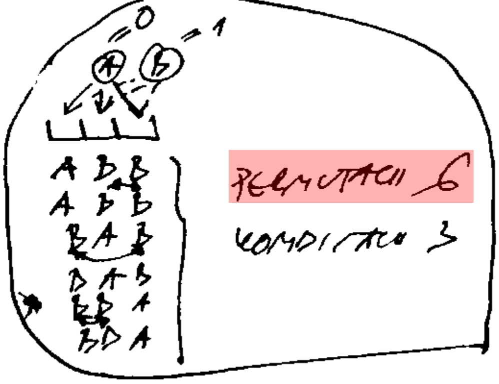
BY STATIONARITY $\boxed{H(X_n) = H(X_{n-1})}$ $\Rightarrow$

$$H(X_{n-1}|X_1) \leq H(X_n|X_1) \quad - \quad \boxed{H(X_n|X_1) \geq H(X_{n-1}|X_1)}$$

⑤ SHUFFLES INCREASE ENTROPY. IF T IS A SHUFFLE (PERMUTATIONS) OF A DECK OF CARDS AND X IS THE INITIAL (RANDOM) POSITION OF THE CARDS IN THE DECK, AND IF THE CHOICE OF THE SHUFFLE T IS INDEPENDENT OF X, THEN

$$H(TX) \geq H(X)$$

WHERE TX IS PERMUTATION OF THE DECK INDUCED BY THE SHUFFLE T ON THE INITIAL PERMUTATION X. PROBLEM 4.3 OUTLINES THE PROOF.



4.5 FUNCTIONS OF MARKOV CHAINS

LET $X_1, X_2, \ldots X_n$ BE STATIONARY MARKOV CHAIN, AND LET $Y_i = \phi(X_i)$ BE A PROCESS EACH TERM OF WHICH IS A FUNCTION OF THE CORRESPONDING STATE IN THE MARKOV CHAIN. WHAT IS $H(Y)$? IT WOULD SIMPLIFY MATTERS IF $Y_1, Y_2, \ldots Y_n$ ALSO FORMED MARKOV CHAIN (MC), BUT IN MANY CASES IT IS NOT TRUE. SINCE MC IS STATIONARY SO IS $Y_1, Y_2, \ldots Y_n$

$$H(Y_n|Y_{n-1}, Y_{n-2}, \ldots Y_1) \rightarrow H(Y)$$

FOR STATIONARY PROCESSES $H'(Y) = H(Y)$ TH. 4.2.1

- FOR LOWER BOUND WE WILL USE

$$H(Y_n|Y_{n-1}, \ldots, Y_1, X_1)$$

BASED ON THE IDEA THAT X_1 CONTAINS AS MUCH INFORMATION FOR Y_n AS $Y_1, Y_0, Y_{-1}, \ldots$

LEMMA 4.5.1 $\boxed{H(Y_n|Y_{n-1}, \ldots Y_2, X_1) \leq H(Y)}$

PROOF: for $k=1,2,\dots$ $H(Y_n|Y_{n-1},\dots Y_2,X_1) \overset{\text{SINCE } Y_1=f(X_1)}{=\!=}$

$H(Y_n|Y_{n-1},\dots Y_2,Y_1,X_1) = ⊛$

$$\boxed{H(Y,Z|X) = H(Y|X) + H(Z|X,Y)}$$

$Y=f(X)$ $\qquad H(Y|X)=0$

$$\boxed{H(\overset{Y_n\ X_1\ Y_1}{Z|X,Y}) = H(Y,Z|X) - \underbrace{H(Y|X)}_{0}}$$

$H(Z,Y|X) = H(Z|X) + \underbrace{H(Y|\otimes,Z)}_{0} = H(Z|X) + \cancel{H(Y|Z)} \rightarrow 0$

AKO GO ZNAEŠ X GO ZNAEŠ Y $\}0$

$\rightarrow H(Z|X,Y) = \underline{H(Z|X)}$ [illegible] $= H(Y,Z|X)$

$H(Y_n|X_1,Y_1) = \underline{H(Y_n|\bigcirc X_1)} + H(Y_1|Y_n)$

$\rightarrow H(Y_n|X_1) = H(Y_1,Y_n|X_1) - \cancel{H(Y_1|X_1,Y_n)} \rightarrow 0$

AKO GO ZNAEŠ X1 GO ZNAEŠ I Y1

IL DIREKTNO $= H(Y_n|X_1 Y_1)$

$H(Y_1,Y_n|X_1) = H(Y_1|X_1) + H(Y_n|X_1 Y_1) =$

$\rightarrow H(Y_n,Y_1|X_1) = H(Y_n|X_1) + \underbrace{H(Y_1|X_1,Y_n)}_{0} = \underline{\underline{H(Y_n|X_1)}}$

$H(Y_n|X_1,Y_1) = -H(Y_n|X_1) + H(X_1,Y_n|Y_1)$

USTE EDNAŠ: $\boxed{Y_1=f(X_1)}$ $\quad H(Y_n|X_1,Y_1) = H(Y_1,Y_n|X_1) - \underbrace{H(Y_1|X_1)}_{0}$ ←

SO ROTIRANJE NA ČLENOVITE:

$H(Y_n,Y_1|X_1) = H(Y_n|X_1) + \underbrace{H(Y_1|X_1,Y_n)}_{0}$ AKO GO ZNAEŠ X1 GO ZNAEŠ Y1

$$\Rightarrow \boxed{H(Y_n|X_1,Y_1) = H(Y_n|X_1)} \text{ IF } Y_1=f(X_1)$$

MMV

$⊛ = H(Y_n|Y_{n-1},\dots Y_1,\underbrace{X_1,X_0,X_{-1},\dots X_{-k}}_{\text{DUE TO MARKOVITY OF } X}) =$

$= H(Y_n|Y_{n-1},\dots Y_1,X_1,X_0,X_{-1},\dots,X_{-k},\underbrace{Y_0,Y_{-1},\dots,Y_{-k}}_{\text{SINCE } Y_i=f(X_i)}) \underset{\text{CONDITIONING REDUCES ENTROPY}}{\leq}$

$\leq H(Y_n|Y_{n-1},\dots Y_1,Y_0,Y_{-1},\dots Y_{-k}) \underset{\text{STATIONARITY } +k+1}{=} H(Y_{n+k+1}|\underbrace{Y_{n+k},\dots Y_1}_{\text{FOR ALL } k})$ ➡

$H(Y_n|Y_{n-1},\dots,Y_2,X_1) \leq \lim_{k\to\infty} H(Y_{n+k+1}|Y_{n+k},\dots Y_1) = H(Y)$

LEMMA 4.5.2 $\boxed{H(Y_n|Y_{n-1},\dots Y_1) - H(Y_n|Y_{n-1},\dots Y_1,X_1) \to 0}$

PROOF: $H(Y_n|Y_{n-1},\dots,Y_1) - H(Y_n|Y_{n-1},\dots,Y_1,X_1) =$

$= I(X_1; Y_n | Y_{n-1},\dots Y_1)$

$$I(X_1; \underbrace{Y_1, Y_2, \dots Y_n}_{Y}) = I(X_1;Y_1) + I(X_1;Y_2|Y_1) + \dots + I(X_1;Y_n|Y_{n-1},\dots Y_1) = \sum_{i=1}^{n} I(X_1;Y_i|Y_{i-1},\dots Y_1)$$

$I(X_1;Y) = H(X_1) - H(X_1|Y)$ $\quad I(X_1;Y) \le H(X_1)$

$$\boxed{I(X_1;Y_1,Y_2\dots Y_n) \le H(X_1)}$$

$I(X_1;Y_1,Y_2,\dots Y_n)$ ⇒ INCREASES WITH „n" THUS $\lim_{n\to\infty} I(X_1;Y_1,\dots Y_n)$ EXISTS AND

$$\boxed{\lim_{n\to\infty} I(X_1;Y_1,Y_2,\dots Y_n) \le H(X_1)}$$

$$H(X_1) \ge \lim_{n\to\infty} I(X_1;Y_1Y_2,\dots Y_n) = \lim_{n\to\infty} \sum_{i=1}^{n} I(X_1;Y_i|Y_{i-1},\dots Y_1)$$

$$\boxed{H(X_1) \ge \sum_{i=1}^{\infty} I(X_1;Y_i|Y_{i-1},\dots Y_1)}$$

SINCE THIS INFINITE SUM IS FINITE AND THE TERMS ARE NON-NEGATIVE, TERMS MUST TEND TO „0", i.e.:

$\lim_{n\to\infty} I(X_1;Y_n|Y_{n-1},\dots Y_1) = 0$ ZNAČI:

$H(Y_n|Y_{n-1},\dots Y_1) - H(Y_n|Y_{n-1},\dots Y_1,X_1) \to 0$

DOKAZANO!!!

- COMBINING LEMA 4.5.1 AND 4.5.2 WE HAVE:

THEOREM 4.5.1 IF $X_1, X_2, \dots X_n$ FORM A STATIONARY MARKOV CHAIN, AND $Y_i = \phi(X_i)$, THEN:

$$\boxed{H(Y_n|Y_{n-1},\dots,Y_1,X_1) \le H(Y) \le H(Y_n|Y_{n-1},\dots Y_1)}$$

AND: $\lim_{n\to\infty} H(Y_n|Y_{n-1},\dots Y_1,X_1) = H(Y) = \lim_{n\to\infty} H(Y_n|Y_{n-1},\dots Y_1)$

← IN GENERAL WE COULD ALSO CONSIDER CASE WHERE Y_i IS STOCHASTIC FUNCTION (AS OPPOSED TO A DETERMINISTIC FUNCTION OF X_i.

CONSIDER A MARKOV PROCESS $X_1, X_2, \dots X_n$ AND DEFINE A NEW PROCESS $Y_1, Y_2, \dots, Y_n$, WHERE EACH Y_i IS DRAWN ACCORDING $p(y_i|x_i)$, CONDITIONALLY INDEPENDENT OF ALL THE OTHER X_j, $j \neq i$, THAT IS:

$$\boxed{p(x^n, y^n) = p(x_1) \prod_{i=1}^{n-1} p(x_{i+1}/x_i) \prod_{i=1}^{n} p(y_i|x_i)}$$

SUCH PROCESS IS CALLED Hidden MARKOV MODEL (HMM) AND IS USED IN SPEECH AND HANDWRITING RECOGNITION.

SUMMARY

▣ ENTROPY RATE: TWO DEFINITIONS

$$H(\mathcal{X}) = \lim_{n\to\infty} \frac{1}{n} H(X_1, X_2, \dots X_n)$$

$$H'(\mathcal{X}) = \lim_{n\to\infty} H(X_n|X_{n-1}, \dots X_1)$$

– FOR STATIONARY STOCHASTIC PROCESS

$$H(\mathcal{X}) = H'(\mathcal{X})$$

- ENTROPY RATE OF STATIONARY MARKOV CHAIN

$$\boxed{H(\mathcal{X}) = -\sum_{ij} \mu_i P_{ij} \,\mathrm{ld}\, P_{ij}}$$

▣ SECOND LAW OF THERMODYNAMICS. FOR MARKOV CHAIN:

1.) RELATIVE ENTROPY $D(\mu_n||\mu_n')$ DECREASES WITH TIME

2.) RELATIVE ENTROPY BETWEEN DISTRIBUTION AND THE STATIONARY DISTRIBUTION $\underbrace{}_{D(\mu_n||\mu)}$ DECREASES WITH TIME.

3.) ENTROPY INCREASES IF STATIONARY DISTRIBUTION IS UNIFORM

4.) $H(X_n|X_1)$ INCREASES WITH TIME.

