PROBLEM 4.6 MONOTONICITY OF ENTROPY PER ELEMENT. FOR A STATIONARY STOCHASTIC PROCESS $x_1, x_2, \dots x_n$ SHOW THAT:

(a.) $$\frac{H(x_1, x_2, \dots x_n)}{n} \leq \frac{H(x_1, x_2, \dots x_{n-1})}{n-1}$$

(b.) $$\frac{H(x_1 x_2, \dots x_n)}{n} \geq H(x_n | x_{n-1}, \dots x_1)$$

(a) $$\frac{H(x_1, x_2, \dots x_n)}{n} = \frac{H(x_1, x_2, \dots x_{n-1}) + H(x_n | x_1, x_2, \dots x_{n-1})}{n}$$

$$H(x_n | x_1, \dots x_{n-1}) \geq 0$$

$$H(x_1, x_2, \dots x_{n-1}) = \sum_{i=1}^{n-1} H(x_i | x_{i-1}, \dots 1)$$

$$\underline{H(x_2, x_3, \dots x_n)} = \sum_{i=1}^{n-1} H(x_i | x_{i-1}, \dots 1)$$

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e.g. $x_i \in \{0, 1\}$ $\quad n = 2 \quad P(x_i) = \{\frac{1}{2}, \frac{1}{2}\}$

$P(0,0) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \quad P(0,1) = \frac{1}{4} \quad -- \quad P(1,1) = \frac{1}{4}$

x_1 & x_2 ARE INDEPENDENT

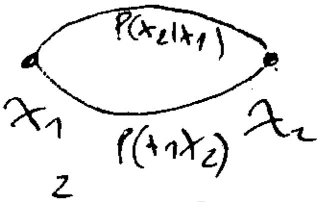
$$\boxed{H(x_1 x_2) = 4 \cdot \left(\frac{1}{4} ld 4\right) = 2}$$

$$\frac{H(x_1, x_2)}{n} = \frac{2}{2} = 1$$

$$\frac{H(x_1)}{1} = ?$$

$$H(x_1) = \frac{1}{2} ld 2 + \frac{1}{2} ld 2 = 1$$

$x_1 \backslash x_2$	0	1
0	0.9	0.1
1	0.1	0.3



APSTRAHIRA > SO ENTANGLED PARTICLES OD NUCLEARNA FIZIKA

MMV

SO OVOJ PRIMER GO SHVATIV HAJKOVIOT DEL VO CELOST!!

$$\mu_j = \sum_{i=1}^{2} \mu_i P_{ij}$$

$\mu_1 = \mu_1 \cdot P_{11} + \mu_2 \cdot P_{21} \qquad (1-\mu_1) P_{11} = \mu_2 P_{21}$

$\mu_2 = \mu_1 \cdot P_{12} + \mu_2 P_{22} \qquad (1-\mu_2) P_{22} = \mu_1 P_{12}$

$$\mu_2 = \frac{(1-\mu_1)}{} \frac{P_{11}}{P_{21}} \qquad \left[1 - (1-\mu_1)\frac{P_{11}}{P_{21}}\right] \frac{P_{22}}{P_{12}} = \mu_1$$

$$\frac{P_{21} - P_{11} + \mu_1 P_{11}}{P_{21}} \cdot \frac{P_{22}}{P_{12}} = \mu_1 \qquad P_{21} - P_{11} + \mu_1 P_{11} = \frac{P_{12} P_{21}}{P_{22}} \mu_1$$

$$\mu_1 \left(\frac{P_{11} P_{22} - P_{12} P_{21}}{P_{22}}\right) = -P_{21} + P_{11} \qquad \boxed{\mu_1 = \frac{P_{22}(P_{11} - P_{21})}{P_{11} P_{22} - P_{12} P_{21}}}$$

$$\mu_1 = \frac{p_{22}(p_{11}-p_{21})}{p_{11}p_{22}-p_{12}p_{21}} = \frac{0.9\,(0.9-0.1)}{0.81-0.01} = \frac{0.9}{0.8}\cdot 0.8 = \underline{\underline{0.9}}$$

$$\mu_2 = (1-\mu_1)\frac{p_{11}}{p_{21}} = \frac{\cancel{p_{11}p_{22}}-p_{12}\cancel{p_{21}}-\cancel{p_{11}p_{22}}+p_{22}\cancel{p_{21}}}{p_{11}p_{22}-p_{12}p_{21}}\cdot\frac{p_{11}}{\cancel{p_{21}}}$$

$$\mu_2 = \frac{p_{11}(p_{22}-p_{12})}{p_{11}p_{22}-p_{12}p_{21}} = \frac{0.9\,(0.9-0.1)}{0.81-0.01} = \underline{\underline{0.9}}$$

$P(0)=P(1)$
$=\frac{1}{2}$

MMV

$$\boxed{\begin{aligned}&P(0,0) = \underbrace{0.9}_{P(0|0)}\cdot 0.5 = 0.45 \quad P(1,1) = \underbrace{0.9}_{P(1|1)}\cdot 0.5 = 0.45 = P(1)\cdot P(1|1)\\ &P(0,1) = 0.1\cdot 0.5 = 0.05 \quad P(1,0) = 0.1\cdot 0.5 = 0.05 = P(1)\cdot P(0|1)\end{aligned}}$$

$$H(x_1,x_2) = 2\cdot 0.45\,\mathrm{ld}\frac{1}{0.45} + 2\cdot 0.05\,\mathrm{ld}\frac{1}{0.05} = \underline{\underline{1.46900}}$$

$$H(x_1) = 0.5\,\mathrm{ld}\,2 + 0.5\,\mathrm{ld}\,2 = 1$$

KOGA IMA ZAVISNOST POMALA E NEIZVESNOSTA.

$$\boxed{\frac{H(x_1,x_2)}{2} = 0.73450 \leq \frac{H(x_1)}{1} = 1}$$

GO DOKAZAV EMPIRISKI.

$$\frac{H(x_1,x_2,\dots x_n)}{\boxed{n}} \leq \frac{H(x_1,x_2,\dots x_{n-1})}{\boxed{n-1}}$$

$$H(x_1 x_2,\dots x_n) = \sum_{i=1}^{n} H(x_i|x_{i-1},\dots x_1) = H(x_1)+H(x_2/x_1)+$$

$$+ H(x_3|x_1,x_2) + \dots \underline{\underline{H(x_n|x_{n-1}\dots x_1)}} \leq H(x_1) +$$

$$H(x_2)+\dots +H(x_n)$$

$$\underline{H(x_1,x_2,\dots x_{n-1})} = -\underline{H(x_n|x_{n-1}\dots x_1)} + \underline{\underline{H(x_1,x_2,\dots x_n)}}$$

$$H(x_n|x_{n-1}\dots x_1) = H(x_{n+1}|x_n,\dots x_2) \geq \underline{H(x_{n+1}|x_n,\dots x_1)}$$

$$H(x_1,x_2,\dots x_{n-1}) = H(x_1,x_2,\dots x_{n-2}) + H(x_{n-1}|x_{n-2},\dots x_1)$$

$$= H(x_1 x_2,\dots x_{n-2}) + H(x_n|x_{n-1},\dots x_2) \geq H(x_1,x_2,\dots x_{n-2})$$

$$+ H(x_n|x_{n-1},\dots x_1)$$

$$\frac{H(x_1,x_2,\dots x_n)}{n} = \frac{1}{n}\sum_{i=1}^{n} H(x_i|x_{i-1},\dots x_1) = \frac{1}{n}\sum_{i=1}^{n-1} H(x_i|x_{i-1},\dots x_1)$$

$$+\frac{1}{n}H(x_n|x_{n-1},\dots x_1) \leq \frac{1}{n}(n-1)\,H(x_{n-1}|x_{n-2},\dots x_1)$$

$$+\frac{1}{n}H(x_n|x_{n-1},\dots x_1) = ⊛$$

$$H(x_n|x_{n-1},\dots x_1) = H(x_{n-1}|x_{n-2},\dots x_0) \leq H(x_{n-1}|x_{n-2},\dots x_1)$$

$$⊛ \leq \frac{1}{n}(n-1)\,H(x_{n-1}|x_{n-2},\dots x_1) + \frac{1}{n}H(x_{n-1}|x_{n-2},\dots x_1)$$

$$\frac{H(x_1,x_2,\dots x_{n-1})}{n-1} = \frac{1}{n-1}\sum_{i=1}^{n-1} H(x_i|x_{i-1},\dots x_1)$$

$$\geq \frac{n-1}{n-1} H(x_{n-1}|x_{n-2},\dots x_1) = H(x_n|x_{n-1},\dots x_2) \overset{\uparrow}{\geq}$$

CONDITIONING REDUCES ENTR.

$$\underline{H(x_n|x_{n-1},\dots x_2 x_1)}$$

$$\frac{H(x_1,x_2,\dots x_n)}{n} = \frac{1}{n}\sum_{i=1}^{n} H(x_i|x_{i-1}\dots x_1)$$

$$\frac{H(x_1,x_2,\dots x_n)}{n} \geq \frac{n}{n} H(x_n|x_{n-1},\dots x_1)$$

$$\boxed{H(x_n|x_{n-1},\dots x_1) \leq \frac{H(x_1,x_2,\dots x_n)}{n}}$$

$$\frac{H(x_1,x_2,\dots x_{n-1})}{n-1} \geq H(x_{n-1}|x_{n-2},\dots x_1) \geq \underline{H(x_n|x_{n-1},\dots x_1)}$$

$$\frac{H(x_1 x_2,\dots x_n)}{n} \geq H(x_n|x_{n-1},\dots x_1)$$

$$\frac{H(x_1,x_2,\dots x_{n-1}) + H(x_n|x_{n-1}\dots x_1)}{n} \geq H(x_n|x_{n-1},\dots x_1)$$

$$H(x_n|x_{n-1},\dots x_1) \geq n\,H(x_n|x_{n-1},\dots x_1) - \underline{H(x_1,x_2,\dots x_{n-1})}$$

$$\frac{\cancel{H(x_1,x_2,\dots x_{n-1})} + n\,H(x_1,x_2,\dots x_{n-1}) - \cancel{H(x_1,x_2\dots x_{n-1})}}{n-1} \geq n\,H(x_n|x_{n-1},\dots$$

$$\frac{H(x_1,x_2,\dots x_{n-1})}{n-1} \geq H(x_n|x_{n-1},\dots$$

$$\frac{H(x_1,x_2,\dots x_n)}{n} \leq \frac{H(x_1,x_2,\dots x_{n-1})}{n-1}$$

CONDITIONING REDUCES ENTROPY

$$H(x_n|x_{n-1},\dots x_1) = H(x_{n+1}|x_n,\dots x_2) \geq H(x_n|x_n,\dots x_2 x_1)$$

$$H(x_{n+1}|x_n,\dots x_1) \leq H(x_n|x_{n-1},\dots x_1)$$

SO DRUGI ZDOLOV!

$$\underline{H(x_{n-1}|x_{n-2},\dots x_1)} = H(x_n|x_{n-1},\dots x_2) \geq \underline{H(x_n|x_{n-1}\dots x_1)}$$

$$\boxed{H(x_n|x_{n-1},\dots x_1) \leq H(x_{n-1}|x_{n-2},\dots x_1)}$$

ACORDING THEOREM T.4.22 $H(X_n|X_{n-1}, X_{n-2}, \dots X_1)$ IS NONINCREASING (OVA E SUŠTINSKA ~~[illegible]~~ ZA OVAA ZADAČA)

$$\underline{H(X_n|X_{n-1}, \dots X_1) \leq H(X_{n-1}|X_{n-2}, \dots X_1)}$$

$$\frac{H(X_n|X_{n-1}, \dots X_1)}{n} \leq \frac{H(X_{n-1}|X_{n-2}, \dots X_1)}{n} \leq \frac{H(X_{n-1}|X_{n-2}, \dots X_1)}{n-1}$$

$$\frac{H(X_1, X_2, \dots X_n)}{n} = \frac{1}{n} \sum_{i=1}^{n} H(X_i|X_{i-1} \dots X_1) =$$

$$= \sum_{i=1}^{n} \frac{H(X_i|X_{i-1}, \dots X_1)}{n} \leq \sum_{i=1}^{n} \frac{H(X_i|X_{i-1}, \dots X_1)}{n-1} =$$

$$\frac{H(X_3|X_2)}{n} \leq \frac{H(X_2|X_1)}{n} \leq \frac{H(X_2|X_1)}{n-1}; \frac{H(X_2|X_1)}{n} \leq \frac{H(X_2)}{n} \leq \frac{H(X_2)}{n-1}$$

$$= \sum_{i=1}^{n} \frac{H(X_{i+1}|X_i, \dots X_2)}{\boxed{n-\cancel{1}}}$$

$(n-1)$ e SINGULARITY ZOTOA NEKA E, n =

$$\frac{H(X_2)}{1} + \frac{H(X_3|X_2)}{2} + \dots + \frac{H(X_n|X_{n-1}, \dots 2)}{n}$$

$$\frac{H(X_1, X_2, \dots X_{n-1})}{n-1} = \frac{1}{n-1} \sum_{i=1}^{n-1} H(X_i|X_{i-1}, \dots X_1) =$$

$$= \frac{1}{n-1} \sum_{i=1}^{n-1} H(X_{i+1}|X_i, \dots X_2) = \frac{1}{n-1} \left(H(X_2) + H(X_3|X_2 + \dots + H(X_n|X_{n-1}, \dots X_2) \right)$$

$$\geq \frac{1}{n-1} \sum_{i=1}^{n-1} H(X_{i+1}|X_i, \dots X_2, X_1) = \frac{1}{n-1} \left(H(X_2|X_1) + H(X_3|X_2 X_1) + \dots + H(X_n|X_{n-1} \dots X_1) \right)$$

(a) $$\frac{H(X_1, X_2, \dots X_n)}{n} = \frac{1}{n} \sum_{i=1}^{n} H(X_i|X_{i-1}, \dots X_1) \leq \frac{1}{n} \sum_{i=1}^{n} H(X_i|X_{i-1}, \dots X_2)$$

$$H(X_1) + H(X_2|X_1) + H(X_3|X_2 X_1) \dots \qquad H(X_1) + H(X_2) + H(X_3|X_2)$$

BY USE OF STATION.

$$= \frac{1}{n} \underbrace{\sum_{i=1}^{n} H(X_{i-1}|X_{i-2}, \dots X_1)}_{H(X_1, X_2, \dots X_{n-1}) = \sum_{i=1}^{n-1} H(X_i|X_{i-1}, \dots X_1)} = \frac{1}{n} \sum_{j=0}^{n-1} H(X_j|X_{j-1}, \dots X_1) = \left| \begin{matrix} X_0 \\ \in \emptyset \end{matrix} \right|$$

$j = i-1 \qquad \begin{matrix} i=1 & j=0 \\ i=n & j=n-1 \end{matrix}$

$$= \frac{1}{n} \sum_{j=1}^{n-1} H(x_j | x_{j-1}, \dots x_1) = \frac{1}{n} H(x_1, x_2 \dots x_{n-1}) \leq \frac{H(x_1, x_2, \dots x_{n-1})}{n-1}$$

PROVED!!!

$\boxed{\text{MNOGU ELEGANTEN DOKAZ}}$

(b) $$\frac{H(x_1, x_2, \dots x_n)}{n} \geq H(x_n | x_{n-1}, \dots x_1)$$

$\boxed{\text{THEOR. 4.2.2}}$

$\to$ $\boxed{H(x_n|x_{n-1}, \dots x_1) \text{ IS NONINCREASING IN } n}$

$$\boxed{H(x_n|x_{n-1}, \dots x_1) \leq H(x_{n-1}|x_{n-2}, \dots x_1) \leq H(x_{n-3}|x_{n-2}, \dots x_1)}$$

$$H(x_{n+1}|x_n, \dots x_2 x_1) \leq H(x_{n+1}|x_n, \dots x_2) \underset{\uparrow \text{ STATIONARITY}}{=} H(x_n|x_{n-1}, \dots x_1)$$

$$\frac{1}{n} H(x_1, x_2, \dots x_n) = \frac{1}{n} \sum_{i=1}^{n} H(x_i|x_{i-1}, \dots x_1) =$$

$$= \frac{1}{n} \Big[\underset{H(x_n|x_{n-1}, \dots x_1)}{\underset{\geq}{H(x_1)}} + \underset{H(x_n|x_{n-1}, \dots x_1)}{\underset{\geq}{H(x_2|x_1)}} + \dots + \underset{H(x_n|x_{n-1}, \dots x_1)}{\underset{=}{H(x_n|x_{n-1} \dots x_1)}} \Big] ⓶$$

$$\textcircled{\geq} \frac{1}{\cancel{n}} \cdot \cancel{n}\, H(x_n|x_{n-1}, \dots x_1) = H(x_n|x_{n-1}, \dots x_1)$$

PROVED!!!

$\boxed{\text{Edition 1 Solution}}$ (b) - ISTO GO IMAM REŠENO

(a) $$\frac{H(x_1, x_2, \dots x_n)}{n} = \frac{\sum_{i=1}^{n} H(x_i|x_{i-1}, \dots x_1)}{n} = \frac{\sum_{i=1}^{n} H(x_i|x^{i-1})}{n}$$

$$= \frac{H(x_n|x^{n-1}) + \sum_{i=1}^{n-1} H(x_i|x^{i-1})}{n} = \frac{H(x_n|x^{n-1}) + H(x_1, x_2, \dots x_{n-1})}{n}$$

- STATIONARITY $\Rightarrow$ NONINCREASING THEOREM

KOMPAKTNA FORMA NA THEOREM T.4.2.2

$$\boxed{H(x_n|x^{n-1}) \leq H(x_i|x^{i-1}) \quad i = 1, 2, \dots n}$$

$$\frac{1}{n-1} \underbrace{\sum_{i=1}^{n-1} H(x_i|x^{i-1})}_{= H(x_1, x_2, \dots x_n)} \geq \frac{\cancel{n-1}}{\cancel{n-1}} H(x_n|x^{n-1})$$

$$\Rightarrow \boxed{\frac{H(x_1, x_2, \dots x_n)}{n-1} \geq H(x_n|x^{n-1})}$$

$$\frac{H(x_n|x^{n-1}) + H(x_1, x_2, \dots x_{n-1})}{n} \leq \frac{\frac{H(x_1, x_2, \dots x_n)}{n-1} + H(x_1, x_2, \dots x_{n-1})}{n}$$

$$= \frac{\cancel{n}\, H(x_1, x_2, \dots x_{n-1})}{(n-1)\, \cancel{n}} = \frac{H(x_1, x_2, \dots x_{n-1})}{(n-1)}$$

PROVED!!!

$\boxed{4.7}$ ENTROPY RATES OF MARKOV CHAINS

(a) FIND THE ENTROPY RATE OF TWO STATE MARKOV CHAIN WITH TRANSITION MATRIX:

$$P = \begin{array}{c|cc} & 0 & 1 \\ 0 & 1-p_{01} & p_{01} \\ 1 & p_{10} & 1-p_{10} \end{array}$$

(b) WHAT VALUES OF p_{01}, p_{10} MAXIMIZE THE ENTROPY RATE?

(c) FIND THE ENTROPY RATE OF TWO STATE MARKOV CHAIN WITH TRANSITION MATRIX

$$P = \begin{array}{c|cc} & 0 & 1 \\ 0 & 1-p & p \\ 1 & 1 & 0 \end{array}$$

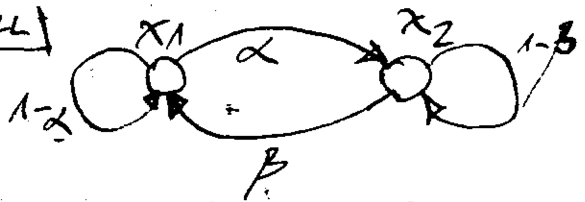
(d) FIND THE MAXIMUM VALUE OF ENTROPY RATE OF MARKOV CHAIN OF PART (c). WE EXPECT THAT MAXIMIZING VALUE OF „p" SHOULD BE LESS THEN 1/2, SINCE THE 0 STATE PERMITS MORE INFORMATION TO BE GENERATED THEN THE 1 STATE.

(e) LET N(t) BE THE NUMBER OF ALLOWABLE STATE SEQUENCES OF LENGTH „t" FOR MARKOV CHAIN OF PART (c). FIND N(t) AND CALCULATE:

$$H_0 = \lim_{t\to\infty} \frac{1}{t} \, ld \, N(T)$$

[HINT: FIND A LINEAR RECURRENCE THAT EXPRESSES N(T) IN TERMS OF N(t-1) & N(t-2). WHY H_0 IS AN UPPER BOUND ON THE ENTROPY RATE OF MARKOV CHAIN? COMPARE H_0 WITH MAXIMUM ENTROPY FOUND IN PART (d)

RECALL



$$\mu_j = \sum_{i=1}^{2} \mu_i P_{ij} \qquad j = 1,2$$

$$\mu_1 = \mu_1 \cdot P_{11} + \mu_2 P_{21} = \mu_1(1-\alpha) + \mu_2 \cdot \beta$$

$$\mu_2 = \mu_1 \cdot P_{12} + \mu_2 P_{22} = \mu_1 \alpha + \mu_2(1-\beta)$$

$$\mu_1(1-1+\alpha) = \mu_2 \cdot \beta \qquad \mu_2(1-1+\beta) = \mu_1 \alpha$$

$$\boxed{\mu_1 \alpha = \mu_2 \beta} \qquad \mu_1 + \mu_2 = 1$$

$$\mu_2 = \frac{\alpha}{\beta} \cdot \mu_1 \qquad \mu_1 + \frac{\alpha}{\beta}\mu_1 = 1 \qquad \boxed{\mu_1 = \frac{1}{\frac{\alpha+\beta}{\beta}} = \frac{\beta}{\alpha+\beta}}$$

$$\boxed{\mu_2 = \frac{\alpha}{\alpha+\beta}} \qquad H(X) = \mu_1 \, ld \frac{1}{\mu_1} + \mu_2 \, ld \frac{1}{\mu_2}$$

$$H(X) = \frac{\beta}{\alpha+\beta} \, ld \frac{\alpha+\beta}{\beta} + \frac{\alpha}{\alpha+\beta} \, ld \frac{\alpha+\beta}{\alpha} \Bigg\} \text{ STATIONARY MARKOV CHAIN}$$

$$H(X) = \lim_{n\to\infty} \frac{1}{n} H(x_1 x_2 \dots x_n)$$

$$H'(X) = \lim_{n\to\infty} H(x_n | x_{n-1}, \dots x_1)$$

$$p(x_1 x_2 \dots x_n) = p(x_1) \cdot p(x_2|x_1) \, p(x_3|x_1) \cdots p(x_n|x_{n-1})$$

$$H(\mathcal{X}) = \lim_{n\to\infty} \frac{1}{n}\left(H(X_1) + H(X_2|X_1) + H(X_3|X_2) + \dots + H(X_n|X_{n-1})\right)$$

$$= \frac{\cancel{n}}{\cancel{n}} H(X_n|X_{n-1}) \Rightarrow H(X_n|X_{n-1}) = \underline{H(X_2|X_1)}$$

$$= H'(\mathcal{X}) \qquad \underline{\text{FOR STATIONARY MARKOV CHAIN}}$$

$$I(X_1;X_2) \geq I(X_1;X_3) \Rightarrow \underline{H(X_2|X_1) \leq H(X_3|X_1)}$$

$$\underline{H(X_1|X_2) \leq H(X_1|X_3)}$$

- ENTROPY RATE OF MARKOV CHAIN

$X_1 \sim \mu$.

$$H(\mathcal{X}) = H(X_2|X_1) \qquad H(Y|X=x) = \sum_y p(y|X=x)\,\mathrm{ld}\frac{1}{p(y|x)}$$

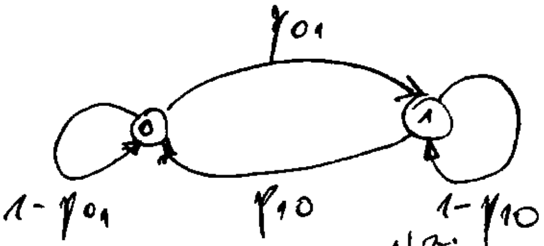
$$H(Y|X) = \overline{H(Y|X=x)} = \sum_x p(x) \sum_y p(y|x)\,\mathrm{ld}\frac{1}{p(y|x)}$$

$$= \sum_{x,y} p(x,y)\,\mathrm{ld}\frac{1}{p(y|x)} \Rightarrow$$

$$H(\mathcal{X}) = -\sum_{i=1}^{n} \mu_i \sum_{j=1}^{n} P_{ij}\,\mathrm{ld}\,P_{ij} = -\sum_{i=1}^{n}\sum_{j=1}^{n} \mu_i P_{ij}\,\mathrm{ld}\,P_{ij}$$

FOR STATIONARY-MARKOV CHAIN !!!

(a)



$$\mu_0 = \frac{p_{10}}{p_{01}+p_{10}} \qquad \mu_1 = \frac{p_{01}}{p_{01}+p_{10}}.$$

$$H(\mathcal{X}) = -\sum_{\substack{i=0\\j=0}}^{1} \mu_i P_{ij}\,\overset{\mathrm{ld}\,P_{ij}}{\vee} = -\mu_0 \sum_{j=0}^{1} P_{0j}\,\mathrm{ld}\,P_{0j} - \mu_1 \sum_{j=0}^{1} P_{1j}\,\mathrm{ld}\,P_{1j} =$$

$p_{00} = 1 - p_{01}$ $\qquad$ $p_{11} = 1 - p_{10}$

$$-\mu_0 \underbrace{\left[p_{00}\,\mathrm{ld}\,p_{00} + p_{01}\,\mathrm{ld}\,p_{01}\right]}_{-H(p_{01})} - \mu_1 \underbrace{\left[p_{10}\,\mathrm{ld}\,p_{10} + p_{11}\,\mathrm{ld}\,p_{11}\right]}_{-H(p_{10})}.$$

$$\boxed{H(\mathcal{X}) = \mu_0 H(p_{01}) + \mu_1 H(p_{10})} = \underline{\frac{p_{10}}{p_{01}+p_{10}} H(p_{01}) + \frac{p_{01}}{p_{01}+p_{10}} H(p_{10})}$$

$$\max\left[H(p_{01})\right] = 1 \ \text{if} \ \boxed{p_{01} = \tfrac{1}{2}}$$

$$\max\left[H(p_{10})\right] = 1 \qquad \boxed{p_{10} = \tfrac{1}{2}}$$

$$\boxed{\begin{matrix} p_{10} = p \\ p_{01} = \alpha \end{matrix}}$$

$$\frac{d\,H(x)}{d\,p_{01}} = -p\,\frac{\left[-(1+p)\,\mathrm{ld}(1-\alpha) + (1-p)\,\mathrm{ld}(1-p) + p(\mathrm{ld}\,\alpha + \mathrm{ld}\,p)\right]}{(\alpha+p)^2} = 0$$

$$-(1+p)\,\mathrm{ld}(1-\alpha) + p\,\mathrm{ld}\,\alpha = -(1-p)\,\mathrm{ld}(1-p) - p\,\mathrm{ld}\,p = H(p)$$

$$-p\,\mathrm{ld}\,\alpha + (1+p)\,\mathrm{ld}(1-\alpha) = (1-p)\,\mathrm{ld}(1-p) + p\,\mathrm{ld}\,p = -H(p)$$

$\mathrm{ld}\,\alpha^{p} + \mathrm{ld}\,(1-\alpha)^{-(1+p)} = -H(p)$ $\qquad$ $\mathrm{ld}\,\frac{\alpha^{p}}{(1-\alpha)^{1+p}} = H(p)$

$$\frac{\alpha^{p}}{(1-\alpha)^{1+p}} = 2^{H(p)} \qquad 2^{-H(p)} = \frac{1}{\alpha^{p}} \sum_{i=0}^{p} (-1)^{i} \binom{p}{i} \alpha^{i}$$

$$(1-\alpha)^{p} = \sum_{i=0}^{p} \binom{p}{i} \alpha^{i}$$

$$(1-\alpha)^{2} = 1 - 2\alpha + \alpha^{2}$$

$$\binom{r}{k} = \frac{(r)_k}{k!} = \frac{r(r-1)\cdots(r-k+1)}{k!}$$

VO GENERAZEN SLUČAJ

$$\boxed{(x+y)^{r} = \sum_{k=0}^{\infty} \binom{r}{k} x^{r-k} y^{k}}$$

NEWTON'S GENERALIZATION

$$(1-\alpha)\left(\frac{1-\alpha}{\alpha}\right)^{p} = 2^{-H(p)}$$

$$H(x) = \frac{p}{\alpha+p} H(\alpha) + \frac{\alpha}{\alpha+p} H(p) \qquad \frac{d\,H(x)}{d\alpha} = 0$$

$$\frac{d}{d\alpha}\left[\frac{p}{\alpha+p} \cdot H(\alpha)\right] + H(p)\,\frac{d}{d\alpha}\left(\frac{\alpha}{\alpha+p}\right) = 0$$

$$\frac{d\,H(\alpha)}{d\alpha} = -\mathrm{ld}\,\alpha + \mathrm{ld}\,(1-\alpha) \qquad \frac{d}{d\alpha}\left(\frac{\alpha}{\alpha+p}\right) = \frac{p}{(\alpha+p)^{2}}$$

$$\frac{d}{d\alpha}\left(\frac{p}{\alpha+p}\right) = -\frac{p}{(\alpha+p)^{2}}$$

$$-\frac{\cancel{p}}{(\alpha+p)^{2}} H(\alpha) + \frac{\cancel{p}}{(\alpha+p)}\left[-\mathrm{ld}\,\alpha + \mathrm{ld}\,(1-\alpha)\right] + H(p)\,\frac{p}{(\alpha+p)^{2}} = 0$$

$$-\frac{H(\alpha)}{(\alpha+p)} - \left[\mathrm{ld}\,\alpha - \mathrm{ld}\,(1-\alpha)\right] + \frac{H(p)}{(\alpha+p)} = 0$$

$$-(\alpha+p)\left[\mathrm{ld}\,\alpha - \mathrm{ld}\,(1-\alpha)\right] + \underline{H(p)} = \underline{H(\alpha)}$$

$$-\alpha\,\cancel{\mathrm{ld}\,\alpha} + \alpha\,\mathrm{ld}\,(1-\alpha) - p\,\mathrm{ld}\,\alpha + p\,\mathrm{ld}\,(1-\alpha) + H(p) = -\alpha\,\cancel{\mathrm{ld}\,\alpha} - (1-\alpha)\,\mathrm{ld}\,(1-\alpha)$$

$$-p\,\mathrm{ld}\,\alpha + p\,\mathrm{ld}\,(1-\alpha) + H(p) = -\mathrm{ld}\,(1-\alpha)$$

$$-p\,\mathrm{ld}\,\alpha + (1+p)\,\mathrm{ld}\,(1-\alpha) = -H(p)$$

$$-p\,\mathrm{ld}\,\alpha + (1+p)\,\mathrm{ld}\,(1-\alpha) = +p\,\mathrm{ld}\,p + (1-p)\,\mathrm{ld}\,p$$

$$p\,\mathrm{ld}\,\alpha \cdot p = (1+p)\,\mathrm{ld}\,(1-\alpha) + (p-1)\,\mathrm{ld}\,p$$

$$p\,\mathrm{ld}\,\alpha \cdot p = p\,\mathrm{ld}\,(1-\alpha) + p\,\mathrm{ld}\,p + \mathrm{ld}\,(1-\alpha) - \mathrm{ld}\,p$$

$$p\,\mathrm{ld}(\alpha \cdot p) = p\,\mathrm{ld}\,(1-\alpha)p + \mathrm{ld}\left(\frac{1-\alpha}{p}\right) \qquad \mathrm{ld}\,(\alpha \cdot p)^{p} = \mathrm{ld}\,(1-\alpha)^{p} \cdot p^{p} \cdot \frac{1-\alpha}{p}$$

$$(\alpha \cdot p)^{p} = (1-\alpha)^{p+1}\,p^{p-1} \qquad \alpha^{p} \cdot \cancel{p^{p}} = (1-\alpha)^{p+1} \cdot \frac{\cancel{p^{p}}}{p}$$

$$\frac{\alpha^{p}}{(1+\alpha)^{p}} = \frac{1-\alpha}{p}$$

$$\boxed{\left(\frac{\alpha}{1-\alpha}\right)^{\beta} = \left(\frac{1-\alpha}{\beta}\right)} \quad \left(\frac{1-\alpha}{\alpha}\right)^{\beta} = \frac{\beta}{(1-\alpha)} \quad \left(\frac{1}{\alpha}-1\right)^{\beta} = \frac{\beta}{1-\alpha}$$

$$x = \frac{1}{\alpha} \quad (x-1)^{\beta} = \frac{\beta}{\alpha(x-1)} \quad \boxed{(x-1)^{\beta+1} = x \cdot \beta}$$

$$\alpha + (1-\alpha) + \beta + (1-\beta) = 2$$

$$\left(\frac{\alpha}{1-\alpha}\right)^{\beta} = \left(\frac{1-\alpha}{\beta}\right) \qquad \frac{\alpha^{\beta}}{(1-\alpha)^{\beta+1}} = \frac{1}{\beta} \quad / \cdot \alpha$$

$$\frac{\alpha^{\beta+1}}{(1-\alpha)^{\beta+1}} = \frac{\alpha}{\beta} \qquad \boxed{\left(\frac{\alpha}{1-\alpha}\right)^{\beta+1} = \frac{\alpha}{\beta}}$$

$$\boxed{\beta = 1-\alpha} \qquad \left(\frac{\alpha}{1-\alpha}\right)^{\beta+1} = \left(\frac{\alpha}{1-\alpha}\right) \quad \left(\frac{\alpha}{1-\alpha}\right)^{\beta} = 1$$

$$\boxed{\alpha = 1-\alpha} \quad \boxed{\alpha = \frac{1}{2}} \quad \boxed{\beta = 1 - \frac{1}{2} = \frac{1}{2}}$$

$$\boxed{\beta = 1} \quad \left(\frac{\alpha}{1-\alpha}\right)^{2} = 1 \qquad \frac{\alpha}{1-\alpha} = 1 \qquad \begin{array}{l} \alpha = 1-\alpha \\ 2\alpha = 1 \quad \boxed{\alpha = \frac{1}{2}} \end{array}$$

$$f(\alpha,\beta) = \left(\frac{\alpha}{1-\alpha}\right)^{\beta+1} - \frac{\alpha}{\beta}$$

$$g(\alpha,\beta) = c$$

LAGRANGE MULTIPLIERS

$$\Lambda(\alpha,\beta,\lambda) = f(\alpha,\beta) + \lambda(\underbrace{g(\alpha,\beta)}_{g(\alpha,\beta)=0} - c)$$

$$\nabla_{\alpha,\beta,\lambda}\left[\Lambda(\alpha,\beta,\lambda)\right] = 0$$

$$1^{0} \quad -(1+\beta)\ln(1-\alpha) + (1-\beta)\ln(1-\beta) + \beta(\ln(\alpha) + \ln(\beta)) = 0 = \frac{dH}{d\alpha}$$

$$2^{0} \quad (1-\alpha)\ln(1-\alpha) + (1+\alpha)\ln(1-\beta) + \alpha(\ln(\alpha) + \ln(\beta)) = 0 = \frac{dH}{d\beta}$$

$$1^{0} \Rightarrow \left(\frac{\alpha}{1-\alpha}\right)^{\beta+1} = \frac{\alpha}{\beta} \qquad 2^{0} \quad \left(\frac{\beta}{1-\beta}\right)^{\alpha+1} = \frac{\beta}{\alpha}$$

$$2^{0} \Rightarrow \left(\frac{\alpha}{\beta}\right) = \left(\frac{1-\beta}{\beta}\right)^{\alpha+1} \qquad \left(\frac{\alpha}{1-\alpha}\right)^{\beta+1} = \left(\frac{1-\beta}{\beta}\right)^{\alpha+1}$$

$$\left(\frac{\alpha}{1-\alpha}\right)^{\beta+1} + \left(\frac{\beta}{1-\beta}\right)^{\alpha+1} = 1+1$$