5.) $H(X_0|X_n)$ INCREASES FOR ANY MARKOV CHAIN.

▣ FUNCTIONS OF MARKOV CHAIN. IF $X_1, X_2, \dots X_n$ FORMS A STATIONARY MARKOV CHAIN AND $Y_i = \phi(X_i)$

$$H(Y_n) Y_{n-1}, \dots Y_1 X_1) \leq H(\mathcal{Y}) \leq H(Y_n|Y_{n-1}, \dots Y_1)$$

$$\underline{\lim H(Y_n|Y_{n-1}, \dots Y_1, X_1) = H(\mathcal{Y}) = \lim_{n\to\infty} H(Y_n|Y_{n-1} \dots Y_1)}$$

PROBLEMS

4.1 DOUBLY STOHASTIC MATRICES. An $n \times n$ matrix $P = [p_{ij}]$ is said to be doubly stohastic if $p_{ij} \geq 0$ and $\sum_j p_{ij} = 1$ for all "i" and $\sum_i p_{ij} = 1$ for all "j". An $n \times n$ matrix P is said to be permutation matrix if it is doubly stohastic and there is precisely one $p_{ij} = 1$ in each row and each column. It can be shown that every doubly stohastic matrix can be written as the convex combination of permutation matrices.

(a) Let $a^t = (a_1, a_2, \dots, a_n)$, $a_i \geq 0$ $\sum a_i = 1$ be a probability vector. Let $b = a.P$ where P is doubly stohastic. Show that b is probability vector and that:

$$H(b_1, b_2, \dots b_n) \geq H(a_1, a_2, \dots, a_n)$$

Thus stohastic mixing increases the entropy.

ⓑ Show that stationary distribution μ for doubly stohastic matrix P is the uniform distribution

ⓒ Conversely, prove that if the uniform distribution is stationary distribution for Markov transition matrix P, then P is doubly stohastic.

(a)

$$[a_1, a_2, \dots a_n] \begin{bmatrix} p_{11} & p_{12} & \dots & p_{1n} \\ p_{21} & p_{22} & \dots & p_{2n} \\ \vdots & & & \\ p_{n1} & p_{n2} & \dots & p_{nn} \end{bmatrix} = \Big[\overbrace{\sum_{i=1}^{n} a_i p_{i1}}^{b_1}, \overbrace{\sum_{i=1}^{n} a_i p_{i2}}^{b_2}, \dots, \overbrace{\sum_{i=1}^{n} a_i p_{in}}^{b_n}\Big]$$

Gilbert se

$$b_1 = \sum_{i=1}^{n} a_i p_{i1} = a_1 \cdot p_{11} + a_2 p_{21} + \dots + a_n p_{n1}$$

$$p_{11} = 1 - \sum_{i=2}^{n} p_{i1} \quad p_{21} = 1 - \sum_{\substack{i=1 \\ i \neq 2}}^{n} p_{i1} \quad \dots \quad p_{n1} = 1 - \sum_{i=1}^{n-1} p_{i1}$$

b - is probability vector hence:

$$b = [b_1, b_2, \dots b_n] \qquad b_1 + b_2 + \dots + b_n = 1$$

$$\sum_{i=1}^{n} a_i p_{i1} + \sum_{i=1}^{n} a_i p_{i2} + \dots + \sum_{i=1}^{n} a_i p_{in} = \sum_{j=1}^{n} \sum_{i=1}^{n} a_i p_{ij} =$$

$$= \sum_{i=1}^{n} a_i \underbrace{\sum_{j=1}^{n} p_{ij}}_{1 \left(\text{DUE TO DOUBLY STOHASTIC}\right)} = \underbrace{\sum_{i=1}^{n} a_i}_{1 \left(\text{DUE TO PROBABILITY VECTOR}\right)} = 1$$

PROVED!

ⓑ $\boxed{\sum_{i=1}^{4} \mu_i P_{ij} = \mu_j}$ $\boxed{\mu_1 = \sum_{i=1}^{4} \mu_i P_{i1}}$

$$[\mu_1 \ \ \mu_2] \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = [\mu_1 \ \ \mu_2]$$

(*) $\boxed{\begin{array}{l} \mu_1(1-P_{11}) = \mu_2 P_{21} \\ \mu_2(1-P_{22}) = \mu_1 P_{12} \end{array}}$

$\mu_1 = \mu_1 P_{11} + \mu_2 P_{21}$

$\mu_2 = \mu_1 P_{12} + \mu_2 P_{22}$

$\mu_1 = \frac{\mu_2 P_{21}}{1-P_{11}}$ $\quad \mu_2(1-P_{22}) = \frac{\mu_2 P_{21} P_{12}}{(1-P_{11})}$

~~$P_{ij} = \begin{cases} P_{21} + P_{22} = 1 \Rightarrow P_{21} = 1 - P_{22} \\ P_{11} + P_{12} = 1 \Rightarrow P_{12} = 1 - P_{11} \end{cases}$~~

~~$\mu_1 P_{12} = \mu_2 P_{21}$~~

~~$\mu_2 P_{21} = \mu_1 P_{12}$~~

~~$\mu_1(1-P_{21}) = \mu_2 P_{21}$~~

~~$= 1$~~

~~$P_{12} + P_{21} = 1$~~ ~~$P_{12} = 1 - P_{21}$~~

(*) ⇒ $\left| \begin{array}{l} P_{11} + P_{21} = 1 \\ P_{21} = 1 - P_{11} \end{array} \right|$ $\underbrace{\mu_1(1-P_{11}) = \mu_2(1-P_{11})}_{\mu_1 = \mu_2}$

OVA E DOKAZ ZA 2×2 MATRICA

- VO GENERALEN SLUCAJ

$\mu_1 = \sum_{i=1}^{4} \mu_i P_{i1} = \mu_1 P_{11} + \mu_2 P_{21} + \mu_3 P_{31} + \dots \mu_4 P_{34}$

$\mu_1(1-P_{11}) = \mu_2 P_{21} + \mu_3 P_{31} + \dots + \mu_4 P_{34}$

$1 - P_{11} = P_{21} + P_{31} + \dots P_{34}$

(#)

$\mu_1 P_{21} + \mu_1 P_{31} + \mu_1 P_{34} = \mu_2 P_{21} + \mu_3 P_{31} + \dots + \mu_4 P_{34}$

↳ $\boxed{\mu_1 = \mu_2 \quad \mu_1 = \mu_3 \quad \mu_1 = \mu_4}$

CONTINUE ON PP.100

ⓒ

$$[\mu_1 \mu_1 \cdots \mu] \begin{bmatrix} P_{11} & P_{12} & \dots & P_{14} \\ P_{21} & P_{22} & \dots & P_{24} \\ P_{41} & P_{42} & \dots & P_{44} \end{bmatrix} = [\mu_1 \mu_1 \cdots \mu]$$

$$\cancel{\mu}\cdot\sum_{i=1}^{n} P_{i1} = \cancel{\mu} \qquad \sum_{i=1}^{n} P_{i1} = 1$$

$$\mu\cdot\sum_{i=1}^{n} P_{i2} = \mu \qquad \sum_{i=1}^{n} P_{i2} = 1$$

$$- - - - - \qquad - - - -$$

$$\mu\cdot\sum_{i=1}^{n} P_{in} = \mu \qquad \sum_{i=1}^{n} P_{in} = 1$$

$$\sum_{i=1}^{n} P_{ij} = 1 \quad \text{for} \quad j = 1, \dots n$$

ⓑ <u>EVEN GREATER GENERALIZATION:</u>

[illegible]

$$\mu_1 P_{21} + \mu_1 P_{31} + \dots \mu_1 P_{n1} = \mu_2 P_{21} + \mu_3 P_{31} + \dots + \mu_n P_{3n}$$

$$\mu_1 \sum_{i=2}^{n} P_{i1} = \sum_{i=2}^{n} \mu_i P_{i1} \qquad \sum_{i=2}^{n} \mu_i P_{i1} - \sum_{i=2}^{n} \mu_1 P_{i1} = 0$$

$$\sum_{i>2}^{n} (\mu_i - \mu_1) P_{i1} = 0 \quad \Rightarrow \quad \underline{\mu_i = \mu_1} \quad i = 2 \dots n$$

ⓐ

$$H(b_1, b_2, \dots b_n) = -\left(\sum_{i=1}^{n} a_i P_{i1}\, ld \sum_{i=1}^{n} a_i \underline{P_{i1}} + \sum_{i=1}^{n} a_i P_{i2}\, ld \sum_{1}^{n} a_i P_{i2} + \dots + \sum_{i=1}^{n} a_i P_{in}\, ld \sum_{i=1}^{n} a_i P_{in}\right) \boxed{\geq}$$

ZARADI

$\Rightarrow$ CONCAVE FUNCTION

$$\sum_{i=1}^{n} a_i P_{i1} = E[a_i] \qquad ld\, E[a_i] \geqslant E[ld\, a_i]$$

$$\sum_{i=1}^{n} a_i P_{i1}\, ld \sum_{i=1}^{n} a_i P_{i1} \geqslant \sum_{i=1}^{n} a_i P_{i1} \left(\sum_{i=1}^{n} ld(a_i) P_{i1}\right)$$

$$ld a_1 P_{11} + ld a_2 P_{21} + \dots ld a_n P_{n1}$$

$$H(a_1, a_2, \dots, a_n) = -\sum_{i=1}^{n} a_i\, ld\, a_i$$

LOGSUM INEQUALITY

$$\sum_{i=1}^{n} a_i\, ld a_i \geq \left(\sum a_i\right) ld \sum_{i=1}^{n} a_i$$

$$\geqslant -\sum_{i=1}^{n} (P_{i1} a_i)\, ld (P_{i1} a_i) - \sum_{i=1}^{n} (P_{i2} a_i)\, ld (P_{i2} a_i) - \dots - \sum_{i=1}^{n} (P_{in} a_i)\, ld (P_{in} a_i)$$

$$= -\sum_{j=1}^{n} \sum_{i=1}^{n} P_{ij} a_i\, ld (P_{ij} a_i) = -\sum_{j=1}^{n} \sum_{i=1}^{n} P_{ij} a_i\, ld P_{ij} - \sum_{j=1}^{n} \sum_{i=1}^{n} P_{ij} a_i\, ld a_i$$

$$= -\underbrace{\sum_{i=1}^{n} a_i \sum_{j=1}^{n} P_{ij}\,\mathrm{ld}\,P_{ij}} - \sum_{i=1}^{n} a_i\,\mathrm{ld}\,a_i \underbrace{\sum_{j=1}^{n} P_{ij}}_{1} =$$

$$= \underbrace{\overset{1}{\sum_{i=1}^{n}} P_{ij}\,\mathrm{ld}\frac{1}{P_{ij}}}_{\geq 0} + \sum_{i=1}^{n} a_i\,\mathrm{ld}\frac{1}{a_i} \geq \sum_{i=1}^{n} a_i\,\mathrm{ld}\frac{1}{a_i} =$$

$$= H(a_1, a_2, \dots a_n)$$

PROVED !!!