→ SO REŠAVAЊE VO MAPLE SE DOBIVA : $\alpha = \frac{1}{2}$ $\beta = \frac{1}{2}$

$$\boxed{p_{01} = \alpha = \frac{1}{2} \quad p_{10} = \beta = \frac{1}{2}}$$

(c) (d) $P = \begin{bmatrix} 1-\gamma & \gamma \\ 1 & 0 \end{bmatrix}$ $\qquad \alpha = \gamma$, $\beta = 1$

$H(x) = \frac{\beta}{\alpha+\beta} H(\alpha) + \frac{\alpha}{\alpha+\beta} H(\beta)$

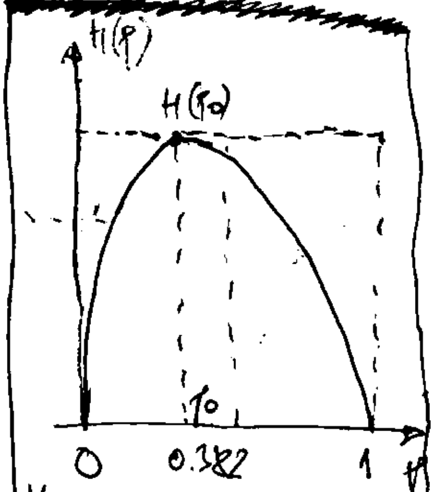
$H(\alpha) = \gamma \, ld \frac{1}{\gamma} + (1-\gamma) \, ld \frac{1}{1-\gamma} = H(\gamma)$

$H(\beta) \overset{?}{=} \underbrace{1 \, ld \, 1}_{0} - \underbrace{(1-1) \, ld \, 0}_{0} = 0$

$$\boxed{H(x) = \frac{1}{\gamma+1}\left[\gamma \, ld \frac{1}{\gamma} + (1-\gamma) \, ld \frac{1}{1-\gamma}\right]}$$

$\frac{dH(x)}{d\gamma} = \frac{2 \cdot \ln(1-\gamma) - \ln \gamma}{(\gamma+1)^2 \ln 2}$

$\Rightarrow \boxed{\gamma_0 = 0.38197}$ $\qquad \boxed{H(\gamma_0) = 0.69424}$



VO DEFINICIJATA NA IZVOROT GO PRETPOSTAVUVAME OVA. // MAKSIMUMOT E POMESTEN KON γ=0 ZATOA ŠTO 0-TA NOSI POGOLEMA INFORMACIJA.

$$P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0.35245 & 0.64755 \\ 1 & 1 & 0 \end{array}$$

$\gamma_0 = \frac{3-\sqrt{5}}{2}$ $\qquad 1-\gamma_0 = \frac{2-3+\sqrt{5}}{2} = \frac{\sqrt{5}-1}{2}$

$\frac{1}{\frac{\sqrt{5}-1}{2}} = \frac{2(\sqrt{5}+1)}{(\sqrt{5})^2-1^2} = \frac{\cancel{2}(1+\sqrt{5})}{\cancel{4}_2} = \frac{1+\sqrt{5}}{2}$

GOLDEN RATIO (FIBONACI)

$P(0,0) = P(0) \cdot P(0|0) = \frac{1}{2} \cdot 0.35245 = 0.17623$

$P(0,1) = P(0) \cdot P(1|0) = \frac{1}{2} \cdot 0.64755 = 0.32378$

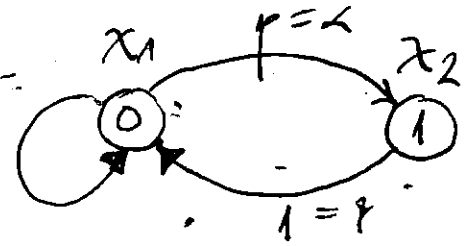
$P(1,0) = P(1) \cdot P(0|1) = \frac{1}{2} \cdot 1 = 1/2$

$P(1,1) = P(1) \cdot P(1|1) = \frac{1}{2} \cdot 0 = 0$

$P(0) = P(0,0) + P(1,0) = 0.5+0.176$

$x_1 \in \{0,1\}$ $x_2 \in \{0,1\}$

• (e) N(t) - NUMBER OF ALLOWABLE STATE SEQUENCES OF 'LENGHT' "t". FIND N(t)!



$$P = \begin{array}{c|cc} x_1 \backslash x_2 & 1 & 2 \\ \hline 1 & 1-\gamma & \gamma \\ 2 & 1 & 0 \end{array} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

$P(0,0) = \underline{\underline{P(0)}} \cdot \underline{\underline{P(0|0)}} = \frac{1}{2}(1-\gamma)$

$P(0,1) = P(0) \cdot P(1|0) = \frac{1}{2} \cdot \gamma$

$P(1,0) = P(1) \cdot P(0|1) = 1 \cdot 1$

$\underline{P(1,1) = P(1) \cdot P(1|1) = \frac{1}{2} \cdot 0}$

$N(t) = 3$ $\qquad P(0,0) \overset{?}{+} P(0,1) + P(1,0) = \frac{1}{2}(1-\gamma) + \frac{1}{2}\gamma + \frac{1}{2} = 1$

$1 - \cancel{\gamma} + \cancel{\gamma} + 1 = 2$ $\qquad \boxed{2 = 2}$

$N(0) = (0,0) \rightarrow (1,0),\ (0,1),\ (0,0)$ $\Big\}$ $N(0) = 3$

$N(1)=(0,1) \to 00,\ 10,\ 01$ $\qquad N(2)=(1,0) \to (0\ 0),\ (1\ 0)$

$N(1)=3 \qquad\qquad N(2)=2$

АКО МОЖЕ ДА ОСТАНЕ ВО ПОСТОЈНАТА СОСТОЈБА
- АКО НЕ ТОГАШ Е ПОСЛЕДОВАТНО -

$N(1)=(0,0) \to (1\ 0),\ (0\ 1) \qquad N(2)=(01) \to 10$

$$\boxed{\mu_1 = \frac{\beta}{\alpha+\beta} = \frac{1}{1+p} \qquad \mu_2 = \frac{\alpha}{\alpha+\beta} = \frac{p}{1+p}}$$

$$\mu_1 = \sum_{i=1}^{2} \mu_i P_{i1} = \mu_1 \cdot P_{11} + \mu_2 P_{21}$$

$$\mu_2 = \sum_{i=1}^{2} \mu_i P_{i2} = \mu_1 P_{12} + \cancel{\mu_2 P_{22}} \xrightarrow{} 0$$

$$\boxed{\mu_2 = \mu_1 \cdot P_{12}} \quad ① \qquad \mu_1 = \mu_1 P_{11} + \mu_1 P_{12} P_{21}$$

$$1 - P_{11} = P_{12} \cancel{P_{21}} \to 1 \qquad P_{12} = 1-(1-p) = p$$

$$P(x_2=1) = P(x_1=1)\cdot P_{12} = \frac{1}{2}\cdot p \quad P(1|1)$$

$$-P(x_2=0) - P(x_1=0)\cdot P_{12} = \frac{1}{2}p \quad \to P(0|0)$$

$P(x_1 =$

$P(x_2|x_1)$

$x_1 \backslash x_2$	0	1
0	$1-p$	p
1	1	0

$$\boxed{P(x_1,x_2) = P(x_1)\cdot P(x_2|x_1)}$$

$x_1 \backslash x_2$	0	1	$P(X1)$
0	$\frac{1}{2}(1-p)$	$\frac{1}{2}\cdot p$	$\frac{1}{2}$
①	$\frac{1}{2}$	0	$\frac{1}{2}$

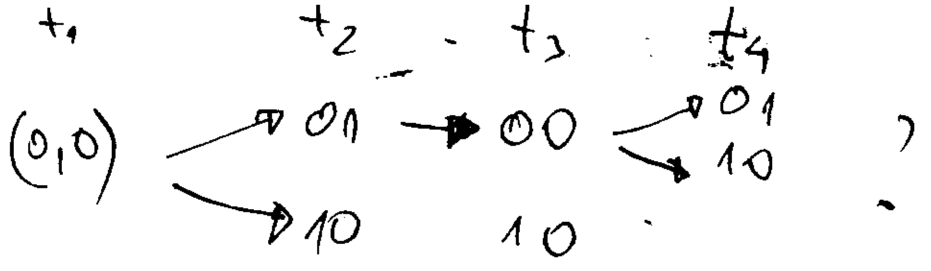
$$P(0,0) = \frac{1}{2}(1-p)$$
$$P(0,1) = \frac{1}{2}p$$
$$P(1,0) = \frac{1}{2}$$

ЛАНЕЦОТ НЕ МОЖЕ ДА СЕ НАЈДЕ ВО (1, 1)

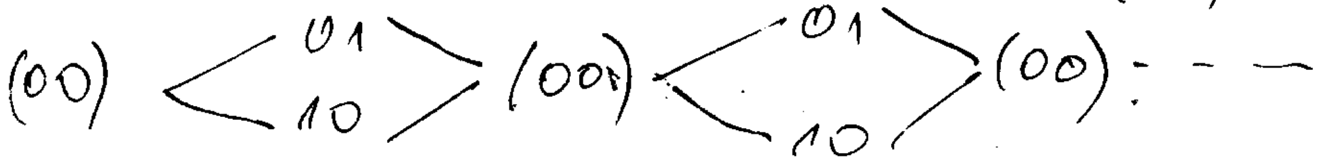
$$N_1=(0,0) \to (0,1) \to (0,0),\ (1,0)\ ?\ ①$$
$$N_1=(0,0) \to (1,0) \to (0,0)$$

$(t-2) \qquad (t-1) \qquad (t)$

$N(t)=1$
$N(t-1)=2$
$N(t-2)=1$



$N(t)=1$
$N(t+1)=2$
$N(t+2)=1$
$N(t+3)=2$

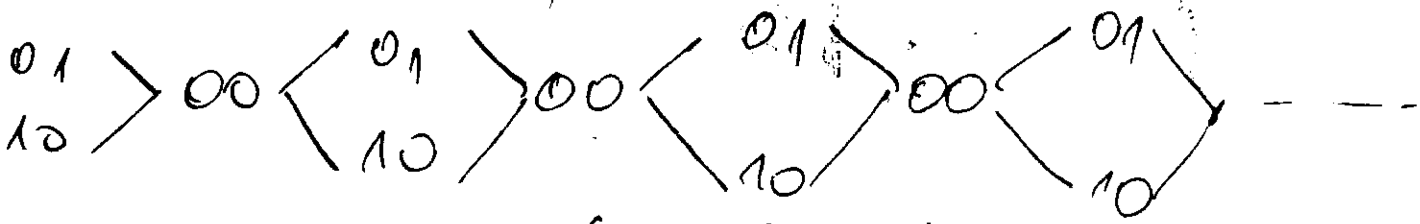


N(t)	1	3	4	6	7	9	10	12
t	1	2	3	4	5	6	7	8

$N(t_1)=1$ $N(t_2)=3$ $N(t_3)=4$ $N(t_4)=6$ $N(t_5)=7$

$$\lim_{t\to\infty}\frac{1}{t}\,ld\,N(t)$$

$$N(t)=(-1)^{t-1}N(t_1)+2(t-1)$$



t	1	2	3	4	5	6	7				
N(t)	2	3	5	6	8	9	11	12	14	15	~~17~~

18, 20, 21, 23, 24

$$N(t)=N_1(t)+N_0(t)$$

$$N_1(t)=2+6+8+12+14+18+20+24$$

1 11 1 11

$$s=\sum_{i=1}^{n}N(t_i)+N(t_{i+1})+\underline{\underline{N(t_{i+2})}}$$

$\frac{1}{8}\,ld\,12=0.44812$ $t=1e4$ $\frac{1}{1e4}\,ld(15000)=0.0013$

$t=1e5 \Rightarrow \frac{1}{1e5}\,ld\,1.5\cdot10^5=0.0001719$

$N(t)=\frac{3}{2}\cdot t$ $t=6$ $N(t)=\frac{3\cdot\overset{7}{\cancel{6}}}{\cancel{2}}=9$

$$\lim_{t\to\infty}\frac{1}{t}\,ld\left(\frac{3t}{2}\right)=\lim_{t\to\infty}\left[\frac{ld\,t}{t}+ld\left(\frac{3}{2}\right)\frac{1}{t}\right]=0$$

$N(t) = \{(x_1, x_2, \ldots x_t)\}$ $\quad (\forall x_i \in \{00, 01, 10\})$

$$H(x) = \lim_{t \to \infty} \frac{H(x_1, x_2, \ldots x_t)}{t}$$

$$P(x_1, x_2, \ldots x_t) = P(x_1) \cdot P(x_2|x_1)\, P(x_3|x_2) \cdots P(x_t|x_{t-1})$$

$$H(x) = \underline{H(x_2|x_1)} = H(x_t|x_{t-1})$$

NEMA NETEVE-SNOST. 001 VO 00 = $\emptyset$

$$H(x_2|x_1) = P(x_1=00) \cdot H(x_2|x_1=00) + P(X_1=01) \cdot \cancel{H(X_2|X_1=01)}$$
$$+ P(X_1=10) \cdot H(x_2|X_1=10) \quad \emptyset$$

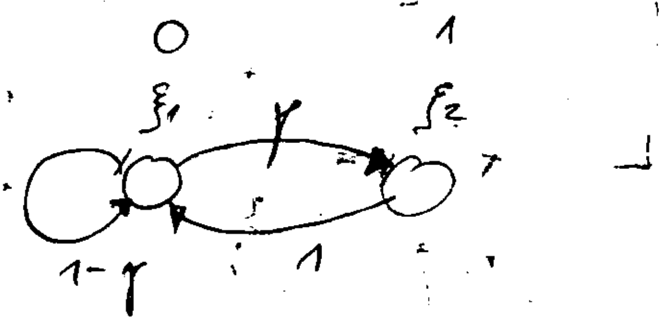
(NEMA 001 VO 00 NETEVEJNOST)

$$\underbrace{P(00) = \frac{1}{2}(1-\gamma) \quad P(01) = \frac{1}{2}\gamma}_{1/2} \qquad P(1,0) = \frac{1}{2}$$

$\xi_1, \xi_2 \in \{0, 1\}$

$P(\xi_1) = \{\frac{1}{2}, \frac{1}{2}\}$

$P(\xi_2) = \{\frac{1}{2}, \frac{1}{2}\}$



$P(\xi_2|\xi_1)$

$\xi_1 \backslash \xi_2$	0	1
0	$1-\gamma$	γ
1	1	0

⬅

$P(\xi_1 \xi_2)$

$\xi_1 \backslash \xi_2$	0	1	$P(\xi_1)$
0	$\frac{1}{2}(1-\gamma)$	$\frac{1}{2}\gamma$	$\frac{1}{2}$
1	$\frac{1}{2}$	0	$\frac{1}{2}$
$P(\xi_2)$	$1-\frac{\gamma}{2}$	$\frac{\gamma}{2}$	▨

$$H(x_2|x_1) = P(x_1=00) \cdot H(x_2|x_1=00) = \frac{1-\gamma}{2} H(x_2|x_1=00)$$

$$H(x_2|x_1=00) = -\underline{P(01)}\, ld\, P(01) - P(10)\, ld\, P(10) = 1$$

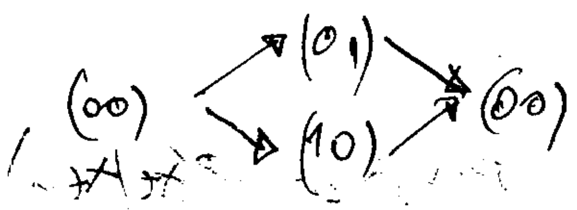
$$= -\left[\frac{\gamma}{2} ld \frac{\gamma}{2} + \frac{1}{2} ld \frac{1}{2}\right] = \frac{\gamma}{2} ld \frac{2}{\gamma} + \frac{1}{2} \cancel{ld\, 2}$$

$$= \frac{\gamma}{2} ld \frac{1}{\gamma} + \frac{\gamma}{2} \cancel{ld\, 2} + \frac{1}{2} = \underline{-\frac{\gamma}{2} ld\, \gamma + \frac{1+\gamma}{2}}$$

$$\boxed{H(x_2|x_1) = \frac{1-\gamma}{2}\left[\frac{1+\gamma}{2} - \frac{\gamma}{2} ld\, \gamma\right] = \frac{1-\gamma^2}{4} - \frac{\gamma-\gamma}{4} ld\, \gamma}$$

$N(t) = (x_1, x_2, \dots x_t)$

$x_i = \begin{cases} 1\,(00) & \text{with } q = \frac{1}{2}(1-\gamma) \\ 2\,(01) & \text{with } r = \frac{1}{2}\gamma \\ 3\,(10) & \text{with } \gamma = \frac{1}{2} \end{cases}$



PRIMER : $t = 3$

1 —
2 =
3 - - -
4 = = =
5 ~~~

$\{1, 1, 1\}$
1, 1, 2
1, 2, 1
1, 2, 2
1, 1, 3

1, 3, 1
1, 3, 3
2, 1, 1
2, 1, 2
2, 1, 3
2, 2, 1

2, 2, 2
3, 1, 1
3, 1, 2
3, 1, 3
3, 3, 1
3, 3, 3

$H(N(3))$: $\left[\frac{1}{2}(1-\gamma)\right]^3$; $\left[\underbrace{\frac{1}{4}(1-\gamma)^2 \cdot \frac{1}{2}\gamma}_{\times 3\,,\text{—}"}\,;\ \underbrace{\frac{1}{2}(1-\gamma)\cdot\frac{1}{4}\gamma^2}_{\times 3\,,\text{- - -}"}\,;\right]$

$\left[\underbrace{\frac{1}{4}(1-\gamma)^2 \cdot \frac{1}{2}}_{\times 3\,,=\,"}\,;\ \underbrace{\frac{1}{2}(1-\gamma)\cdot\frac{1}{4}}_{\times 3\,,\equiv"}\,;\right]$ $\underbrace{\frac{1}{2}(1-\gamma)\cdot\frac{2}{2}\gamma\cdot\frac{1}{2}}_{\times 2\,,\sim"}$;

$\left(\frac{1}{2}\gamma\right)^3$; $\left(\frac{1}{2}\right)^3$;

"

PRIMER : $t = 2$

11 22 33
12 21 31
13

USTE EDNAJ: $t = 3$

	111	211	311	
	112	212	312	
	121	213	313	
	122	221	321	
	113	222	333	
	131			
	133			
	A	B	C	TOT
REDUCED BY	2	4	4	10

$N(2) = 7$ $N(3) = 17$

$H(N(2)) = -\left[\frac{1}{2}(1-\gamma)\right]^2 ld\,\frac{(1-\gamma)^2}{4}$
$- \frac{2}{2}(1-\gamma)\frac{1}{2}\cdot\gamma\ ld\ \frac{1}{4}(1-\gamma)\cdot\gamma -$
$- \frac{2}{4}(1-\gamma) ld\left(\frac{1-\gamma}{4}\right) - \frac{\gamma^2}{4} ld \frac{\gamma^2}{4} -$
$- \frac{1}{4}\, ld\, \frac{1}{4}$

$\boxed{3^3 = 27}$ $3^2 = 9$

$\frac{1}{2} ld\,7 = 1.4037$ | $\frac{1}{3} ld\,17 = 1.363$ | $\frac{1}{4} ld\,41 = 1.339$ | $\frac{1}{5} ld\,99 = 1.3259$

1111
1112
1121
1211
1212
1213
1113
1131
1311
1312

11313
1221
1222

2113

2111
2112
2121
2122
2131
2133
2211
2212
2213
2221

3111
3112
3113
3121
3122
3131
3133
3211
3212
3213

3221
3222
3311
3312
3313
3331
3333

$\} \nearrow N(4) = 41$ }

$\{$ CO INKREMENTIRAV I VO NAZAR. $\}$

$N(7) = 576$
$\frac{1}{7} ld\,577 = 1.3103$

$N(5) = 99$
$N(6) = 239$
$N(8) = 1323$

$\boxed{ld\,N(8)/8 = 1,2963}$

$\frac{1}{6} ld\,239 = 1,317$

$H(x) = -\frac{1}{2}(1-q)\,ld\,\frac{1-q}{2} - \frac{1}{2}q\,ld\,\frac{1}{2}q - \cancel{\frac{1}{2}\,ld\,\frac{1}{2}} \xrightarrow{} -\frac{1}{2}$

$= -\frac{1}{2}(1-q)\,ld\,\frac{1-q}{2} - \frac{1}{2}\,ld\,\frac{q}{2} + \frac{1}{2}$

$\boxed{q_0 = \frac{1}{2}}$

$\boxed{H\left(\frac{1}{2}\right) = \frac{3}{2}}$

- SAKAM DA NAJDAM FORMULA:

$\boxed{t=3}$ Ⓐ $3^2 = 9 \quad N_A = 3^2 - 2 = 7$

$\boxed{N = N_A + N_B + N_C = t^2 - 2 + [t^2 - 2(t-1)]2}$

Ⓑ $N_B = 3^2 - 2\cdot 2 = 3^2 - 2(t-1) = 9 - 4 = 5$

Ⓒ $N_C = 3^2 - 2\cdot 2 = 3^2 - 2(t-1) = 9 - 2 = 5$

$N = t^2 - 2 + [t^2 - 2(t-1)](t-1) = \cancel{t^2} - 2 - \cancel{t^2} + 2(t-1) +$
$+ t^3 - 2t(t-1) = -2 + 2t - 2 + t^3 - 2t^2 + 2t$

$N(t) = t^3 - 2t^2 + 4t - 4 \qquad N(3) = 17$

$\boxed{t=2} \quad N = 3^2 - t = 3^t - t = 9 - 2 =$

$C_k^n = \binom{3}{2} = \frac{3!}{1!\,2!} = \frac{6}{2} = 3$

DOKOLKU NEMA USLOVA REDOSLEDNOST

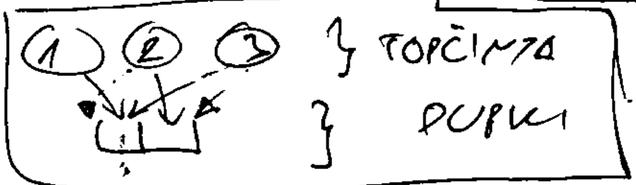
11	21	31
12	22	32
13	23	33

$K_k^n = \frac{3!}{(3-2)!} = 6$

$\boxed{P = 3^2 = 9}$

${}_k^nC = \frac{3!}{1!\,2!} = 3$

$\boxed{x \in \{12, 13, 23\}} \Rightarrow$ BEZ POVTORUVANJE



} TOPČINJA

} DUPKI

$\boxed{{}_k^nC}$

$\boxed{n = 3 \quad k = 2}$

[illegible]

$N(t) = {}_k^nC + t$

111	211	311
112	212	312
113	213	313
121	221	321
122	222	322
123	223	323
131	231	331
132	232	332
133	233	333

$\boxed{3^3 = 27}$

123
132
213
231
312
321

$\boxed{{}_k^nK = {}_3^3K = \frac{3!}{0!} = 6}$

n	n	
123	223	321
132	231	322
	232	323
	233	332

$\boxed{{}_k^nC = \frac{6!}{0!\,6!} = {}_3^3C = 1 \ldots \quad \boxed{123}}$

1111
1112
1113
1121
1122
1123
1131
1132
1133
1211
1212
1213
1221
1222
1223
1231

1232
1233
1311
1312
1313
1323
1331
1332
1333
...
...

$\boxed{3^4 = 81}$

${}_k^nP = {}_2^2P = \frac{n!}{(n-k)!} =$
$= \frac{2!}{0!} = 2$

$\textcircled{{}_2^2P} + ({}_2^2P + {}_2^2C + 1)2 =$
$= 2 + (2 + 1 + 1)2 = 2 + 8 = 10$

15

$\boxed{t=4}$

1111
1112
…
1333

$\underbrace{}_{3^3-10}$

2111
2112
…
2222

$\underbrace{}$ ⓢ#

3^3-15

3111
3112
…
3333

$\underbrace{}$ ⓢ$

3^3-15

1123.
1132.
1223.
1231.
1232.
1233.
1321.
1322.
1323.
1332.

IZEMI GA!
1223.

t	N(t)	$\frac{1}{t}\,\log N(t)$
2	7	1.4037
3	17	1.363
4	41	1.339
5	99	1.326
6	239	1.317
7	577	1.3103
8	1323	1.2962

[illegible] 1.3185
N(t)

$$\begin{array}{r} 2187 \\ -\ 577 \\ \hline 1610 \end{array} \quad \begin{array}{r} 243 \\ -\ 99 \\ \hline 144 \end{array} \quad \begin{array}{r} 729 \\ -\ 239 \\ \hline 490 \end{array}$$

$17-7=10$
$41-17=24$
$99-41=48$
$239-99=140$
$577-239=338$
$1323-577=746$

	3^t	N(t)	R
t=2	9	7	2
t=3	27	17	10
t=4	81	41	40
t=5	243	99	144
t=6	729	239	490
t=7	2187	577	1610
t=8	6561	1323	1.246

$3^3=27 \quad 3^4=81 \quad 3^5=243$

$3^4-(10+24)+1=81-34+1= \quad 47+1=48$

$3^5-(10+24+48)+1=243-82+1=161+1$
$=162$

KUSTNOM ESTIM: 462

POTOCNI SE NA PP.21 !!!

ⓢ#

2223

2123.
2132.
2231.
2232.
2233.
2311.
2312.
2313.

2321.
2322.
2323.
2331.
2332.
2333.

$\}\ 15$

$3\cdot 3^3-6-14-14=$
$=3^4-34=81-34=\underline{\underline{47}}$
$=3^4-10-15-15=81-50=\underline{\underline{41}}$

$$\boxed{\begin{aligned} 10 &\doteq 2^{t-1}+2 \\ 15 &= 2^{t-1}+8 \end{aligned}}$$

ⓢ$

ZAMENI KO — PERMUTAT

3123.
3132.
3211
3212
3213.
3221.
3222.
3223.

3231.
3232.
3233.
3321.
3322.
3323.
3332.

$\}\ 15$

${}^3_2P = \frac{3!}{1!} = ⑥$

${}^3_2C = \frac{3!}{1!\,2!} = \frac{6}{2} = ③$

01
11
000
001
010
011
100
101
110
111

$\boxed{t>4}$ $R = {}^3_2P + ({}^3_2P + C + 1)\cdot 2$

3211
3212
3221
3222

3223
3232
3233
3322
3323
3332

$\underbrace{}_{{}^3_2P}$

$R_{15} = \underline{{}^3_2P-1} + \underbrace{2^3-2}_{6} + \underbrace{2^3-4}_{4}$

⇓
5

111
112
113
121
122
123 ×
131
132 ×
133

211 ×
212 ×
213 ×
221 ×
222 ×
223 ×
231 ×
232 ×
233 ×

311
312
313
321 ×
322 ×
323 ×
331
332 ×
333

$9\times 3=27$
$27-7=20$

111
112
113
121
122
131
133

NEODSTOTUVA
$\boxed{123 \quad 132}$

$R = 3^3-(3^2-2) = 27-7=20$

111
112
113
121
122
123
131
132
133

211
212
213
221
222
223
231
232
233

311
312
313
321
322
323
331
332
333

123
132

223
231
232
233

321
322
323
332

$^3_2P = \frac{3!}{1!} = 6$

$R_4 = {}^3_2P -$

14
14
18
46

144
99
243

27
9
3
39

- MATZAD

t:	$N(t)$	$\frac{1}{t}\,ld\,N(t)$
9	2652	1.263
10	3237	1.166
11	3921	1.0852
12	4143	1.0014

$= 243 - 121 = \boxed{122}$

ZASEGA NEODVISNA EST.

$$\boxed{N(t) \doteq 3^t - 3^{t-1} - 3^{t-2} - 3^{t-3} = 3^t - \sum_{i=1}^{t-1} 3^i}$$

$t = 4$ $3^4 - 3^3 - 3^2 - 3 - 1 = 81 - 27 - 9 - 3 - 1 = 41$

$t = 3$ $3^3 - 3^2 - 3^1 - 1 = 27 - 9 - 3 - 1 = 14$

$t = 5$ $3^5 - 3^4 - 3^3 - 3^2 - 3^1 - 1 =$

$243 - (81 + \underbrace{27 + 9 + 3 + 1}_{40}) =$

$$\boxed{N(t) = 3^t - \frac{1 - 3^t}{1 - 3}}$$

$$\boxed{N(t) = 3^t - \frac{1 - 3^t}{2}}$$

$S = 1 + 2 + 2^2 + \cdots 2^{t-1}$

$2S = 2 + 2^2 + 2^3 + \cdots 2^t$

$S - 2S | = 1 - 2^t$ $S = \frac{1 - 2^t}{1 - 2}$

14·3
42

$N(t) = \frac{2 \cdot 3^t - 1 + 3^t}{2} = \frac{3 \cdot 3^t - 1}{2} = \frac{1}{2} 3^{t+1} - \frac{1}{2}$

18·3
54

$N(t) = \frac{1}{2}\left(3^{t+1} - 1\right)^2$

$H_0 = \lim_{t \to \infty} \frac{1}{t}\, ld\, \frac{1}{2}\left(3^{t+1} - 1\right) =$

$= \lim_{t \to \infty} \frac{1}{t}\, ld \frac{1}{2} + \lim_{t \to \infty} \frac{1}{t}\, ld\left(3^{t+1} - 1\right) = ld\,3 = \underline{1.58496}$

19
18
39

$ld \lim_{t \to \infty}\left(3^{t+1} - 1\right)^{\frac{1}{t}} = ld \lim_{t \to \infty}\left(3^{\frac{t+1}{t}}\right) = ld\,3^1 = ld\,3$

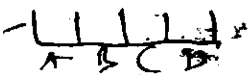
81
54
27

81
37
44

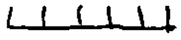
81
29
52

t	$N(t)$	$\frac{1}{t}\,ld\,N(t)$
13	4199	0.9283

1 2 3



$\left(3^2 - 2\right)\left(3^2 - 2\right) = 7 \cdot 7 = 49$



$N(t) = 3\left(3^2 - 2\right)\left(3^2 - 2\right) = 3 \cdot 49 = \underline{147}$

$\left(3^2 - 2\right)\left(3^2 - 2\right) - 2 \cdot \overset{BC}{3^2} = 49 - 18 = \underline{31}$

$3 + \left(3^2 - 2\right) + 3^3 - 2 \cdot 3 - 2 + 3^4 - \left(3^3 - 2 \cdot 3 - 2\right) - 2 \cdot 3^2 =$

$= 3 + 7 + 27 - 8 + 81 - (27 - 8) - 18 = 10 + 19 + 44$

$= 29 + 44 = 73$

$3^4 - 2 \cdot 3^2 - 2 \cdot 3^2 - 2 3^2 = 81 - 18 - 18 - 18 = 81 - 54 = 27$

1 2 3

$\sqcup\sqcup\sqcup$

$3^3 - 2\cdot 3 \quad - \quad R(3)$

$3^3 - \underbrace{2} - 4 - 4 = 27 - 10 = 17$

1XX 2XX 3XX

223	312
232	323
233	322
231	321

$N(t) = 3^t - 2 - 4 - 4 \quad \} \; t = 3$

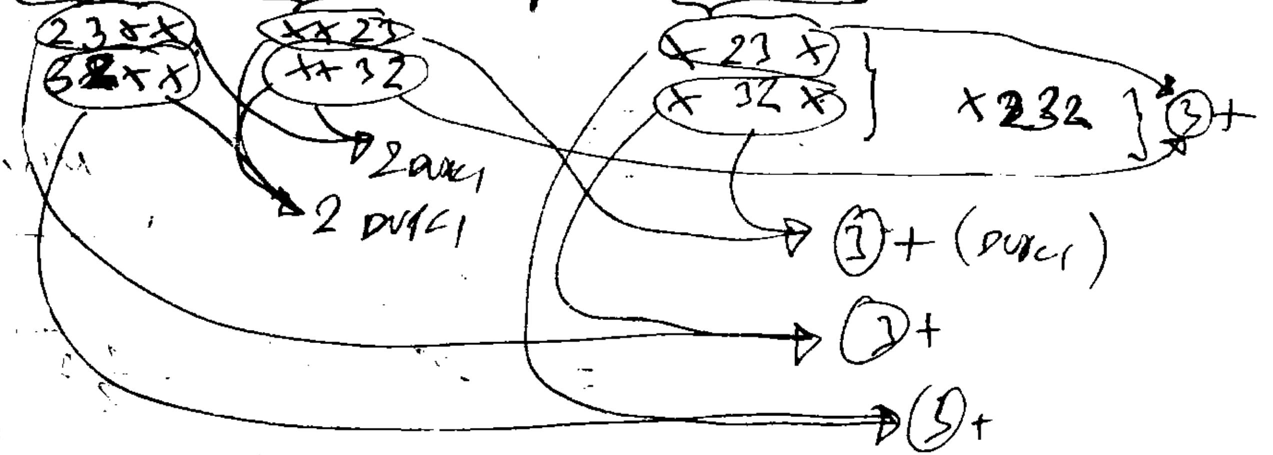
18
18
10
40

①②③ $\sqcup\sqcup\sqcup\sqcup$

$3^4 - R(3) - R(3) -$

1XXX 2XXX 23

$3^4 - 2\cdot 3^2 - 2\cdot 3^2 + 4 - 2\cdot 3^2 + 2 =$



PRAVILNOTO PRAŠANJE E KOLKU PATI MOŽE POSLEDOVATELNO DA SE JAVI 23 VO SEKVENCA DOLŽINA BEZ KONTROLIRANJE

$t = 0$

$\boxed{t = 3}$

$\sqcup\sqcup\sqcup$

231 1 23
232 2 23
233 3 23

ZAMISLI GO KAKO EDEN EVENT !!!

4

	44
41	14
42	24
43	34

$C_t^4 = \binom{4}{2} = \frac{4!}{2!\,2!} = \frac{1\cdot2\cdot3\cdot4}{4} = 6$

GO LEJIV VO MATCE SO MULTINOMIJAL.

NE E VOZMOŽEN !!! $= 3!$

SEGA ZA 32

321 132
322 232
323 332

DUPLIKAT !!!

DUPLO!!

$C_k^4 = \binom{4}{2} = \frac{4!}{2\cdot 2} = 3! = 6$

$C_{44} =$

$N(3) = 3^3 - 2\binom{4}{2} + ② = 27 - 12 + 2 = 27 - 10 = 17$

$\boxed{t = 4}$

$N(t) = 3^t - 2\binom{t+1}{t-1}$ → (+1) PODOLGA AZBUKA VO (-1) POMALKU SLOTOVI

$t - 2$

$N(4) = 81 - 2\binom{5}{3} + 2 = 81 - 20 + 2 =$

$_k^nP \Rightarrow {}_2^4P = \frac{4!}{2!} = \frac{1\cdot2\cdot3\cdot4}{2!} = \underline{\underline{2\cdot6}} = \underbrace{12}_{}$ СИТЕ ПЕРМУТАЦ.

ПЕРМУТАЦИ САМО СО „4"

$\boxed{3!=6}$ = t=4 $\boxed{4!=24}$

$N(t) = 3^t - 2\cdot t! + 2^{t-2}$ [illegible]

$\boxed{t=5}$ $\underbrace{23}_{W=4}+YZ$ $\boxed{W+YZ}$

ВАЖНО ПЕРМУТАЦИ:

$\frac{4!}{(4-4)!} = 24$

-ВАЌАЊЕ НАЗАД НА $\boxed{t=3}$

$N(t) = 3^t - 4^{t-1} = |t=3| = 27 - 4^2 = 27-16 = 11$

→ ДА СЕ ОДЗЕМАТ САМО КОМБИНАЦИИТЕ СО 4"

Ⓐ 4123, 1423, 1243 … $\frac{4!}{(4-4)!} = \frac{4!}{1} = 24$ 4+YZ

Ⓑ

44XY 44YX

X44Y Y44X

XY44 YX44

4X4Y 4Y4X

4XY4 4YX4

+4x „XX" +4 „YY"

24 = 11

+4x „XZ" + 4x „YZ" + 4x ZZ

+ 12

TOTAL: 28

Ⓒ

444X

X444

4X44

44X4

×3 X, Y, Z

TOTAL: 12

24 + 28 + 12 = 74

$N(t) = 3^3 - 4^4$

0 1 n=2

⊔⊔⊔ n=3 $2^3 = 8$

$\binom{3}{2} = \frac{3!}{2!\cdot1!} = \frac{6}{2} = 3$ → ЕДИНИЦИ

001
010
100

000
111
011
001

0×3
1×3
1+2
0+2

$\binom{3}{1} = \frac{3!}{2!\,1!} = 3$ $\binom{3}{0} = 1$ $\binom{3}{3} = 1$

$\boxed{t=4}$ DA VIDIME KOCKU DUPLIKATI [illegible]

$\boxed{4 \triangleq 23}$

2311
2312
2313
× 2321
× 2322
× 2323

2331
× 2332
2333

1123
1223
× 1323
2123
2223
2323

3123
× 3223
× 3323

1231
× 1232
1233
2231
× 2232
2233

× 3231
× 3232
× 3233

3^1+3^0 } DUPLIKATI

$4 \triangleq 32$

3211
3212
3213
3221
3222
× 3223

3231 ×
× 3232 ×
3233 ×

1132
1232 ×
1332
2132
2232 ×
× 2332

3132
× 3232 ×
3332

1321
1322
1323 ×
2321 ×
2322 ×
2323 ×

3321
× 3322
3323 ×

12 DUPLIKATI $\boxed{3232}$ je POSEBAN $(11+1)$

$N(4)=3^4-2f(4)+12 = 81-2{*}27+\overset{3\times 4}{12} = 93-54 = 39$

4×18

$\boxed{t=5}$ $N(4)=3^5-2f(5)+72 = 27+72 = 99$

$\boxed{t=6}$ $N(4)=3^6-2f(6)+\underbrace{290}_{3\cdot 5!}$

4abc | a4bc | ab4c | abc4

23111
23112
23113
23211

23212
23213
23221
23222
23223
$\boxed{3^2=9}$

23231
23232
23233

23321
23322
23323
3^1

23132
23332
$2\cdot 3^0$

$4\times 14 = 46$ $\quad 4\times 18 = 72$ } FALI JOŠ 4

DUPLIKATI : $4(3^2+3^1+6\cdot 3^0) = 4\cdot(9+3+6)$

$4 = 1\cdot 2^2+0\cdot 2^1+0\cdot 2^0 = 100$ BINARNI

$4 = 0\cdot 3^3+0\cdot 3^2+1\cdot 3^1+1\cdot 3^0 = 11$ TERNARNI

$3^4+3^3+3^2+3^1+3^0 =$

$5 = 1\cdot 3^1+2\cdot 3^0 = 3+2=5$

- SO NOVATA SKRIPTA VO MATLAB & MAPLE SE DOBIVA

t	$N(t)$	$\frac{1}{t}\ln N$
3	17	
4	41	
5	99	
6	239	
7	577	
8	1393	1.305
9	3363	1.30179
10	8119	1.2987
11	19601	1.2962

t	$N(t)$	$\frac{1}{t}\ln N$
11	19601	1.2962
12	47321	1.29418
13	114243	1.2924
14	275807	1.2909
15		
16		
17		
18		

NOVATA SKRIPTA SE BAZIRA NA TERCIJAREN NUMERIČKI SISTEM:
Problem4_7_final.m

MMV

- DA SE OBIDAM DA JA NAJDAM GENERALIZIRANATA FORMULA SO KOMPETENDE NA TERCIJAREN SISTEM

$\boxed{t=3}$ $$2 = 2\cdot3^2 + 2\cdot3^1 + 2\cdot3^0+1 = 2(9+3+1)+1 = 26+1$$

TREBA DA SE OTPLATAT SITE SEKVENCI ŠTO SODRŽAT 12

1 21 — $R_1 = 1\cdot3^1$

2 1 X = 5

1 2 X — $R_2 = 1\cdot3^1$

X 2 1 — $R_3 = 3$

X 1 2 — $R_4 = 3$

210
211
212

120
121
122

021
121
221

012
112
212

$\boxed{4\cdot3-2 = 12-2=10}$

$$\binom{4}{2} = \frac{4!}{(4-2)!} = \frac{1\cdot2\cdot3\cdot4}{2!} = 12$$

$t=3$ ② $R_1=6$, $R_2=4$, $T=10$

30 03
31 13
32 23

$\boxed{t=4}$ $R_1 = 26$, $R_2 = 14$ } 10, $T_1 = 41$

$\boxed{t=5}$ $R_1 = 99$, $R_2 = 45$ } 44, $T_1 = 99$

$\boxed{t=6}$ $R_1 = 352$, $R_2 = 138$ } 214, $T = 239$

$$\binom{4}{3} = \frac{4!}{1!} = \frac{1\cdot2\cdot3\cdot4}{1} = 24$$

$$(a+b+c+d)^t = \sum_{i,j,k,m=1}^{t} \binom{t}{i,j,k,m} a^i b^j c^k d^m$$

①②③
1231
1332
21
23

$\boxed{j=0}$ $k = 3-1-0 = 2$ $m = 3-1-0-2 = 0$

Ⓚ ① ②
④

111 212
112 221
121 222
122
211

$\boxed{k^3}$

$$\frac{4!}{(4-k)!} = \frac{3!}{1!} = 6$$

123 312
213 321
231

$$\sum_{j=1}^{3}\sum_{k=j}^{3}\sum_{m=k}^{3}\binom{3}{j\,k\,m} \qquad \sum_{j=1}^{3}\sum_{k=3-j}^{3}\sum_{m=3-j-k}^{3}\binom{3}{j\,k\,m}.$$

$j=1 \quad k=2:3 \quad m=0:$

$$\sum_{j=0}^{3}\sum_{k=0}^{3-j}\sum_{m=0}^{3-j-k}\binom{3}{j\,k\,m}$$

$j=1 \quad k=0:2 \quad m=0:0$ $=0:$

$$\sum_{j=0}^{3}\sum_{k=0}^{3-j}\sum_{m=3-j-k}^{3}\binom{3}{j\,k\,m}$$

$j=0 \quad k=0:3 \quad m=3:0$

$$\sum_{j=0}^{3}\sum_{k=j}^{3-j}\sum_{m=3-k}^{3-j}\binom{3}{j\,k\,m}$$

$j=0 \quad k=0:3 \quad m:3$

$$\sum_{j=0}^{3}\sum_{k=j}^{3-j}\sum_{m=k}$$

$$\sum_{k_1+k_2+\dots+k_m=n}\binom{n}{k_1,k_2,\dots,k_m} = m^n$$

$$\#_{n,m} = \binom{n+m-1}{m-1} \qquad \begin{matrix} n=3 \\ m=4 \end{matrix}$$

$$\boxed{\#_{n,m} = \binom{7-1}{3} = \binom{6}{3} = \frac{6!}{3!\cdot 3!} = \frac{4\cdot 5\cdot 6}{1\cdot 2\cdot 3} = \frac{120}{6} = 20}$$

~~VIDJEN BROJ NA KOEFICIJENTI~~

☐ **C.M. GRINSTEAD PROBLEM 16.** (SLIČNA ZA MULTINOMIAL THEOREM)

THE SIWASH UNIVERSITY FOOTBALL TEAM PLAYS 8 GAMES IN THE SEASON, WINNING 3, LOSING 3 AND ENDING 2 IN A TIE. SHOW THAT NUMBER OF WAYS THAT THIS CAN HAPPEN IS:

$$\binom{8}{3}\binom{5}{3} = \frac{8!}{3!3!2!}$$

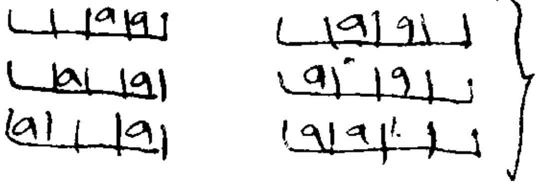
|1|1|1|2|2|2|3|3|

$(a+b+c)^8$

PRVO SE $(a+b+c)^4$

DA SE JAVI 2 a

DA SE JAVI 1 b



$$\binom{4}{2} = 6$$

OSTANUVAAT 2 PRAZNI MESTA OD TIE „2“ MESTA „b“ MOŽE DA ODI NA

$$\binom{2}{1} = \frac{2!}{(2-1)!\,1!} = ②$$

bcaa	baac
cbaa	caab
baca	abac
caba	acab
abca	aabc
acba	aacb

VKUPNIOT BROJ NA NOVI KOMBINACII E:

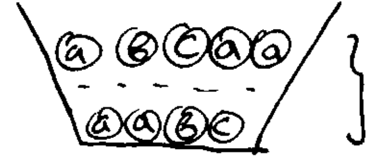
$$\binom{4}{2}\cdot\binom{2}{1} = 6\cdot 2 = 12$$

- SLIČNOTO VAŽI I ZA „8“ SLOTA I MOŽNOST a DA SLOŽI 3 PATI, b 3 PATI, I c 2 pati

$\binom{8}{3} \to$ KAŽUVA KOLKU KOMBINACII IMA KOI SODRŽAT 3a VO NIV OSTANUVAAT 5 SLOBODNI ZA NA TIE POZICII SI STAVIS „b“, DVETE MESTA ŠTO ŠE OSTANAT SE ZA „c“

$$\binom{8}{3}\cdot\binom{5}{3} = \frac{8!}{\cancel{5!}\cdot 3!}\cdot\frac{\cancel{5!}}{2!\cdot 3!} = \frac{8!}{2!3!3!}$$

PROBLEM ⑰ | MMV



USING THE TECHNIQUE FROM PROBLEM ⑯ SHOW THAT NUMBER OF WAYS THAT ONE CAN PUT „4“ DIFFERENT OBJECTS INTO TREE BOXES WITH „a“ IN THE FIRST „b“ IN THE SECOND AND „c“ IN THE THIRD IS $4!/a!b!c!$.

- SOLUTION: THERE ARE $\binom{4}{a}$ DIFFERENT WAYS OF PUTING „a“ DIFFERENT OBJECTS IN FIRST BOX AND THEN $\binom{4-a}{b}$ WAYS OF PUTING „b“ DIFFERENT OBJECTS INTO SECOND, AND ONE WAY TO PUT REMAINING OBJECTS INTO THE THIRD BOX.

$$\binom{4}{a}\binom{4-a}{b} = \frac{4!}{\cancel{(4-a)!}\,a!}\cdot\frac{\cancel{(4-a)!}}{(\underbrace{4-a-b}_{c})!\cdot b!} = \frac{4!}{a!b!c!}$$

- SEGA SE NAVRAĆAM NA PROBLEMOT

→ BEZ 23 & 32

$\boxed{t=3}$ ① ② ③ $3^3 = 27$ $k=10$ $\boxed{T = 27-10 = 17}$

SPORNI SE:

231	123	321	132
232	223	322	232
233	323	323	332

GO KORISTEV [illegible] ŠO

$$\binom{4}{2} = \frac{4!}{2!2!} = \frac{12}{4} = 3$$

$$\boxed{\begin{matrix}44\\ \times 4\\ 4\times\end{matrix}}$$

23 1 1 23 32 1 1 32

~~22 2~~ 2 23 32 2 2 32

23 3 3 23 32 3 3 32

$\boxed{2\times2}$ $\boxed{\begin{matrix}22\times\\ \times22\end{matrix}}$

$$\binom{3}{2} = \frac{3!}{2!} = 3$$

$$\frac{3!}{1!\cdot1!\,1!} + \frac{3!}{0!\,2!\,1!} + \frac{3!}{0!\,1!\,2!}$$

2, 6, 1, 6

2, 6, 1, 6

a b c

①②③ ②②② ③③③

$$\boxed{\frac{4!}{a!\cdot b!\,c!}}$$

b = 90

"IMA EDNO b = 2"

$$\left.\right\} : \frac{3!}{1!\,1!\,1!} + \frac{3!}{1!\,\cancel{2!}}\cdot\cancel{2} = 6 + 6 = 12$$

$\boxed{t=3}$

OVA E ISKLUČOST $f(t)$ OD MATLE SAMO KOGA PUILIVADITE NE ZNAM KAKO DA GI OTFRLAM

$\boxed{t=4}$

$$\frac{4!}{\underbrace{1!}_{a}\cdot\underbrace{1!}_{b}\cdot\underbrace{2!}_{c}} + \frac{4!}{\underbrace{1!}_{b}\,\underbrace{3!}_{c}} + \frac{4!}{\underbrace{2!}_{a}\,\underbrace{1!}_{b}\,\underbrace{1!}_{c}}$$

$$+ \frac{4!}{\underbrace{3!}_{b}\,\underbrace{1!}_{c}} + \frac{4!}{\underbrace{2!}_{b}\,\underbrace{2!}_{c}} + \frac{4!}{\underbrace{2!}_{c}\,\underbrace{2!}_{b}} + \frac{4!}{\underbrace{1!}_{b}\,\underbrace{1!}_{c}\,\underbrace{2!}_{a}}$$

00, 01, 10, 100, 101, 1000, 1001, 1010,

0, 1, 2, 4, 5, 8, 9, 10,

$P(x_2|x_1)$

$x_1 \backslash x_2$	0	1
0	$1-\gamma$	γ
1	1	0

■ SOLUTION (EDITION 1)

1 10 100 101 1000 1001

1 2 4 5 8 9

$N(t) = c\cdot\lambda^t$

λ IS MAXIMUM MAGNITUDE SOLUTION OF THE CHARACTERISTIC EQUATION:

$$1 = \lambda^{-1} + \lambda^{-2} \qquad 1 = \frac{1}{\lambda} + \frac{1}{\lambda^2} \qquad \frac{\lambda+1}{\lambda^2} = 1.$$

$$z^2 - z - 1 = 0 \qquad z = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{1 \pm \sqrt{1+4}}{2}$$

$$\boxed{z = \frac{1 \oplus \sqrt{5}}{2}}$$ — $\boxed{\text{GOLDEN RATIO}}$!!! $\left(\boxed{\frac{a+b}{a} = \frac{a}{b}}\right)$

$$N(t) = \left(\frac{1 \oplus \sqrt{5}}{2}\right)^t \qquad \lim_{t\to\infty} \frac{1}{t}\, ld\left(\frac{1+\sqrt{5}}{2}\right)^t =$$

$$= \lim_{t\to\infty} ld\left(\frac{1+\sqrt{5}}{\cancel{1+2}}\right) = ld\left(\frac{1+\sqrt{5}}{2}\right) = \underline{0.69424} = H(\gamma_0)$$

} UPPER BOUND KONSTANTA

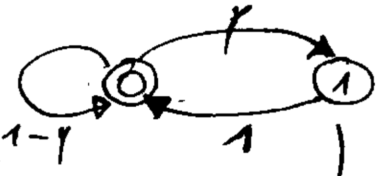
PR.10

$\boxed{t=1}$ 0; 1; $N(1)=2$

MMV

$\boxed{t=2}$: 00, 01, 10 $\quad N(2)=3$

$$\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+2\sqrt{5}+5}{4}$$



$\boxed{t=3}$ 000, 001, 010, 100, 101 $\quad N(3)=5$

$\boxed{0 \quad 1 \quad 2 \quad 4 \quad 5}$

$\boxed{t=4}$ 0000, 0001, 0010, 0100, 0101, 1000, 1001, 1010 $\}$ $N(4)=8$

APSTRAHIRAV GO KAKO DVIŽENJE NA ČESTIČKA. OD SOSTOJBA „0" MOŽE DA OTIDE VO SOSTOJBA „1" NO TAMU NE MOŽE DA SE ZADRŽI, I SE VRAĆA VO NAREDNIOT ČEKOR VO SOSTOJBA 0. ISTO TAKA OD SOSTOJBA „0" MOŽE DA SE ZADRŽI (VO NAREDNIOT ČEKOR DA PREMINE) NA 0. ZATOA SEKVENCIJATA „11" NE E DOZVOLIVA.

$\boxed{N(t) = 2, 3, 5, 8, 13 \dots}$ $\quad N(t) = N(t-1) + N(t-2)$

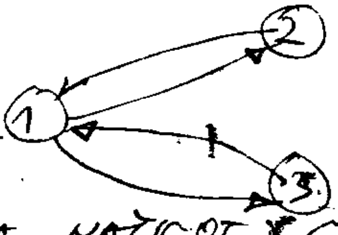
FIBONACI $\qquad$ FIBONACI SEQUENCE

$\boxed{t=5}$ 00000, 00001, 00010, 00100, 00101, 01000, 01001, 01010, 10000, 10001, 10010, 10100, 10101, $\quad \boxed{N(5) = 13}$ $\quad$ MMV

$\boxed{N(5) = N(4) + N(3) = 8+5 = 13}$

SINCE $(x_1, x_2, \dots x_t)$ CAN TAKE $N(t)$ POSIBLE OUTCOMES THE UPPER BOUND OF $H(x_1, x_2, \dots x_t) = \log N(t)$

- MOJATA GREŠKA BEŠE ŠTO MOMENTALNATA SOSTOJBA JA OPIŠUVAV SO DVE PROMENLIVI $x_1 x_2$ KOI MOŽAT DA ZAFATAT SOSTOJBA ILI „0" ILI „1" NO NE MOŽAT I DVETE DA BIDAT VO SOSTOJBA „11". TOA BI ODGOVARALO POVEĆE NA MARKOV LANEC SO TRI SOSTOJBI.



$t = 1, 2, 3, 4, 5; 6, 7, 8$

$N(t) = 3, 7, 17, 41, 99, 239, 577, 1393, \dots$

HALF COMPANION PELL'S NUMB. PP. 16

SUŠTINATA NA NAČINOT NA DOBIVANJE NA N(t) SUM GO SVATIV, NO POGREŠNO SUM GO DEFINIRAL MARKOVIOT LANEC!!!

SE OBIDOV ZA MNOGU KOMPLIKUVANA VRSKA MEĐU BROJOT NA SOSTOJBI I GENERIRANITE SEKVENCI, SLIČNO NA FIBONAČI SEKVENCATA. PODOLU ZA NAJDOV MOJATA SEKVENCA NA OEIS:

OEIS - ONLINE ENCICLOPEDIA OF INTEGER SEQUENCES

$a(n) = 6 \cdot a(n-1) - a(n-2) - 2$

$a(3) = 6 \cdot a(2) - a(1) - 2 = (6 \cdot 7) - 3 - 2 = 42 - 5 = 37$

$a(2) = 6 \cdot a(1) - 0 - 2 = 18 - 2 = 16$

+ Blone's Puzzle Bag

$\boxed{\text{Pell's Equation}}$ $\quad \boxed{a^2 - 2b^2 = -1}$

0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, $\quad \boxed{a(n) = 2a(n-1) + a(n-2)}$ MMV

$= N(n-1) + N(n-2)$ PELL'S NUMBERS

$\boxed{P_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}}$

$n: 0, 1, 2, 3, 4, 5, 6$

$a(n): 0, 1, 2, 5, 12, 29, 70 \ldots$

$b(n): 0, 0, 1, 3, 7, 17, 41, 99$

$a(n) = 2a(n-1) + a(n-2)$

$b(n) = a(n-1) + a(n-2)$

$$b(n) = a(n-2) + \underline{a(n-3)} + 2\underline{a(n-3)} + a(n-4) =$$

$$= 2a(n-2) + 3\underset{5}{\underline{a(n-3)}} + a(n-4)$$

$$\begin{array}{r} 99 \\ 99 \\ \hline 198 \end{array}$$

$$b(n) = \overset{1}{1}\ \overset{2}{0}\ \overset{3}{0}\ \overset{4}{3}\ \overset{5}{}$$

$$b(n): \overset{0}{0}, \overset{1}{1}, \overset{2}{3}, \overset{3}{7}, \overset{4}{17}, \overset{5}{41}, \overset{6}{99}, \ldots$$

$b(n) = a(n) + a(n-1)$

$b(n) = 2a(n-1) + a(n-2) + 2a(n-2) + a(n-3)$

$$\boxed{b(n) = 2a(n-1) + 3a(n-2) + a(n-3)}$$

$$\boxed{\frac{99}{70} = 1.41429 \doteq \sqrt{2}}$$

$\frac{1}{b} = \frac{1}{1}, \frac{1}{3}, \frac{1}{7}, \frac{1}{}$

$\frac{n}{d}$ $\quad n^2 = 2d^2 +/- 1$

$n^2 = 2 \cdot 1 - 1 = 1 \Rightarrow n = 1$

$n^2 = 2 \cdot 4 + 1 = 9 \quad \boxed{n = 3}$

$n^2 = \boxed{2 \cdot 9} - 1 = 17$

$\frac{1}{1}; \frac{3}{2}; \frac{}{3}$

$\frac{1}{1}; \frac{3}{2}; \frac{7}{5}; \frac{17}{12}; \frac{41}{29}; \frac{\boxed{99}}{70} \longrightarrow$ HALF THE COMPANION PELL'S NUMBERS

$2, 6, 14, 34, 82, 198, \ldots$ } COMPANION PELL'S NUMBER (PELL'S LUCAS NUMBERS)

$b(n) = [2b(n-1) + b(n-2)]$.

$2 + 12 = 14$

$6 + 28 = 34 \quad \ldots$

$$\boxed{Q_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n}$$

CLOSSED FORMULA FOR COMPANION PELL'S NUMBERS.

$n = 0 \quad Q_0 = 1 + 1 = 2$

$n = 1 \quad Q_1 = (1+\sqrt{2}) + (1-\sqrt{2}) = 2$

$n = 2 \quad Q_2 = (1+\sqrt{2})^2 + (1-\sqrt{2})^2 = 1 + \cancel{2\sqrt{2}} + 2 + 1 - \cancel{2\sqrt{2}} + 2 =$

$= 3 + 3 = 6$

$$\boxed{\frac{2a+b}{a} = \frac{a}{b}}$$

- MOJOT PROBLEM SE SVEDE NA **SILVER RATIO**: $(1+\sqrt{2})$

ZNAČI MNOGU SLIČNO NA REŠENIETO OD EDITION 1 !!!

$$\lim_{t \to \infty} \frac{1}{t} \operatorname{ld}\left[\frac{(1+\sqrt{2})^{t+1} + (1-\sqrt{2})^{t+1}}{2}\right] = \operatorname{ld}(1+\sqrt{2}) = 1.27155$$

- ZNAČI: SOGLASNO MOJATA REPREZENTACIJA NA MARKOVIOT LANEC SE DOBIVA!

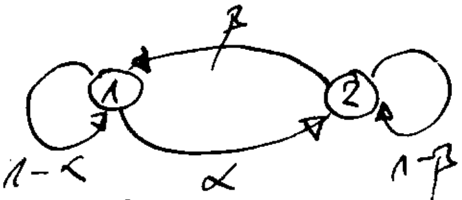
$$\boxed{N(t) = \frac{(1+\sqrt{2})^{t+1} + (1-\sqrt{2})^{t+1}}{2} = 3, 7, 17, 41, 99, 239, 577 \ldots}$$

26 OVA SE MAČEV DA GO DOBIJAM 3 NEDELI !!!

4.8 MAXIMUM ENTROPY PROCESS. A DISCRETE MEMORYLESS SOURCE HAS THE ALPHABET $\{1,2\}$, WHERE THE SYMBOL 2 HAS DURATION 2. THE PROBABILITIES OF 1 AND 2 ARE p_1, p_2, RESPECTIVELY. FIND THE VALUE OF p_1 THAT MAXIMIZES THE SOURCE ENTROPY PER UNIT TIME $\mathcal{H}(x) = \frac{\boxed{H(x)}}{ET}$. WHAT IS THE MAXIMUM VALUE $\mathcal{H}(x)$?

ENTROPY PER SYMBOL!!!

ŠTOM E MEMORYLESS TOGAŠ NE E MARKOV PROCESS.



TRAENJETO NA SIMBOLITE!!

$$\{X_i\} = \{x_1, x_2, x_3, \dots, x_n\}$$

$$\{T_i\} = \{1, 2, 2, 1, \dots, 2\}$$

$$p = \{p_1, p_2, p_2, p_1, \dots, p_2\}$$

$$\boxed{p_k = p_1^{k} \cdot p_2^{n-k}, \forall k}$$

($)

i.e. AVERAGE SYMBOL DURATION

$$E[T] = \sum_{i=1}^{n} p(T_i) \cdot T_i = \frac{p_1 + 2p_2}{n}$$

BY LAW OF LARGE NUMBERS!!!

$$-\frac{1}{n} \, ld \, p(x_1, x_2, \dots x_n) = -\frac{1}{n} \sum_{i=1} ld \, p(x_i) \xrightarrow[n\to\infty]{}$$

$$-E[ld \, p(x_i)] = H(x)$$

$$2^{-n(H(x)+\varepsilon)} \le p(x_1, x_2, \dots x_n) \le 2^{-n(H(x)-\varepsilon)}$$

$$p(x_1, x_2, \dots x_n) \to 2^{-nH(x)}$$

$$\boxed{\underbrace{\frac{1}{n}\sum_{i=1}^{n} x_i}_{\text{SAMPLE AVERAGE}} \xrightarrow[n\to\infty]{} \underbrace{E[x_i]}_{\text{EXPECTED VALUE}} = \sum_{i=1}^{n} p_i x_i}$$

LAW OF LARGE NUMBERS.

SUŠTINATA E VO ($)

$$H(x) = p_1 \, ld \frac{1}{p_1} + p_2 \, ld \frac{1}{p_2}$$

$$H(X) = -\sum_{i=1}^{2^n} \underbrace{p(x_1, x_2, \dots, x_n)}_{x_i} \, ld \, p(x_1, x_2, \dots, x_n)$$

$$= -\sum_{i=1}^{2^{nH(x)}} 2^{nH(x)} \cdot ld \, 2^{nH(x)} = + \cancel{2^{nH(x)}} \cdot \cancel{2^{-nH(x)}} \cdot \underline{n \cdot H(x)}$$

$$\mathcal{H}(x) = \lim_{n\to\infty} \frac{1}{n} \cdot H(X) = \lim \frac{1}{\cancel{n}} \cdot \cancel{n} H(x) = H(x)$$

$$= -(p_1 \, ld \, p_1 + p_2 \, ld \, p_2)$$

NEMA POTREBA ZOŠTO SOURCE ENTROPY E POZNATA DEKA E $H(x) = -p_1 \, ld \, p_1 - p_2 \, ld \, p_2$

$$\frac{d}{d\gamma_1}\left[-\frac{\gamma_1\,\mathrm{ld}\,\gamma_1+\gamma_2\,\mathrm{ld}\,\gamma_2}{\gamma_1+2\gamma_2}\right]=0$$

$$\frac{\left(\mathrm{ld}\,\gamma_1+\gamma_1\cdot\frac{1}{\gamma_1}\right)(\gamma_1+2\gamma_2)-(\gamma_1\,\mathrm{ld}\,\gamma_1+\gamma_2\,\mathrm{ld}\,\gamma_2)\cdot 1}{(\gamma_1+2\gamma_2)^2}=0$$

$-\gamma_2\,\mathrm{ld}\,\gamma_2$

$$(\mathrm{ld}\,\gamma_1+1)(\gamma_1+2\gamma_2)-\gamma_1\,\mathrm{ld}\,\gamma_1 = \cancel{\gamma_1\,\mathrm{ld}\,\gamma_1}+\gamma_1+2\gamma_2\,\mathrm{ld}\,\gamma_1+2\gamma_2 \cancel{-\gamma_1\,\mathrm{ld}\,\gamma_1} -\gamma_2\,\mathrm{ld}\,\gamma_2$$

$$=\gamma_1+2\gamma_2\,\mathrm{ld}\,\gamma_1+2\gamma_2-\gamma_2\,\mathrm{ld}\,\gamma_2=0$$

~~[illegible]~~

$$\boxed{H(x)=-\frac{\gamma_1\,\mathrm{ld}\,\gamma_1+\gamma_2\,\mathrm{ld}\,\gamma_2}{\gamma_1+2\gamma_2}} \qquad \frac{dH(\gamma)}{d\gamma}=0$$

SOURCE ENTROPY PER UNIT TIME

$\gamma=\gamma_1$ $\quad\boxed{\gamma_2=1-\gamma}$

$$\boxed{H(\gamma)=\frac{\gamma\,\mathrm{ld}\,\gamma+(1-\gamma)\,\mathrm{ld}(1-\gamma)}{\gamma+2(1-\gamma)}} \quad ④ \quad \Rightarrow \quad \gamma_0=\frac{\sqrt{5}-1}{2}=0.61803$$

$$\boxed{H(\gamma_0)=0.69424}$$ → ISTATA VREDNOST OD PRETHODNATA ZADACA.

$\gamma_2=1-\gamma$

~~$\gamma+2(1-\gamma)\,\mathrm{ld}\,\gamma+2(1-\gamma)-(1-\gamma)\,\mathrm{ld}(1-\gamma)=$~~

~~$\gamma+2(1-\gamma)\,\mathrm{ld}\,\gamma+2-2\gamma=0$ $\quad 2(1-\gamma)\,\mathrm{ld}\,\gamma+2-\gamma=0$~~

~~[illegible]~~

~~$2(1-\gamma)\,\mathrm{ld}\,\gamma+2-\gamma-(1-\gamma)\,\mathrm{ld}(1-\gamma)=0$~~

~~$(1-\gamma)\left[\mathrm{ld}\,\gamma^2-\mathrm{ld}(1-\gamma)\right]+2-\gamma=0$ $\quad \mathrm{ld}\frac{\gamma^2}{1-\gamma}=\frac{\gamma-2}{1-\gamma}$~~

④ + MAPLE $\quad \frac{dH(\gamma)}{d\gamma}=\frac{\ln(1-\gamma)-2\ln\gamma}{\ln 2\,(\gamma-2)^2}=0$

⇒ SE DOBIVA ISTA VREDNOST KAKO ENTROPIJATA OD PROBLEM 4.7 ZATOA ŠTO IZVOR VO KOJ SEKOJA „1" MORA DA BIDE SLEDENA SO „0" E EKVIVALENTEN NA IZVOR VO KOJ SIMBOLOT „1" IMA TRAENJE „2" A SIMBOLOT „0" IMA TRAENJE „1"

4.9 INITIAL CONDITIONS. SHOW FOR MARKOV CHAIN THAT! $H(X_0|X_n) \geq H(X_0|X_{n-1})$

THUS INITIAL CONDITIONS X_0 BECOME MORE DIFFICULT TO RECOVER AS THE FUTURE X_n UNFOLDS.

$$I(X_0;X_1) \geq I(X_0;X_2) \qquad X \to Y \to Z$$

i.e $I(X;Y) \geq I(X;Z)$

$$I(X;Y,Z) = I(X;Y) + \underbrace{I(X;Z|Y)}_{0} = I(X;Z) + \underbrace{I(X;Y|Z)}_{\geq 0}$$

$$\boxed{I(X;Y) \geq I(X;Z)}$$ DATA PROCESSING INEQUALITY

$$I(X_0;X_{n-1}) \geq I(X_0;X_n)$$

$$I(X_0;X_{n-1}) = H(X_0) - H(X_0|X_{n-1}) = H(X_{n-1}) - H(X_{n-1}|X_0)$$

$$I(X_0;X_n) = H(X_0) - H(X_0|X_n) = H(X_n) - H(X_n|X_0)$$

$$\cancel{H(X_0)} - H(X_0|X_{n-1}) \geq \cancel{H(X_0)} - H(X_0|X_n) \qquad \boxed{H(X_0|X_{n-1}) \leq H(X_0|X_n)}$$

PROVED . !!!

ALTERN: $H(\cancel{X_{n-1}}) - H(X_{n-1}|X_0) \geq H(\cancel{X_n}) - H(X_n|X_0)$

$H(X_n) = H(X_{n-1})$ STATIONARITY

$$\boxed{H(X_{n-1}|X_0) \leq H(X_n|X_0)}$$

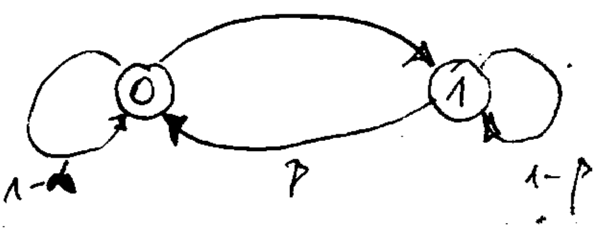
4.10 PAIRWISE INDEPENDENCE. LET $X_1, X_2, \ldots X_{n-1}$ BE I.I.D RANDOM VARIABLES TAKING VALUES $\{0,1\}$ WITH $Pr\{X_i = 1\} = \frac{1}{2}$. LET $X_n = 1$ if $\sum_{i=1}^{n-1} X_i$ IS ODD AND $X_n = 0$ OTHERWISE. LET $n \geq 3$.

(a) SHOW THAT X_i & X_j ARE INDEPENDENT FOR $i \neq j$, $i,j \in \{1,2,\ldots,n\}$.

(b) FIND $H(X_i,X_j)$. for $i \neq j$

(c) FIND $H(X_1,X_2,\ldots,X_n)$. IS THIS EQUAL TO $nH(X_1)$?