EDITION 1 SOLUTIONS

SKORO IDENTIČNI SO DOKAZITE 11.99

(a) $H(b) - H(a) = -\sum_{i=1}^{n} \textcircled{b_i}\,\mathrm{ld}\,b_i + \sum_{i=1}^{n} a_i\,\mathrm{ld}\,a_i =$

$$= -\sum_{j=1}^{n}\sum_{i=1}^{n} a_i P_{ij}\,\mathrm{ld}\underbrace{\sum_{k=1}^{n} a_k P_{kj}} + \sum_{i=1}^{n} a_i\,\mathrm{ld}\,a_i =$$

VOVEDUVAJ „K"
ZA DA NE SE MEŠA
SO „i" (VIDI N100)

$$\sum_{i=1}^{n} a_i\,\mathrm{ld}\,a_i = \sum_{i=1}^{n} a_i\,\mathrm{ld}\,a_i \sum_{j=1}^{n} P_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i P_{ij}\,\mathrm{ld}\,a_i$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i P_{ij}\,\mathrm{ld}\frac{a_i}{\underbrace{\sum_{k=1}^{n} a_k P_{kj}}_{=b_j}} \textcircled{$\geq$} \left(\sum_{ij} a_i P_{ij}\right)\mathrm{ld}\frac{\sum_{ij} a_i \nearrow n}{\sum_{ij} \cancel{b_i} \searrow n} =$$

LOGSUM INEQUALITY:

$$\sum_{i=}^{n} a_i\,\mathrm{ld}\frac{a_i}{b_i} \geq \left(\sum_{i=1}^{n} a_i\right)\mathrm{ld}\frac{\sum_{i=1} a_i}{\sum_{i=1} b_i}$$

$$= \underbrace{\sum_{i=1}^{n} a_i \sum_{j=1}^{n} P_{ij}}_{1} \cdot \mathrm{lo}\frac{\cancel{n}}{\cancel{n}} = 1 \cdot \mathrm{ld}\,1 = 0$$

4.2 TIMES ARROW. LET $\{x_i\}_{i=-\infty}^{\infty}$ BE A STATIONARY STOCHASTIC PROCESS PROVE THAT:

$$\boxed{H(X_0|X_{-1}, X_{-2}, \dots, X_{-n}) = H(X_0|X_1, X_2, \dots X_n)}$$

- $H(X_{0+n+1}|X_1, X_2, \dots \overset{\#}{X_n}) = H(X_0|X_1, X_2, \dots X_n)$

$H(X_{n+1}|X_1, X_2, \dots X_n) = H(X_0|X_1, X_2, \dots X_n)$

$H(\underline{x_{n+1}}|x_1,\dots x_n) = H(x_0|x_1,\dots x_n)$

$$\boxed{H(x_1|x_0) = H(x_0|x_{-1})}$$

$\underline{H(x_1,x_2,\dots x_{n+1})} = \sum_{i=1}^{n+1} H(x_i|x_{i-1},\dots x_1) =$

$= H(x_1) + H(x_2|x_1) + H(x_3|x_1x_2) \dots + \underline{H(x_{n+1}|x_1,x_2,\dots x_n)}$

$H(x_0,x_1,\dots x_n) = \sum_{i=0}^{n} H(x_i|x_{i-1},\dots x_0) =$

$= H(x_0) + H(x_1|x_0) + \dots H(x_n|x_{n-1},\dots x_0) =$

$= \underline{H(x_1,\dots x_n,x_0)} = H(x_1) + H(x_2|x_1) + H(x_3|x_1x_2) + \dots$

$+ \dots + H(x_0|x_1,x_2,\dots x_n)$

$\underline{H(x_0,x_1,\dots x_n)} = H(x_{0+t},x_{0+t},\dots x_{n+t}) = |t=1| =$

$= \underline{H(x_1,x_2,\dots x_{n+1})}$

ZNAČI: $\underline{H(x_1,x_2,\dots x_{n+1}) = H(x_1,x_2,\dots x_n,x_0)}$

$\Rightarrow \cancel{H(x_1)} + \cancel{H(x_2|x_1)} + \cancel{H(x_3|x_1,x_2)} \dots \cancel{H(x_n|x_1,x_2,\dots x_{n-1})} +$

$+ \underline{H(x_{n+1}|x_1,x_2,\dots x_n)} = \cancel{H(x_1)} + \cancel{H(x_2|x_1)} + \cancel{H(x_3|x_1,x_2)} + \dots$

$+ \cancel{H(x_n|x_{n-1},x_{n-2}\dots x_1)} + \underline{H(x_0|x_n,x_{n-1},\dots x_1)}$

ZNAČI: $\boxed{H(x_{n+1}|x_1,x_2,\dots x_n) = H(x_0|x_n,x_{n-1},\dots x_1)}$

PROVED !!! $= H(x_0|x_1,x_2,\dots x_n)$

T.e. BIDEJĆI $H(x_{n+1}|x_1,x_2,\dots x_n) = H(x_0|x_{-1},x_{-2},\dots x_{-n})$.

$$\Rightarrow \boxed{H(x_0|x_{-1},x_{-2},\dots x_{-n}) = H(x_0|x_1,x_2,\dots x_n)}.$$

EDITIONAL SOLUTION: $\underline{H(x_0|x_{-1},x_{-2},\dots x_{-n})} = H(x_0,x_{-1},\dots x_{-n})$

$- H(x_{-1},x_{-2},\dots x_{-n}) = H(x_n,\dots x_1,x_0) - H(x_1,x_2,\dots x_n)$

$= \underline{H(x_0|x_1,x_2,\dots x_n)}$

4.3 Shuffles increase ENTROPY. ARGUE THAT FOR ANY DISTRIBUTION ON SHUFFLES T AND ANY DISTRIBUTION ON CARD POSITIONS X THAT:

$$H(TX) \geq H(TX|T) = H(T^{-1}TX|T) = H(X|T) = H(X)$$

IF X AND T ARE INDEPENDENT

$$X = \{x_1, x_2, \dots, x_i, \dots x_{51}, x_{52}\} \qquad X_i \in \{\underline{1, 2, \dots 52}\}$$

e.g $X = \{\underline{3, 4, 1, 2, \dots 52, 51}\}$

INITIAL RANDOM POSITION OF THE CARDS IN THE DECK

$$\underline{X_1 \to X_2 \to X_3 \to \dots \to X_n}$$

$$\boxed{X_2 = TX_2}$$

$$I(X_1; X_2) \geq I(X_1; X_n)$$

$$X_1 \to TX_1 \to TX_2 \to \dots \to TX_n$$

$$I(X_1; TX_1) \geq I(X_1; TX_n)$$

$$H(X_1) - H(X_1|TX_1) \geq H(X_1) - H(X_1|TX_n) \Rightarrow$$

$$\underline{H(X_1|TX_1) \leq H(X_1|TX_n)}$$

THE CONDITIONAL ENTROPY DECREASES.

$$\cancel{H(TX_1)} - H(TX_1|X_1) \geq \cancel{H(TX_n)} - H(TX_n|X_1)$$

$$\boxed{H(TX_1|X_1) \leq H(TX_n|X_1)}$$

$$\underbrace{[1, 2, 3]}_{X} \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{T} = \underline{[2, 1, 3]}$$


PERMUTATION MATRIX

$$[x_1, x_2, \dots x_{52}] \begin{bmatrix} p_{11} & p_{12} & \dots & p_{1,52} \\ p_{21} & p_{22} & \dots & p_{2,52} \\ \dots & \dots & \dots & \dots \\ p_{52,1} & p_{52,2}, & \dots & p_{52,52} \end{bmatrix} = [x_{i1}, x_{i2}, \dots x_{i52}]$$

$$x_{ik} \in \{1, 2, \dots 52\}$$

$$\boxed{H(X) = \sum_{i=1}^{n!} \frac{1}{n!} \, ld(n!) = ld(n!)} \quad \boxed{n = 52}$$

$$\underline{H(X) = 225.581}$$

$$\to p(x) = ? \qquad p(x_i) = \frac{1}{n!}$$

- АКО ИМАЙ 3 КАРТИ $[1, 2, 3]$

$x_1 = 1, 2, 3;$ $x_2 = 2 1 3;$ $x_3 = 2 3 1;$ $x_4 = 3 2 1;$

$x_5 = 3 1 2;$ $x_6 = 1, 3, 2$

$$\boxed{P^3_3 = \frac{n!}{(n-k)!} = \frac{3!}{0!} = 6}$$

$$P^n_k = \frac{n!}{(n-k)!} \qquad P^n_n = \frac{n!}{0!} = n!$$

$T_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $T_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $T_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$T_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $T_5 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $T_6 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$T_7 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $T_8 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $T_9 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$T_{10} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $T_{11} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $T_{12} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$T_{13} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $T_{14} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $T_{15} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$T_{16} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- BEZ POVTORUVANJE

$t_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $t_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $t_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $t_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$t_5 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $t_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$P^n_k = P^3_3 = \frac{3!}{0!} = 6$

$H(x) = \text{ld}\, n! = \text{ld} 6 = 2.585$

- UŠTE POEDOSTAVEN PRIMER (n=2 MUTEN AKO NA ~~[illegible]~~ VOSHIZOT)

$[X_1 \; X_2]$ $\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$ $T_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $T_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$X_1^n = [1 \; 2]$ $X_2^n = [2 \; 1]$ $H(X^n) = \sum_1^{n!} \frac{1}{n!} \text{ld}\, n! = \text{ld}\, n!$

$\boxed{H(X^n) = \text{ld}\, 2 = 1}$

$$H(TX|T) = \underbrace{P(T=t_1)}_{1/2} \cdot H(t_1X|t_1) + \underbrace{P(T=t_2)}_{1/2} \cdot H(t_2X|t_2)$$

PERMUTACIONITE MATRICI SE GENERIRAAT SO UNIFORMNA RASPREDELBA.

- $H(t_1 X|t_1) = ?$

$x_1^n = \{1, 2\}$ $\quad x_2^n = [2, 1]$

$t_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad t_1 \cdot x_1^n = [2 \quad 1]$ $\quad p(t_1 x_1^n) = \frac{1}{2!}$

$t_1 \, x_2^n = [1 \quad 2]$ $\quad p(t_1 x_2^n) = \frac{1}{2!}$

$$H(t_1X|t_1) = \frac{1}{2!}\, ld\, 2! + \frac{1}{2!} \cdot ld\, 2! = \frac{1}{2} + \frac{1}{2} = 1$$

$$H(t_2X|t_2) = \frac{1}{2}\, ld\, 2 + \frac{1}{2}\, ld\, 2 = 1$$

$$H(tX|T) = \frac{1}{2} + \frac{1}{2} = 1$$

- SE NAVRAĆAM NA PRIMEROT SO $n=3$ KAKO VO ISPITOT

$$H(TX|T) = \sum_{i=1}^{6} P(T=t_i)\, \underline{H(t_iX|t_i)}$$

AKO PERMUTACIONITE MATRICI SE SO UNIFORMNA RASPREDELBA:

$$H(TX|T) = \frac{1}{6} \sum_{i=1}^{6} H(t_iX|t_i) \qquad x^n = [x_1, x_2, x_3]$$

$$H(t_1X|t_1) = ?$$

$x_1^n = [1, 2, 3]$ $\quad x_2^n = [1, 3, 2]$ $\quad x_3^n = [2, 1, 3]$ $\quad x_4^n = [2, 3, 1]$

$x_5^n = [3, 1, 2]$ $\quad x_6^n = [3, 2, 1]$

$t_1 x_1^n = [2, 1, 3]$; $\quad t_1 x_2^n = [3, 1, 2]$; $\quad t_1 x_3^n = [1, 2, 3]$; $\quad t_1 x_4^n = [3, 2, 1$

$t_1 x_5^n = [1, 3, 2]$; $\quad t_1 x_6^n = [2, 3, 1]$

$$H(t_1X|t_1) = \left(\frac{1}{6}\, ld\, 6\right) \cdot 6 = ld\, 6 = \underline{ld(3!)}$$

- DUH NE MORA UNIFORMNO DA SE RASPOREDENI PERMUTACIONITE MATRICI, KOJA

$$H(TX|T) = \sum_{i=1}^{6} P(T=t_i) \cdot \underbrace{H(t_iX|t_i)}_{=ld\,6} = ld\,6 \cdot \sum_{i=1}^{6} P(T=t_i) = \underline{ld\,6} = H(X)$$

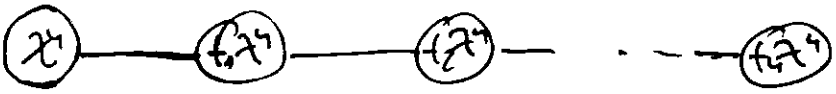
OVOJ DOKAZ MOŽE DA SE PROŠIRI VO GENERALEN SLUČAJ !!

$$H(TX|T) = \sum_{i=1}^{4!} P(T=t_i)\, \underbrace{H(TX|T=t_i)}_{= ld(4!)} = \underline{ld(4!)} \underbrace{\sum_{i=1}^{4!} P(T=t_i)}_{=1}$$

④ $\underbrace{H(TX|T) = ld(4!) = H(X)}$!!! $\qquad H(TX) \geq H(TX|T)$

JA DOKAŽAV EDNAKVOSTA !!!