(a)



- $X_n \in \{0,1\}$

$$P(X_n=0) = \left(\frac{1}{2}\right)^{n-1}$$

$$P(X_n=1) = \left(\frac{1}{2}\right)^{n-1}$$

$$P(x_n | x_1, x_2, \dots x_{n-1}) = ?$$

$\frac{1}{2} \cdot 2^{n-1} = 2^{n-2}$ → VKUPNO PARNI SEQUENCI

$\frac{1}{2} \cdot 2^{n-1} = 2^{n-2}$ → VKUPNO NEPARNI " "

$$\boxed{H(\underbrace{x_1, x_2, \dots x_{n-1}}_{X}, x_n) = H(X, x_n) = H(X) + \cancel{H(x_n|X)}} \quad 0$$

$\boxed{x_{n-1}}$ → e VAZNA PROMENLIVA KOJA KAZUVA DALI SEKVENCATA E PARNA ILI NEPARNA

00	0 000	5 101	11 = 1011
01	1 001	6 110	13 = 1101
10	2 010	7 111	$\boxed{15 = 1111}$ — 4 EDINICI
11	3 011		
	4 100		

$$\boxed{H(x_n | x_1, x_2, \dots x_{n-1}) = \underbrace{H(x_n | x_{n-1})}}$$

TIPICNO ZA MARKOV LANEC PRVA [illegible]

TEKOVNATA SOSTOJBA ZAVISI SAMO OD PRETHODNATA SOSTOJBA NO NE I OD SOSTOJBITE PRED TOA.

$$H(x_n|x_{n-1}) = P(x_{n-1} = 1) \cdot H(x_n | x_{n-1} = 1) + P(x_{n-1} = 0) \cdot \underbrace{H(x_n | x_{n-1} = 0)}_{0}$$

NEMA NEIZVESNOST

AKO GO ZNAES x_{n-1} GO ZNAES I x_n

$$\boxed{x_n = \begin{cases} 1 & \text{if } x_{n-1} = 1 \\ 0 & \text{if } x_{n-1} = 0 \end{cases}}$$

(g) x_i, x_j → INDEPENDENT FOR $\forall i, j \in \{1, 2, \dots n\}$ $i \neq j$

$00|\overset{x_n}{0}$ $\quad$ $01|\overset{x_n}{1}$

$\boxed{n = 3}$ $\quad P(x_1, x_2, x_3) = \underbrace{P(x_1 x_2)}_{P(x_1, x_2)} \cdot \underbrace{P(x_3 | x_1 x_2)}_{P(x_3|x_2)}$

$P(x_1 x_2) = \frac{1}{4}$

$\boxed{P(x_2, x_3) = P(x_2) P(x_3|x_2)}$

$x_2 \backslash x_3$	0	1	$P(x_2)$
0	1/2	0	1/2
1	0	1/2	1/2
$P(x_3)$	1/2	1/2	▨

$x_1 \backslash x_2$	0	1	$P(x_1)$
0	$\frac{1}{4}$	1/4	$\frac{1}{2}$
1	$\frac{1}{4}$	1/4	1/2
$P(x_2)$	1/2	1/2	

$x_2 \backslash x_3$	0	1
0	1	0
1	0	1
	1	

$$H(x_3|x_2) = \sum P(x_2 x_3) \operatorname{ld} \frac{1}{P(x_3|x_2)}$$

$$H(X_3|X_2) = P(00)\,ld\frac{1}{P(0|0)} + P(11)\,ld\frac{1}{P(1|1)} =$$

$$= \frac{1}{2}\,ld\,1 + \frac{1}{2}\,ld\,1 = 0$$

$$\boxed{\frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}}$$

$$H(X_2,X_3) = \frac{1}{2}\,ld\,2 + \frac{1}{2}\,ld\,2 = 1$$

$$H(X_1,X_2) = 4\left(\frac{1}{4}\,ld\,4\right) = 4\cdot\frac{1}{2} = 2$$

$$P(X_2|X_1) = P(X_1,X_2)/P(X_1)$$

$\boxed{P(X_2|X_1)}$

$X_1 \backslash X_2$	0	1
0	1/2	1/2
1	1/2	1/2

$$H(X_2|X_1) = \sum_{X_1X_2} P(X_1,X_2)\,\underbrace{ld\frac{1}{P(X_2|X_1)}}_{H(X_2)} = 4\left(\frac{1}{4}\,ld\,2\right) = 1$$

$$H(X_1,X_2) = H(X_1) + H(X_2|X_1) = 1 + 1 = 2$$

- ИСТИОТ ИЗРАЗ ВАЖИ ЗА СЕКОЕ $i, j \in \{1,..n-1\}$ КАДЕ $i \neq j$ Т.Е:

$$H(X_i,X_j) = 2 \qquad i \neq j \qquad i,j \in \{1,2,\dots n-1\}$$

(c) $H(X_1,X_2,\dots X_n) = H(X_1,X_2,\dots X_{n-1}) = (n-1)\,H(X_1)$

$$H(X_1,\dots X_{n-1}) = H(X_1) + \underbrace{H(X_2|X_1)}_{H(X_2)} + \dots + \underbrace{H(X_{n-1}|X_{n-2},\dots X_1)}_{H(X_{n-1})} =$$

$$= \sum_{i=1}^{n-1} H(X_i) = (n-1)H(X_1)$$

$$\boxed{H(X_2,X_3) = H(X_2) + \underbrace{H(X_3|X_2)}_{1} = 2}$$

$$\boxed{H(X_1,X_3) = H(X_1) + H(X_3) = 2}$$

- ВАЖИ И ОБРАТНО АКО ГО ЗНАЕШ
- БРОЈОТ НА ЕДИНИЦИ НЕМА ВРСКА СО ПАРНОСТА НА БРОЈОТ ШТО ТАА СЕКВЕНЦА ГО ПРЕТСТАВУВА!!!

	X_3
00	0
01	1
10	1
11	0

$$H(X_3|X_1X_2) = P(00)\cdot H(X_3|\underline{00}) + P(01)\,H(X_3|01) + P(10)\,H(X_3|10) + P(11)\,H(X_3|11) = 0$$

$$H(X_1,X_2,X_3,\dots X_n) = \underbrace{H(X_1)}\cdot\underbrace{H(X_2|X_1)}_{1}\cdot\underbrace{H(X_3|X_2)}_{1}\cdots H(X_n|X_{n-1})$$

$$H(X_n|X_{n-1}) = P(X_{n-1}=0)\,\underbrace{H(X_n|0)}_{?} + P(1)\cdot\underbrace{H(X_n|1)}_{?}$$

$$\boxed{P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \longrightarrow$$ ТРАНЗИЦИОНА МАТРИЦА

$x_1, x_2, \ldots x_{n-1}, x_n$ . . $x_i \in \{0, 1\}$

$p(x_i) = \{p, 2\}$

$P(x_1, x_2, \ldots x_{n-1}) = p^{m} \cdot 2^{n-1-m}$ $m = 2K$ $K = 1, 2, \ldots$

$P(x_1, x_2, \ldots x_{n-1}) = p^{n} \cdot 2^{n-1-m}$ $m = 2k+1$.

(b) $H(x_i, x_j) = \begin{cases} 2 & i,j \in \{1,2,\ldots n\} \\ 1 & j = n \quad i = n-1 \end{cases}$ $i \neq j$ $\boxed{(i \neq n-1) = n}$

ONA NEE DODIO!!

■ $(x_1 x_2 x_3)$

000
011
101
110

$H(x_1, x_3) = ?$

$H(x_1, x_2, x_3) = H(x_1 x_2) + \underbrace{H(x_3 | x_1, x_2)}_{0} = H(x_1 x_3) +$

$+ \underbrace{H(x_2 | x_1 x_3)}_{0} = H(x_2 x_3) + \underbrace{H(x_1 | x_2 x_3)}_{0}$

$\boxed{H(x_1 x_3) = H(x_1 x_2) = H(x_2 x_3) = H(x_1) + H(x_2) = 2}$

PROVE FOR INDEPENDANCE

■ $(x_1 x_2 x_3, \ldots x_{n-1}, x_n)$

$H(x_1 x_2, \ldots x_{n-1} x_n) = H(x_1, x_n) + H(x_2 x_3, \ldots x_{n-1} | x_1 x_n)$

$= H(x_1 x_2, \ldots x_{n-1}) + \underbrace{H(x_n | x_1 x_2, \ldots, x_{n-1})}_{0} =$

$= H(x_1 x_2, \ldots x_{n-2}, x_n) + \underbrace{H(x_{n-1} | x_1 x_2, \ldots x_{n-2} x_n)}_{0}$

$= H(x_2, x_3, \ldots x_{n-2}, x_{n-1} x_n) + \underbrace{H(x_1 | x_2 x_3, \ldots x_n)}_{0}$

$\boxed{H(x_1 x_2 \ldots x_n) = H(x_1 x_2 \ldots x_{n-1}) = H(x_2, \ldots x_n) = (n-1) H(x_1)}$

DOKAZ ZA (C) ★

$(x_1 x_2 x_3 x_4)$

$H(x_1 x_2 x_3 x_4) = H(x_1 x_4) + H(x_2 x_3 | x_1 x_4) = H(x_2 x_3) + H(x_1 x_4 | x_2 x_3)$

$H(x_2 x_3 | x_1 x_4) = \underline{H(x_2 | x_1 x_4)} + \underbrace{H(x_3 | x_1 x_2 x_4)}_{0}$

~~$H(x_2 x_3 | x_1 x_4) = P(00) \cdot H(x_2 x_3 | 00) + P(11) H(x_2 x_3 | 11) = 2$~~

~~$\frac{1}{2} ld2 + \frac{1}{2} ld2 = 1$~~

$H(x_1 x_2 x_3 x_4) = H(x_1 \overset{1}{x_2} x_3) + \underbrace{H(x_4 | x_1 x_2 x_3)} = H(x_1 x_2 x_4) + \underbrace{H(x_3 | x_1 x_2 x_4)}$

$= H(x_1 x_3 x_4) + \underbrace{H(x_2 | x_1 x_3 x_4)}_{0} = \overset{0}{H}(x_2 x_3 x_4) + \underbrace{H(x_1 | x_2 x_3 x_4)}_{0}$.

$$H(x_1x_2x_3x_4) = H(\underline{x_1x_2x_3}) = H(\underline{x_1x_3x_4}) = H(x_2x_3x_4) = H(x_1x_2x_4)$$

$$H(x_1x_2x_3) = H(x_1x_2) + H(x_3|x_1x_2) = H(x_1,x_2) + H(x_3) = 3$$

$$H(x_1x_3x_4) = H(x_1x_3) + H(x_4|x_1x_3) = \underline{H(x_1,x_3)} + H(x_4|x_1x_3)$$

$$H(x_2x_3x_4) = H(x_2x_3) + H(x_4|x_2x_3) = \underline{H(x_2x_3)} + H(x_4|x_2x_3)$$

$$H(x_1x_2x_4) = H(x_1x_4) + H(x_2|x_1x_4) = \underline{H(x_1x_2)} + H(x_4|x_1x_2)$$

$$\underline{I(X_{n-1}; X_n | x_1x_2, \dots x_{n-2})} = H(X_n | x_1x_2 \dots x_{n-2}) -$$

$$\underbrace{H(X_n | x_1x_2, \dots x_{n-1})}_{0} = \underbrace{H(x_1x_2, \dots x_{n-1})}_{(n-1)} - \underbrace{H(x_1x_2, \dots x_{n-2})}_{(n-2)}$$

$$= \cancel{n}-1-\cancel{n}+2 = 1$$

$$\boxed{I(x_{n-1}; x_n | x_1x_2, \dots x_{n-2}) = 1 \text{ bit}}$$

$$I(x_1; x_n | x_2, x_3, \dots x_{n-1}) \overset{!}{=} H(x_n|x_2x_3 \dots x_{n-1})$$

$$- \underbrace{H(x_n|x_1x_2, \dots x_{n-1})}_{\cancel{0}} = \underbrace{H(x_2x_3 \dots x_n)}_{?} - \underbrace{H(x_2x_3 \dots x_{n-1})}_{n-2}$$

$$\to H(x_1, x_2, \dots x_n) = H(x_2, \dots x_n) + \underbrace{H(x_1|x_2x_3 \dots x_n)}_{0}$$

VIDI: ★

$$\boxed{I(X_1; x_n | x_2x_3, \dots x_{n-1}) = H(x_1x_2, \dots x_n) - n + 2}$$

$$I(x_1; x_n | x_2x_3, \dots x_{n-1}) = \underbrace{H(x_1 | x_2x_3, \dots x_{n-1})}_{H(X_1)} - \underbrace{\underline{H(x_1|x_2, \dots x_n)}}_{0}$$

$$\boxed{H(x_1|x_2, \dots x_n) = H(x_1x_2, \dots x_n) - H(x_2, \dots x_n) = 0}$$

ako gi znaeš $\underline{x_2, \dots x_n}$ go znaeš i x_1 !!!

$$H(x_1) = H(x_1x_2, \dots x_n) - (n-2) \qquad H(x_1x_2, \dots x_n) = H(x_1) + n - 2$$

$$\boxed{H(x_1) = 1} \Rightarrow \boxed{H(x_1, x_2, \dots x_n) = n-1}$$

$$\boxed{I(x_1; x_n | x_2x_3, \dots x_{n-1}) = H(x_1)}$$

DALI E OVA DOVOLEN DOKAZ DEKA x_1, x_n SE NEZAVISNI ???

$$I(x_1; x_n | x_2x_3 \dots x_{n-1})$$

$$I(x_1, x_2, \dots x_{n-1}; X_n) = \sum_{i=1}^{n} I(x_i; x_n | x_{i-1}, x_{i-2}, \dots x_1) =$$

$$= I(x_1; x_n) + I(x_2; x_n|x_1) + I(x_3; x_n|x_1x_2) + \dots +$$

$$+ \underbrace{I(x_{n-1}; x_n | X_{n-2}, x_{n-3}, \dots x_1)}_{1}$$

$$I(x_2; x_n|x_1) = \underline{H(x_2|x_1)} - H(x_2|x_1, x_n)$$

~~[illegible]~~ $n=(n)H=$ [illegible]

$H(x_2|x_1x_n) = ?$ x_2, x_1 INDEPENDENT

$H(x_1x_2x_n) = H(x_1) + \underbrace{H(x_2|x_1)}_{H(x_2)} + H(x_n|x_1x_2) =$

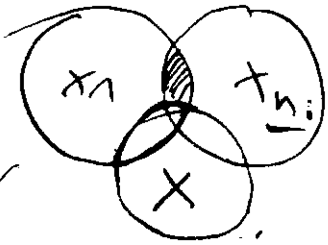
$= H(x_1x_2) + H(x_n|x_1x_2)$

$H(x_1x_2x_n) = H(x_1) + H(x_n|x_1) + H(x_2|x_1x_n)$

$I(x_1, x_2, \dots x_{n-1}; x_n) = I(x_1; x_n) + H(x_2) - H(x_2|x_1,x_n) +$

$+ H(x_3) - H(x_3|x_1x_2x_n) + H(x_{n-2}) - H(x_{n-2}|x_1x_2,\dots x_{n-3}x_n) + 1$

$I(x_1; x_n|\underbrace{x_2, x_3, \dots x_{n-1}}_{X}) = 1$



$\underbrace{I(x_1;x_n|X)}_{1} = I(x_1,X;X_n) - I(X;x_n)$

$I(x_1x_2,\dots x_{n-1};x_n) = 1 - I(X;x_n)$

[redacted]

$I(x_1X;x_n) = I(x_1;x_n) + I(X;x_n|x_1)$

$I(x_1;x_n) = I(x_1X;x_n) - I(X;x_n|x_1) =$

$= I(x_1x_2,\dots x_{n-1};x_n) - I(x_2\dots x_{n-1};x_n|x_1) =$

$= H(x_n) - \underbrace{H(x_n|x_1x_2,\dots x_{n-1})}_{0} - H(x_n|x_1) + \underbrace{H(x_n|x_1\dots x_{n-1})}_{0}$

$I(x_1;x_n) = H(x_n) - H(x_n|x_1) = I(x_1;x_n)$

$= H(x_1x_2,\dots x_{n-1}) - H(x_1x_2,\dots x_{n-1}|x_n) -$

$- \underbrace{H(x_2\dots x_{n-1}|x_1)}_{H(x_2,\dots x_{n-1})} + H(x_2,\dots x_{n-1}|x_1x_n) =$

$= (n-1) - (n-2) - (\overbrace{H(x_1x_2,\dots x_{n-1}|\boxed{x_n})}^{H(x_1x_2,\dots x_{n-1})}) - H(x_2,\dots x_{n-1}|x_1x_n)$

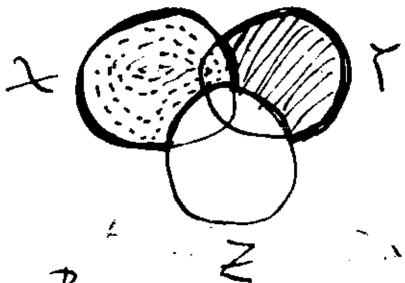
$= 1 - H(\cancel{x_1x_2},\dots x_{n-1}) + H(\cancel{x_1x_2}\dots x_{n-1}) - H(x_1|x_n) = 1-1=0$

DOKAZANO!!!

$H(x_2,\dots x_{n-1}|x_1x_n) = ?$

$\underbrace{H(x_1x_2,\dots x_{n-1}|x_n)}_{H(x_1x_2\dots x_{n-1})} = \underbrace{H(x_1|x_n)}_{=H(x_1)=1} + H(x_2,x_3,\dots x_{n-1}|x_1x_n)$

$$H(X,Y|Z) = \underbrace{H(X|Z)} + \underbrace{H(Y|X,Z)}$$



$$I(X,Y;Z) = I(X;Z) + I(Y;Z|X)$$

- ПОВТОРУВАЊЕ НА ДОКАЗОТ ⓐ : **РЕШЕНИЕ НА (а)** MMV

$$I(x_1,\underbrace{x_2,\dots x_{n-1}}_{x};x_n) = I(x_1;x_n) + I(x_2,x_3,\dots x_{n-1};x_n|x_1)$$

$$I(x_1;x_n) = I(x_1x_2,\dots x_{n-1};x_n) - I(x_2x_3,\dots x_{n-1};x_n|x_1) =$$
$$H(x_1x_2,\dots x_{n-1}) - H(x_1x_2,\dots x_{n-1}|x_n) - H(x_2x_3,\dots x_{n-1}|x_1) +$$
$$H(x_2x_3,\dots x_{n-1}|x_1,x_n) = (n-1) - H(x_1x_2,\dots x_{n-1}) -$$
$$H(x_2x_3;\dots x_{n-1}) + H(x_2x_3,\dots x_{n-1}|x_1x_n)$$

x_1 НЕ ЗАВИСИ ОД x_n

$$H(x_1x_2,\dots x_{n-1}|x_n) = H(x_1|x_n) + H(x_2,\dots x_{n-1}|x_1x_n)$$
$$= H(x_1) + H(x_2,\dots x_{n-1}|x_1x_n)$$

$$\boxed{H(x_2,\dots x_{n-1}|x_1x_n) = \underbrace{H(x_1x_2,\dots x_{n-1}|x_n)}_{=H(x_1x_2,\dots x_{n-1})} - H(x_1)}$$

$$I(x_1;x_n) = (n-1) - \cancel{(n-1)} - (n-2) + (n-1) - H(x_1) = 2-1-1 = 0$$

- НЕ МОРА ДА ГО ИЗМЕНУВАШ ПРЕКУ „n" ЗОШТО СТАНАЛО ЌЕ КРАТАТ

$$\boxed{\begin{aligned}I(x_1;x_n) &= \cancel{H(x_1x_2,\dots x_{n-1})} - \cancel{H(x_1x_2,\dots x_{n-1})} - H(x_2x_3,\dots x_{n-1}) +\\ &+ H(x_1x_2,\dots x_{n-1}) - H(x_1) = -(n-2) + (n-1) - H(x_1) = 1-1 = 0\end{aligned}}$$

ДОКАЖАНО!!!

$\boxed{I(x_1;x_n)=0}$ $\boxed{x_1 \,\&\, x_n \text{ ARE INDEPENDENT}}$

ИСТИОТ ДОКАЗ ВАЖИ ЗА БИЛО КОЕ $\boxed{i=1,2,\dots n-1}$

$$I(x_2x_1x_3x_4,\dots x_{n-1};x_n) = I(x_2;x_n) + I(x_1x_3\dots x_{n-1};x_n|x_2)$$
$$I(x_2;x_n) = I(x_1x_2,\dots x_{n-1};x_n) - I(x_1x_3\dots x_{n-1};x_n|x_2) =$$
$$= H(x_1x_2,\dots x_{n-1}) - \overset{H(x_1x_2,\dots x_{n-1})}{H(x_1x_2,\dots x_{n-1}|x_n)} - \overset{H(x_1x_3x_4,\dots x_{n-1})}{H(x_1x_3\dots x_{n-1}|x_2)} +$$
$$H(x_1x_3\dots x_{n-1}|x_2x_n) \qquad H(x_2)=1$$

$$H(x_2x_1x_3,\dots x_{n-1}|x_n) = H(x_2|x_n) - H(x_1x_3\dots x_{n-1}|x_nx_2)$$
$$H(x_1x_3,\dots x_{n-1}|x_nx_2) = H(x_2|x_n) - H(x_1x_2\dots x_{n-1})$$
$$I(x_2;x_n) = H(x_1x_3\dots x_{n-1}|x_2) + H(x_2|x_n) - H(x_1x_2\dots x_{n-1})$$
$$= H(x_1x_3\dots x_{n-1}) + H(x_2) - (n-1) = (\cancel{n}-2) + 1 - \cancel{n} + 1 = 2-2 = 0$$

ДОКАЗ ЗА $\boxed{i=2}$ ИСТОТО МОЖЕ ДА СЕ НАПРАВИ ЗА БИЛО КОЕ i

(6) $H(X_i, X_i) = ?$ $\boxed{\text{РЕШЕНИЕ НА (6)}}$

$$H(X_1, X_2) = H(X_1) + H(X_2|X_1) = H(X_1) + H(X_2) = 1+1 = 2$$

$$H(X_1, X_4) = H(X_1) + H(X_4|X_1) = H(X_4) + \overbrace{H(X_1|X_4)}^{H(X_1)}$$

$$H(X_1 X_2 \dots X_4 | X_4) = H(X_1|X_4) + H(X_2 X_3 \dots X_{4-1} | X_4 X_1)$$

$$\rightarrow H(X_1, X_4) = H(X_1) + H(X_4|X_1) = H(X_4) + H(X_1) \Rightarrow$$

$$\cancel{H(X_1)} + H(X_4|X_1) = H(X_4) + \cancel{H(X_1)} \Rightarrow \boxed{H(X_4) = H(X_4|X_1)}$$

$$\Rightarrow \boxed{H(X_1, X_4) = H(X_1) + H(X_4)}$$

ЛОГИЧНО ВО (9)

ДОКАЖАВ ДЕКА X_1 & X_4 СЕ НЕЗАВИСНИ. СЕГА ОСТАНУВА ДА СЕ НАЈДЕ $H(X_4)$!!!

$$I(X_1; X_4) = 0 = H(X_1) - H(X_1|X_4) = H(X_4) - H(X_4|X_1)$$

• $\{X_i\}$ $\{X_1, X_2, \dots X_{4-1}\}$ И СО ОГЛЕД НА ТОА ШТО $\boxed{P(X_i) = [\frac{1}{2}, \frac{1}{2}] \quad X \in [0,1]}$

ЗБИРОТ НА СЕКВЕНЦИ СО ПАРЕН И НЕПАРЕН БРОЈ НА ЕДИНИЦИ Е ИСТ Т.Е ИЗНЕСУВА 2^{4-1}

$$\boxed{P(X_i = ODD) = \frac{1}{2^{4-1}} = P(X_4=1) \;\Big|\; P(X_i = even) = \frac{1}{2^{4-1}} = P(X_4=0)}$$

$$H(X_4) = 2^4 \left[P(X_4=1) \cdot ld \frac{1}{P(X_4=1)} + P(X_4=0)\, ld \frac{1}{P(X_4=0)} \right]$$

$$= 2^4 \left[\frac{1}{2^{4-1}} ld\, 2^{4-1} + \frac{1}{2^{4-1}} ld\, 2^{4-1} \right] = 2^{4+1} \frac{4-1}{2^{4-1}} = 4(4-1)$$

— НЕ Е ОВА ДОБРА МАТЕМАТИКА !!! . СУШТИНАТА Е ДЕКА ВЕРОЈАТНОСТА СЕКВЕНЦАТА ИЗВЛЕЧЕНА ОД ЗБИРОТ НА СЕКВЕНЦИ $\{X_1 X_2, \dots X_{4-1}\}$ ИМА НЕПАРЕН БРОЈ ЕДИНИЦИ Е : $1/2$, А ДА ИМА ПАРЕН БРОЈ НА ЕДИНИЦИ Е $1/2$ ЗАТОА : (*1)

$$P(X_4=1) = \frac{1}{2} \qquad P(X_4=0) = 1/2$$

$$\boxed{H(X_4) = \frac{1}{2} ld 2 + \frac{1}{2} ld 2 = 1}$$

$$H(X_1, X_n) = 2 \quad \text{i.e} \quad H(X_i, X_4) = 2 \quad \forall i \neq 4$$

ОВА Е РЕШЕНИЕ ЗА (6) !!!

(c) USTE EDNAŠ !!!

$$H(x_1 x_2, \dots x_n) = H(x_1 x_2 \dots x_{n-1}) + \underbrace{H(x_n | x_1 x_2 \dots x_{n-1})}_{\emptyset}$$

$$\rightarrow \boxed{H(x_1 x_2, \dots x_n) = H(x_1, x_2, \dots x_{n-1}) = (n-1)\, H(x_1) = (n-1)}$$

AKO SE ~~BELOSINO~~ NEZAVISNI SITE x_i TOGAŠ OVAA ENTROPIJA ĆE BIDE „n"

PAIRWISE INDEPENDENCE DOESN'T IMPLY COMPLETE INDEPENDENCE!!! HMV

⊡ Edition 1 Solution (⊛4 GO DOKAŽUVAAT SO INDUKCIJA)

- PROOF THAT FOR ANY $k \leq n-1$, THE PROBABILITY THAT $\sum_{i=1} x_i$ IS ODD IS 1/(2), BY USING THE INDUCTION.
- CLEARLY IT IS TRUE FOR $k=1$ (DEFINITION OF THE PROBLEM)
- ASUME IT IS TRUE FOR $\boxed{k-1}$
- FOR „k": $S_k = \sum_{i=1}^{k} x_i$

$$\boxed{\begin{aligned} P(S_k \text{ odd}) &= P(S_{k-1} \text{ odd}) \cdot P(x_k = 0) + P(S_{k-1} \text{ even}) \cdot P(x_k = 1) \\ &= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2} \end{aligned}} \quad \boxed{HMV}$$

- MNOGU JAK I LOGIČEN DOKAZ!!!

HENCE FOR ~~R~~ $k \leq n-1$ PROBABILITY THAT S_k IS ODD IS EQUAL WITH PROBABILITY S_k I EVEN, HENCE:

$$\boxed{P(x_n = 1) = \frac{1}{2} \qquad P(x_n = 0) = \frac{1}{2}}$$

(a) $\boxed{j=n}$ $\boxed{i=1}$ WITHOUT THE LOSS OF GENERALITY.

$$P(x_1 = 1, x_n = 1) = P\left(x_1 = 1, \sum_{i=2}^{n-1} x_i \text{ even}\right) = P(x_1 = 1) \cdot P\left(\sum_{i=2} \text{even}\right)$$

$$= \frac{1}{2} \cdot \frac{1}{2} \overset{⊛}{=} P(x_1 = 1) \cdot P(x_n = 1) \rightarrow \text{HENCE } x_1 \& x_n$$

ARE INDEPENDENT.

- DOKAZOT NE MI DELUVA UBEDLIV!!! MOJOT DOKAZ NA STR. 35 MI DELUVA POUBEDLIV.

→ SEPAK REZONOT E JAK!! PRAŠAS SE KOJA E VEROJATNOSTA $x_1 = 1$ i $x_n = 1$. TREBA x_1 DA E 1 I SUMATA OD x_i $i = 2 \dots n-1$ DA BIDE PARNA. VO OVOJ KONTEKST, OVAA FORMULACIJA E MNOGU JAKA!!! ONA ŠTO NE E 100%. JASNO E ⊛

(b) SINCE $x_i \& x_j$ $i,j \in 1,2,\dots,n$ ARE INDEPENDENT ⇒

$$H(x_i, x_j) = H(x_i) + H(x_j) = 1 + 1 = 2 \text{ bits}$$

$\boxed{4.11}$ STATIONARY PROCESSES

LET $\ldots, X_{-1}, X_0, X_1, \ldots$ BE A STATIONARY (NOT NECESSARY MARKOV) STOCHASTIC PROCESS. WHICH OF THE FOLLOWING STATEMENTS ARE TRUE? PROVE OR PROVIDE THE COUNTEREXAMPLES.

✓(a) $H(X_n|X_0) = H(X_{-n}|X_0)$

✓(b) $H(X_n|X_0) \geq H(X_{n-1}|X_0)$

✓(c) $H(X_n|X_1 X_2, \ldots X_{n-1}, X_{n+1})$ IS NONINCREASING IN n?

✓(d) $H(X_n|X_1, \ldots, X_{n-1}, X_{n+1}, \ldots X_{2n})$ IS NONINCREASING IN "n"?

(b) FOR MARKOV CHAIN — DATA PROCESSING

$$I(X_0, X_{n-1}) \geq I(X_0, X_n)$$
$$\geq H(X_n) - H(X_n|X_0)$$

$$H(X_{n-1}) - H(X_{n-1}|X_0) \geq$$
$$\boxed{H(X_{n-1}|X_0) \leq H(X_n|X_0)}$$

(b) e točno no za stacionaren Markov lanec

$$H(X_{n+1}|X_1 X_2 \ldots X_n) \leq H(X_{n+1}|X_2 X_3 X_4 \ldots X_n) =$$
$$= H(X_n|X_1 X_2 X_3, \ldots X_{n-1})$$
$$\boxed{H(X_n|X_1 \ldots X_{n-1}) \geq H(X_{n+1}|X_1 X_2, \ldots X_n)}$$

$$H(X_n|X_1) \geq H(X_n|X_1 X_2) = H(X_n|X_2) = H(X_{n-1}|X_1)$$
$$H(X_n|X_1) \geq H(X_{n-1}|X_1) \Rightarrow$$ ENTROPY INCREASES WITH "n" = FOR MARKOV CHAIN

(UŠTE EDEN DOKAZ NA "b" NO ZA MARKOV L.

(c) $H(X_n|X_1 X_2, \ldots X_{n-1}) \geq H(X_{n+1}|X_1 X_2, \ldots X_n)$

↳ NONINCREASING FOR STATIONARY PROCES

$$H(X_n|X_1 X_2 \ldots X_{n-1}, X_{n+1})$$
$$= H(X_{n+1}|X_2 X_3, \ldots X_n, X_{n+2}) \geq H(X_{n+1}|X_1 X_2 \ldots X_n X_{n+2})$$

ZNAČI:

$$H(X_n|X_1 X_2 \ldots X_{n-1}, X_{n+1}) \geq H(X_{n+1}|X_1 X_2, \ldots X_n, X_{n+2})$$

↳ NONINCREASING WITH "n" i.e. TRUE !!!

(a) $X_{-n}, X_{-n+1}, \ldots, X_{-1}, X_0, X_1, \ldots X_{n-1}, X_n$

$$H(X_{-n}|X_0) = H(X_{-n+n}|X_{0+n}) = H(X_0|X_n)$$
$$H(X_0|X_n) \neq H(X_n|X_0)$$

VIDI NOVO RAZMISLUVANJE NA PP. 39

(b) REVISITED

$$H(X_n / X_0 X_1, \ldots X_{n-1}) = H(X_{n+1} | X_1, \ldots X_n)$$

$$H(X_n | X_0) \geq H(X_n | X_0, X_1, \ldots X_{n-1}) = H(\underline{X_{n+1}} | X_1, \ldots X_n)$$

↑ CONDITIONING REDUCES ENTROPY

$$= H(X_{n-1} | X_{-1}, X_0, X_1, \ldots X_{n-1}) \leq H(X_{n-1} | X_0)$$

NE MOŽE DA SE POKAŽE ZA NON-MARKOV STOHASTIC PROCESS, SAMO ZA MARKOV VAŽI

(d) $H(X_n | X_1, X_2, \ldots, X_{n-1}, X_{n+1}, \ldots X_{2n}) =$

$$H(X_n | X_1, X_n) = H(X_1 X_n | \underset{=}{X_n}) - H(X_1 | X_n)$$

$$H(X_1 X_n | X_n) = H(X_1 | X_n) + \underbrace{H(X_n | X_1 X_n)}_{0}$$

$$= H(X_{n+1} | X_2, X_3, \ldots X_n, X_{n+2}, \ldots X_{2n+1}) \geq$$

$$H(X_{n+1} | X_1, X_2, \ldots X_n, X_{n+2}, \ldots X_{2n+1}) \Rightarrow$$

$$H(X_n | X_1, X_2, \ldots, X_{n-1}, X_{n+1}, \ldots X_{2n}) \geq H(X_{n+1} | X_1, X_2 \ldots X_n, X_{n+2}, \ldots X_{2n+1})$$

⇒ (d) IS TRUE!!!

(a) OVA MOŽE DA BIDE, TOČNO ZOŠTO ZA STACIONAREN PROCESS $H(X_0) = H(X_n) = H(X_t)$

$$\left.\begin{aligned} H(X_n / X_0) &= H(X_0, X_n) - H(X_0) \\ H(X_0 / X_n) &= H(X_0, X_n) - H(X_n) \end{aligned}\right\} \quad H(X_0) = H(X_n) \Rightarrow H(X_0 | X_n) = H(X_n | X_0)$$

MI, TEKNA TOČNO ŠTO SEFAK!!! AKO MI TEKNEŠE POLAKO ĆE GO REŠIV ISTO KAKO VO EDITION 1.