- SO OGLED NA TOA ŠTO ~~SE~~ KAŽUVA ZA MARKOV LANCI (MOŽE I RANDOM WALK APSTRAKCIJA DA SE NAPRAVI KADE T = (VERIJATNOST TRANZICIONATA MATRICA)



$$H(t_1x^4|x^4) \leq H(t_nx^4|x^4)$$

$$I(X;Y) \geq I(X;Z) \qquad H(X) - H(X|Y) \geq H(X) - H(X|Z)$$

$$H(X|Y) \leq H(X|Z)$$

$$H(Y) - H(Y|X) \geq H(Z) - H(Z|X)$$

$H(Y) = H(Z) \rightarrow$ STATIONARY MARKOV CHAIN

$$H(Y|X) \leq H(Z|X)$$

$$H(TX) \underset{(*)}{\geq} \underbrace{H(TX|T) = H(T^{-1}TX|T) = H(X|T)}_{Ⓑ\ \text{ENTROPY}} \underset{①}{=} H(X)$$

(*) ⇒ CONDITIONING REDUCES

$$H(X,Y) = H(X) + H(Y|X)$$

① ⇒ X & T ARE STATISTICALY INDEPENDENT

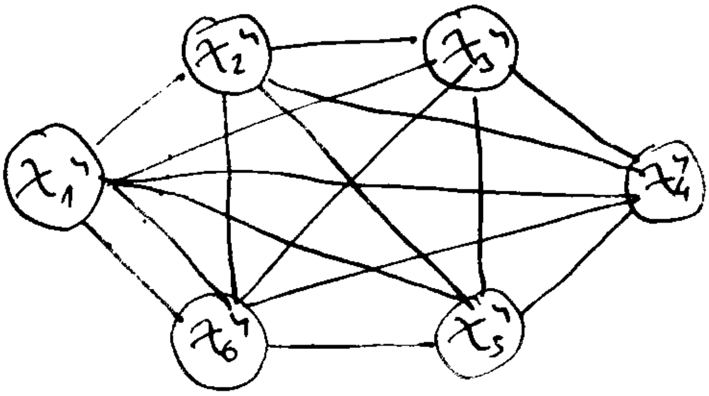
$$H(TX) = \sum_{tx} p(T,x)\, ld\, p(T,x)$$

$$p(T,x) = p(t)\cdot p(x) = \frac{1}{4!}\cdot\frac{1}{4!}$$

$$H(TX) = \sum_{t,x} \frac{1}{(4!)^2}\, ld(4!)^2 = \cancel{(4!)^2}\cdot\frac{1}{\cancel{(4!)^2}}\, ld(4!)^2$$

$$H(TX) = 2\underbrace{ld(4!)}_{H(X)} = 2H(X)$$

?



$$x_1^4 = \{\overset{x_1}{1}, \overset{x_2}{2}, \overset{x_3}{3}\}$$

$$x_2^4 = \{1, 3, 2\}$$
$$x_3^4 = \{2, 1, 3\}$$
$$x_4^4 = \{2, 3, 1\}$$
$$x_5^4 = \{3, 1, 2\}$$
$$x_6^4 = \{3, 2, 1\}$$

$$\mu_j = \sum_{i=1}^{6} \underline{\mu_i P_{ij}} \qquad j = 1, 2, \ldots 4$$

$$P_{ij} = P(t_i)$$

$$H(X) = \sum_{i=1}^{6} \mu_i \sum_{j=1}^{6} P_{ij} \, ld P_{ij} \qquad \sum_{i=1}^{6} \sum_{j=1}^{6} \mu_i P_{ij} \, ld P_{ij}$$

$$= \sum_{i=1}^{6} \sum_{j=1}^{6} \frac{1}{6!} \frac{1}{6!} \cdot ld 6! = \frac{\cancel{6!^2}}{\cancel{6!^2}} ld 6! = ld(6!)$$

- АКО ОДИШ СО ФОРМУЛАТА ЗА RANDOM WALK!

$$H(X) = ld 2E - H\left(\frac{E_1}{2E}, \frac{E_2}{2E}, \ldots \frac{E_7}{2E}\right).$$

$$\boxed{2E = 6 \cdot 6 = 36} \qquad \frac{E_1}{2E} = \frac{E_2}{2E} = \ldots \frac{E_6}{2E} = \frac{6}{36} = \frac{1}{6}$$

$$H(X) = ld 36 - \cancel{6} \cdot \frac{1}{\cancel{6}} \cdot ld 6 = ld 6 \cdot 6 - ld 6 =$$
$$= ld 6 + \cancel{ld 6} - \cancel{ld 6} = ld 6$$

СУШТИНСКАТА КОНСТАТАЦИЈА КОЈА СЛЕДИ ОД АНАЛИЗАТА НА СТР. 105/106 Е ДЕКА:

$\boxed{H(tX) = H(X)}$, ПРЕРЕДЕНИ СОСТОЈБИТЕ, А ЗАТОА ШТО САМО СЕ ПРЕРЕДЕНИ ВЕРОЈАТНОСТИТЕ НА СОСТОЈБИТЕ. НЕ СЕ ПРОМЕНЕТИ

$$H(TX) \geq H(TX|T) = \sum_t p(t) \underbrace{H(t \cdot X|T=t)}_{\text{ИСТА ВРЕДНОСТ ЗА СЕКОЕ } t \text{ Т.Е. } = H(X)} =$$

$$= \sum_t p(t) H(tX) = H(X) \underbrace{\sum_t p(t)}_{1} = H(X)$$

DUE TO CONDITIONING REDUCES ENTROPY.

- ОВА Е ВСУШНОСТ ФОРМИРАН ДОКАЗ НА ОНА ШТО СУМ ГО ДОКАЖАЛ НА СТР. 105-106

4.4 SECOND LAW OF THERMODYNAMICS. LET X_1, X_2, X_3 ... BE A STATIONARY FIRST-ORDER MARKOV CHAIN. IN SECTION 4.4 IT WAS SHOWN THAT:

$$H(X_n|X_1) \geq H(X_{n-1}|X_1) \quad \text{for } n=2,3,\dots$$

THUS CONDITIONAL UNCERTAINTY ABOUT THE FUTURE GROWS WITH TIME. THIS IS TRUE ALTHOUGH THE UNCONDITIONAL UNCERTAINTY $H(X_n)$ REMAINS CONSTANT. HOWEVER, SHOW BY EXAMPLE THAT $H(X_n|X_1=x_1)$ DOES NOT NECESSARILY GROW WITH "n" FOR EVERY x_1.

$$H(X_n|X_1) = p(X_1=x_1)\,H(X_n|X_1=x_1) + p(X_1=x_2)\,H(X_n|X_1=x_2)$$

$$+ \dots + p(X_1=x_n)\,H(X_n|X_1=x_1)$$

EXAMPLE IF $n=3$

$$X_1 \to X_2 \to X_3 \qquad I(X_1;X_2) \geq I(X_1;X_3)$$

$$\cancel{H(X_2)} - H(X_2|X_1) \geq \cancel{H(X_3)} - H(X_3|X_1)$$

STATIONARY

$$\boxed{H(X_3|X_1) \geq H(X_2|X_1)}$$

if $p(x_1) = \frac{1}{|X_1|}$ $\Rightarrow$ UNIFORM DISTRO

$$H(X_2|X_1) = \frac{1}{|X_1|}\,H\left(X_2|X_1=\frac{1}{|X_1|}\right) + \frac{1}{|A_1|}\cdot H\left(X_2|X_1=\frac{1}{|A_1|}\right) + \dots$$

$$H(X_2|X_1) = \cancel{|X_1|}\cdot\frac{1}{\cancel{|X_1|}}\,H\left(X_2|X_1=\frac{1}{|X_1|}\right) = H\left(X_2|X_1=\frac{1}{|A_1|}\right) \;!$$

$$H(X_3|X_1) = H\left(X_3|X_1=\frac{1}{|X_1|}\right) \Rightarrow \underbrace{H\left(X_3\Big|\frac{1}{|X_1|}\right) = H\left(X_2\Big|\frac{1}{|X_1|}\right)}_{\text{(STATIONARY MARKOV CHAIN)}} \;!$$

$$D(p\|q) = \sum_x p \,\mathrm{ld}\frac{p}{\frac{1}{|Q|}} = -\sum_x p\,\mathrm{ld}\frac{1}{p} + \underbrace{\sum_x p\,\mathrm{ld}|Q|}_{\mathrm{ld}|Q|}$$

$$= \mathrm{ld}|Q| - H(X^n) = \mathrm{ld}|X| - \underline{H(X^n)} \quad ■$$

POTSEĆNANJE:

$$\underline{\mu_n}, \underline{\mu_n'} \;;\; \mu_{n+1}, \mu_{n+1}'$$

TRANSITION PROBABILITY

$$p(x_n, x_{n+1}) = p(x_n)\cdot r(x_{n+1}|x_n)$$

$$q(x_n, x_{n+1}) = q(x_n)\cdot r(x_{n+1}|x_n)$$

$$D(p(x_n, x_{n+1})\,\|\,q(x_n, x_{n+1})) = D(p(x_n)\,\|\,q(x_n)) +$$

$$\underbrace{D(p(x_{n+1}|x_n) \| q(x_{n+1}|x_n))}_{0} = D(p(x_{n+1}) \| q(x_{n+1})) +$$

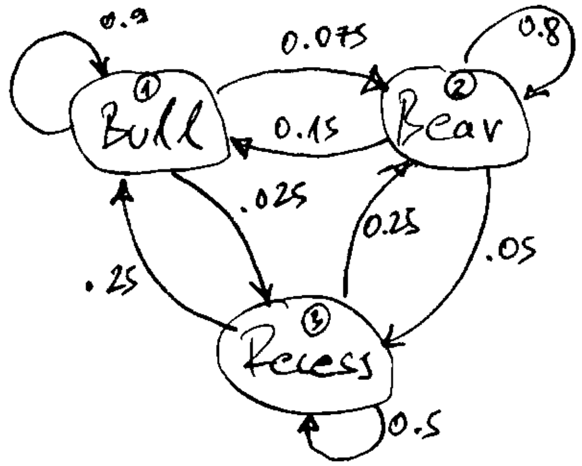
$$+ D(p(x_n|x_{n+1}) \| q(x_n|x_{n+1}))$$

$$p(x_{n+1}|x_n) = q(x_{n+1}|x_n) = r(x_{n+1}|x_n) \Rightarrow \underline{D(p(x_{n+1}|x_n) \| q(x_{n+1}|x_n)) =}$$

$$\boxed{D(p(x_n)\|q(x_n)) \geq D(p(x_{n+1})\|q(x_{n+1}))} \quad \text{i.e}$$

$$D(\mu_n \| \mu_n') \geq D(\mu_{n+1} \| \mu_{n+1}')$$

- ПРИМЕРОТ ЗА МАРКОВ CHAIN ОД ВИКИПЕДИЈА



- BULL WEEK IS FOLLOWED BY ANOTHER BULL WEEK 90% OF TIME

$$\mu_j = \sum_{i=1}^{3} \mu_i \underline{P_{ij}} \qquad j = 1$$

$$\mu_1 = \mu_1 \cdot P_{11} + \mu_2 \cdot P_{21} + \mu_3 P_{31}$$

$$\mu_1(1 - P_{11}) = \mu_2 P_{21} + \mu_3 P_{31}$$

$$\boxed{0.1\mu_1 = 0.15\mu_2 + 0.25\mu_3}$$

$$\mu_2 = \sum_{i=1}^{3} \mu_i P_{i2} = \mu_1 \underline{P_{12}} + \mu_2 \underline{P_{22}} + \mu_3 P_{32}$$

$$\mu_2(1 - P_{22}) = 0.075\mu_1 + 0.25\mu_3 \qquad \boxed{0.2\mu_2 = 0.075\mu_1 + 0.25\mu_3}$$

$$\mu_3 = \mu_1 \cdot P_{13} + \mu_2 P_{23} + \mu_3 P_{33} \qquad \boxed{0.5\mu_3 = 0.025\mu_1 + 0.05\mu_2}$$

$$\underline{\mu_1 = 0.625 \quad \mu_2 = 0.3125 \quad \mu_3 = 0.0625}$$

$$D(\mu_n \| \mu_n') \geq D(\mu_{n+1} \| \mu_{n+1}') \swarrow$$ IF STATIONARY DISTRIBUTION IS UNIFORM

② $\Rightarrow \quad \operatorname{ld}|X| - H(\mu_n) \geq \operatorname{ld}|X| - H(\mu_{n+1})$

IF STATIONARY DISTRIBUTION IS $p(x) = \mu = \frac{1}{2^n}$

$$D(\mu_n \| \mu) \geq D(\mu_{n+1} \| \mu)$$

$$D(\mu_n\|\mu) = \sum_{(x\in X)} p(x)\,\mathrm{ld}\left(\frac{p(x)}{q(x)}\right) \qquad q(x) = \frac{1}{2^n}$$

$$= -H(X^n) + \sum_{x\in X} p(x)\,\mathrm{ld}\,2^n = n - \underline{H(X^n)}$$

CONST

$$D(\mu_{n+1}\|\mu) = (n+1) - H(X^n) \qquad \underline{D(\mu_{n+1}\|\mu) \geq D(\mu_n\|\mu)}$$

NOT POSSIBLE!!!

$$\underline{H(X_n|X_1)} \geq H(X_n|X_1,X_2) \underset{\uparrow \text{MARKOVITY}}{=} H(X_n|X_2) \underset{\uparrow}{=} \underline{H(X_{n-1}|X_1)}$$

STATIONARITY.

$$H(X_n|X_{n-1}) = H(X_n|X_{n-1},X_{n-2}) =$$

DU TO MARKOVITY i.e KNOWING THE PRESENCE FUTURE DOESNT DEPEND FROM PAST

$$= H(X_{n-1}|\underline{X_{n-2}},X_{n-3}) = H(X_{n-1}|X_{n-2})$$

IF: $p(x_n) = p(x_{n-1}) \Rightarrow$

$$H(X_n|X_1) = H(X_{n-1}|X_{n-2})$$

NE E TOA TOA !!!

IF: $\boxed{X_{n-1} = f(X_1)}$ IMAJ JA VO PREDVID KAKO E OPTIMIRANA ZADACATA BI REKOL OVA E PRAVILNO RESENIE

$$\boxed{H(X_n|X_1) = H(X_n|X_1,\underline{X_{n-1}}) \underset{\uparrow \boxed{\text{MARKOVITY}}}{=} H(X_n|X_{n-1})}$$

SE KLADAM VO OVA !!! PRASANJE E ARZI VO OVOJ SLUCAJ IMAME MARKOV CHAIN DOKAZANO!!!

CDITION 1: $X_1 \to X_{n-1} \to X_n$

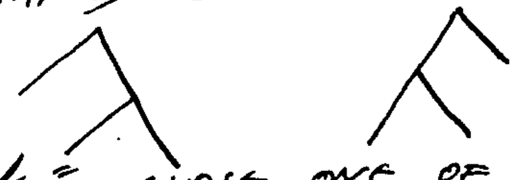
$$I(X_1;X_{n-1}) \geq I(X_1;X_n) \qquad H(X_{n-1}) - H(X_{n-1}|X_1) \geq$$

$$\geq H(X_n) - H(X_n|X_1); \quad H(X_n) = H(X_{n-1}) \Rightarrow$$

$$H(X_{n-1}|X_1) \leq H(X_n|X_1) \quad \text{i.e.} \quad H(X_n|X_1) \geq H(X_{n-1}|X_1)$$

4.5 <u>ENTROPY OF RANDOM TREE</u> CONSIDER THE FOLLOWING METHOD OF GENERATING A RANDOM TREE WITH „n" NODES. FIRST EXPAND THE ROOT NODE:

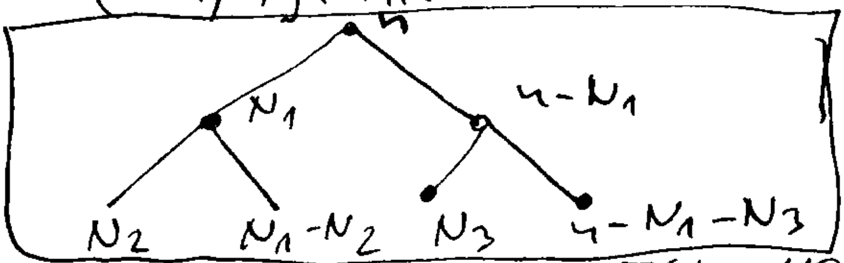
THEN EXPAND ONE OF THE TERMINAL NODES AT RANDOM:



AT TIME „k", CHOSE ONE OF THE $k-1$ TERMINAL NODES ACCORDING TO THE UNIFORM DISTRIBUTION AND EXPAND IT. CONTINUE UNTIL „n" TERMINAL NODES HAVE BEEN GENERATED. THUS SEQUENCE LEADING TO A FIFE-NODE TREE MIGHT LOOK LIKE THIS:

SURPRISINGLY, THE FOLLOWING METHOD OF GENERATING RANDOM TREES YIELDS THE SAME PROBABILITY DISTRIBUTION ON TREES WITH „n" TERMINAL NODES. FIRST CHOSE AN INTEGER N_1 UNIFORMLY DISTRIBUTED ON $\{1,2,\dots,n-1\}$. WE THEN HAVE PICTURE:

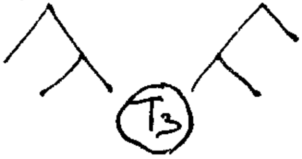
THEN CHOSE AN INTEGER N_2 UNIFORMLY DISTRIBUTED OVER $\{1,2,\dots N_1-1\}$, AND INDEPENDENTLY CHOSE ANOTHER INTEGER N_3 UNIFORMLY OVER $\{1,2,\dots(n-N_1)-1\}$. THE PICTURE IS NOW



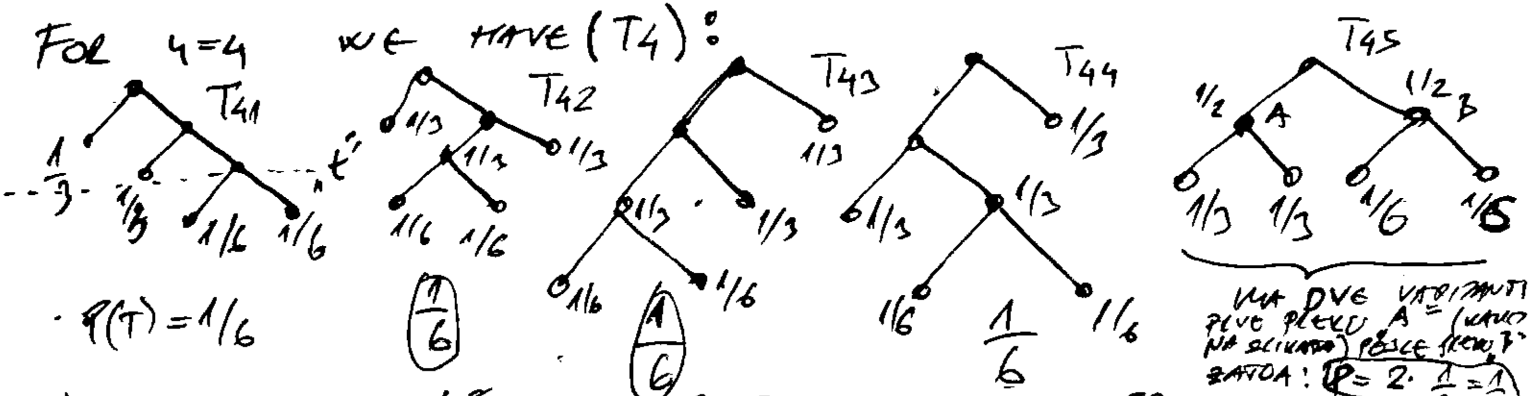
CONTINUE THE PROCESS UNTILL NO FURTHER SUBDIVISION CAN BE MADE. (THE EQUIVALENCE OF THISE TWO TREE GENERATION SCHEMES FOLLOWS, FOR EXAMPLE FROM POLYA'S URN MODEL.)

NOW LET T_n DENOTE A RANDOM n-NODE TREE GENERATED AS DESCRIBED. THE PROBABILITY OF SUCH TREES SEEMS DIFFICULT TO DESCRIBE, BUT WE CAN FIND THE ENTROPY OF THIS DISTRIBUTION IN RECURSIVE FORM. FIRST SOME EXAMPLES. FOR $n=2$ WE HAVE ONLY ONE TREE THUS $H(T_2)=1\cdot ld1=0$. FOR $n=3$ WE HAVE TWO EQUALLY PROBABLE TREES:

THUS $H(T_3)=\left(\frac{1}{2}\,ld2\right)\cdot 2 = ld2$



FOR $n=4$ WE HAVE $(T4)$:



IMA DVE VARIJANTI PRVO PREKU A (KAKO NA SLIKATA) POSLE PREKU B, ZATOA: $\boxed{P = 2 \cdot \frac{1}{6} = \frac{1}{3}}$

- VO MOMENTOT „t''" IMA 3 JAZLI OD KOI, TO UNIFORMNA DISTRIBUCIJA BIRA EDEN I GO EKSPANDIRA

$$\boxed{8 = 4 \cdot \frac{1}{3} + \frac{1}{3} = \frac{2}{3} + \frac{1}{3} = 1}$$

$P(T_4)$ NEMA VRSKA SO GRANENJETO NA DRVOTO TUKU SO VKUPNIOT BROJ NA DRVA I BROJOT NA IOPTOLUVANJE NA DR.