EDITION II SOLUTIONS

(a)
$$H(X_n | X_0) = -H(X_0) + H(X_0, X_n)$$
$$H(X_{-n} | X_0) = -H(X_0) + H(X_0, X_{-n})$$

$$\boxed{H(X_0, X_n) = H(X_n, X_0)}$$

$$\boxed{H(X_0, X_{-n}) = H(X_{0+n}, X_{-n+n}) = H(X_n, X_0)}$$

$$\Rightarrow H(X_n | X_0) = H(X_{-n} | X_0)$$

(b) $H(X_n | X_0) \geq H(X_{n-1} | X_0)$ } TRUE FOR MARKOV CHAIN OF FIRST ORDER

EXAMPLE: SIMPLE PERIODIC PROCESS WITH PERIOD „n"

$X_0, X_1, \ldots X_{n-1} \qquad k \geq n \qquad X_k = X_{k-n}$

$X_0, X_1, \ldots X_{n-1}, X_0, X_1, \ldots X_{n-1}$

$\Rightarrow H(X_n | X_0) = H(X_0 | X_0) = 0 \qquad \underline{H(X_{n-1} | X_0) = H(X_{n-1})}$

BINARY RANDOM VARIABLES $\Rightarrow H(X_{n-1}) = 1$

$H(X_{n-1}|X_0) = 1 > H(X_n|X_0) = 0$ WHICH IS CONTRADICTING THE STATEMENT $H(X_n|X_0) \geq H(X_{n-1}|X_0)$

- ПРИМЕР КОЈ ПОКАЖУВА ДЕКА НЕ Е ТОЧНО ТВРДЕЊЕТО

(б) ОД <u>solns3_229.pdf</u> ОД Berkley Solutions Е УШТЕ ПОТВРДЕНО.

$$P((X_0,X_1,X_2)=(0,0,1)) = P((X_0,X_1,X_2)=(0,1,0))$$
$$= P((X_0,X_1,X_2)=(1,0,0)) = 1/3$$

$X_n = X_{n+3k}$ for all $n \in \mathbb{Z}$ & ALL $k \in \mathbb{Z}$

→ DISCRETE TIME RENEWAL PROCESS CORRESPONDING TO A RENEWAL TIME OF 3. THEN $H(X_3|X_0)=0$

$$H(X_2|X_0) = \frac{1}{3}\underbrace{H(X_2|X_0=1)} + \frac{2}{3}H(X_2|X_0=0)$$

НЕМА НЕИЗВЕСНОСТ АКО $X_0=1$ $X_2=0$

$$H(X_2|X_0) = \frac{1}{3}\cdot 0 + \frac{2}{3}\Big(\underbrace{\frac{1}{2}ld2 + \frac{1}{2}ld2}_{1}\Big) = \frac{2}{3}$$

$$H(X_3|X_0) = H(X_0|X_0) = 0 < H(X_2|X_0) = \frac{2}{3}$$

(c) $H(X_n|X_1X_2,\dots X_{n-1},X_{n+1}) = H(X_n|X_1^{n-1},X_{n+1})$

ВЕРОЈАТНО ИСТ Е ДОКАЗОТ КАКО МОЈОТ НО [illegible] ДА СЕ НАВИКНАМ НА НОТАЦИЈАТА:

$$= H(X_{n+1}|X_2,\dots X_n,X_{n+2}) = H(X_{n+1}|X_2^n,X_{n+2}) \geq$$
$$\geq H(X_{n+1}|X_1^n,X_{n+2})$$ ⇒ NONINCREASING WITH TIME !!!

(d) ИСТА ФИНТА КОРИСТАТ ЗА (d) САМО НОТАЦИЈАТА СЕ КОМПЛИКУВА:

$2n-n+1 = n+1$

$$\underline{H(X_n|X_1^{n-1},X_{n+1}^{2n})} = H(X_1|X_{-n}^{0},X_2^{n+1}) \geq H(X_1|X_{-(n+1)}^{0},X_2^{n+1})$$
$$= \underline{\underline{H(X_{n+1}|X_1^n,X_{n+2}^{2n+1})}}$$ <u>PROVED !!!</u>

$\boxed{\text{PROBLEM 4.12}}$ ENTROPY RATE OF A DOG LOOKING FOR A BONE. A DOG WALKS ON THE INTEGERS, POSSIBLY REVERSING DIRECTION AT EACH STEP WITH PROBABILITY $p=0.1$. LET $X_0=0$. THE FIRST STEP IS EQUALLY LIKELY TO BE POSITIVE OR NEGATIVE. A TYPICAL WALK MIGHT LOOK LIKE THIS:

$$(X_0,X_1,\dots) = (0,-1,-2,-3,-4,-3,2,-1,0,1,\dots)$$

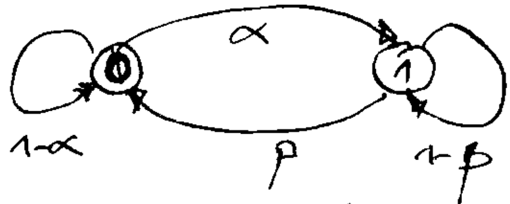
(a) FIND $H(X_1,X_2,\dots,X_n)$

(B) FIND ENTROPY RATE OF THE DOG

(C) WHAT IS EXPECTED NUMBER OF STEPS THAT THE DOG TAKES BEFORE REVERSING THE DIRECTION.

RECALL:

$$H(\mathcal{X}) = \sum_{i} \mu_i \sum_{j=1}^{n} p_{ij} \operatorname{ld} \frac{1}{p_{ij}}$$



$$\mu \circ P = \mu$$

$$[\mu_1 \; \mu_2] = [\mu_1 \; \mu_2] \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

$$\mu_1 = \mu_1(1-\alpha) + \beta \cdot \mu_2$$

$$\mu_2 = \alpha \cdot \mu_1 + (1-\beta)\mu_2$$

$$\alpha\mu_1 = \mu_2 - \mu_2 + \beta\mu_2$$

$$\boxed{\alpha\mu_1 = \beta\mu_2}$$

$$\mu_1 = \mu_1(1-\alpha) + \cancel{\beta} \cdot \frac{\alpha}{\cancel{\beta}} \cdot \mu_1$$

$$\boxed{\mu_1 + \mu_2 = 1} \qquad \alpha\mu_1 = \beta \cdot (1-\mu_1) \qquad \alpha\mu_1 = \beta - \beta\mu_1$$

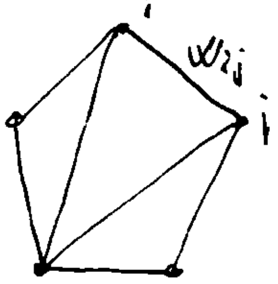
$$(\alpha+\beta)\mu_1 = \beta \qquad \boxed{\mu_1 = \frac{\beta}{\alpha+\beta}} \qquad \boxed{\mu_2 = \frac{\alpha}{\alpha+\beta}}$$

$$H(\mathcal{X}) = \frac{\beta}{\alpha+\beta} \cdot \left(p_{11} \operatorname{ld} \frac{1}{p_{11}} + p_{12} \operatorname{ld} \frac{1}{p_{12}} \right) +$$

$$+ \frac{\alpha}{\alpha+\beta} \left(p_{21} \operatorname{ld} \frac{1}{p_{21}} + p_{22} \operatorname{ld} \frac{1}{p_{22}} \right) =$$

$$= \frac{\beta}{\alpha+\beta} \left[(1-\alpha) \operatorname{ld} \frac{1}{1-\alpha} + \alpha \operatorname{ld} \frac{1}{\alpha} \right] + \frac{\alpha}{\alpha+\beta} \left[\beta \operatorname{ld} \frac{1}{\beta} + (1-\beta) \operatorname{ld} \frac{1}{1-\beta} \right.$$

$$\boxed{H(\mathcal{X}) = \frac{\beta}{\alpha+\beta} H(\alpha) + \frac{\alpha}{\alpha+\beta} H(\beta)}$$



$$\boxed{W_i = \sum_{j=1}^{n} W_{ij}} \qquad \boxed{\mu_i}$$

E - NUMBER OF EDGES

$$2E = \sum_{i=1}^{n} \sum_{j=1}^{n} W_{ij}$$

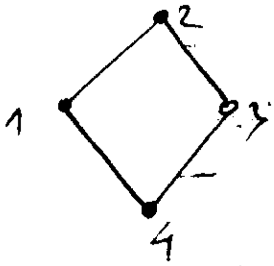
$$\mu_i = \frac{W_i}{2E} \qquad H(\mathcal{X}) = \sum_{i=1}^{n} \mu_i \sum_{j=1}^{n} \frac{W_{ij}}{W_i} \cdot \operatorname{ld} \frac{W_i}{W_{ij}}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\cancel{W_i}}{2E} \cdot \frac{W_{ij}}{\cancel{W_i}} \operatorname{ld} \frac{W_i}{W_{ij}} = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{W_{ij}}{2E} \operatorname{ld} \frac{W_i}{W_{ij}}$$

$$\boxed{H(x) = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{W_{ij}}{2E} \cdot ld \frac{W_i}{W_{ij}}} \quad ⊛ \qquad \boxed{W_{ij} = w}$$

- FOR EQUAL WEIGHTS OF EDGES

$$\boxed{H(x) = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{w}{2E} \cdot ld \frac{W_i}{w}}$$



$$H(x) = \sum_{i=1}^{n} \left(\frac{w}{2E} ld \frac{W_i}{w} + \frac{w}{2E} ld \frac{W_i}{w} + \dots + \frac{w}{2E} ld \frac{W_i}{w} \right) = \sum_{i=1}^{n} n \cdot \frac{w}{2E} ld \frac{W_i}{w}$$

$$= n \frac{w}{2E} \left[\sum_{i=1}^{n} ld W_i - ld w \right] = -\frac{nw}{2E} ld w + \frac{nw}{2E} \sum_{i=1}^{n} ld W_i$$

$$⊛ \downarrow \quad \boxed{\mu_j = \sum_{i=1}^{n} \mu_i P_{ij} = \sum_i \frac{W_i}{2E} \frac{W_{ij}}{\cancel{W_i}} = \frac{W_j}{2E}}$$

$$H(x) = \sum_{\substack{i=1 \\ j=1}}^{n} \frac{W_{ij}}{2E} ld \frac{W_i}{2E} - \sum_{i,j} \frac{W_{ij}}{2E} ld \frac{W_{ij}}{2E} =$$

$$= -H\left(\dots, \frac{W_{ij}}{2}, \dots\right) + H\left(\dots, \frac{W_i}{2E}, \dots\right)$$

- FOR EQUAL WEIGHT: $\frac{E_i}{2E} = \mu_i$

E_i - NUMBER OF EDGES EMANATING FROM NODE i
E - TOTAL NUMBER OF EDGES

$$H(x) = -\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{W_{ij}}{2W} ld \frac{W_{ij}}{W_i} = -\sum_{i,j} \frac{W_{ij}}{2E} ld \frac{W_{ij}}{E_i}$$

$$= \underbrace{-\sum_{ij} \frac{W_{ij}}{2E} ld \frac{W_{ij}}{2E}}_{⊛} + \sum_{i,i} \frac{W_{ij}}{2E} ld \frac{E_i}{2E}$$

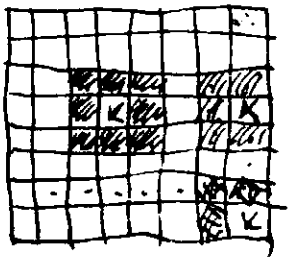
$$⊛ = -\sum_{i=1}^{E}\sum_{j=1}^{E} \frac{W_{ij}}{2E} ld \frac{W_{ij}}{2E} = -\sum_{i=1}^{E}\sum_{j=1}^{E} \frac{w}{2E} ld \frac{w}{2E} =$$

$$= -2E \cdot \frac{w}{2E} ld \frac{w}{2E} = w \, ld \frac{2E}{w} = \left| w=1 \right| = ld \, 2E$$

$$H(X) = H(X_2|X_1) = \text{ld}\, 2E + \sum_i \frac{\overset{=E_i}{\boxed{W_i}}}{2E} \text{ld} \frac{\overset{\to =E_i}{\boxed{W_i}}}{2E}$$

$$H(X) = \text{ld}\, 2E - H\left(\dots, \frac{E_i}{2E}, \dots\right) = \text{ld}\, 2E - H\left(\frac{E_1}{2E}, \dots \frac{E_n}{2E}\right)$$

- KING REVISITED



$$2E = 24 \cdot 5 + 36 \cdot 8 + 4 \cdot 3 =$$
$$= 120 + 288 + 12 = 420$$

$$\mu_1 = \frac{5}{420} \quad \mu_2 = \frac{8}{420} \quad \mu_3 = \frac{3}{420}$$

$$H(X) = \text{ld}\, 420 + 24 \frac{5}{420} \text{ld} \frac{5}{420} + 36 \cdot \frac{8}{420} \text{ld} \frac{8}{420} + 4 \cdot \frac{3}{420} \text{ld} \frac{3}{420}$$

$$\boxed{H(X) = 2.76584}$$

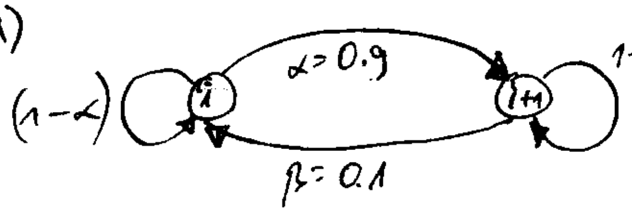
MIGHT LOOK LIKE

$$(X_0, X_1, X_2, \dots) = (0, -1, -2, \dots -1, 0, 1, \dots)$$

$$H(X_1, X_2, \dots X_n) = \,?$$

$$\boxed{H(X_1, X_2, \dots X_n) = H(X_2|X_1)}$$

(a)



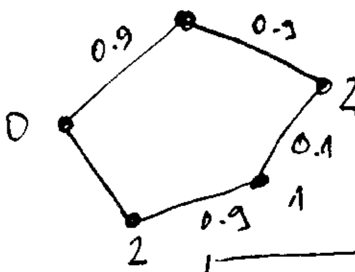
$$P_{ij} = \begin{bmatrix} 0 & 0.9 \\ 0.1 & 0 \end{bmatrix}$$

$$H(X) = \frac{\beta}{\alpha+\beta} H(\alpha) + \frac{\alpha}{\alpha+\beta} H(\beta) \qquad (H(\alpha) = H(\beta))$$

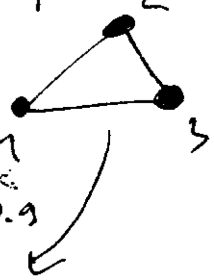
$$\boxed{H(X) = \frac{\cancel{\beta+\alpha}}{\cancel{\alpha+\beta}} \cdot H(\alpha) = \alpha\, \text{ld} \frac{1}{\alpha} + \beta\, \text{ld} \frac{1}{\beta} = \\ = -0.9\, \text{ld}\, 0.9 + 0.1\, \text{ld}\, 0.1 = 0.469}$$

$$H(X) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{W_{ij}}{\boxed{2W}} \cdot \text{ld} \frac{W_i}{W_{ij}} =$$

$$\boxed{E = 4}$$



$$: \quad \gamma = 0.1 \quad q = 0.9$$



1 2 3 = q^2
1 2 1 = γq
1 0 1 = γq
1 0 −1 = γq
−1 0 1 = q^2
−1 0 −1 = γq
−1 −2 −1 = γq
−1 −2 −3 = γ^2

$$\boxed{H(X_1 X_2 X_3) = 4\gamma q\, \text{ld} \frac{1}{\gamma q} + 2\gamma^2\, \text{ld} \frac{1}{\gamma^2} + 2 q^2\, \text{ld} \frac{1}{q^2}}$$

$$H(X_1 X_2 X_3) = 1.87598$$

$H(x_2|x_1) = H(x_1,x_2) - H(x_1)$ $\quad x_1 \in \{1,-1\}$

$H(x_1) = \frac{1}{2} ld 2 + \frac{1}{2} ld 2 = 1$ $\quad p(\underbrace{x_1}_{0.5}) = \{\frac{1}{2}, \frac{1}{2}\}$

$H(x_1,x_2) = ?$ $\quad \boxed{P(x_2|x_1)}$ $\quad \boxed{P(x_1x_2)} = \overbrace{P(x_1)} \cdot P(x_2|x_1)$

$x_1 \backslash x_2$	0	2	~~[illegible]~~	-2	~~p(x_1)~~
1	p=0.1	q=0.9	~~0~~	0	~~1~~
-1	q=0.9	0	[illegible]	p=0.1	~~1~~
~~P(x_2)~~	~~1~~	~~0.9~~	[illegible]	~~0.1~~	

$x_1 \backslash x_2$	0	2	-2	$p(x_1)$
1	0.05	0.45	0	0.5
-1	0.45	0	0.05	0.5
$P(x_2)$	0.5	0.45	0.05	

$H(x_1x_2) = 2 \cdot 0.45 \, ld \frac{1}{0.45} + 2 \; 0.05 \, ld \frac{1}{0.05} = \underline{\underline{1.469}}$

$\boxed{H(x_2|x_1) = H(x_1x_2) - H(x_1) = 0.469}$ $= H(\mathcal{X}) \rightarrow$ FOR MARKOV CHAIN!!!

MOŽE DA GO DOBIEŠ I KAKO € NA 1.46 SOLUTION OF (b)

$H(\mathcal{X}) = \lim_{n\to\infty} \frac{1}{n} H(x_1,x_2,\dots x_n) = \lim_{n\to\infty} H(x_n|x_{n-1},\dots x_1)$

ENTROPY RATE

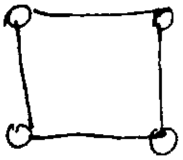
MHW ④

- RETURN TO (a) FOR $\boxed{n=4}$

PRETPOSTAVUVAM DEKA:

$$H(x_1,x_2,\dots x_n) = \cancel{2}\frac{1}{2}\sum_{k=0}^{n-1} \binom{n-1}{k} p^{n-k-1} q^k \, ld \frac{2}{p^{n-k-1} q^k}$$

$\boxed{n=4}$



(1*)

-1 0 1 2	$\frac{1}{2} q^3$	
-1 0 1 0	$\frac{1}{2} p q^2$	|
-1 -2 -3 -4	$\frac{1}{2} p^3$	
-1 -2 -3 -2	$\frac{1}{2} p^2 q$	||
-1 -2 -1 0	$\frac{1}{2} p q^2$	|
-1 -2 -1 -2	$\frac{1}{2} p^2 q$	||
-1 0 -1 -2	$\frac{1}{2} p^2 q$	||
-1 0 -1 0	$\frac{1}{2} p q^2$	|

1 2 3 4	$\frac{1}{2} q^3$	
1 2 3 2	$\frac{1}{2} p q^2$	|
1 2 1 2	$\frac{1}{2} p q^2$	|
1 2 1 0	$\frac{1}{2} p^2 q$	||
1 0 1 2	$\frac{1}{2} p q^2$	|
1 0 1 0	0.5 $p^2 q$	||
1 0 -1 -2	0.5 p^3	
1 0 -1 0	0.5 $p^2 q$	||

$H(x_1x_2x_3x_4) = 2 q^3 ld \frac{2}{q^3} + 6 p q^2 ld \frac{2}{pq^2} + 6 q^2 p \, ld \frac{2}{q^2 p} + q^3 ld \frac{2}{p^3}$

$\boxed{\binom{3}{1} = \frac{3!}{2! \cdot 1!} = \frac{6}{2} = 3}$ JA POTVRDUVA FORMULATA

- MISLAM DEKA GREŠKATA BEŠE VO PRESMETKATA NA $P(-1,0,1,2)$ SUM PIŠAL DEKA E: q^3. TREBA
- DA BIDE: $\frac{1}{2} q^3 \rightarrow P(-1)$
- DA DEFINITIVNO FORMULATA ZA $H(x_1,x_2,\dots x_n)$ E:

(11)

$$\boxed{H(x_1,x_2,\ldots,x_n) = \sum_{k=0}^{n-1}\binom{n-1}{k} p^{n-k-1} q^k \, ld \frac{1}{p^{n-k-1}q^k}}$$

- VO MAPLE:

$$evalf\left(\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\binom{n-1}{k} p^{n-k-1} q^k \, ld \frac{1}{p^{n-k-1}q^k}\right) = \underline{0,46900}$$

SE DOBIVA ISTATA VREDNOST OD ENTROPY RATE $H(\mathcal{X})$ ZNAČI NE MORA DA PRETPOSTAVUVAŠ DEKA E MARKOV LANEC GO DOBIVAŠ SO LIMESI.

AKO ZEMEŠ DEKA SE RABOTI ZA MARKOV LANEC (ŠTO E EVIDENTNO) TOGAŠ MOŽEŠ DA ODIŠ SO GOTOVA FORMULA ⓐ ILI SO FORMULATA $H(\mathcal{X}) = H(X_2|X_1)$ T.E. ①

(c) WHAT IS THE EXPECTED NUMBER OF STEPS BEFORE THE DOG REVERSE THE DIRECTION?

$$E[Y] = \sum_{i=1}^{n} i \cdot q^i \qquad Y \in [\xrightarrow{1}, \xrightarrow{2}, \xrightarrow{3}, \xrightarrow{4}, \ldots \xrightarrow{n}]$$

$$p(Y) = [q, q^2, q^3, q^4 \ldots q^n]$$

$$n\to\infty \quad \sum_{i=1}^{\infty} i q^i = E[Y]$$

$$S = x\sum_{i=1}^{n} i x^{i-1} \qquad \int \frac{S}{x}dx = \sum_{i=1}^{n} \cancel{i}\frac{x^i}{\cancel{i}} = \sum_{i=1}^{n} x^i = \frac{x(1-x^n)}{1-x} \quad S_1$$

$$S_1 = x + x^2 + \ldots - x^n \qquad S_1(1-x) = x - x^{n+1}$$

$$xS_1 = x^2 + x^3 + \ldots x^{n+1}$$

$$S_1 = \frac{x(1-x^n)}{1-x} \qquad S_1' = \frac{(1-x)(1-(n+1)x^n) + x(1-x^n)}{(1-x)^2}$$

$$S_1' = \frac{(1-x)(1-nx^n - x^n) + x - x^{n+1}}{(1-x)^2} = \frac{1 - nx^n - x^n - \cancel{x} + nx^{n+1} + \cancel{x^{n+1}} + \cancel{x} - \cancel{x^{n+1}}}{(1-x)^2} =$$

$$= \frac{1 - x^n + nx^n(x-1)}{(1-x)^2}$$

$$\boxed{S = \frac{x(1-x^n + nx^n(x-1))}{(1-x)^2}} \qquad n\to\infty \qquad \boxed{S = \frac{x}{(1-x)^2}}$$

$$S = x\sum_{i=1}^{\infty} i x^{i-1} \qquad \int \frac{S}{x}dx = \sum_{i=1}^{\infty} x^i = x + x^2 + \ldots = x(1 + x + x^2 + \ldots$$

$$= x\frac{1}{1-x}$$

$$\frac{S}{x} = \left(\frac{x}{1-x}\right)' = \frac{1-\cancel{x}+\cancel{x}}{(1-x)^2} \Rightarrow \boxed{S = \frac{x}{(1-x)^2}}$$

$$\Rightarrow E[Y] = \frac{q}{(1-q)^2} = \frac{0.9}{(0.1)^2} = \frac{0.9}{0.01} = \frac{90}{1} = 90 \text{ STEPS.}$$

• EDITION 1 SOLUTION:

(a) $H(x_0,x_1,\dots x_n) := \sum_{i=0}^{n} H(x_i|x_0^{i-1}) = \underbrace{H(x_0)}_{0} + \underbrace{H(x_1|x_0)}_{H(x_1)=1}$

$+ \sum_{i=2}^{n} H(x_i|x_{i-1},x_{i-2})$

$$\boxed{H(x_i|x_{i-1},x_{i-2}) = H(0.1,0.9) = 0.1\,ld\frac{1}{0.1} + 0.9\,ld\frac{1}{0.9}}$$

$H(x_0,x_1,\dots x_n) = 1 + (n-1)H(0.1,0.9)$

(b) $\frac{H(x_0,\dots x_n)}{n+1} = \frac{1+(n-1)H(0.1,0.9)}{n+1} \xrightarrow[n\to\infty]{} H(0.1,0.9)$

(c) $\boxed{E(S) = \sum_{s=1}^{\infty} s(0.9)^{s-1}\cdot 0.1}$ (**)

$\sum_{i=1}^{\infty} i q^i = \frac{q}{(1-q)^2}$ $\sum_{i=1} i q^{i-1} = \frac{1}{\cancel{q}}\frac{\cancel{q}}{(1-q)^2} = \frac{1}{(1-q)^2}$

$E(s) = \frac{1}{(1-0.9)^2}\cdot 0.1 = \frac{1}{0.01}\cdot 0.1 = \frac{10}{1} = 10$

▶ DOKAZ NA OVA TVRDENJE MOŽE DA SE VIDI OD (1*) NA ST. 44:

- DEFINITIVNO ZAVISI SAMO OD EDNA PRETHODNA SOSTOJBA !!!

$H(x_0,x_1,x_2,\dots x_n) = H(x_1,x_2,\dots x_n|x_0) + \overset{0}{H(x_0)}$

$= H(x_1,x_2\dots x_n) = H(x_1) + H(x_2|x_1) + \sum_{i=2} H(x_i|x_{i-1})$

$$\boxed{\begin{aligned} H(x_2|x_1) &= P(x_1=-1)\cdot H(x_2|x_1=-1) + P(x_1=1)\cdot H(x_2|x_1=1) = \\ &= \frac{1}{2}\cdot\left[0.1\,ld\frac{1}{0.1} + 0.9\,ld\frac{1}{0.9}\right] + \frac{1}{2}\left[0.1\,ld\frac{1}{0.1} + 0.9\,ld\frac{1}{0.9}\right] = \\ &= H(0.1,0.9)\end{aligned}}$$ (E)

$H(x_4|x_3) = P(x_3=1)\cdot[H(0.1,0.9)] + P(x_3=-1)\cdot[H(0.1,0.9)] + \dots$

SEGA MISLAM MORA DA SE ODI SO MOJOT DOKAZ!!!

- ME UOPŠTA ZOŠTO VAŽI (**). DA VIDAM ŠTO SE SLUČUVA ZA $\boxed{n=4}$

-1012
-1010
-1-2-3-4
-1-2-3-2
-1-2-1 0
-1-2-1-2

1232
-1010
2 2 4
s 1 2 3

$$\boxed{E(s) = \sum_{s=1}^{5} s\cdot(0.9)^2\cdot 0.1}$$

$\boxed{4.13}$ THE PAST HAS LITTLE TO SAY ABOUT THE FUTURE

FOR STATIONARY STOCHASTIC PROCESS $X_1, X_2 \dots X_n, \dots,$ SHOW THAT:

$$\lim_{n\to\infty} \frac{1}{2n} I(X_1 X_2 \dots X_n; X_{n+1}, X_{n+2}, \dots, X_{2n}) = 0$$

THUS, THE DEPENDENCE BETWEEN ADJACENT n-blocks OF STATIONARY PROCESS DOES NOT GROW LINEARLY WITH n.

$$I(X_1 X_2, \dots X_n; X_{n+1}^{2n}) = I(X_1^n; X_{n+1}^{2n}) = \cancel{H(X_1^n) - H(X_1^n|X_{n+1}^{2n})}$$

$$= H(X_{n+1}^{2n}) - H(X_{n+1}^{2n}|X_1^n)$$

$$H(X_1 X_2, \dots X_n) = H(X_{n+1}, \dots X_{2n})$$

$$\left. H(X_1^n|X_{n+1}^{2n}) = H(X_{n+1}^{2n}|X_1^n) \right\}$$ ISTOTO SLEDI OD DEFINICIJATA NA STACIONARNOST

$$I(X_1^n; X_{n+1}^{2n}) = H(X_1^n) - H(X_1^n|X_{n+1}^{2n})$$

$$I(X_1 X_2; X_3 X_4) = I(X_1; X_3 X_4) + I(X_2; X_3 X_4|X_1) =$$

$$= H(X_1) - H(X_1|X_3 X_4) + H(X_2|X_1) - H(X_2|X_1 X_3 X_4)$$

$$= H(X_1 X_2) - [H(X_1|X_3 X_4) + H(X_2|X_1 X_3 X_4)] =$$

$$= H(X_1 X_2) - H(X_1 X_2|X_3 X_4) \leq H(X_1 X_2) - \underbrace{H(X_1 X_2|X_3 X_4 X_1 X_2)}_{0}$$

$$\boxed{H(X_1 X_2) = H(X_3 X_4)}$$

$$D(p\|q) = \sum p \operatorname{ld} \frac{p}{q}$$

$$I(X;Y) = \sum_x \sum_y p(xy) \operatorname{ld} \frac{p(xy)}{p(x) \cdot p(y)} = \sum_x \sum_y p(x,y) \operatorname{ld} \frac{1}{p(x)}$$

$$+ \sum_{xy} p(x,y) \operatorname{ld} \frac{p(x,y)}{p(y)} = + \underbrace{\sum_x p(x) \operatorname{ld} \frac{1}{p(x)}}_{H(X)} + \underbrace{\sum p(xy) \operatorname{ld} p(x|y)}_{-H(X|Y)}$$

$$= H(X) - H(X|Y)$$

$$I(X;Y) = D(p(xy) \| p(x)q(y))$$

$$D(p(X_n) \| q(X_n)) \geq \underbrace{D(p(X_{n+1}) \| q(X_{n+1}))}_{H(X)}$$

$$\lim_{n\to\infty} \frac{I(X_1^n; X_{n+1}^{2n})}{2n} = \frac{1}{2} \lim_{n\to\infty} \frac{H(X_1 X_2 \dots X_n)}{n} -$$

$$- \frac{1}{2} \lim_{n\to\infty} \frac{H(X_1 X_2 \dots X_n|X_{n+1} \dots X_{2n})}{n}$$

$$\lim_{n\to\infty}\frac{1}{2n} I(X_1^n; X_{n+1}^{2n}) = \frac{1}{2}H(\mathcal{X}) - \frac{1}{2}\lim_{n\to\infty}\frac{H(X_1^n|X_{n+1}^{2n})}{n}$$

$$H(X_1,\dots X_n|X_{n+1}\dots X_{2n}) = H(X_{n+1},\dots X_{2n}|X_1,\dots X_n)$$

$$H(X_{n+1},\dots X_{2n}|X_1\dots X_n) = H(X_{n+1}|X_1^n) + H(X_{n+2}|X_1^{n+1}) +$$
$$+ H(X_{n+3}|X_1^{n+2}) + \dots H(X_{2n}|X_1^{2n-1})$$

$$\lim_{n\to\infty} H(X_1,\dots X_n|X_{n+1}\dots X_{2n}) = \sum_{i=0}^{n-1}\underbrace{\lim_{n\to\infty} H(X_{n+1+i}|X_1^{n+i})}_{H(\mathcal{X})}$$

$$= n\cdot H(\mathcal{X})$$

$$\lim_{n\to\infty}\frac{1}{2n} I(X_1^n; X_{n+1}^{2n}) = \frac{1}{2}H(X) - \frac{1}{2}\frac{\cancel{n}\cdot H(\mathcal{X})}{\cancel{n}} = \frac{H(\mathcal{X})}{2} - \frac{H(X)}{2} = 0$$

- ODID ~~ZA~~ VSTE POPRECIZEN DOKAZ :

$$H(X_1^n|X_{n+1}^{2n}) = \sum_{i=0}^{n-1} H(X_{n+1+i}|X_1^{n+i})$$

$$\lim_{n\to\infty}\frac{H(X_1^n|X_{n+1}^{2n})}{n} = \lim_{n\to\infty}\frac{\sum_{i=0}^{n-1} H(X_{n+1+i}|X_1^{n+i})}{n} =$$

$$= \frac{\sum_{i=0}^{\infty}\lim_{n\to\infty} H(X_{n+1+i}|X_1^{n+i})}{\lim_{n\to\infty} n} = \frac{\sum_{i=0}^{\overset{n}{\infty}} H(X)}{\lim_{n\to\infty}(n)} = \frac{\cancel{\lim_{n\to\infty}(n)}H(X)}{\cancel{\lim_{n\to\infty}(n)}}$$

$$= H(X) \Rightarrow \boxed{\lim_{n\to\infty}\frac{1}{2n} I(X_1^n; X_{n+1}^{2n}) = \frac{H(X)}{2} - \frac{H(X)}{2} = 0}$$

VSTE PODOBNO: $\frac{\lim_{n\to\infty}\sum_{i=0}^{n} H(X)}{\lim_{n\to\infty}(n)} = \lim_{n\to\infty}\frac{\sum_{i=0}^{n} H(X)}{n} =$

$= \lim_{n\to\infty}\frac{(n+1)H(X)}{n} = \underline{H(X)}$ DOKAZANO!!

$\boxed{4.14}$ FUNCTIONS OF STOCHASTIC PROCESS

(a) CONSIDER A STATIONARY STOCHASTIC PROCESS $X_1 X_2 \cdots X_n$ AND LET $Y_1, Y_2, \dots Y_n$ BE DEFINED BY:

$Y_i = \phi(X_i)$ $i = 1, 2, \dots$ FOR FUNCTION ϕ.

PROVE THAT: $H(\mathcal{Y}) \le H(\mathcal{X})$

(b) WHAT IS THE RELATIONSHIP BETWEEN ENTROPY RATES $H(\mathcal{Z})$ AND $H(\mathcal{X})$ IF: $Z_i = \psi(X_i, X_{i+1})$ $i = 1, 2, \dots$

RECALL:

LEMA 4.5.1 $H(Y_n|Y_{n-1},Y_{n-2},\dots Y_2,X_1) \le H(\mathcal{Y})$

$Y_i = f(X_i)$

$H(Y_n|X_1) = H(Y_n|Y_1,X_1)$

~~$H(Y_n|Y_{n-1},\dots Y_2,X_1) = H(Y_n|Y_2^{n-1},X_1,X_2) \le H(Y_n|X_1)$~~

~~$H(\mathcal{Y}) = \lim_{n\to\infty} H(Y_n|Y_{n-1},\dots X_1)$~~

$H(Y_n|Y_{n-1},Y_{n-2},\dots Y_2,X_1) = H(Y_n|Y_2^{n-1},X_1) = H(Y_n|Y_1^{n-1},X_1) =$

$= /\text{MARKOVITE}/ = H(Y_n|Y_1^{n-1}X_1,X_0,\dots X_{-k}) = H(Y_n|Y_1^{n-1},X_{-k}^1,Y_{-k}^0)$

$= H(Y_n|Y_{-k}^{n-1},X_{-k}^1) \le H(Y_n|Y_{-k}^{n-1}) = H(Y_{n+k+1}|Y_1^{n+k+1-1})$

$= H(Y_{n+k+1}|Y_1^{n+k})$ VAŽI ZA ∀k, ZA k→∞

$H(Y_n|Y_2^{n-1},X_1) \le \lim_{k\to\infty} H(Y_{n+k+1}|Y_1^{n+k}) = H(\mathcal{Y})$

LEMA 4.5.2 $H(Y_n|Y_{n-1},\dots Y_1) - H(Y_n|Y_{n-1},\dots Y_1,X_1) = 0$

$H(Y_n|Y_1^{n-1}) - H(Y_n|Y_1^{n-1}X_1) = I(X_1;Y_n|Y_1^{n-1}) \to 0$

$I(X_1;Y_1^n) = H(X_1) - H(Y_1^n|X_1) \le H(X_1)$

$\lim_{n\to\infty} I(X_1;Y_1^n) \le H(X_1)$

$I(X_1;Y_1^n)$ INCREASES WITH n

$I(X_1;Y_1^n) = I(Y_1;X_1) + I(Y_2;X_1|Y_1) \cdots + I(Y_n;X_1|Y_1^{n-1})$

$H(X_1) \ge \lim_{n\to\infty} I(X_1;Y_1^n) = \lim_{n\to\infty}\sum_{i=1}^{n} I(Y_i;X_1|Y_1^{i-1}) =$

$= \sum_{i=1}^{\infty} I(Y_i;X_1|Y_1^{i-1})$ SINCE INFINITE SUM IS FINITE AND TERMS ARE NONNEGATIVE, THE TERM MUST TEND TO 0 i.e.