NOW FOR THE RECCURENCE RELATION. LET $N_1(T_n)$ DENOTE THE NUMBER OF TERMINAL NODES OF T_n IN THE RIGHT HALF OF THE TREE

$$H(T_n) \overset{(a)}{=} H(N_1, T_n) \overset{(e)}{=} H(N_1) + H(T_n|N_1) \overset{(c)}{=} ld(n-1) + H(T_n|N_1) \overset{(d)}{=}$$

$$\overset{(d)}{=} ld(n-1) + \frac{1}{n-1} \sum^{n-1} \left(H(T_k) + H(T_{n-k})\right) \overset{(e)}{=} ld(n-1) + \frac{2}{n-1} \sum_{k=1}^{n-1} H(T_k)$$

$$= ld(n-1) + \frac{2}{n-1} \sum_{k=1}^{k=1_{n-1}} H_k.$$

(f) USE THIS TO SHOW THAT:

$$(n-1)H_n = nH_{n-1} + (n-1)\,ld(n-1) - (n-2)\,ld(n-2) \quad \text{OR}$$

$$\frac{H_n}{n} = \frac{H_{n-1}}{n-1} + C_n \quad \text{FOR APPROPRIATELY DEFINED } C_n.$$

SINCE $\sum C_n = C < \infty$, YOU HAVE PROVED THAT $\frac{1}{n}H(T_n)$ CONVERGES TO A CONSTANT. THUS, THE EXPECTED NUMBER OF BITS NECESSARY TO DESCRIBE RANDOM TREE T_n GROWS LINEARLY WITH n.

(e) AT USAGE OF CHAIN FORMULA

$$H(N_1, T_n) = H(N_1) + H(T_n|N_1)$$

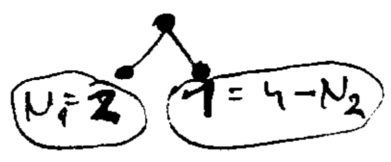
(c) N_1 IS UNIFORMLY DISTRIBUTED OVER $\{1, 2, \ldots n-1)$

i.e. $\boxed{P(N_1) = \frac{1}{n-1}} \Rightarrow H(N_1) = \sum_{i=1}^{n-1} \frac{1}{n-1}\, ld\, n-1$

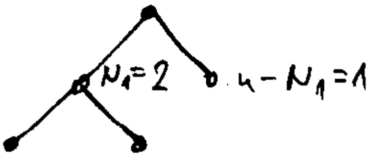
$N_1 \in \{1, 2, \ldots n-1\}$

- PROBA NA RECURENCE PRISTAPOT ZA $n=3$

1°) $N_1 \in \{1, 2\}$ $\quad P(N_1) = \frac{1}{2}$ $\quad$ e.g $\boxed{N_1 = 2}$



2^0 $N_2 \in \{1\}$ $N_3 \in \{1 .. 3-2-1\} \in \{0\}$



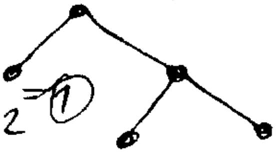
$N_2=1$ $1=2-1=N_1-N_2$

- DRUGIOT PAT

$1^0)$ $N_1 \in \{1, 2\}$ $P(N_1) = \frac{1}{2}$ $N_1 = 1$

$N_1=1$ $n-N_1 = 2$

$2^0)$ $N_2 \in \{1, .. \overset{1}{\cancel{N_1}}-1\} \in \{\}$ $\underline{\underline{N_3}} = \{1, ... \overset{3-1-1}{\underline{\underline{n-N_1-1}}}\} \in \{1\}$



$N_2=\textcircled{1}$

$N_3=\textcircled{1}$ $n-\underline{\underline{N_1-N_3}}=\textcircled{1}$

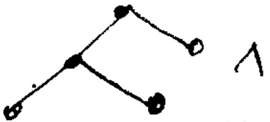
- RECURENCE ~~PRISTAP~~ ~~2A~~ $n=4$

$1^0)$ $N_1 \in \{1, 2, 3\}$ $\boxed{N_1 = 3}$ $\boxed{P(N_1) = \frac{1}{3}}$

$N_1=3$ $n-N=1$

$2^0_a)$ $N_2 \in \{1, 2\}$; $P(N_2) = \frac{1}{2}$; $N_3 \in \{1 .. 4-3-1\} = \{\}$

4.) $\boxed{N_2 = 2}$

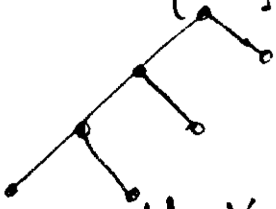


$N_2=2$ $N_1-N_2=1$

6.) $\boxed{N_2 = 1}$

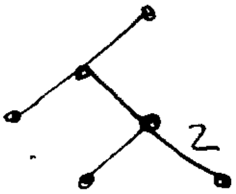
$3^0_a)$ $N_4 \in \{1, 2, .. N_2-1\}$

$N_4 \in \{1\}$



$N_4=1$ $\cancel{N}_2-N_4=1$

$3^0_e)$ $N_4 \in \{1, 2, .. N_1-N_2-1\} \in \{1\}$



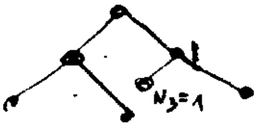
$N_4=1$ $N_1-N_2-N_4=1$

$1^\circ)$ $\boxed{N_1 = 2}$

$N_1=2$ $n-N_1=2$

$2^\circ)$ $N_2 \in \{1\}$

$N_3 \in \{1\}$



$n-N_1-N_3=1$

$N_2=1$ $N_1-N_2=1$

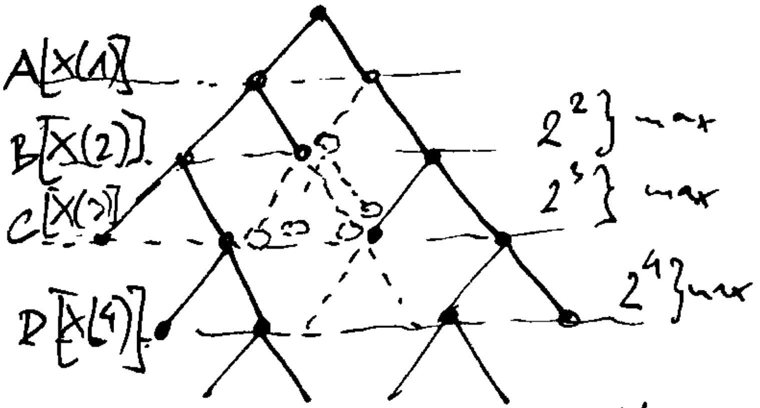
$\boxed{N_1 = 1}$ $1^0)$

$N_1=1$ $n-N_1=3$

$2^0)$ $N_2 \in \{\}$ $N_3 \in \{1 .. 4-1-1\} \in \{1, 2\}$ $N_3=1$

$N_4 \in \{\}$ $N_5 \in \{1 .. 2-1\} = \{1\}$

$n-N_1-N_3-N_5 = 4-3=1$



$\boxed{n=3}$ $\overbrace{\qquad\qquad}^{2^{n-1}}$

$$X'=\begin{array}{cccc|c} 1 & 2 & 3 & 4 & \\ 0 & 2 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 2 \end{array}\Big\}\, n-1$$

$$X''=\begin{array}{cccc|c} 1 & 2 & 3 & 4 & \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{array}$$

3

$\boxed{n=4}$

N_1

$$T_{41}=\begin{array}{cccccccc|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\ & & & 1 & \textcircled{3} & & & & 1 \\ & & & & 1 & 2 & & & 2 \\ & & & & & 1 & 1 & & 3 \end{array}$$

N_1

$$T_{42}=\begin{array}{cccccccc|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\ & & & 1 & \textcircled{3} & & & & \\ & & & & 2 & 1 & & & \\ & & & 1 & 1 & & & & \end{array}\ \cdots$$

N_1

$$T_{45}=\begin{array}{cccccccc|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\ & & & 2 & \textcircled{2} & & & & 1 \\ & & 1 & 1 & 1 & 1 & & & 2 \\ & & & & & & & & 3 \end{array}$$

$$\boxed{\begin{aligned} H(N_1)&=P(N_1=1)\,ld\frac{1}{P(N_1=1)}+\\ &P(N_1=2)\,ld\frac{1}{P(N_1=2)}+P(N_1=3)\,ld\frac{1}{P(N_1=3)}\\ &=\left(\frac{1}{3}\,ld\,3\right)\cdot 3=ld\,3\end{aligned}}$$

ELIMINACIJA NA DUPLIKATI

1 2 2 3 4 3   1 2 1 2 1 2   2 1 2 1 2 ; 1 2 1 2 ; 2 1 2

N_1

$$T_{43}=\begin{array}{cccccccc|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\ & & & 3 & \textcircled{1} & & & & 1 \\ & & 2 & & & & & & 2 \\ & 1 & 1 & & & & & & 3 \end{array}$$

N_1

$$T_{44}=\begin{array}{cccccccc|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\ & & & 3 & \textcircled{1} & & & & 1 \\ & & 1 & 2 & & & & & 2 \\ & & & 1 & 1 & & & & 3 \end{array}$$

- GI IMPLEMENTIRAV T_3 I T_4 VO MATLAB. TREBA DA GI GENERALIZIRAM.

(a) $H(N_1,T_4)=H(T_4)+H(N_1|T_4)=\left|\begin{array}{l} N_1=f(T_4)\\ \Rightarrow H(N_1|T_4)=0\end{array}\right|=H(T_4)$

$H(T_4)=H(N_1,T_4)=H(N_1)+H(T_4|N_1)$

(d) $H(T_4|N_1)=?$ $\quad H(T_4|N_1)=\frac{1}{n-1}\sum_{k=1}^{n-1}\left(H(T_k)+H(T_{n-k})\right)$

e.g $N_1=3$ $\boxed{n=4}$ (FIG ON PP.112)

$$H(T_4|N_1)=\frac{1}{3}\sum_{k=1}^{3}H(T_k)+H(T_{n-k})=\frac{1}{3}\left[\underline{H(T_1)+H(T_3)}+\right.$$

$$\left.+H(T_2)+H(T_2)+\underline{\underline{H(T_3)}}+\underline{H(T_1)}\right]=\frac{2}{3}\left[H(T_1)+H(T_2)+H(T_3)\right]$$

• ДИРЕКТНА ПРЕСМЕТКА НА :

$H(T_4|N_1=3) = H(T_{41},T_{42}) = \frac{1}{2} ld2 + \frac{1}{2} ld2 = 1$

T_{43} $H(T_4|N_1=2) = 1\, ld1 = 0$

$\boxed{H(T_4|N_1=1) = \frac{1}{2} ld2 + \frac{1}{2} ld2 = 1}$ $\nearrow$ T_{43}, T_{44}

$$\boxed{H(T_4|N_1) = P(N_1=3)\cdot H(T_4|N_1=3) + P(N_1=2)\cdot H(T_4|N_1=2) + P(N_1=1)\cdot H(T_4|N_1=1) = \frac{1}{3}\cdot 1 + \frac{1}{3}\cdot 1 + \frac{1}{3}\cdot 0 = \frac{2}{3}}$$

$\boxed{MMV}$ $\nearrow 1$

$H(T_4) = 4\cdot\frac{1}{6} ld6 + \frac{1}{3} ld3 = \frac{2}{3}\cancel{ld}2 + \frac{2}{3} ld3 + \frac{1}{3} ld3$

$= \frac{2}{3} + ld3$ $\quad H(T_4) = H(N_1) + H(T_4|N_1) = \left(\frac{1}{3} ld3\right)\cdot 3 + \frac{2}{3} = ld3 + \frac{2}{3}$