$\lim_{n\to\infty} I(X_1;Y_n|Y_1^{n-1}) = 0$

THEOREM 4.5.1: If $X_1,X_2\dots X_n$ FORM STATIONARY MARKOV CHAIN, AND $Y_i = \phi(X_i)$:

$$\boxed{H(Y_n|Y_2^{n-1}X_1) \le H(\mathcal{Y}) \le H(Y_n|Y_{n-1},\dots Y_1)}$$

i.e. $\lim_{n\to\infty} H(Y_n|Y_2^{n-1}X_1) = H(Y) = \lim_{n\to\infty} H(Y_n|Y_1^{n-1})$

- If Y_i IS STOHASTIC FUNCTION OF X_i

$p(x^n,y^n) = p(x_1^n,y_1^n) = p(x_1^n)\cdot p(y_1^n|x_1^n) = p(x_1)\prod_{i=1}^{n-1} p(x_{i+1}|x_i)$

- $\prod_1^n p(y_i|x_i) \Rightarrow Y_i = f(X_i)$ i.e. NE ZAVISI OD X_j, $j \ne i$

(a) $x_1, x_2, \ldots x_n$ $\quad Y_i = \phi(x_i)$ i=1..n PROVE: $H(\mathcal{Y}) \le H(\mathcal{X})$

- IMAŠE ZADAĆA $\quad Y = f(x) \quad H(Y) \le H(x)$

$$I(x;Y) = H(x) - H(X|Y) = H(Y) - \underbrace{H(Y|X)}_{0} \Rightarrow$$

$$H(x) - H(X|Y) = H(Y)$$

$$H(x) = H(Y) + H(X|Y) \Rightarrow \underline{\underline{H(X) \ge H(Y)}}$$

- MOŽE I PREKO ENTROPIJA

$$H(X,Y) = H(X) + \underbrace{H(Y|X)}_{0} = H(Y) + H(X|Y)$$

$$\Rightarrow \boxed{H(X) \ge H(Y)}$$

$$H(Y) = \lim_{n\to\infty} \frac{1}{n} H(Y_1, Y_2, \ldots Y_n) = \lim_{n\to\infty} (Y_n | Y_1^{n-1})$$

$$H(Y_n|Y_1^{n-1}X_1) \le H(\mathcal{Y}) \le H(Y_n|Y_1^{n-1})$$

$$H(X_1 X_2 \ldots X_n, Y_1 \ldots Y_n) = H(X_1^n, Y_1^n) =$$

$$= H(X_1^n) + \underbrace{H(Y_1^n|X_1^n)}_{0} = H(Y_1^n) + H(X_1^n|Y_1^n)$$

$$\Rightarrow H(X_1^n) \ge H(Y_1^n) \Rightarrow \lim_{n\to\infty} \frac{H(X_1^n)}{n} \ge \lim_{n\to\infty} \frac{H(Y_1^n)}{n}$$

$\Rightarrow H(\mathcal{X}) \ge H(\mathcal{Y})$ DOKAZANO !!!

(b) $H(\mathcal{Z})$ & $H(\mathcal{X})$ $\quad Z_i = \psi(x_i, X_{i+1})$

$$H(x_1, x_2, \ldots x_n, Z_1, \ldots Z_n) = H(x_1^n, Z_1^n) =$$

$$= H(x_1^n) + \underbrace{H(Z_1^n|x_1^n)}_{0} = H(Z_1^n) + \underbrace{H(x_1^n|Z_1^n)}_{\neq 0}$$

$$H(x_1^n) \ge H(Z_1^n) \Rightarrow \lim_{n\to\infty} \frac{H(x_1^n)}{n} \ge \lim_{n\to\infty} \frac{H(Z_1^n)}{n}$$

$$\Rightarrow \boxed{H(\mathcal{X}) \ge H(\mathcal{Z})}$$

4.15 ENTROPY RATE: let $\{X_i\}$ be a DISCRETE STATIONARY STOCHASTIC PROCESS WITH ENTROPY RATE $H(\mathcal{X})$. SHOW THAT

$$\frac{1}{n} H(X_n, \ldots X_1 | X_0 X_{-1}, \ldots, X_{-k}) \to H(X) \quad \text{FOR } k = 1, 2, \ldots$$

$$\lim_{n\to\infty} H(X_n|X_{n-1}, \ldots X_1) = H(\mathcal{X})$$

$$\lim_{n\to\infty} \frac{H(X_1 X_2, \ldots X_n)}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} H(X_i|X_{i-1}, \ldots 1) =$$

/ - CESARO mean $\quad a_n \xrightarrow[n\to\infty]{} a \quad b_n = \frac{1}{n}\sum_{i=1}^{n} a_i \to a$ /

$$= \lim_{n\to\infty} \frac{1}{\cancel{n}} \cancel{n} \cdot H(X_n|X_{n-1}, \ldots 1) = H(X_n|X_{n-1}, \ldots 1)$$

$\boxed{K=1}$ $\frac{1}{n} H(X_n, \dots X_1 | X_0 X_{-1})$

$$H(X_n, \dots X_1 X_0 X_{-1}) = H(X_n, \dots X_1 | X_0 X_{-1}) + H(X_0 X_{-1}) =$$
$$= H(X_n, \dots X_1) + H(X_0 X_{-1} | X_1^n)$$

$$\lim_{n\to\infty} \frac{1}{n} H(X_1^n | X_0 X_{-1}) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} H(X_i | X_1^{i-1}, X_0 X_{-1}) = ⊛$$

$$H(X_n, \dots X_1 | X_0 X_{-1}) = \underline{H(X_{n+2} \dots X_3 | X_2 X_1)}$$

$$H(X_{n+2}, \dots X_2 X_1) = H(X_2 X_1) + H(X_{n+2} \dots X_3 | X_2 X_1)$$
$$H(X_{n+2}, \dots X_3 | X_2 X_1) = \underline{H(X_1^{n+2}) - H(X_1^2)}$$

$$H(X_n, \dots X_1 | X_0 X_{-1}) = H(X_1 | X_0 X_{-1}) + H(X_2 | X_1 X_0 X_{-1}) + \dots + H(X_n | X_{n-1} \dots X_{-1})$$
$$= H(X_n | X_0 X_{-1}) + H(X_{n-1} | X_n X_0 X_{-1}) + \dots + H(X_1 | X_n X_{n-1}, \dots X_2 X_0 X_{-1})$$
$$= H(X_3 | X_2 X_1) + H(X_4 | X_3 X_2 X_1) + \dots + H(X_{n+2} | X_{n+1} \dots X_1) =$$
$$= \sum_{i=1}^{n} H(X_{i+2} | X_{i+1}, \dots X_1) = \sum_{i=1}^{n} H(X_{i+2} | X_1^{i+1}) \quad \boxed{\text{FOR } K=1}$$

$$⊛ = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} H(X_i | X_{-1}^{i-1}) = \underbrace{\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} H(X_{i+2} | X_1^{i+1})}$$

OVA SE DOBIVA I OD SVOJSTVOTO NA STACIONAROST ($l=2$).

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} H(X_{i+2} | X_1^{i+1}) = \lim_{n\to\infty} \left[H(X_{n+2} | X_1^{n+1}) \right] = \underline{H(X)}$$

CESARO MEAN

- VO GENERALEN SLUČAJ!

$$\lim_{n\to\infty} \frac{1}{n} H(X_1^n | X_{-k}^0) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} H(X_i | X_{-k}^{i-1}) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} H(X_{i+k+1} / X_1^{i+k}$$

$$= \left| \text{CAESARO MEAN} \right| = \lim_{n\to\infty} H(X_{n+k+1} | X_1^{n+k}) = H(X)$$

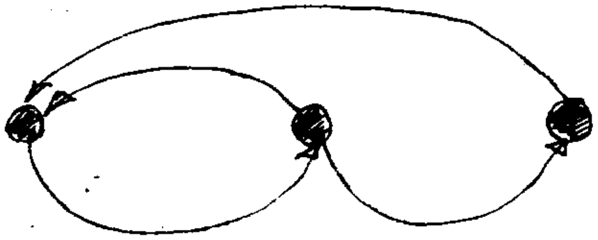
} IF TRUE FOR $K=1$ THEN TRUE FOR $\forall K$!!!

$\boxed{4.16}$ ENTROPY RATE OF CONSTRAINED SEQUENCES

IN MAGNETIC RECORDING, MECHANISM OF READING AND RECORDING THE BITS IMPOSES CONSTRAINTS ON THE SEQUENCES OF BITS THAT CAN BE RECORDED. FOR EXAMPLE, TO ENSURE PROPER SYNCHRONIZATION, IT IS OFTEN NECESSARY TO LIMIT THE LENGHT OF RUNS OF 0'S BETWEEN TWO 1'S. ALSO TO REDUCE INTERSYMBOL INTERFERENCE, IT MAY BE NECESSARY TO REQUIRE AT LEAST

ONE ZERO BETWEEN ANY TWO 1'S. WE CONSIDER SIMPLE EXAMPLE OF SUCH CONSTRAINT. SUPPOSE WE ARE REQUIRED TO HAVE AT LEAST ONE 0 AND AT MOST TWO 0'S BETWEEN ANY PAIR OF 1'S IN A SEQUENCE. THUS SEQUENCES LIKE 101001 AND 0100101 ARE VALID SEQUENCES, BUT 0110010 AND 0000101 ARE NOT. WE WISH TO CALCULATE THE NUMBER OF VALID SEQUENCES OF LENGTH „n".

(a) SHOW THAT THE SET OF CONSTRAINED SEQUENCES IS THE SAME AS THE SET OF ALLOWED PATHS ON THE FOLLOWING STATE DIAGRAM.



(b) LET $x_i(n)$ BE NUMBER OF VALID PATHS OF LENGHT „n" ENDING AT STATE „i". ARGUE THAT $x(n) = [x_1(n), x_2(n), x_3(n)]^t$ SATISFIES:

$$\begin{bmatrix} x_1(n) \\ x_2(n) \\ x_3(n) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(n-1) \\ x_2(n-1) \\ x_3(n-1) \end{bmatrix}$$

WITH INITIAL CONDITION $x(1) = [1 1 0]^t$

(c) LET:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

THEN WE HAVE BY INDUCTION:

$$x(n) = Ax(n-1) = A^2 x(n-2) = \dots = A^{n-1} x(1)$$

USING EIGEN VALUE DECOMPOSITION OF A FOR THE CASE OF DISTINCT EIGENVALUES, WE CAN WRITE $A = U^{-1}\Lambda U$ WHERE Λ IS DIAGONAL MATRIX OF EIGEN VALUES. THEN $A^{n-1} = U^{-1}\Lambda^{n-1}U$. SHOW THAT WE CAN WRITE:

$$x(n) = \lambda_1^{n-1} Y_1 + \lambda_2^{n-1} Y_2 + \lambda_3^{n-1} Y_3,$$

WHERE Y_1, Y_2, Y_3 DO NOT DEPEND ON n. FOR LARGE n, THIS SUM IS DOMINATED BY THE LARGEST TERM. THEREFORE, ARGUE THAT FOR $i = 1,2,3$ WE HAVE

$$\frac{1}{n} \log x_i(n) \to \log \lambda$$

WHERE „λ" IS THE LARGEST (POSITIVE) EIGENVALUE.

THUS THE NUMBER OF SEQUENCES OF LENGHT „n" GROWS AS λ^n FOR LARGE „n". CALCULATE λ FOR MATRIX A ABOVE. (THE CASE WHERE THE EIGENVALUES ARE NOT DISTINCT CAN BE HANDLED IN SIMILAR MANNER.)

(d) WE NOW TAKE A DIFFERENT APPROACH. CONSIDER MARKOV CHAIN WHOSE STATE DIAGRAM IS THE ONE GIVEN IN PART (a) BUT WITH ARBITRARY TRANSITION PROBABILITIES. THEREFORE THE PROBABILITY TRANSITION MATRIX OF THIS MARKOV CHAIN IS:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ \alpha & 0 & 1-\alpha \\ 1 & 0 & 0 \end{bmatrix}$$

SHOW THAT STATIONARY DISTRIBUTION OF THIS MARKOV CHAIN IS:

$$\left[\frac{1}{3-\alpha} , \frac{1}{3-\alpha} , \frac{1-\alpha}{3-\alpha} \right]$$

(e) MAXIMIZE THE ENTROPY RATE OF THE MARKOV CHAIN OVER CHOICES OF "α". WHAT IS THE MAXIMUM ENTROPY RATE OF THE CHAIN?

(f) COMPARE THE MAXIMUM ENTROPY RATE IN PART (e) WITH $\log \lambda$ IN PART (c). WHY ARE THE TWO ANSWERS THE SAME?

(d)

$$\begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \alpha & 0 & 1-\alpha \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}$$

$\mu_1 = \mu_2$

$\mu_2 = \alpha\mu_1 + (1-\alpha)\mu_3$

$\mu_3 = \mu_1$ Ⓓ

$\mu_2 = \alpha\mu_1 + (1-\alpha)\mu_1$ $\mu_1 = ?$

$$[\mu_1\ \mu_2\ \mu_3] = [\mu_1\ \mu_2\ \mu_3] \begin{bmatrix} 0 & 1 & 0 \\ \alpha & 0 & 1-\alpha \\ 1 & 0 & 0 \end{bmatrix}$$

$\mu_1 = \alpha\mu_2 + \mu_3$

$\mu_2 = \mu_1$

$\mu_3 = (1-\alpha)\mu_2$

$\boxed{\mu_1 + \mu_2 + \mu_3 = 1}$

GOTO PP54

$\mu_1 = \alpha\mu_1 + (1-\alpha)\mu_1$

$\mu_1 (1 - \alpha - 1 + \alpha) = 0$

$\mu_j = \sum_{i=1}^{4} \mu_i P_{ij}$

$\mu_2 = \alpha\mu_2 + (1-\alpha)\mu_2 = \mu_2$

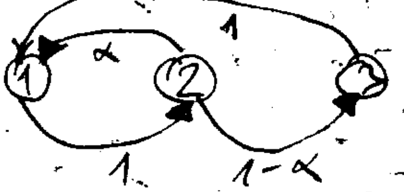
$\mu_3 = \mu_1 - \alpha\mu_2 = \mu_2(1-\alpha)$

$\mu_2 = \frac{\mu_3}{1-\alpha}$

$$[\mu_1 \mu_2 \mu_3] = [\mu_1 \mu_2 \mu_3] \cdot \begin{bmatrix} 0 & \alpha & 1 \\ 1 & 0 & 0 \\ 0 & 1-\alpha & 0 \end{bmatrix}$$

ISTO KAO Ⓓ

$\mu_1 = \mu_2$; $\mu_2 = \alpha\mu_1 + (1-\alpha)\mu_3$ $\mu_3 = \mu_1$



$\mu_1 = \sum_{i=1}^{3} \mu_i P_{i1} =$

$= \mu_1 \cdot \underset{0}{P_{11}} + \mu_2 \cdot \underset{\alpha}{P_{21}} + \mu_3 \cdot \underset{\mu_3}{P_{31}} = \alpha\mu_2 + \mu_3$

$\mu_2 = \sum_{i=1}^{3} \mu_i P_{i2} = \mu_1 P_{12} + \mu_2 P_{22} + \mu_3 P_{32} = \mu_1$

$$\left.\begin{aligned}\mu_1 &= \alpha\mu_2 + \mu_3\\ \mu_2 &= \mu_1\\ \mu_3 &= (1-\alpha)\mu_2\\ \mu_1+\mu_2+\mu_3 &= 1\end{aligned}\right\}$$

$$\mu_1 + \mu_1 + (1-\alpha)\mu_1 = 1$$

$$3\mu_1 - \alpha\mu_1 = 1$$

$$\boxed{\mu_1 = \frac{1}{3-\alpha} \qquad \mu_2 = \frac{1}{3-\alpha}}$$

$$\boxed{\mu_3 = \frac{1-\alpha}{3-\alpha}}$$

$$\boxed{p_{ij} = P(j|i)}$$

(e)

$$H(X) = \sum_{i=1}^{3}\mu_i \sum_{j=1}^{3} p_{ij}\,\text{ld}\frac{1}{p_{ij}} = \sum_{i=1}^{3}\sum_{j=1}^{3}\mu_i p_{ij}\,\text{ld}\frac{1}{p_{ij}} =$$

$$= \frac{1}{3-\alpha}\left(1\,\text{ld}\,1\right) + \frac{1}{3-\alpha}\left[\alpha\,\text{ld}\frac{1}{\alpha} + (1-\alpha)\,\text{ld}\frac{1}{1-\alpha}\right] + \frac{1-\alpha}{3-\alpha}(1$$ [illegible]

$$= -\frac{\alpha}{3-\alpha}\,\text{ld}\,\alpha - \frac{1-\alpha}{3-\alpha}\,\text{ld}(1-\alpha)$$

$$\frac{dH(\alpha)}{d\alpha} = \frac{2\ln(1-\alpha) - 3\ln(\alpha)}{(\alpha-3)^2\ln(2)}$$

$$\left.\frac{dH(\alpha)}{d\alpha}\right|_{\alpha=\alpha_0} = 0$$

[illegible]

$$\text{ld}(1-\alpha)^2 = \text{ld}\,\alpha^3$$

$$\alpha^3 - 1 + 2\alpha - \alpha^2 = 0$$

$$(1-\alpha)^2 = \alpha^3$$

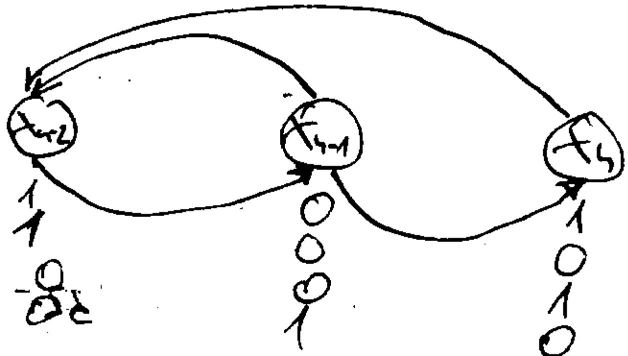
$$\boxed{\alpha^3 - \alpha^2 + 2\alpha - 1 = 0}$$

$$\boxed{\alpha_0 = \left(\frac{1}{600}\left(11 - 3\sqrt{69}\right)\left(44 + 12\sqrt{69}\right)^{2/3} + \frac{1}{6}\left(44 + 12\sqrt{69}\right)^{1/3} + \frac{1}{3}\right.}$$

$$\alpha_0 = 0.56984$$

$$\boxed{H(\alpha_0) = 0.40569}$$

(a)

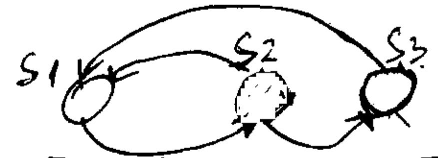


	1	0	0
	Y_1	Y_2	Y_3
1 X_1	0	1	
0 X_2	α	0	$1-\alpha$
0 X_3	1	0	0

$\boxed{n=3}$ THE NUMBER OF ALLOWED PATHS:

$\boxed{100, 010, 001, 101}$ $\boxed{N_p=4}$

$\boxed{n=4}$

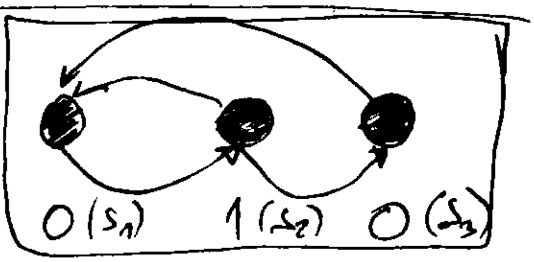


$\boxed{N_p=5}$

0	0	1	0
0	1	0	0
1	0	0	1
1	0	1	0
0	1	0	1
t_1	t_2	t_3	t_4

✓ ✓ ✓ ✓

t



✓ 1|0|0|1
✓ 0|1|0|0
✓ 1
✓ 0|0|1|0
✓ 1|0|1|0

$\boxed{n=2}$

00
10
01

$N_p=3$

ZA $\boxed{n=4}$ t_3 SOSTOJBATA ZAVISI OD DVETE DVE PRETHODNI MOMENTI T.E. SOSTOJBI (t_1, t_2)
- t_4 ZAVISI SAMO OD t_2 I t_3 SOSTOJBITE (DVETE PRETHODNI)

t_2	t_3	t_4
0	0	1
0	1	0
1	0	0
=	=	1

t_1	t_2	t_3	t_4
0	0	1	0
0	1	0	0 1
1	0	0	1
1	0	1	0

(b) $x_i(n)$ NUMBER OF VALID PATHS OF LENGHT „n" ENDING AT STATE „i"

$$\begin{bmatrix} x_1(n)\\ x_2(n)\\ x_3(n)\end{bmatrix} = \begin{bmatrix} 0 & 1 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix} x_1(n-1)\\ x_2(n-1)\\ x_3(n-1)\end{bmatrix}$$

OD „0" EDEN PAT VO „1" OD „1" EDEN PAT VO „0"

SOSTOJBITE SE POČETNI: $\boxed{n=1}$

$x(1) = [1\,1\,0]^t$ → LOGIČNO: VO PRVATA SOSTOJBA ($S_1=0$) ZAVRŠUVA EDEN PAT, VO VTORATA 1 PAT. SOSTOJBA ① ZAVRŠUVA (ŠTO SAMO

$$\begin{bmatrix} x_1(2)\\ x_2(2)\\ x_3(2)\end{bmatrix} = \begin{bmatrix} 0 & 1 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix} 1\\ 1\\ 0\end{bmatrix} = \begin{bmatrix} 1\\ 1\\ 1\end{bmatrix}$$

$x(2)= \begin{bmatrix} 1\\ 1\\ 1\end{bmatrix}$

$\boxed{00}$ TRGNUVA VO S_2 ODI VO S_3 (0) A OD S_3 VO S_1 (0)

$\boxed{n=5}$

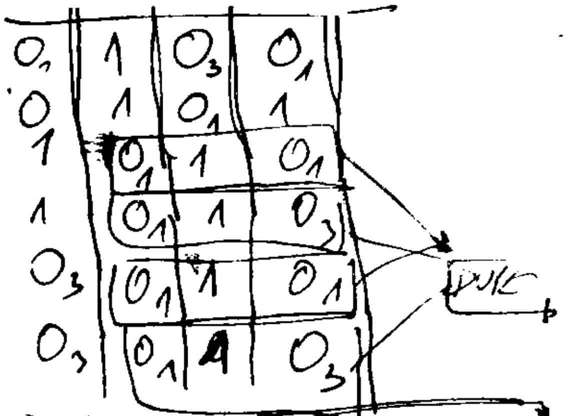
$$N_f=7\;\left\{\begin{matrix} 0&0&1&0&0\\ 0&0&1&0&1\\ 0&1&0&0&1\\ 0&1&0&1&0\\ 1&0&0&1&0\\ 1&0&1&0&0\\ 1&0&1&0&1 \end{matrix}\right.$$

$$x(3)=\begin{bmatrix}2\\1\\1\end{bmatrix}\quad x(4)=\begin{bmatrix}2\\2\\1\end{bmatrix}$$

$\boxed{\text{MATLAB}}$

$$x(5)=\begin{bmatrix}3\\2\\2\end{bmatrix}$$

$\boxed{n=3}$



0_1	1	0_3	0_1
0	1	0_1	1
1	0_1	1	0_1
1	0_1	1	0_3
0_3	0_1	1	0_1
0_3	0_1	1	0_3

ZNAČI VKUPNO SE:

$1\;0_3\;0_1$ → $\boxed{\begin{matrix}2\times s_1\\ 2\times s_2\\ s_3\end{matrix}}$
$1\;0_1\;1$
$0_1\;1\;0_1$
$0_1\;1\;0_3$

FALI $\boxed{0_3 0_1 1}$

$\boxed{\text{GREŠKA JE ŠTO NE SAM IZBACIO } S_1=1}$

(c) $x(2)=A\cdot x(1)$

$\boxed{n=k}$

$x(k)=A^{k-1}\cdot x(k-1)$ TRUE

$n=k+1 \quad x(k+1) = A^{k+1-1}x(k+1-1) \rightarrow$

$\boxed{x(k+1)=A^k x(k)}$ PROVED BY INDUCTION

$$\boxed{\begin{aligned}&x(n)=A\,x(n-1)=A^2x(n-2)=\ldots=A^{n-1}x(1)\\ &x(2)=A(x(1)) \rightarrow \text{PROVED}\\ &x(k)=A^{k-1}x(1) \rightarrow \text{ASUMED}\\ &x(k+1)=A^k\,x(1) \rightarrow \text{PROVED}\end{aligned}}$$

$$\boxed{\begin{aligned}x(k+1)&=A\cdot x(k)=\\ &=/x(k)=A^{k-1}x(1)/=\\ &=A\cdot A^{k-1}x(1)=A^k\cdot x(1)\end{aligned}}$$

$A=U^{-1}\Lambda U$

$$\Lambda=\begin{bmatrix}1.3247\\ -0.66236+0.56228i\\ -0.66236-0.56228i\end{bmatrix}$$

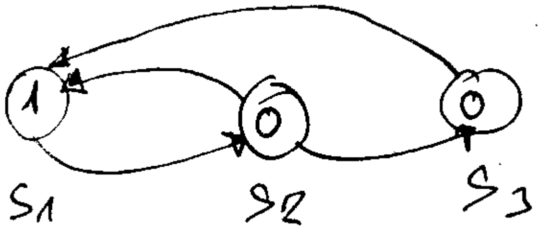
$\boxed{\mathrm{ld}\,\Lambda(1)=\mathrm{ld}\,(1.3247)=0.4056}$

$x(n)=\lambda_1^{n-1}Y_1+\lambda_2^{n-1}Y_2+\lambda_3^{n-1}Y_3$

$\frac{1}{n}\mathrm{ld}\,x_i(n)\rightarrow \mathrm{ld}\,\lambda$

EXAMPLE 1 SOLUTION:

1) LET THE STATE OF THE SYSTEM BE THE NUMBER OF 0's THAT HAS BEEN SEEN SINCE THE LAST "1"
- SEQUENCE ENDING IN STATE 1 IS IN S1
- SEQUENCE ENDING IN STATE 10 IS IN S2
- SEQUENCE ENDING IN STATE 100 IS IN S3



- FROM S1 IT IS ONLY POSSIBLE TO GO TO S2, SINCE THERE HAS TO BE AT LEAST ONE 0 BEFORE NEXT 1. FROM STATE 2 WE CAN GO TO EITHER STATE 1 OR STATE 3. FROM STATE 3, WE HAVE TO GO TO SATE 1 SINCE THERE CANNOT BE MORE THAN TWO ZEROS IN A ROW

(b) $x_1(n)=x_2(n-1)+x_3(n-1)$

ANY VALID SEQUENCE OF LENGTH „n" THAT ENDS IN A „1" MUST BE FORMED BY TAKING A VALID SEQUENCE OF LENGTHTH n-1 THAT ENDS IN 0 AND ADDING „1" AT THE END. THE NUMBER OF VALID SEQUENCES OF LENGTH n-1 THAT ENDS IN 0 IS EQUAL TO $x_2(n-1)+x_3(n-1)$ AND THEREFORE:

$$x_1(n)=x_2(n-1)+x_3(n-1)$$

$\boxed{\text{(1) SO DOLZINA „n"}}$

- OVDE NA POIMAKVA LOGIKA OD MOMTATA:

SO GORNAVA FORMULA SAKA DA KAZE DEKA BROJOT NA POTENCIJALNI SEQUENCI (1) KOI ZAVRŠUVAAT SO „1" ZAVISI OD BROJOT NA SEKVENCI SO DOLZINA „n-1" KOI ZAVRŠILE SO „0" (S2, S3) T.E NIVNA SUMA!!

$\boxed{n=2}$

00
10
01

$\boxed{n=3}$

00 1 — S1
10 1
10 0 — S3
01 0 — S2

$$x(3)=\begin{bmatrix}x_1(3)\\x_2(3)\\x_3(3)\end{bmatrix}=\begin{bmatrix}2\\1\\1\end{bmatrix}$$

- SLIČKA E LOGIKATA ZA SOSTOJBA „2" (T.E POSLE EDINICA DA SLEDI 0). BROJOT NA SEKVENCI KOI ZAVRŠUVAAT VO SOSTOJBA „S2" E EDNAKOV NA BROJOT NA SEKVENCI SO DOLZINA („n-1") KOI ZAVRŠUVAAT VO SOSTOJBA $\boxed{S_1=1}$:

$$x_2(n)=x_1(n-1)$$

- ZA S3 LOGIKATA VELI DEKA BROJOT NA SEKVENCI SO DOLZINA „n" KOI ZAVRŠUVAAT VO SOSTOJBA „S3" E EDNAKOV NA BROJOT NA SEKVENCI SO DOLZINA „(n-1)" KOI ZAVRŠUVAAT VO SOSTOJBA S2 T.E.:

$$x_3(n)=x_2(n-1)$$

- ILI VO OBLIK NA MATRICA:

$$\begin{bmatrix}x_1(n)\\x_2(n)\\x_3(n)\end{bmatrix}=\begin{bmatrix}0&1&1\\1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}x_1(n-1)\\x_2(n-1)\\x_3(n-1)\end{bmatrix}$$

(c) REVISITED: $x(2) = A\cdot x(1)$ WITH NO GENERATION [illegible] i.e. $x(k) = A\cdot x(k-1)$

- THEMA [illegible] DOUBLE [illegible]:

$$x(n) = A\cdot x(n-1) = A^2 x(n-1) = \dots A^{n-1}\cdot x(n)$$

- ASSUMPTION $\boxed{x(k) = A^{k-1} x(1)}$

$$x(k+1) = A\cdot x(k) = A\cdot \boxed{x(1)\cdot A^{k-1}} = A^k\cdot x(1)$$

PROVED !!!

- USING THE EIGENVALUE DECOMPOSITION: $A = U^{-1}\Lambda U$

$$A^2 = U^{-1}\Lambda U\cdot U^{-1}\Lambda U = U^{-1}\Lambda^2 U$$ AND SOFOR

$$x(n) = A^{n-1} x(1) = U^{-1}\Lambda U\cdot x(1) =$$

$$= U^{-1}\begin{bmatrix}\lambda_1^{n-1} & 0 & 0\\ 0 & \lambda_2^{n-1} & 0\\ 0 & 0 & \lambda_3^{n-1}\end{bmatrix} U \begin{bmatrix}1\\1\\0\end{bmatrix} = (*)$$

$$\begin{bmatrix}a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{bmatrix}\begin{bmatrix}x_{11} & x_{12} & x_{13}\\ x_{21} & x_{22} & x_{23}\\ x_{31} & x_{32} & x_{33}\end{bmatrix} = \begin{bmatrix}a x_{11} & a x_{12} & a x_{13}\\ b x_{21} & b x_{22} & b x_{23}\\ c x_{31} & c x_{32} & c x_{33}\end{bmatrix} =$$

$$= a\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_{11} & x_{12} & x_{13}\\ x_{21} & x_{22} & x_{23}\\ x_{31} & x_{32} & x_{33}\end{bmatrix} + b\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}\dots\dots$$

$$(*) = \lambda_1^{n-1}U^{-1}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}U\begin{bmatrix}1\\1\\0\end{bmatrix} + \lambda_2^{n-1}U^{-1}\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}U\begin{bmatrix}1\\1\\0\end{bmatrix} +$$

$$+ \lambda_3^{n-1}U^{-1}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}U\begin{bmatrix}1\\1\\0\end{bmatrix} = \lambda_1^{n-1}\gamma_1 + \lambda_2^{n-1}\gamma_2 + \lambda_3^{n-1}\gamma_3$$

$$\gamma_1, \gamma_2, \gamma_3 \neq f(n)$$

$$\begin{bmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{bmatrix} = \begin{bmatrix}x_{11} & \cancelto{0}{x_{12}}\\ \cancelto{0}{x_{21}} & x_{22}\end{bmatrix}\begin{bmatrix}x_{11} & \cancelto{0}{x_{12}}\\ \cancelto{0}{x_{21}} & x_{22}\end{bmatrix}$$

$$= \begin{bmatrix}x_{11}^2 & 0\\ 0 & x_{22}^2\end{bmatrix}$$

- WITHOUT LOSS OF GENERALITY, WE CAN ASSUME THAT $\lambda_1 > \lambda_2 > \lambda_3$, THUS:

$$x_1(n) = \lambda_1^{n-1}\gamma_{11} + \lambda_2^{n-1}\gamma_{21} + \lambda_3^{n-1}\gamma_{31}$$
$$x_2(n) = \lambda_1^{n-1}\gamma_{12} + \lambda_2^{n-1}\gamma_{22} + \lambda_3^{n-1}\gamma_{32}$$
$$x_3(n) = \lambda_1^{n-1}\gamma_{13} + \lambda_2^{n-1}\gamma_{23} + \lambda_3^{n-1}\gamma_{33}$$

- FOR LARGE „n" THIS SUM IS DOMINATED BY THE LARGEST TERM. THUS IF $\lambda_1 i > 0$ WE HAVE

$$\frac{1}{n} \operatorname{ld} x_i(n) \to \frac{1}{n} \operatorname{ld} \lambda_1^{n-1} \to \operatorname{ld} \lambda_1$$

PROBLEM 4.17 Recurrence times are insensitive to distributions. Let $x_0, x_1, x_2, \ldots$ be DRAWN i.i.d $\sim p(x)$, $x \in \mathcal{X} = \{1, 2, \ldots, m\}$ AND N BE WAITING TIME TO NEXT OCCERENCE OF x_0.

THUS $N = \min_n \{x_n = x_0\}$

(a) SHOW THAT $E[N] = m$

(b) SHOW THAT $E[\operatorname{ld}(N)] \leq H(X)$

(c) (optional) PROVE THAT (a) FOR $\{X_i\}$ STATIONARY AND ERGODIC.