ПРОДОЛЖЕНА НА Л. 117

– DREAM ABOUT CONDITIONAL PROBABILITY $\boxed{MMV}$

$P(X,Y)$

X \ Y	♢	♡	♠	♣	P(X)
RED	$\frac{1}{2}\cdot\frac{1}{2}$	$\frac{1}{4}$	0	0	1/2
BLACK	0	0	$\frac{1}{4}$	$\frac{1}{4}$	1/2
P(Y)	$\frac{1}{4}$	1/4	1/4	1/4	

X – COLOUR

Y – CARD FIGURE

e.g. $P(X,Y) = (RED, ♢) =$

$= P(RED)\cdot P(♢/RED) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$

$\boxed{P(Y|X)}$

X \ Y	♢	♡	♠	♣	
RED	$2\cdot\frac{1}{4}$	$\frac{1}{2}$	0	0	
BLCK			$\frac{1}{2}$	1/2	

• Ако земам дека N_1 не е само за десната страна од првото се добива:

$H(T_4|N_1=3) = H(T_{41},T_{42},T_{43},T_{44}) = \cancel{4}\,\frac{1}{4} ld4 = 2$ #

• Ако овам со веројатности: $\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}$ #

$H(T_4) = \left(\frac{1}{5} ld5\right)5 = ld5$

$H(T_4|N_1=3) = \left(\frac{1}{2} ld2\right)\cdot 2 = 1 \quad H(T_4|N_1=2) = 1\, ld1 = 0$

$H(T_4|N_1=1) = \left(\frac{1}{2} ld2\right)2 = 1$ $\nearrow 1$

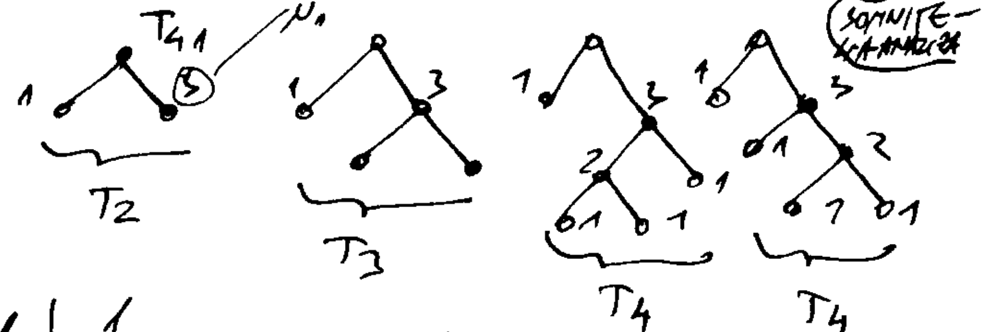
$H(T_4,N_1) \overset{?}{=} P(N_1=3)\cdot H(T_4|N_1=3) + P(N_1=2)\cdot H(T_4|N_1=2) +$

$+ P(N_1=1)\cdot H(T_4|N_1=1) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$ ✓

• AKO SE ZEMAT VKUPNOT BROJ NA DRVA KOI SE IZGLADENI :

SOMNITE- LNA ANALIZA

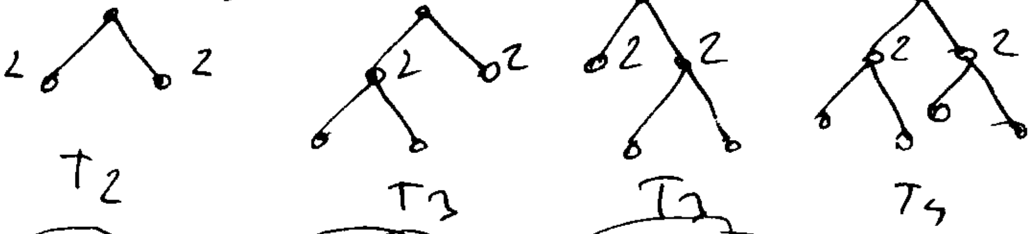
$\boxed{N_1=3}$ T_{41}, T_{42} (?)



$$H(T_4|N_1=3) = P(T_2)\,ld\frac{1}{P(T_2)} + P(T_3)\,ld\frac{1}{P(T_3)} + P(T_4)\,ld\frac{1}{P(T_4)}$$

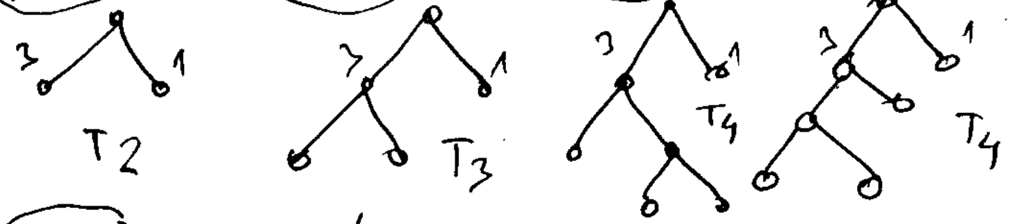
$$= \frac14\,\overset{2}{\cancel{ld4}} + \frac14\,\overset{2}{\cancel{ld4}} + \frac12\,\overset{1}{\cancel{ld2}} = \frac32$$

$\boxed{N_1=2}$ T_{45}



$$H(T_4|N_1=2) = \boxed{\frac14\,ld4} + \boxed{\frac12\,ld2} + \boxed{\frac14\,ld4} = \frac32$$

$\boxed{N_1=1}$ T_{43}, T_{44}



$$H(T_4|N_1=1) = \underbrace{\boxed{\frac14\,ld4}}_{H(T_2)} + \underbrace{\frac14\,ld4}_{H(T_3)} + \underbrace{\frac12\,ld2}_{H(T_4)} = \frac32$$

$$H(T_4|N_1) = P(N_1=3)\,\underline{H(T_4|\boxed{N_1=3})} + P(N_1=2)\,H(T_4|N_1=2) + P(N_1=1)\,H(T_4|N_1=1)$$

$$= \frac13\,\frac{2}{4}\,ld4 + \frac13\cdot\frac14\,ld4 + \frac13\,\frac24\,ld4 = \frac13 + \frac16 + \frac13 = \frac56$$

$$H(T_3|N_1) = \frac13\cdot\Big(\underset{\frac14 ld4}{H(T_3|N_1=3)} + \underset{\frac24 ld4}{H(T_3|N_1=2)} + \underset{\frac14 ld4}{H(T_3|N_1=1)}\Big) = 2/3$$

$$T(T_2|N_1) = \frac13\Big(\underset{1/2}{H(T_2|N_1=3)} + \underset{1/2}{H(T_2|N_1=2)} + \underset{1/2}{H(T_3|N_1=1)}\Big) = \frac12$$

$$H(T_3) = \frac{4}{12}\,ld3 = \frac13\,ld3 \qquad T(T_2) = \frac14\,ld4 = \frac12$$

$$\boxed{H(T_4) = \frac{5}{12}\,ld12} \qquad H(T_4) = H(N_1) + H(T_4|N_1) \to$$

$$= \cancel{\frac14}\,ld3 + \frac56$$

$$\boxed{H(T_4) = \frac{5}{12}\,ld(4\cdot3) = \frac{5}{12}\,ld4 + \frac{5}{12}\,ld3 = 5/6 + 5/12\,ld3}$$ (?)

• AKO SE ZEME VO PREDVID I „T_1"

$$H(T_4|N_1) = \frac{1}{3}\Big(\underbrace{H(T_4|N_1=3)}_{2/5\, ld \frac{5}{2}} + \underbrace{H(T_4|N_1=2)}_{1/5\, ld 5} + \underbrace{H(T_4|N_1=1)}_{2/5\, ld \frac{5}{2}}\Big) =$$

$$= \frac{1}{3}\Big(ld 5 \left(2/5 \cdot 2 + \frac{1}{5}\right) - ld 2 \left(\frac{2}{5} + \frac{2}{5}\right)\Big) = \frac{1}{3}\left(ld 5 - \frac{4}{5} \overset{1}{\cancel{ld 2}}\right)$$

$$= \frac{1}{3}\left(ld 5 - \frac{4}{5}\right)$$

?

$$H(T_4) = \frac{1}{3} ld 3 + \frac{1}{3} ld 5 - \frac{4}{15}$$

$$H(T_4) = \frac{1}{3} \cdot ld 3$$

$$H(N_1) = P(N_1=3)\, ld \frac{1}{P(N_1=3)} + P(N_1=2)\, ld \frac{1}{P(N_1=2)} + P(N_1=1)\, ld \frac{1}{P(N_1=1)} = 3 \frac{1}{3} ld 3 = ld 3$$

(SOMNITELNA ANALIZA !!!)

PRODOLZUVA OD PR.15

$$H(T_2) = 0 \qquad H(T_3) = \left(\frac{1}{2} ld 2\right) \cdot 2 = ld 2$$

$$H(T_4) = 4 \frac{1}{6} ld 6 + \frac{1}{3} ld 3 = \frac{2}{3} \overset{1}{\cancel{ld 2}} + \frac{2}{3} ld 3 + \frac{1}{3} ld 3$$

$$\boxed{H(T_4) = \frac{2}{3} + ld 3}$$

$$H(T_4|N_1) = \frac{2}{3}\left[\overset{0}{\cancel{H(T_1)}} + \overset{0}{\cancel{H(T_2)}} + \overset{ld 2}{\cancel{H(T_3)}}\right] = \frac{2}{3} \overset{1}{\cancel{ld 2}} = \frac{2}{3}$$

$$H(T_4|N_1) = \frac{1}{N-1} \sum_{k=1}^{4-1} \left[H(T_k) + H(T_{4-k})\right]$$

$$H(T_4|N_1) = P(N_1=4-1)\, H(T_4|N_1=4-1) + P(N_1=4-2)\, H(T_4|N_1=4-2) + \dots + P(N_1=1)\, H(T_4|N_1=1)$$

$$\boxed{P(N_1) = \frac{1}{4-1}} \qquad \boxed{H(T_4|N_1) = \frac{1}{4-1} \sum_{i=1}^{4-1} H(T_4|N_1=i)} \quad H(T_2) \quad H(T_1)$$

e.g. $H(T_4|N_1) = P(N_1=3)\, H(T_4|N_1=3) + P(N_1=2)\, H(T_4|N_1=2) + P(N_1=1)\, H(T_4|N_1=1)$

$$= \frac{1}{3}\left[\underbrace{2 \frac{1}{2} ld 2}_{1} + 1 \cdot \overset{0}{\cancel{ld 1}} + \underbrace{2 \frac{1}{2} ld 2}_{1}\right] = \frac{2}{3}$$