$x \in \mathcal{X} = \{1, 2, 3\}$ $\qquad p(x) = \left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}$

$$p(l=3) = p(x=2) \cdot p(x=2) \cdot p(x=1) + p(x=2) \cdot p(x=3) \cdot p(x=1) + p(x=3) \cdot p(x=2) \cdot p(x=1) + p(x=3) \cdot p(x=3) \cdot p(x=1) = p(x_0) \cdot p(x_1) \cdot p(x_1) + p(x_0) \cdot p(x_1) \cdot p(x_2) + p(x_0)\, p(x_2)\, p(x_1) + p(x_0)\, p(x_2)\, p(x_2) =$$

$$= p(x_0) \Big[\underbrace{p^2(x_1) + 2\, p(x_1)\, p(x_2) + p^2(x_2)}_{\to (p(x_1) + p(x_2))^2} \Big] = p(x_0) \sum_{i=0}^{2} \binom{2}{i} p^i(x_1) \cdot p^{2-i}(x_2)$$

- VO GENERALEN SLUČAJ:

$$p(l=n) = p(x_0) \Big[\underbrace{p(x_1) + p(x_2) + \cdots + p(x_{n-1})}_{= 1 - p(x_0)} \Big]^{n-1}$$

e.g. AKO SE SITE SO ISTA VEROJATNOST

$$\boxed{p(l=n) = \frac{1}{m} \left[(m-1) \cdot \frac{1}{m} \right]^{n-1}}$$

GEOMETRIC DISTRIBUTION

$$p(l=n) = \frac{1}{m} (m-1)^{n-1} \cdot \frac{1}{(m)^{n-1}} = \frac{(m-1)^{n-1}}{m^n}$$

$$E[N] = \sum_{n=1}^{\infty} \frac{(m-1)^{n-1}}{m^n} \cdot n$$

$$\left(1 - \frac{1}{m}\right)^{n-1} = \sum_{i=0}^{n-1} \binom{n-1}{i} \left(\frac{-1}{m}\right)^i$$

$$E[N] = \sum_{n=1}^{\infty} \frac{n}{m} \left(1 - \frac{1}{m}\right)^{n-1} = \sum_{n=1}^{\infty} \frac{n}{m} \sum_{i=0}^{n-1} \binom{n-1}{i} \left(\frac{-1}{m}\right)^i$$

$$\boxed{E[N] = \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \binom{n-1}{i} \frac{(-1)^i}{m^{i+1}}\, n}$$

$$\boxed{E[N] = m}$$

MAPLE

$x \in \mathcal{X} = \{1,2,3\}$ $\qquad p(x) = \frac{1}{n}$

21 31

221, 231, 321, 331

$$p(l=n) = p(x_0)\underbrace{\left[\sum_{i>1}^{n-1} p(x_i)\right]^{n-1}}_{(1-p(x_0))^{n-1}}$$

$$\boxed{E[N/_{x_0=i}] = \sum_{n=1}^{\infty} p(x_0)(1-p(x_0))^{n-1}\cdot n}$$

NAUM TRENOVSKI

$$p(x_0)\sum_{n=1}^{\infty}(1-\underbrace{p(x_0)}_{=\alpha})^{n-1}\cdot n = \alpha\sum_{n=1}^{\infty}(\underbrace{1-\alpha}_{q=1-\alpha})^{n-1}\cdot n = \alpha\sum_{n=1}^{\infty} n\cdot q^{n-1} = ⊛$$

$$S = \sum_{n=1}^{N} q^n = 1 + q + q^2 + \cdots q^N$$

$$qS = q^{N+1} + \cdots + q$$

$$S(1-q) = 1 - q^{N+1}$$

$$\boxed{S = \frac{1-q^{N+1}}{1-q}}$$

$N \to \infty \quad 0 \le q \le 1 \quad \Rightarrow \quad \boxed{S = \frac{1}{1-q}}$

$$S = \sum_{n=1}^{\infty} n\, q^{n-1} \qquad \int S\, dq = \sum_{n=1}^{\infty} \cancel{n}\cdot\frac{q^n}{\cancel{n}} = \frac{1}{1-q}$$

$$S = \left(\frac{1}{1-q}\right)' = \frac{1}{(1-q)^2} = \frac{1}{\alpha^2}$$

$$⊛ = \alpha\cdot\frac{1}{\alpha^2} = \frac{1}{\alpha}$$

$$\boxed{E[N/_{x_0=i}] = \frac{1}{\alpha} = \frac{1}{p(x_0=i)}} \quad Ⓢ$$

- АКО СЕ ~~Р~~АБОТИ ЗА УНИФОРМНА РАСПРЕДЕЛБА ТОГАШ

$$\boxed{p(x_i) = \frac{1}{n} \quad i = 0, 1, \ldots n-1}$$

$E[N] = n$

(б) $E[\mathrm{ld} N] \le H'(X)$ $\qquad p(l=n) = \frac{(n-1)^{n-1}}{n^n}$

$$E[\mathrm{ld}N] = \sum_{n=1}^{\infty} \mathrm{ld}(n)\cdot\frac{(n-1)^{n-1}}{n^n}$$

$$H(\dot{X}) = \lim_{n\to\infty}\frac{1}{n}H(x_1, x_2, \ldots x_n) = \lim_{n\to\infty} H(x_n|x_1\ldots x_{n-1})$$

$$H(\mathcal{X}) = \frac{\cancel{n}}{\cancel{n}}\cdot\frac{1}{n}\mathrm{ld}\,n = \underline{\frac{1}{n}\mathrm{ld}\,n} \qquad \ln(x) \le x-1$$

$$\mathrm{ld}(x) = \frac{\ln(x)}{\ln 2} \le \frac{x-1}{\ln 2}$$

$$E[\mathrm{ld}N] = \sum_{n=1}^{\infty}\left[n\ln^{-1}(2) - \ln^{-1}(2)\right]\frac{(n-1)^{n-1}}{n^n} \le \ln^{-1}(2)\sum_{n=1}^{\infty}(n-1)\frac{(n-1)^{n-1}}{n^n}$$

$$E[\operatorname{ld} N] \le \underbrace{\ln^{-1}(2) \sum_{n=1}^{\infty} (n-1)\frac{(m-1)^{n-1}}{m^n}}_{①} = \frac{(m-1)}{\ln 2}$$

[redacted]

200737 SAŠE DOJKOVSKI

070201017 | GJOKO IUŠKOVSKI

$$\sum_{n=1}^{\infty}(n-1)\frac{(m-1)^{n-1}}{m^n} = -1 + \sum_{n=1}^{\infty} n \frac{(m-1)^{n-1}}{m^n} = -1 + \underbrace{\frac{1}{m}}_{\alpha}\sum_{n=1} n\Big(1-\underbrace{\frac{1}{m}}_{\alpha}\Big)^{n-1}$$

$$= -1 + \alpha\sum_{n=1}^{\infty} n(1-\alpha)^{n-1} = -1 + \frac{1}{\alpha} = \left|\alpha = \frac{1}{m}\right| = \underline{\underline{m-1}}$$

EDITION 1 SOLUTION: ⓐ GIVEN $x_0 = i$, EXPECTED TIME UNTIL WE SEE IT AGAIN IS $1/p(i)$. (④ VIDI 80 PP.60)

$$E[N] = E[E[N|x_0]] = \sum_{i=1}^{m} p(x_0=i)\cdot\frac{1}{p(i)} = \sum_{i=1}^{m} 1 = m$$

④ VAŽI ZA BILO KOJA VREDNOST NA $x_0 \in X_0 = \{1,2,\dots m\}$ PA ZATOA GO USREDNUVA ZA SEKOE i

$$E[N] = E[E[N|X_0]] = \sum_{i=1}^{m} p(x_0=i)\cdot\underbrace{\frac{1}{p(x_0=i)}}$$

COMPARE !!!

EXPECTED TIME OF RECCURENCE OF i

ⓑ SINCE GIVEN $x_0 = i$ N HAS GEOMETRIC DISTRIBUTION WITH MEAN $p(i)$ AND:

$$E[N|x_0=i] = \frac{1}{p(i)}$$

$$E[\operatorname{ld} N] = E[E[\operatorname{ld}(N|x_0)]] = \sum_{i=1}^{m} p(x_0=i)E[\operatorname{ld}(N|x_0=i)]$$

$$\le \sum_{i=1}^{m} p(x_0=i)\operatorname{ld}(E[N|x_0=i]) = \sum_{i=1}^{m} p(x_0=i)\operatorname{ld}\frac{1}{p(x_0=i)} =$$

$$= \sum_{i=1}^{m} p(i)\operatorname{ld}\frac{1}{p(i)} = H(X)$$

(c) THE PROPERTY THAT $E[N] = m$ is essentially A COMBINATORIAL PROPERTY RATHER THAN A STATEMENT ABOUT EXPECTATIONS. WE PROVE THIS FOR STATIONARY ERGODIC SOURCES. IN ESSENCE, WE WILL CALCULATE THE EMPIRICAL AVERAGE OF THE WAITING TIME, AND SHOW

THAT THIS CONVERGES TO „n“. SINCE THE PROCESS IS ERGODIC, THE EMPIRICAL AVERAGE CONVERGES TO THE EXPECTED VALUE, AND THUS EXPECTED VALUE MUST BE „n“. TO SIMPLIFY MATTERS, WE WILL CONSIDER $x_1 x_2, \dots x_n$ ARRANGED IN CIRCLE SO THAT x_1 FOLLOWS x_n. THEN WE CAN GET RID OF THE EDGE EFFECTS (NAMELY THAT WAITING TIME IS NOT DEFINED FOR x_n, ETC) AND WE CAN DEFINE THAT WAITING TIME N_k AT TIME „k“ AS $\min\{n>k: x_n=x_k\}$. WITH THIS DEFINITION WE CAN WRITE THE EMPIRICAL AVERAGE OF N_k FOR PARTICULAR SAMPLE SEQUENCE:

$$\overline{N} = \frac{1}{n}\sum_{i=1}^{n} N_i = \frac{1}{n}\cdot\sum_{i=1}^{n}\left(\sum_{j=i+1}^{\min\{n>i:x_n=x_i\}} 1\right)$$

NA MOMENT POMISLIV DEKA TREBA DA BIDE „j“

NOW WE CAN REWRITE THE OUTER SUM BY GROUPING TOGETHER ALL THE TERMS WHICH CORRESPOND TO $x_i=l$. THUS WE OBTAIN:

$$\overline{N} = \frac{1}{n}\sum_{l=1}^{n}\underbrace{\sum_{i:x_i=l}\left(\sum_{j=i+1}^{\min\{n>i:x_n=l\}} 1\right)}_{=\,„n“\,=} = \frac{1}{n}\sum_{l=1}^{n} n = \frac{1}{n}\cdot n\cdot n = n$$

THUS EMPIRICAL AVERAGE OF N OVER ANY SAMPLE SEQUENCE IS „n“ AND THUS THE EXPECTED VALUE OF „N“ MUST ALSO BE „n“. EDINICI

- SAKA DA KAŽE DEKA SUMIRA ODDEKA NE STANE $X_n=X_i$. ANALOGNO NA IMPLEMENTACIJA NA PROGRAMOT VO MATLAB
- SAKA DA KAŽE DEKA VO SAMPLE-OT SO DOLŽINA „n“ MOŽEŠ DA IMAŠ 4 PATI POVTORUVANJE NA „1“ 10 PATI POVTORUVANJE NA „2“, 7 PATI POVTORUVANJE NA „3“, ... X PATI POVTORUVANJE NA „n“. VKUPNIOT BROJ NA OVIE „POVTORUVANJA“ E EDNAKOV NA „n“ KOLKU ŠTO E DOLGA SEKVENCIJATA T.E. SAMPLE-OT.

4.18 STATIONARY BUT NOT ERGODIC PROCESS. A BIN HAS TWO BIASED COINS, ONE WITH PROBABILITY OF HEADS $1-p$, AND THE OTHER WITH PROBABILITY OF HEADS p. ONE OF THIS COINS IS CHOSEN AT RANDOM (I.E. WITH PROBABILITY $\frac{1}{2}$) AND IS THEN TOSSED „n“ TIMES. LET X DENOTE THE IDENTITY OF THE COIN THAT IS

PICKED, AND LET Y_1 AND Y_2 DENOTE RESULTS OF THE FIRST TWO TOSSES.

(a) CALCULATE $I(Y_1;Y_2|X)$.

(b) CALCULATE $I(X;Y_1,Y_2)$.

(c) LET $\mathcal{H}(Y)$ BE THE ENTROPY RATE OF THE Y PROCESS (THE SEQUENCE OF COIN TOSSES). CALCULATE $\mathcal{H}(Y)$. HINT: RELATE THIS TO:

$$\lim \frac{1}{n} H(X, Y_1, Y_2, \dots Y_n)$$

$$\boxed{H(Y_n|Y_{n-1},\dots Y_2 X_1) \leq H(Y)} \Rightarrow \boxed{H(Y_n|Y_{n-1},\dots Y_2 Y_1 X_1) \leq H(Y)}$$

$$H(Y_n|Y_{n-1},\dots Y_2 Y_1 X_1) \leq H(Y)$$

$$\leq H(Y_n|Y_{n-1},\dots Y_2 Y_1)$$

$$H(Y_n|Y_{n-1},\dots Y_2 Y_1 X_1) \leq H(Y) \leq H(Y_n|Y_{n-1},\dots Y_1)$$

$$\lim_{n\to\infty}(Y_n|Y_{n-1},\dots Y_1 X_1) = H(Y) = \lim_{n\to\infty} H(Y_n|Y_{n-1},\dots Y_1)$$

$$Y_1 = f(X)$$

X \ Y_1	H	T	$P(X)$
C_1	p	$(1-p)$	$\frac{1}{2}$
C_2	$(1-p)$	p	$\frac{1}{2}$
$P(Y_1)$	$\frac{1}{2}$	$\frac{1}{2}$	▨

$P(Y_1|X)$

$$I(X;Y) = H(X) - H(X|Y)$$

$$\boxed{H(X) = \frac{1}{2}\,\mathrm{ld}\,2 + \frac{1}{2}\,\mathrm{ld}\,2 = 1}$$

★

$P(X,Y_1) = P(X,Y_2)$

X \ Y_1	H	T	$P(Y_1)$
C_1	$\frac{1}{2}p$	$\frac{1}{2}(1-p)$	1/2
C_2	$\frac{1}{2}(1-p)$	$\frac{1}{2}p$	1/2
$P(X)$	1/2	1/2	

$$P(X,Y_1) = P(X)\cdot P(Y_1|X)$$

~~[illegible]~~

$$p + (1-p) = 1 = \sum_{x\,y} p(x,y)$$

$$P(Y_1 Y_2) \rightarrow$$

Y_1 \ Y_2	H	T	$P(Y_1)$
H	1/4	1/4	1/2
T	1/4	1/4	1/2
$P(Y_2)$	1/2	1/2	▨

Y_1 \ Y_2	H	T
H	1/2	1/2
T	1/2	1/2

$\boxed{P(Y_2|Y_1)}$

$$X \rightarrow Y_1 \rightarrow Y_2 \qquad I(X;Y_2|Y_1) = 0$$

$$I(X;Y_2|Y_1) = H(X|Y_1) - H(X|Y_1 Y_2) = H(Y_2|Y_1) - H(Y_2|X,Y_1)$$

$$= H(Y_2|Y_1) - H(Y_2|Y_1) = 0$$

$$I(Y_1;Y_2|X) = H(Y_1|X) - H(Y_1|X Y_2) = H(Y_2|X) - H(Y_2|X Y_1)$$

$$= H(Y_2|X) - H(Y_2|Y_1)$$

$$I(Y_1;X Y_2) = I(Y_1;X) + I(Y_1;Y_2|X) = I(Y_1;Y_2) + I(Y_1;X|Y_2)$$

$$I(X Y_1;Y_2) = I(X;Y_2) + \underbrace{I(Y_1;Y_2|X)} = I(Y_1;Y_2) + \underbrace{I(X;Y_2|Y_1)}$$

$$-I(Y_1;Y_2|X) = I(Y_1;Y_2) - I(X;Y_2) = \cancel{H(Y_2)} - H(Y_2|Y_1) - \cancel{H(Y_2)} + H(Y_2|X)$$

$$= H(Y_2|X) - H(Y_2|Y_1$$

$I(Y_1;Y_2|X) = H(Y_2|X) - H(Y_2|Y_1)$ (070503550)

$H(Y_2|Y_1) = \underbrace{P(Y_1=H)}_{\frac{1}{2}} \cdot H(Y_2|Y_1=H) + \underbrace{P(Y_1=T)}_{1/2} H(Y_2|Y_1=T) = \frac{1}{2}+\frac{1}{2}=1$

$H(Y_2|Y_1=H) = -P(Y_2=H|Y_1=H)\, ld\, P(Y_2=H|Y_1=H) - P(Y_2=T|Y_1=H) \cdot ld\, P(Y_2=T|Y_1=H) = \frac{1}{2} ld 2 + \frac{1}{2} ld 2 = 1$

ANAZOGANO $H(Y_2|Y_1=T)=1$

$H(Y_2|X) = \overbrace{P(X=C_1)}^{1/2} \cdot H(Y_2|X=C_1) + \overbrace{P(X=C_2)}^{1/2} \cdot H(Y_2|X=C_2)$

$\boxed{X=C_1}$

$P(Y_2|Y_1) = P(Y_2|Y_1 X)$ $P(Y_1 Y_2|X_1=C_1) = P(Y_2|Y_1) \cdot P(Y_1)$

$Y_1 \backslash Y_2$	H	T
H	1/2	1/2
T	1/2	1/2

$Y_1 \backslash Y_2$	H	T	$P(Y_1\|X=C_1)$
H	$\frac{p}{4}$	$\frac{p}{4}$	$\frac{p}{2}$
T	$\frac{1-p}{4}$	$\frac{1-p}{4}$	$\frac{1-p}{2}$
$P(Y_2\|C_1)$	$\frac{1}{4}$	$\frac{1}{4}$	

$P(Y_1) = \left[\frac{p}{2}, \frac{1}{2}(1-p)\right]$

$Y_1 \in [H, T]$

$\boxed{X=C_2}$ $P(Y_1) = \left[\frac{1-p}{2}, \frac{p}{2}\right]$

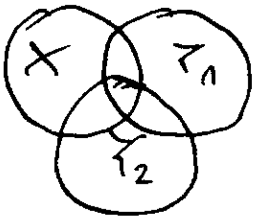
$Y_1 \in [H, T]$

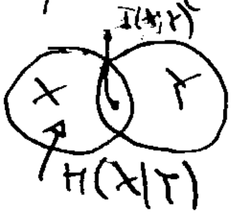
$Y_1 \backslash Y_2$	H	T	$P(Y_1\|X=C_2)$
H	$\frac{1-p}{4}$	$\frac{1-p}{4}$	$\frac{1-p}{2}$
T	$\frac{p}{4}$	$\frac{p}{4}$	$\frac{p}{2}$
$P(Y_2\|X=C_2)$	$\frac{1}{4}$	$\frac{1}{4}$	1

$H(Y_2|X) = \frac{1}{2}\left[\frac{1}{4} ld 4 + \frac{1}{4} ld 4\right] + \frac{1}{2}\left[\frac{1}{4} ld 4 + \frac{1}{4} ld 4\right]$

$= \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1 = 1 \rightarrow \underline{I(Y_1;Y_2|X) = 1-1 = 0}$

(6) $I(X;Y_1,Y_2) = I(X;Y_1) + \overbrace{I(X;Y_2|Y_1)}^{0} = I(X;Y_1)$





$I(X;Y) = H(X) - H(X|Y)$

$I(X;Y_1) = H(X) - H(X|Y_1) = H(Y_1) - H(Y_1|X)$

$H(Y_1) = P(Y_1=T) \cdot ld \frac{1}{P(Y_1=T)} + P(Y_1=H) \cdot ld \frac{1}{P(Y_1=H)} = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1 = 1$

$$\circledast \boxed{\begin{aligned} \underline{H(Y_1|X)} &= \sum_{X Y_1} P(X Y_1) \cdot ld \frac{1}{P(Y_1|X)} = -\frac{p}{2} \cdot ld\, p - \frac{1}{2}(1-p)\, ld (1-p) - \\ &- \frac{1}{2}(1-p)\, ld(1-p) - \frac{p}{2} ld\, p = \underline{-p\, ld\, p - (1-p)\, ld(1-p)} = H(p) \end{aligned}}$$

$I(X;Y_1 Y_2) = I(X;Y_1) = 1 + p\, ld\, p + (1-p)\, ld(1-p) = \underline{1-H(p)}$

(c) $H(Y) = \lim_{n\to\infty} H(Y_n | Y_{n-1}, \dots Y_1 X) = \lim_{n\to\infty} \frac{1}{n} H(X Y_1 Y_2, \dots Y_n)$

$H(X Y_1 Y_2, \dots Y_n) = H(X) + H(Y_1|X) + H(Y_2|X Y_1) + H(Y_3|X Y_1 Y_2) + \dots + H(Y_n|X, Y_1, Y_2, \dots Y_{n-1}) =$

$= \underbrace{H(X)}_{1} + \underbrace{H(Y_1|X)}_{H(p)} + \underbrace{H(Y_2|Y_1)}_{1} + \underbrace{H(Y_3|Y_2)}_{1} \dots + \underbrace{H(Y_n|Y_{n-1})}_{1}$

$= n + H(p)$

$H(Y) = \lim_{n\to\infty} \frac{1}{n}(n + H(p)) = 1$

- ALTERNATIVELY: $H(Y) = \lim_{n\to\infty} \frac{1}{n} H(Y_1 Y_2, \dots Y_n)$

$H(Y_1^n) = \underbrace{H(Y_1)}_{1} + \underbrace{H(Y_2|Y_1)}_{1} + \dots \underbrace{H(Y_n|Y_{n-1})}_{1} = n$

$H(Y) = \lim_{n\to\infty} \frac{1}{n} \cdot n = 1$

- UIC SOLUTION (HW4s2010)

	$X=C1$	$X=C2$	$P(Y_1 Y_2)$
$Y_1=H\ Y_2=H$	$(1/2)p^2$	$(1/2)(1-p)^2$	$(1/2)p^2 + (1/2)(1-p)^2$
$Y_1=T\ Y_2=T$	$\frac{1}{2}(1-p)^2$	$\frac{1}{2}p^2$	$\frac{1}{2}p^2 + \frac{1}{2}(1-p)^2$
$Y_1=H\ Y_2=T$	$\frac{1}{2}p(1-p)$	$\frac{1}{2}(1-p)\cdot p$	$(1-p)\cdot p$
$Y_1=T\ Y_2=H$	$\frac{1}{2}(1-p)\cdot p$	$\frac{1}{2}p(1-p)$	$(1-p)\cdot p$

TABLE.1

IU E IZNAN C_1 IU ∈ ITDRAN C_2

$H(X) = \left[\frac{1}{2} ld 2\right] \cdot 2 = 1$

(a) $Y_1, Y_2, \dots Y_n$ ARE I.I.D WITH KNOWLEDGE OF X HENCE:

$I(Y_1; Y_2|X) = 0$

(b) $I(X; Y_1, Y_2) = H(Y_1 Y_2) - H(Y_1 Y_2 |X) =$

$= H(X) - H(X|Y_1 Y_2) = I(X; Y_1) + I(X; Y_2 | Y_1)$

$= H(Y_1, Y_2) - \underbrace{H(Y_1|X)}_{H(p)} \cdot \underbrace{H(Y_2|X)}_{H(p)} = -[p^2 + (1-p)^2] ld\left(\frac{p^2 + (1-p)^2}{2}\right)$

$- 2(1-p)p\, ld(1-p)p - \underbrace{(p\, ld\, p + (1-p) ld(1-p))^2}_{⊛}$

⊛ $= p^2 ld^2 p + 2p\, ld p \cdot (1-p) \cdot ld(1-p) + (1-p)^2 ld^2 (1-p)$

$I(X; Y_1 Y_2) = H(X) - H(X| Y_1 Y_2)$ $\quad H(X|Y_1 Y_2) = H(X Y_1 Y_2) - H(Y_1 Y_2)$

$H(X Y_1 Y_2) = H(X) + H(Y_1 Y_2|X) \rightarrow I(X; Y_1 Y_2) = \cancel{H(X)} + H(Y_1 Y_2) - \cancel{H(X)} - H(Y_1 Y_2|X)$

$I(X; Y_1 Y_2) = H(Y_1 Y_2) - H(Y_1 Y_2|X)$

$H(X, Y_1 Y_2) = H(X) + H(Y_1 Y_2|X)$

$$H(Y_1Y_2|X) = P(X=c_1)\cdot \underline{H(Y_1Y_2|X=c_1)} + P(X=c_2)\cdot H(Y_1Y_2|X=c_2)$$
$$= \frac{1}{2}\cdot\left[-p^2 \,ld\, p^2 - (1-p)^2\,ld\,(1-p)^2 - 2p(1-p)\,ld\,p(1-p)\right] +$$
$$+ \frac{1}{2}\left[-(1-p)^2\,ld\,(1-p)^2 - p^2\,ld\,p^2 - 2p(1-p)\,ld\,p(1-p)\right] =$$
$$= -p^2\,ld\,p^2 - (1-p)^2\,ld\,(1-p)^2 - 2p(1-p)\,ld\,p(1-p)$$

$$\boxed{H(X,Y_1Y_2) = H(X) + H(Y_1Y_2|X) = 1 - p^2\,ld\,p^2 - (1-p)^2\,ld\,(1-p)^2 - 2p(1-p)\,ld\,p(1-p)}$$

→ GLEDAŠ NA TABLE. 1 SAMO BREZ 1/2 ZOITO TE NAGE SILA Y_1Y_2 ZA DANO X $(p(X) = [1/2, 1/2])$.

$$I(X;Y_1Y_2) = H(Y_1Y_2) - H(Y_1Y_2|X) = -(p^2+(1-p)^2)\,ld\,\frac{p^2+(1-p)^2}{2} -$$
$$- 2(1-p)p\,ld\,p(1-p) - p^2\,ld\,p^2 - (1-p)^2\,ld\,(1-p)^2 - 2p(1-p)\,ld\,p(1-p)$$

(c) $\boxed{P.63-64 ⊛}$ ⇒ $H(Y_1|X) = H(Y_2|X) = H(p) = -p\,ld\,p - (1-p)\,ld\,(1-p)$

$Y_1Y_2,\dots Y_n$ ARE INDEPENDENT WITH KNOWLEDGE OF X

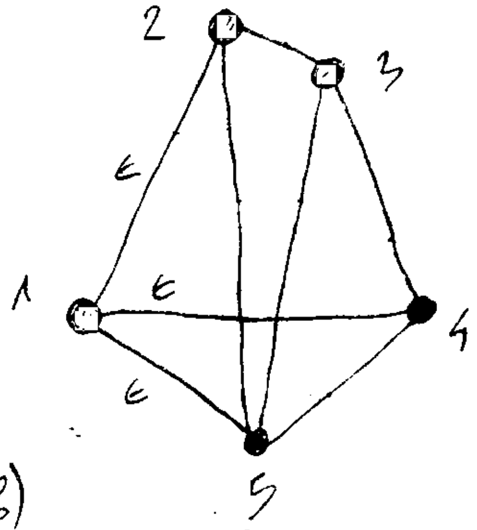
$$H(Y_1Y_2,\dots Y_n,X) = H(X) + H(Y_1Y_2,\dots Y_n|X)$$
$$H(Y_1Y_2\dots Y_n|X) = H(Y_1|X) + H(Y_2|XY_1) + \dots + H(Y_n|XY_1Y_2\dots Y_n)$$
$$= H(Y_1|X) + H(Y_2|X) + \dots + H(Y_n|X) = \underline{n\cdot H(p)}$$
$$H(Y_2|XY_1) = H(Y_2|X)$$ SINCE Y_2 & Y_1 ARE INDEPENDENT GIVEN "X"

$$H(Y_1Y_2,\dots Y_n,X) = H(X) + n\cdot H(p) = 1 + nH(p)$$
$$H(Y) = \lim_{n\to\infty}\frac{1}{n}H(Y_1Y_2,\dots Y_n) = \lim_{n\to\infty}\frac{1}{n}H(Y_1Y_2,\dots Y_nX)$$
$$= \lim_{n\to\infty} H(Y_n|Y_{n-1}\dots Y_1X) = \lim\frac{1}{n}(1+nH(p)) = \underline{H(p)}$$

ALTERNAT:
$$H(p) = \lim_{n\to\infty}\frac{1}{n}H(Y_1Y_2\dots Y_n|X) \le \lim_{n\to\infty}\frac{1}{n}H(Y_1Y_2\dots Y_n)$$
$$\le \lim_{n\to\infty}\frac{1}{n}H(Y_1Y_2,\dots Y_n,X) = H(p) \Rightarrow H(Y) = \lim_{n\to\infty}\frac{1}{n}H(Y_1\dots Y_n) = H(p)$$

PROBLEM 4.19 CONSIDER RANDOM WALK ON FIG. 1.

(a) CALCULATE THE STATIONARY DISTRIBUTION
(b) WHAT IS THE ENTROPY RATE?
(c) FIND THE MUTUAL INFORMATION $I(X_{n-1};X_n)$ ASSUMING THAT THE PROCESS IS STATIONARY.



$$H(X) = \sum_{i=1}^{n} \mu_i \sum_{j=1}^{n} P_{ij}\, ld \frac{1}{P_{ij}}$$

$\mu = \mu \cdot P$ $\boxed{E=8}$ $2\epsilon = 16$

$\mu_1 = \frac{3}{16}$ $\mu_2 = \frac{3}{16}$ $\mu_3 = \frac{3}{16}$ $\mu_4 = \frac{3}{16}$

$\mu_5 = \frac{4}{16} = \frac{1}{4}$

(b)

$H(X) = ld(2\epsilon) - H(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5) =$

$= ld(2\epsilon) - 4 \cdot \frac{3}{16} ld \frac{16}{3} - \frac{1}{4} ld 4 = ld 16 - \frac{3}{4} ld \frac{16}{3} - \frac{1}{2}$

$= 4 - \frac{3}{4} ld 16 + \frac{3}{4} ld 3 - \frac{1}{2} = 4 - 3 + \frac{3}{4} ld 3 - \frac{1}{2}$

$H(X) = \frac{1}{2} + \frac{3}{4} ld 3 = \underline{\frac{3}{4} ld 3 + 0.5} = 1.68872$

(c) $I(X_{n+1}; X_n) = H(X_{n+1}) - H(X_{n+1}|X_n)$

$= 2.06128$

(*) $H(X_{n+1}) = \sum_{i=1}^{5} \mu_i\, ld \frac{1}{\mu_i} = 4 \cdot \frac{3}{\cancel{16}4} ld \frac{16}{3} + \frac{1}{4} \cdot 2 = \underline{\frac{3}{4} ld \frac{16}{3} + \frac{1}{2}}$

~~[illegible]~~ ①

$H(X_{n+1}|X_n) = P(X_n=1)\, H(X_2|X_1) + P(X_2) \cdot H(X_3|X_2) + P(X_3) H(X_4|X_3)$
$+ P(X_4)\, H(X_5|X_4)$

$H(X_2|X_1) = H(X_3|X_2) = H(X_4|X_3) = H(X_5|X_1) \Rightarrow$ BY ~~STATIONARITY~~

$H(X_2|X_1) = -P(X_2|X_1)\, ld P(X_2|X_1) = +\frac{1}{3} ld 3$

$H(X_{n+1}|X_n) = (\mu_1 + \mu_2 + \mu_3 + \mu_4) \cdot H(X_2|X_1) = \cancel{4} \cdot \frac{\cancel{3}}{\cancel{16}4} \frac{1}{\cancel{3}} ld 3$

$H(X_{n+1}|X_n) = \frac{1}{4} ld 3$

$I(X_{n+1}; X_n) = \frac{3}{4} ld \frac{16}{3} + \frac{1}{4} - \frac{1}{4} ld 3 = 1.66504$

OVA NE E DOBRO !!!

- BERKELEY SOLUTIONS (solu_2_229 Apr07.pdf)

$H(X) = H(X_2|X_1) = \frac{3}{4} ld 3 + \frac{1}{2}$ (VIA (*))

$I(X_{n+1}; X_n) = H(X_{n+1}) - H(X_{n+1}|X_n) = \underline{H(X_{n+1}) - H(X_2|X_1)}$

$= /①/ = \frac{3}{4} ld \frac{16}{3} + \cancel{\frac{1}{2}} - \frac{3}{4} ld 3 - \cancel{\frac{1}{2}} = \frac{3}{4} ld 16 - \frac{3}{4} ld 3 - \frac{3}{4} ld 3$

$= 3 - 2 \frac{3}{4} ld 3 = 3 - \frac{3}{2} ld 3 = 0.62256$ ($)

$I(X_{n+1};X_n) = \frac{3}{4} ld \frac{16}{9} = \frac{3}{4} ld 16 - \frac{3}{4} ld 9 = 3 - \frac{3}{4} ld 3^2 =$

$= 3 - \frac{3}{2} ld 3$

1590 50 N14.67 $

$\boxed{420}$ RANDOM WALK ON CHESSBOARD

FIND THE ENTROPY RATE OF THE MARKOV CHAIN ASSOCIATED WITH RANDOM WALK OF THE KING ON THE 3×3 CHESSBOARD

1	2	3
4	5	6
7	8	9

WHAT ABOUT THE ENTROPY RATE OF ROOKS, BISHOPS AND QUEENS?

THERE ARE TWO TYPES OF BISHOPS.

ⓐ **KING**
- 4 NODES WITH 3 EDGES = 12 EDGES
- 1 NODE WITH 8 EDGES = 8 EDGES
- 4 NODES WITH 5 EDGES = 20 EDGES

} 40

$\boxed{2E = 40}$

$$H(X) = ld 40 - 4 \cdot \frac{3}{40} ld \frac{40}{3} - \cancel{\frac{8}{40}} ld \cancel{\frac{40}{8}} - 4 \cdot \frac{5}{40} ld 8$$

$-\frac{1}{5}$ 5

$$H(X) = ld(2^3 \cdot 5) - \frac{3}{10} ld\left(\frac{2^3 \cdot 5}{3}\right) - \frac{1}{5} ld 5 - \frac{5}{10} ld 2^3 =$$

$$= 3 + ld 5 - \frac{3}{10} \cdot 3 - \frac{3}{10} ld 5 + \frac{3}{10} ld 3 - \frac{1}{5} ld 5 - \frac{3}{2} =$$

$$= \frac{30-9-15}{10} + \left(1 - \frac{3}{10} - \frac{1}{5}\right) ld 5 + \frac{3}{10} ld 3 = \frac{3}{5} + \frac{10-3-2}{10} ld 5 + \frac{3}{10} ld 3$$

$$H(X) = \frac{3}{5} + \frac{1}{2} ld 5 + \frac{3}{10} ld 3 = 2.23645$$

ⓑ **ROOKS**
- 9 NODES WITH 4 EDGES = 36 EDGES } 2E=36

$$H(X) = 9 \cdot \frac{4}{36} ld \frac{4}{36} + ld 36 = ld 4 - \cancel{ld 36} + \cancel{ld 36} = 2$$

~~$H(X) = 2 + \frac{3}{10} ld 10 = 2 + \frac{1}{10} + \frac{1}{10} ld 3 = \frac{21}{10} + \frac{1}{10} ld 5$~~

~~$H(X) = 2.33219$~~

ⓒ **BISHOPS**

8 NODES WITH 2 EDGES = 16 EDGES
1 NODE WITH 4 EDGES = 4 EDGES

} 2E=20

$$H(X) = ld 20 + 8 \cdot \frac{2}{20} ld \frac{2}{20} + \frac{4}{20} ld \frac{4}{20} = 2 + ld 5 - \frac{4}{5} ld 10$$

$$- \frac{1}{5} ld 5 = 2 + \cancel{ld 5} - \frac{4}{5} - \cancel{\frac{4}{5} ld 5} - \cancel{\frac{1}{5} ld 5} = 2 - \frac{4}{5} = \frac{10-4}{5} = \frac{6}{5} = 1.2$$

ⓓ **QUEEN**

9 NODES WITH 6 EDGES = 54 EDGES } 2E=42

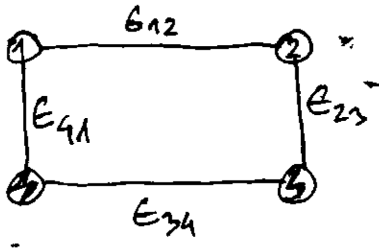
$$H(X) = ld 54 - \cancel{9} \cdot \frac{6}{54} ld \frac{54}{6} = ld(6 \cdot 9) - ld 9$$

$$H(X) = ld 6 + \cancel{ld 9} - \cancel{ld 9} = ld 6 = 1 + ld 3 = \underline{2.58416}$$

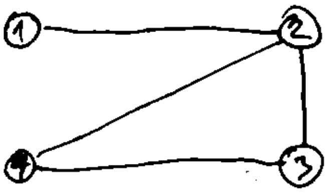
$\boxed{4.21}$ MAXIMAL ENTROPY GRAPHS

[redacted] CONSIDER A RANDOM WALK ON CONNECTED GRAPHS WITH FOUR EDGES.