2*

DODADA ZATOA STO EDNAS SUMIRAS ZA DESNATA I EDNAS ZA LEVATA STRANA

$$H(T_4|N_1) = \frac{1}{3} \sum_{i=1}^{3} H(T_4|N_1=i) = \frac{1}{3} \sum_{i=1}^{3} \left[H(T_{4-i}) + H(T_i)\right]$$

$$H(T_4|N_1=3) = P(N_2=2) \cdot H(T_3|N_2=2) + P(N_2=1) \cdot H(T_3|N_2=1) =$$

$$= \frac{1}{2} = H(T_3) = ld 2 \qquad \boxed{H(T_4|N_1=3) = H(T_3)}$$

SUSTINA

- SUŠTINSKI ZAKLUČOK E DOLA:

$$H(T_n \mid N_1 = i) = H(T_{n-i}) + H(T_i)$$

(*) $\boxed{H(T_4 \mid N_1=3) = H(T_3)}$ GO IMA DVA PATI NA LEVA I DESNA STRANA

$$H(T_5 \mid N_1) = P(N_1=4)\cdot H(T_5 \mid N_1=4) + P(N_1=3)\cdot H(T_5 \mid N_1=3) +$$

$$+ P(N_1=2)\cdot H(T_5 \mid N_1=2) + P(N_1=1)\cdot H(T_5 \mid N_1=1) =$$

$$= \frac{1}{4}\left[H(T_4) + H(T_3) + H(T_2) + H(T_1) \right]\cdot ②$$

GO IMA ZA LEVATA I DESNATA STRANA

$\boxed{n=5}$

$$\sum_{i=1}^{4} H(T_i) + H(T_{n-i}) = H(T_1) + H(T_4) + H(T_2) + H(T_3) +$$

$$+ H(T_3) + H(T_2) + H(T_4) + H(T_1) =$$

$$= 2\left[H(T_1) + H(T_2) + H(T_3) + H(T_4) \right]$$

ISTIOT REZULTAT

(*) OVA E POSLEDICA NA TOA ŠTO SE RABOTI ZA MARKOV CHAIN

ZNAČI: $H(T_n \mid N_1) = \frac{1}{n-1}\sum_{i=1}^{n-1} H(T_i) + H(T_{n-1}) = \frac{2}{n-1}\sum_{i=1}^{n-1} H(T_i)$

$$H(T_3) = \left(\frac{1}{2}\cdot ld 2\right)\cdot 2 = ld\,2$$

$$H(T_4) = H(\textcircled{N_1}) + H(T_4 \mid N_1) = ld3 + \frac{2}{3}$$

$$\boxed{H(T_n) = ld(n-1) + \frac{2}{n-1}\sum_{i=1}^{n-1} H(T_i)}$$

POTSETI SE NA ČLANAKOT OD WIKI-PEDIJA KADE ZBOR ZA RANDOM WALK ON NUMBER SCALE

2 4 5 [MMV]

$$H(T_5) = ld4 + \frac{2}{4}\sum_{i=1}^{4} H(T_i) = ld4 + \frac{1}{2}\left(0+0+1+ + ld3 + \frac{2}{3}\right)$$

$$H(T_5) = 2 + \frac{1}{2} + \frac{1}{3} + \frac{1}{2} ld3 = \frac{12+3+2}{6} + \frac{1}{2} ld3 = \frac{19}{6} + \frac{1}{2} ld3$$

(f) $\underline{H_n} = ld(n-1) + \frac{2}{n-1}\sum_{i=1}^{n-1} H_k$

$$(n-1)H_n = (n-1)ld(n-1) + 2\sum_{k=1}^{n-1} H_k = (n-1)ld(n-1) + 2H_{n-1} + 2\sum_{k=1}^{n-2} H_k$$

$$H_{n-1} = ld(n-2) + \frac{2}{n-2}\sum_{k=1}^{n-2} H_k \Rightarrow 2\sum_{k=1}^{n-2} H_k = (n-2)H_{n-1} - (n-2)ld(n-2)$$

$$(n-1)H_n = (n-1)ld(n-1) + \cancel{2H_{n-1}} + (n-\cancel{2})H_{n-1} - (n-2)ld(n-2) =$$

$$= \underline{(n-1)ld(n-1) - (n-2)ld(n-2) + nH_{n-1}}$$

PROVED !!!

$$(n-1)H_n = (n-1)ld(n-1) - (n-2)ld(n-2) + nH_{n-1}$$

$$H_n = ld(n-1) - \frac{n-2}{n-1} ld(n-2) + \frac{n}{n-1} H_{n-1} \Rightarrow$$

$$\frac{H_n}{n} = \frac{H_{n-1}}{n-1} + \underbrace{\frac{ld(n-1)}{n} - \frac{n-2}{(n-1)\cdot n} ld(n-2)}_{C_n} \quad ▼$$

$$\boxed{C_n = \frac{(n-1) ld(n-1) - (n-2) ld(n-2)}{n(n-1)}} \qquad \boxed{\frac{H_n}{n} = \frac{H_{n-1}}{n-1} + C_n}$$

$$\lim_{n\to\infty} C_n = 0 \qquad \lim_{n\to\infty}\left(\frac{H_n}{n} - \frac{H_{n-1}}{n-1}\right) = 0$$

Edition 1 - SPECIAL FUTURE (JAS ŠTO IMPLICITNO GO UPOTREBIV)

$$H(T_n|N_1=k) = H(T_{nl}, T_{nr}|N_1=k) = H(T_n|N_1=k) + H(T_n|N_1=n-k) = H(T_k) + H(T_{n-k})$$

OVA E VAŽNA ZATOA ŠTO LEVOTO POD-DRVO I DESNOTO POD-DRVO SE IZBIRAAT NEZAVISNO.

SEGA GO OGLED DEFINICIJATA NA USLOVNA ENTROPIJA:

$$H(T_n|N_1) = \sum_{k=1}^{n-1} P(N_1=k)\cdot H(T_n|N_1=k) = \frac{1}{n-1}\sum_{k=1}^{n-1}\left[H(T_n|N_1=k) + H(T_n|N_1=n-k)\right]$$

$$= \frac{1}{n-1}\sum_{k=1}^{n-1}\left[H(T_k) + H(T_{n-k})\right] = \left|\begin{matrix}\text{LEVOTO I}\\ \text{DESNOTO POD-}\\ \text{DRVO SE}\\ \text{SIMETRIČNI}\end{matrix}\right| = \frac{2}{n-1}\sum_{k=1}^{n-1} H(T_k)$$

VO EDDITION 1 UŠTE PODOBRO GO DOKAŽUVAAT SO PROMENA NA IZMENLIVI

$$\frac{1}{n-1}\sum_{k=1}^{n-1}\left[H(T_k) + H(T_{n-k})\right] = \frac{1}{n-1}\sum_{k=1}^{n-1} H(T_k) + \frac{1}{n-1}\sum_{k=1}^{n-1} H(T_{n-k})$$

$$= \left| i = n-k \Rightarrow \begin{matrix} k=1 & i=n-1 \\ k=n-1 & i=1 \end{matrix}\right| = \frac{1}{n-1}\sum_{k=1}^{n-1} H(T_k) + \frac{1}{n-1}\sum_{i=1}^{n-1} H(T_i) =$$

$$= \frac{2}{n-1}\sum_{k=1}^{n-1} H(T_k)$$

▼

(I) AS IN Edition 1:

$$\boxed{C_n = \frac{ld(n-1)}{n} - \frac{ld(n-2)}{n-1} + \frac{2\, ld(n-2)}{n(n-1)}}$$

$$\frac{H_{n-1}}{n-1} = \frac{H_{n-2}}{n-2} + C_{n-1} \qquad C_{n-1} = \frac{ld(n-2)}{n-1} - \frac{ld(n-3)}{n-2} + \frac{2\, ld(n-3)}{(n-2)(n-1)}$$

$$\frac{H_n}{n} = \frac{H_{n-2}}{n-2} + C_n + C_{n-1} = \frac{H_{n-2}}{n-2} + \frac{ld(n-1)}{n} + \frac{2\, ld(n-2)}{n(n-1)} - \frac{ld(n-3)}{n-2} + \frac{2\, ld(n-3)}{(n-2)(n-1)}$$

$$Ⓘ = \frac{-(n-1)\, ld(n-2) + 2\, ld(n-2)}{(n-2)(n-1)} = \frac{n-2}{} \frac{-(n-3)\, ld(n-2)}{(n-2)(n-1)} \cdot n(n-1) \quad Ⓘ$$

070342764 MARKO
072334049
078270741 IGOR

- VMOTEN
- ELDORADO
- ANKETEN 2008
- TRANSFORMACIJA
- ORCUKA 2008

ĆE SE SKRATI VO NAREDNATA ITERACIJA (TELESCOPING THE SUM)

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① $$\frac{H_n}{n} = \frac{H_{n-1}}{n-1} + \frac{ld(n-1)}{n} - \frac{ld(n-2)}{n-1} + \frac{2\,ld(n-2)}{n(n-1)}$$

② $$\frac{H_n}{n} = \frac{H_{n-2}}{n-2} + \frac{ld(n-1)}{n} + \underbrace{\frac{2\,ld(n-2)}{n(n-1)} - \frac{ld(n-3)}{n-2} + \frac{2\,ld(n-3)}{(n-2)(n-1)}}$$

$$c_n = \frac{ld(n-1)}{n} - \frac{ld(n-2)}{n-1} + \frac{2\,ld(n-2)}{n(n-1)}$$

$$\sum_{j=n-1}^{n} \frac{2\,ld(j-2)}{j(j-1)} = \frac{2\,ld(n-3)}{(n-1)(n-2)} + \frac{2\,ld(n-2)}{n(n-1)} \Bigg\}$$ OD ②

$$\sum_{j=n}^{n} \frac{2\,ld(j-2)}{j(j-1)} = \frac{2\,ld(n-2)}{n(n-1)} \Bigg\}$$ OD ①

③ $$\frac{H_{n-2}}{n-2} = \frac{H_{n-3}}{n-3} + C_{n-2} = \frac{H_{n-3}}{n-3} + \frac{ld(n-3)}{n-2} \underbrace{- \frac{ld(n-4)}{n-3}} + \frac{2\,ld(n-4)}{(n-2)(n-3)}$$

ĆE SE SKRATI VO NAREDNATA ITERACIJA

$$\sum_{j=n-2}^{n} \frac{2\,ld(j-2)}{j(j-1)} = \underbrace{\frac{2\,ld(n-4)}{(n-2)(n-3)}}_{\text{POSLEDNIOT ČLEN OD 3°}} + \underbrace{\frac{2\,ld(n-3)}{(n-1)(n-2)}}_{\text{POSLEDNIOT ČLEN OD 2°}} + \underbrace{\frac{2\,ld(n-2)}{n(n-1)}}_{\text{POSLEDNIOT ČLEN OD 1°}}$$

③ ⇒ ②

$$\frac{H_n}{n} = \frac{H_{n-3}}{n-3} + \frac{ld(n-1)}{n} + \frac{2\,ld(n-2)}{n(n-1)} + \frac{2\,ld(n-3)}{(n-1)(n-2)} - \frac{ld(n-4)}{n-3} + \frac{2\,ld(n-4)}{(n-2)(n-3)}$$

ĆE SE SKRATI VO ČEKOR ④

$$\frac{H_n}{n} = \frac{ld(n-1)}{n} + \sum_{j=3}^{n} \frac{2\,ld(j-2)}{j(j-1)}$$

OD 3 POŠTO $\frac{H_2}{2} = 0$

$$\lim_{n\to\infty} \frac{H_n}{n} = \cancel{\lim_{n\to\infty} \frac{ld(n-1)}{n}}^{\to 0} + \sum_{j=3}^{\infty} \frac{2\,ld(j-2)}{j(j-1)} \leq \sum_{j=3}^{\infty} \frac{2\,ld(j-1)}{(j-1)^2}$$

$$= \sum_{k=2}^{\infty} \frac{2\,ld\,k}{k^3} \leq \sum_{j=2}^{\infty} \frac{2\sqrt{j}}{j^2} = \sum_{j=2}^{\infty} 2\cdot j^{-3/2} = \sum_{j=2}^{\infty} \frac{2}{\sqrt{j^3}} \Bigg\}$$ CONVERGIRA!!

$$\boxed{\lim \frac{H_n}{n} = 1.736 \text{ bits}}$$