(a) WHICH GRAPH HAS THE HIGHEST ENTROPY RATE?
(b) WHICH GRAPH HAS THE LOWEST?



$$\rightarrow H(X) = ld\,2E - H\left(\frac{E_1}{2E}, \frac{E_2}{2E}, \frac{E_3}{2E}, \frac{E_4}{2E}\right)$$
$$= ld8 - H\left(\frac{2}{8}, \frac{2}{8}, \frac{2}{8}, \frac{2}{8}\right) =$$
$$= 3 - \left(\frac{1}{\cancel{4}} \cdot ld4\right) \cdot \cancel{4} = ①$$



$$\rightarrow H(X) = ld8 - \frac{1}{8} ld8 - 2\left(\frac{1}{4} ld4\right) - \frac{3}{8} \cdot ld\left(\frac{8}{3}\right) = 3 - \frac{3}{8} - 1 -$$
$$- \frac{3}{8} ld8 + \frac{3}{8} ld3 = 3 - \frac{3}{8} - 1 - \frac{9}{8} + \frac{3}{8} ld3$$

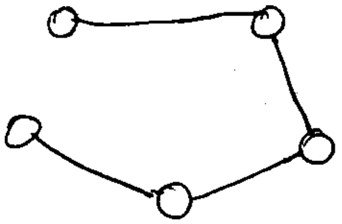
$$H(X) = \frac{24-3-8-9}{8} + \frac{3}{8} ld3 = \frac{24-20}{8} + \frac{3}{8} ld3 = \frac{1}{2} + \frac{3}{8} ld3$$
$$= \underline{1.09436}$$

HIGHEST IN CASE OF SINGLE EDGE CONNECTION

$$E_1 + E_2 + E_3 + E_4 = 4$$

$$H(X) = ld8 - \sum_{i=1}^{4} \frac{E_i}{2E} ld \frac{2E}{E_i}$$

$$H(X) = ld8 - \left(\frac{2}{8} ld4\right) 4$$



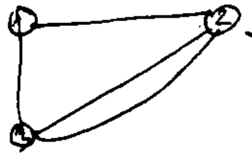
$$\rightarrow H(X) = ld8 - \left(\frac{1}{8} ld8\right) 2 - \left(\frac{1}{4} ld4\right) 3$$
$$= 3 - \frac{3}{8} \cdot 2 - \frac{1}{2} 3 = 3 - \frac{3}{4} - \frac{3}{2}$$
$$= \frac{12-3-6}{4} = \frac{3}{4} = \underline{0,75}$$

LOWEST

$$\rightarrow H(X) = ld8 - \left(\frac{4}{8} ld2\right) \cdot 2 =$$
$$= 3 - \frac{1}{2} \cdot 2 = \underline{2}$$

$$\rightarrow H(X) = ld8 - \overset{0}{\cancel{1\,ld1}} = \underline{\underline{3}}$$

HIGHEST

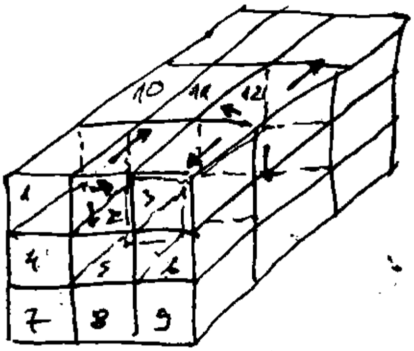


$$\rightarrow H(X) = ld8 - \frac{1}{4} ld4 - 2\left(\frac{3}{8} ld\frac{8}{3}\right) =$$
$$= 3 - \frac{1}{2} - \frac{3}{4} ld\frac{8}{3} = \underline{1,43872}$$

$\boxed{4.22}$ THREE-DIMENSIONAL MAZE

: A BIRD IS LOST IN 3×3×3 CUBICAL MAZE. THE BIRD FLIES FROM ROOM TO ROOM GOING TO ADJOINING ROOMS WITH EQUAL PROBABILITIES THROUGH EACH OF THE WALLS. FOR EXAMPLE CORNER ROOMS HAVE 3 EXITS

(a) WHAT IS THE STATIONARY DISTRIBUTION?
(b) WHAT IS THE ENTROPY RATE OF THE RANDOM WALK?



$\boxed{\Sigma E = 108}$

• JAZZI 3×3×3 = 9·3 = 27

(1) $\underline{2\times 4 \text{ JAZZI}}$ SO 3 EDGE = 8×3 = 24 EDGES
(2) 4×3 JAZZI SO 4 EDGE = 12×4 = 48 EDGES
(3) 2×1 + 4 SO 5 EDGES = 6×5 = 30 EDGES
(4) 1 SO 6 EDGES = 1×6 = 6 EDGES

(1) 3 NAJVOLJŠIM STRANI NAJVOLJŠA
(2) 2 NAJVOLJŠIM STRANI (4 VRATOLJEM)
(3) 1 NAJVOLJŠA STRANA (5 VRATOLJEM)

(a) $\underbrace{\mu_1 = \frac{3}{108}}_{8 \text{ JAZZI}}$ $\underbrace{\mu_2 = \frac{4}{108}}_{12 \text{ JAZZI}}$ $\underbrace{\mu_3 = \frac{5}{108}}_{6 \text{ JAZZI}}$ $\underbrace{\mu_4 = \frac{6}{108}}_{1 \text{ JAZZI}}$

(b) $$H(X) = ld(108) - 8\left(\frac{3}{108} ld \frac{108}{3}\right) - 12\left(\frac{4}{108} ld \frac{108}{3}\right) -$$
$$- 6\left(\frac{5}{108} ld \frac{108}{5}\right) - \frac{6}{108} ld \frac{108}{6} = \underline{\underline{2.02969}}$$

$\boxed{4.20 \text{ REVISITED}}$ (VIDOV DELA NA SVOJSKOV NAČIN ZA REŠILE NO DOBILE IST REZULTAT)

Ⓐ $\underline{\text{QING}}$

$$\mu_1 = \mu_3 = \mu_7 = \mu_9 = 3/40$$
$$\mu_2 = \mu_4 = \mu_6 = \mu_8 = 5/40 = 1/8$$
$$\mu_5 = 8/40 = 1/5$$

$\boxed{\text{MMV}}$

$i = 1,3,7,9$ $H(X_2|X_1=i) = \frac{1}{3} ld3 + \frac{1}{3} ld3 + \frac{1}{3} ld3 = 3\cdot\frac{1}{3} ld3 = ld3$
$i = 2,4,6,8$ $H(X_2|X_1=i) = 5 \cdot \frac{1}{5} ld5 = ld5$
$i = 5$ $H(X_2|X_1=5) = 8 \cdot \frac{1}{8} ld8 = 3$

$$\mathcal{H} = \sum_{i=1}^{9} \mu_i H(X_2|X_1=i) = 4 \cdot \frac{3}{40} \cdot ld3 + 4 \frac{5}{40} ld5 + \frac{1}{5} \cdot 3$$

$$\mathcal{H} = 0.3 ld3 + 0.5 ld5 + 0.2 \cdot 3 = 2.23645 \doteq 2.24$$

$\boxed{4.23}$ $\underline{\text{ENTROPY RATE}}$. LET $\{X_i\}$ BE A STATIONARY STOCHASTIC PROCESS WITH ENTROPY RATE $H(\mathcal{X})$

(a) ARGUE THAT $H(\mathcal{X}) \leq H(X_1)$

(b) WHAT ARE THE CONDITIONS FOR EQUALITY?

(a) $$H(\mathcal{X}) = \lim_{n\to\infty} \frac{1}{n} H(X_1, X_2, \dots X_n) = \lim_{n\to\infty} H(X_n|X_{n-1} \dots X_1)$$

$$H(X_1 X_2 \dots X_n) = H(X_1) + H(X_2|X_1) + H(X_3|X_1 X_2) + \dots$$
$$+ H(X_n|X_1 X_2 \dots X_{n-1})$$

$H(X_n|X_{n-1},\ldots X_1)$ IS NONINCREASING IN n = CONDITIONING REDUCES ENTROPY

PROOF: $H(X_{n+1}|X_n, X_{n-1},\ldots X_1) \le H(X_{n+1}|X_n,\ldots X_2) =$

$= H(X_n|X_{n-1},\ldots X_1)$

$$\boxed{H(X_n|X_{n-1},\ldots X_1) \geqslant H(X_{n+1}|X_1^n)}$$

$H(X_n|X_1) \geqslant H(X_n|X_1X_2) \underset{\text{MARKOVITY}}{=} H(X_n|X_2) = H(X_{n-1}|X_1)$ } FOR MARKOV CHAIN!!!

$H(X_n|X_1) \geqslant H(X_{n-1}|X_1)$

$\Rightarrow H(X_n|X_{n-1}\ldots X_1) \geqslant H(X_{n+1}|X_1^n)$

$H(X_2|X_1) \geqslant H(X_3|X_2^1) \geqslant \ldots \geqslant H(X_{n-2}|X_1^{n-3}) \geqslant H(X_{n-1}|X_1^{n-2}) \geqslant \underline{H(X_n|X_{n-1}\ldots X_1)}$

$$\Rightarrow \boxed{H(X_2|X_1) \geqslant H(X_n|X_{n-1}\ldots X_1)}$$

$H(X_1X_2) = H(X_1) + H(X_2|X_1)$ $\qquad H(X_2|X_1) = H(X_1X_2) - H(X_1)$

$H(X_1X_2) - H(X_1) \geqslant H(X_n|X_{n-1}\ldots X_1)$

- [illegible]

$\underline{H(X_1|X_0) \geqslant H(X_2|X_0^1) \geqslant} \ldots \geqslant H(X_{n-2}|X_0^{n-3}) \geqslant H(X_{n-1}|X_{n-2}^0) \geqslant \underline{H(X_n|X_{n-1}\ldots X_0)}$

$\{ H(X_1|X_0) \geqslant H(X_n|X_{n-1}\ldots X_0)$

$H(X_1) \geqslant H(X_1|X_0)$ $\Rightarrow$ DUE CONDITIONING REDUCES ENTROPY

$$\Rightarrow \boxed{H(X_1) \geqslant H(X_n|X_{n-1}\ldots X_0)} \Rightarrow H(X_1) \geqslant H(\mathcal{X})$$

(b) $H(X_1) = H(\mathcal{X}) = H(X_n|X_{n-1}\ldots X_0) = \frac{1}{n}H(X_1X_2\ldots X_n)$

IF X_i ARE i.i.D ie:

$H(\mathcal{X}) = \frac{1}{n}H(X_1,X_2\ldots X_n) = \frac{1}{n}\left(H(X_1)+H(X_2)+\ldots H(X_n)\right)$

$= \frac{1}{\cancel{n}}\cdot\cancel{n}H(X_1) = H(X_1)$

EDITION 2 SOLUTIONS:

(a) $H(\mathcal{X}) = H(X_n|X_{n-1},\ldots X_1) = H(X_1|X_0\ldots X_{-n}) \leq H(X_1)$

MNOGU ELEGANTNO!!!

(b) $H(\mathcal{X}) = H(X_1)$ IF X_1 INDEPENDENT FROM $X_0, X_{-1}, \ldots$

i.e IF X_i are i.i.d.

$\boxed{4.24}$ ENTROPY RATES. LET $\{X_i\}$ BE A STATIONARY PROCESS. LET $Y_i = (X_i, X_{i+1})$. LET $Z_i = (X_{2i}, X_{2i+1})$. LET $V_i = X_{2i}$. CONSIDER THE ENTROPY RATES $H(X)$, $H(Y)$, $H(Z)$, AND $H(V)$ OF THE PROCESS $\{X_i\}$, $\{Y_i\}$, $\{Z_i\}$, AND $\{V_i\}$. WHAT IS THE INEQUALITY RELATIONSHIP $\le$, $=$, OR $\ge$ BETWEEN EACH OF THE PAIRS LISTED BELOW?

(a) $H(X) \gtreqless H(Y)$ (c) $H(X) \gtreqless H(V)$

(b) $H(X) \gtreqless H(Z)$ (d) $H(Z) \gtreqless H(X)$

(a) $X = \{X_1, X_2, \dots X_n\}$

$Y = \{Y_1 Y_2, \dots Y_n\}$

$$H(Y) = H(Y_n | Y_1 Y_2, \dots Y_{n-1}) = H(Y_n | X_1 X_2 X_2 X_3 \cdots X_{n-1} X_n)$$

$$= H(X_n, X_{n+1} | X_1 X_2 X_2 X_3 \cdots X_{n-1} X_n) =$$

$$= H(X_n | X_1 X_2 X_2 X_3 \cdots X_{n-1} X_n) + H(X_{n+1} | X_1 X_2 X_2 X_3 \cdots X_{n-1} X_n)$$

$$\boxed{\underline{H(X|Y,Y)} = H(X,Y|Y) - \underbrace{H(Y|Y)}_{0} = H(X,Y|Y) \xrightarrow{} = \underline{H(X|Y)}}$$

$$\boxed{H(X,Y|Y) = H(X|Y) + \underbrace{H(Y|Y,X)}_{0}}$$

$$\boxed{H(X,Y,Y) = H(Y) + H(XY|Y)}$$

$$H(XYY) = \underbrace{H(YY)}_{=H(Y)} + H(X|YY)$$

$$\boxed{H(XYY) = \underbrace{H(YY)}_{H(Y)} + H(X|YY)}$$

GI ISPOSTAS DUKICATE.

$$H(Y) = \underbrace{H(X_n | X_1 \dots X_n)}_{0} + \underbrace{H(X_{n+1} | X_1 \dots X_n)}_{= H(X) \text{ FOR } n \to \infty}$$

ZNAČI: $H(X) = H(Y)$

(b) $X = \{X_1 X_2, \dots X_n\}$ $Z = \{Z_1 Z_2 \dots Z_n\}$

$$H(Z) = \lim_{n \to \infty} H(Z_n | Z_1 Z_2 \dots Z_{n-1}) = H(X_{2n}, X_{2n+1} | X_2 X_3 X_4 X_5, X_6, X_7, \dots, X_{2n-2}, X_{2n-1})$$

$$= H(X_{2n} | X_2 X_3, \dots X_{2n-2}, X_{2n-1}) + H(X_{2n+1} | X_2 \dots X_{2n}) \ge$$

$$= H(X_{2n} | X_1 X_2 X_3 \dots X_{2n-1}) + H(X_{2n+1} | X_1 \dots X_{2n})$$

$$H(Z) \geq \underbrace{H(X_{2n} | X_{2n-1} X_{2n-2} \dots X_1)}_{H(X)} + \underbrace{H(X_{2n+1} | X_1^{2n})}_{H(X)}$$

$$H(Z) \geq 2H(X) \Rightarrow \boxed{H(X) \leqslant H(Z)}$$

AND H(Z) [illegible] 2H(X) [illegible] H(X)

(c) $\boxed{Y_i = X_{2i}}$ $Y = \{Y_1, Y_2, \dots Y_n\} = \{X_2 X_4 \dots X_{2n}\}$

$$H(Y_n | Y_1 \dots Y_{n-1}) = H(X_{2n} | X_2^{2n-2}) = H(X_{2n} | X_2, X_4 \dots X_{2n-2})$$

$$\geq H(X_{2n} | X_1, X_2, X_3, \dots X_{2n-2} X_{2n-1}) = H(X)$$

↑ CONDITIONING REDUCES ENTROPY

$$\boxed{H(Y) \geq H(X)}$$

(d) SAME AS (b) $H(Z) \geq H(X)$

- EDITION 2 SOLUTIONS:

(a) $H(Y) = H(Y_1 Y_2 \dots Y_n) = H(\underbrace{X_1 X_2}_{Y_1} \underbrace{X_2 X_3}_{Y_2} \dots \underbrace{X_n, X_{n+1}})$

$$= /H(X_1 X) = H(X) + H(X|X) = H(X)/ = H(X_1 X_2 X_3 \dots X_{n+1})$$

$$\lim_{n\to\infty} \frac{1}{n} H(Y_1 Y_2 \dots Y_n) = \lim_{n\to\infty} \frac{1}{n} H(X_1 \dots X_{n+1})$$

$$H(Y) = H(X)$$

(b) $H(Z) = H(Z_1 Z_2 \dots Z_n) = H(X_2 X_3 X_4 X_5 \dots X_{2n} X_{2n+1}) =$

$$= H(X_1 X_2 \dots X_{2n}) = H(X_1 X_2 \dots X_n) + H(X_{n+1} \dots X_{2n} | X_1^n)$$

$$\frac{1}{n} H(X_1 X_2 \dots X_n) + \frac{1}{n} \left(H(X_{n+1} | X_1^n) + H(X_{n+2} | X_1^{n+1}) + \dots + H(X_{2n} | X_1^{2n-1}) \right)$$

$$\lim_{n\to\infty} \frac{1}{n} H(X_1 X_2 \dots X_n) + \frac{1}{n} \Big[\underbrace{\lim_{n\to\infty} H(X_n | X_1^n) + \lim_{n\to\infty} H(X_{n+2} | X_1^{n+1}) + \dots + \lim_{n\to\infty} H(X_{2n} | X_1^{2n-1})}_{n \cdot H(X)} \Big]$$

$$= H(X) + H(X) = 2H(X) \Rightarrow \boxed{H(Z) = 2H(X)}$$

$$\Rightarrow H(Z) \geq H(X)$$

$\boxed{4.25}$ MONOTONICITY

(a) SHOW THAT $I(X; Y_1, Y_2, \dots Y_n)$ IS NONDECREASING IN n.

(e) UNDER WHICH CONDITIONS IS THE MUTUAL INFORMATION CONSTANT FOR ALL n?

$$I(X; Y_1 Y_2, \dots Y_n) = I(X; Y_1) + I(X; Y_2 | Y_1) + \dots +$$

$$+ I(X; Y_n | Y_1 Y_2 \dots Y_{n-1})$$

$H(X_n|X_1) = H(X_{n+1}|X_2) \leqslant H(X_{n+1}|X_1X_2)$

$H(X_n|X_1) \geqslant H(X_n|X_1X_2) \underset{\text{MARKOVITY}}{=} H(X_n|X_2) \underset{\text{STATIONARITY}}{=} H(X_{n-1}|X_1)$

$\Rightarrow H(X_n|X_1) \geqslant H(X_{n-1}|X_1)$ CONDITIONAL ENTROPY IS INCREASING IN „n"

CONDITIONAL ENTROPY NONDECREASING WITH „n"

- $H(X_n|X_1^{n-1}) = H(X_{n+1}|X_2^n) \geqslant H(X_{n+1}|X_1^n)$

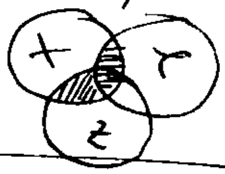
$\boxed{H(X_n|X_1^{n-1})}$ – IS NONINCREASING WITH „n"

- $I(X;Y_1^n) = \sum_{i=1}^{n} I(X;Y_i|Y_1^{i-1}) = I(X;Y_1)+I(X;Y_2|Y_1)+ \dots + I(X;Y_n|Y_1^{n-1})$

$I(X;Y,Z) = I(X;Y) + I(X;Z|Y)$

$= H(X) - H(X|Y) + H(X|Z) - H(X|Y,Z)$

$= H(X) - H(X|Y,Z)$



$I(X;Y_1Y_2\dots Y_n) = H(X) - H(X|Y_1^n) =$

$= H(Y_1,Y_2\dots Y_n) - H(Y_1\dots Y_n|X) =$

$= H(Y_2^{n+1}) - H(Y_2\dots Y_{n+1}|X)$

$I(X;Y_1^n) = I(X;Y_1) + I(X;Y_2|Y_1) + I(X;Y_3|Y_1Y_2) + \dots + I(X;Y_n|Y_1^{n-1})$

$I(X;Y_2|Y_1) = H(X|Y_1) - H(X|Y_1Y_2)$

$\{ I(X;Y_1Y_2\dots Y_{n+1}) = H(X) - H(X|Y_1^{n+1}) \geqslant \underbrace{H(X) - H(X|Y_1^n)}_{= I(X;Y_1\dots Y_n)}$

$H(X|Y_1^{n+1}) \leqslant H(X|Y_1^n)$ CONDITIONING REDUCES ENTROPY

$\boxed{I(X;Y_1^{n+1}) \geqslant I(X;Y_1^n)}$ MUTUAL INFORMATION IS NONDECREASING IN „n" PROVED!!!

(b) $I(X;Y) \geqslant I(X;Z)$ DATA PROCESSING INEQUALITY

$I(X;YZ) = I(X;Y) + \underbrace{I(X;Z|Y)}_{0} = I(X;Z) + \underbrace{I(X;Y|Z)}_{\neq 0} \Rightarrow$

$\boxed{I(X;Y) \geqslant I(X;Z)}$

$I(X;Y_1Y_2\dots Y_{n+1}) = I(X;Y_1^n)$ IF:

$\boxed{H(X|Y_1^{n+1}) = H(X|Y_1^n)}$

$H(X,Y_1^n) = H(Y_1^n) + H(X|Y_1^n)$

$H(X,Y_1^{n+1}) = H(Y_1^{n+1}) + H(X|Y_1^{n+1})$

$I(X;Y_1^{n+1}) = H(X) - H(X|Y_1^{n+1}) = H(Y_1^{n+1}) - H(Y_1^{n+1}|X)$

$H(Y_1^{n+1}|X) = H(Y_1|X) + H(Y_2|XY_1) + \dots + H(Y_n|X,Y_1^{n-1})$

$Y = \{Y_1 Y_2 \dots Y_n\}$ Y_i i.i.d $\quad H(Y_1^n) = n \cdot H(Y_1)$

- e.g. IF $X = f(Y_1) \Rightarrow$

$$\left.\begin{array}{l} I(X;Y_1^{n+1}) = H(X) - \overset{0}{H(X|Y_1^{n+1})} = H(X) \\ I(X;Y_1^{n}) = H(X) - \underset{0}{H(X|Y_1^{n})} = H(X) \end{array}\right]$$

MUTUAL INFORMATION CONSTANT FOR ALL n ?

- e.g. IF „X" IS STATISTICALY INDEPENDANT OF $Y_2, Y_3 \dots Y_n$

$I(X;Y_1^{n+1}) = H(X) - H(X|Y_1Y_2\dots Y_{n+1}) = H(X) - H(X|Y_1)$

$I(X;Y_1^{n}) = H(X) - H(X|Y_1Y_2\dots Y_n) = H(X) - H(X|Y_1)$

i.e. IF X IS CONDITIONALY INDEPENDENT OF $Y_2 Y_3 \dots Y_n$ — GIVEN Y_1.

$\boxed{4.26}$ **TRANSITIONS IN MARKOV CHAINS** SUPPOSE THAT $\{X_i\}$ FORMS IRREDUCIBLE MARKOV CHAIN WITH TRANSITION MATRIX P AND STATIONARY DISTRIBUTION μ. FORM THE ASSOCIATED „EDGE PROCESS" $\{Y_i\}$ BY KEEPING TRACK ONLY OF THE TRANSITIONS. THUS, THE NEW PROCESS $\{Y_i\}$ TAKES VALUES IN $\mathcal{X} \times \mathcal{X}$, AND $Y_i = (X_{i-1}, X_i)$. FOR EXAMPLE,

$X^n = 3, 2, 8, 5, 7 \dots$ BECOMES

$Y^n = (0,3), (3,2), (2,8), (8,5), (5,7), \dots$

FIND ENTROPY RATE OF EDGE PROCESS $\{Y_i\}$.

- $H(X) = -\sum_{i=1}^{n} \mu_i \sum_{j=1}^{n} P_{ij}\, \mathrm{ld}\, P_{ij}$

$X \in \{x_1 x_2 \dots x_n\}$ $\qquad Y \in \{Y_1 Y_2 \dots Y_n\} =$

$= \{x_0 x_1\, x_1 x_2\, x_2 x_3 \dots x_{n-1} x_n\}$

$H(Y_n|Y_1^{n-1}) = H(x_{n-1}, x_n | x_0 x_1\, x_1 x_2\, x_2 x_3\, x_3 \dots x_{n-2} x_{n-2} x_{n-1}\}$

$= H(x_{n-1} x_n | x_1 x_2 \dots x_{n-1}) = \underbrace{H(x_{n-1}| x_1 x_2 \dots x_{n-1})}_{0} +$

$+ H(x_n | x_1 x_2 \dots \underbrace{x_{n-1} x_{n-1}}_{x_{n-1}}) = H(x_n | x_1 x_2 \dots x_{n-1}) =$

$\boxed{H(Y) = H(X)}$ $\qquad = H(X) = H(x_2|x_1) = H(x_n|x_{n-1}$

- OPTION 2 SOLUTION

$H(Y_1 Y_2 \dots Y_n) = H(x_0 x_1\, x_1 x_2\, x_2 x_3 \dots x_{n-1} x_n) =$

$= H(X_1 X_2 \dots X_n)$; $\lim_{n\to\infty} \frac{1}{n} H(Y_1^n) = \lim_{n\to\infty} \frac{1}{n} H(X_1^n)$ $\overset{\leftarrow}{H(Y) = H(X)}$

$\boxed{4.27}$ ENTROPY RATE. LET $\{X_i\}$ BE STATIONARY $\{0,1\}$-VALUED STOCHASTIC PROCESS OBEYING:

$$X_{k+1} = X_k \oplus X_{k-1} \oplus Z_{k+1}$$

$P(Z_i=0) = 1-p$
$P(Z_i=1) = p$

WHERE $\{Z_i\}$ IS Bernoulli(p) AND $\oplus$ DENOTES mod2 ADDITION. WHAT IS THE ENTROPY RATE $H(\mathcal{X})$?

$Z \in \{Z_1, Z_2, \dots Z_n\}$ $\quad H(Z) = \frac{1}{n} H(Z_1, \dots Z_n)$

$$H(Z) = \frac{1}{n} \times H(Z_1) = p \,\mathrm{ld}\frac{1}{p} + (1-p)\,\mathrm{ld}\frac{1}{(1-p)}$$

$Z = X \oplus Y$ $\quad \underline{X \in \{0,1\}} \quad \underline{Y \in \{0,1\}} \quad \boxed{p = \frac{1}{2}}$

$$H(Z) = \left[P(00)\,\mathrm{ld}\frac{1}{P(00)} + P(11)\,\mathrm{ld}\frac{1}{P(11)}\right] \cdot P(Z=0) + P(Z=1)\left[P(10)\,\mathrm{ld}\frac{1}{P(10)} + P(01)\,\mathrm{ld}\frac{1}{P(01)}\right]$$

$P(00) = (1-p)^2 = \frac{1}{4}$ $\quad P(11) = p^2 = \frac{1}{4}$

$P(01) = P(10) = p(1-p) = \frac{1}{4}$

$$H(Z) = P(Z=0)\cdot\left[\frac{1}{4}\cdot 2 + \frac{1}{4}\cdot 2\right] + P(Z=1)\left[\frac{1}{2} + \frac{1}{2}\right]$$
$$= P(Z=0) + P(Z=1) = \underline{1 \text{ bit}}$$

e.g. $Z = \{0, 1, 1, 0, 1\}$

$X =$

$k=0$

$X_1 = X_0 \oplus X_{-1} \oplus Z_1 = Z_1$

$k=1$ $\quad X_2 = X_1 \oplus \cancel{X_0} \oplus Z_2 = X_1 \oplus Z_2 = Z_1 \oplus Z_2$

$k=2$ $\quad X_3 = X_2 \oplus X_1 \oplus Z_3 = Z_1 \oplus Z_2 \oplus Z_1 \oplus Z_3 = \underbrace{Z_1 \oplus Z_1}_{0} \oplus Z_2 \oplus Z_3 = Z_2 \oplus Z_3$

$k=3$ $\quad X_4 = X_3 \oplus X_2 \oplus Z_4 = Z_2 \oplus Z_3 \oplus Z_1 \oplus Z_2 \oplus Z_4$
$= Z_1 \oplus Z_3 \oplus Z_4$

$H(\mathcal{X}) = H(X_n | X_1 X_2 \dots X_{n-1}) = \frac{1}{n} H(X_1 X_2 \dots X_n)$

$H(\mathcal{X}) = H(X_n | X_{n-1} X_{n-2} Z_n)$ $\quad \}$ SECOND ORDER MARKOV CHAIN.

- FIRST ORDER MARKOV CHAIN:

$$H(\mathcal{X}) = \lim_{n\to\infty} H(X_n | X_{n-1} \dots X_1) = \lim_{n\to\infty} H(X_n | X_1^{n-1})$$

$$H(X_1^n) = H(X_1) + H(X_2|X_1) + H(X_3|X_1 X_2) + \dots + H(X_n|X_1^{n-1}) =$$
$$= H(X_1) + H(X_2|X_1) + H(X_3|X_2) + \dots + H(X_n|X_{n-1}) =$$
$$= H(X_1) + H(X_2|X_1) + H(X_2|X_1) + \dots + H(X_2|X_1) =$$
$$= H(X_1) + (n-1) H(X_2|X_1) \quad H(\mathcal{X}) = \lim_{n\to\infty}\left[\frac{H(X_1)}{n} + \frac{n-1}{n} H(X_2|X_1)\right] = H(X_2|X_1)$$

- SECOND ORDER

$$H(X_1^n) = H(x_1) + H(x_2|x_1) + H(x_3|x_1x_2) + H(x_4|x_3x_2) + \dots + H(x_n|x_{n-1}x_{n-2})$$

$$= H(x_1) + H(x_2|x_1) + H(x_3|x_1x_2) + H(x_3|x_2x_1) + \dots + H(x_3|x_2x_1)$$

$$= H(x_1) + H(x_2|x_1) + (n-2)\, H(x_3|x_1x_2)$$

$$H(x) = \lim_{n\to\infty}\left[\frac{H(x_1) + H(x_2|x_1)}{n} + \frac{n-2}{n} H(x_3|x_1x_2)\right] = \underline{H(x_3|x_1x_2)}$$

$$H(x) = H(x_3/x_2x_1) = \underline{H(z_2 \oplus z_3 | z_1 \oplus z_2, z_1)}$$

$$H(x_3|x_1x_2) = P(00)\left[\underbrace{P(x_3=0|00)}\, ld\frac{1}{P(x_3=0|00)} + \underbrace{P(x_3=1|00)}\, ld\frac{1}{P(x_3=1|00)}\right]$$

$$+P(01)\left[\underbrace{P(x_3=0|01)}_{P(z_3=1)=\gamma}\, ld\,\overset{(1-\gamma)}{\frac{1}{P(x_3=0|01)}} + \underbrace{P(x_3=1|01)}_{P(z=0)=1-\gamma}\, ld\frac{1}{P(x_3=1|01)}\right] +$$

$$+P(10)\left[\underbrace{P(x_3=0|10)}_{P(z=1)=\gamma}\, ld\frac{1}{P(x_3=0|10)} + \underbrace{P(x_3=1|10)}_{P(z=0)=1-\gamma}\, ld\frac{1}{P(x_3=1|10)}\right] +$$

$$+P(11)\left[\underbrace{P(x_3=0|11)}_{P(z=0)=1-\gamma}\, ld\frac{1}{P(x_3=0|11)} + \underbrace{P(x_3=1|11)}_{P(z=1)=\gamma}\, ld\frac{1}{P(x_3=1|11)}\right] =$$

$$P(00) = P(z_1=0)\cdot P(z_1\oplus z_2=0) = (1-p)\cdot(1-\gamma) = (1-\gamma)^2$$

$$P(10) = P(z_1=0)\cdot \underbrace{P(z_1\oplus z_2=1)}_{=P(z_2=1)} = (1-\gamma)\cdot\gamma$$

$$P(01) = P(z_1=1)\cdot \underbrace{P(z_1\oplus z_2=0)}_{=P(z_2=1)} = \gamma^2$$

$$P(11) = P(z_1=1)\cdot \underbrace{P(z_1\oplus z_2=1)}_{=P(z_2=0)} = \gamma(1-\gamma)$$

$$H(x) = H(x_3|x_2x_1) = -(1-\gamma)^2\left[\underbrace{(1-\gamma)\,ld(1-\gamma) + \gamma\, ld\gamma}_{-H(\gamma)}\right] + \gamma^2 H(\gamma) +$$

$$+ (1-\gamma)\gamma H(\gamma) + \gamma(1-\gamma)H(\gamma) = (1-\gamma)^2 H(\gamma) + \gamma^2 H(\gamma) + 2(1-\gamma^2)H(\gamma)$$

$$H(x) = \left[1 - \cancel{2\gamma} + \cancel{\gamma^2} + \cancel{\gamma^2} + \cancel{2\gamma} - \cancel{2\gamma^2}\right] H(\gamma) = H(\gamma)$$

$$\boxed{H(x) = H(\gamma)}$$

- АКО ГИ ЗНАЕШ x_k & x_{k-1} ТОГАШ x_{k+1} ЗАВИСИ САМО ОД Z_{k+1} ПА ЗАТОА $H(x_{k+1}|x_k=a, x_{k-1}=b) = H(Z_{k+1})$

- SECOND EDITION SOLUTION,

$$\underline{X_{k+1} = X_k \oplus X_{k-1} \oplus Z_{k+1}}$$

$$H(X) = H(X_{k+1}|X_k, X_{k-1}, \dots X_1) = H(x_{k+1}|x_k, x_{k-1}) = H(Z_{k+1})$$

$$= H(\gamma)$$

$\boxed{4.28}$ **MIXTURE OF PROCESSES** SUPPOSE THAT WE OBSERVE ONE OF TWO STOCHASTIC PROCESSES BUT DON'T KNOW WHICH. WHAT IS THE ENTROPY RATE? SPECIFICALLY, LET $X_{11}, X_{12}, \ldots$ BE BERNOULLI PROCESS WITH PARAMETER p_1, AND LET $X_{21}, X_{22}, \ldots$ BE BERNOULLI(p_2). LET!

$$\theta = \begin{cases} 1 & \text{WITH PROBABILITY } 1/2 \\ 2 & \text{WITH PROBABILITY } 1/2 \end{cases}$$

AND LET $Y_i = X_{\theta i}$, $i = 1, 2, \ldots$, BE THE STOCHASTIC PROCESS OBSERVED. THUS, Y OBSERVES THE PROCESS $\{X_{1i}\}$ OR $\{X_{2i}\}$. EVENTUALLY, Y WILL KNOW WHICH.

(a) IS $\{Y_i\}$ STATIONARY?

(b) IS $\{Y_i\}$ AN i.i.d PROCESS?

(c) WHAT IS THE ENTROPY RATE H OF $\{Y_i\}$?

(d) DOES $-\frac{1}{n}\log p(Y_1, Y_2, \ldots Y_n) \to H$?

(e) IS THERE A CODE THAT ACHIEVES AN EXPECTED PER SYMBOL DESCRIPTION LENGTH $\frac{1}{n}E[L_n] \to H$?

$\boxed{\text{PART 1}}$ $H(X|F) \geq H(X|Y)$ $\quad$ $I(X;F) \leq I(X;Y)$

$\cancel{H(X)} - H(X|F) \leq \cancel{H(X)} - H(X|Y)$ $\quad$ $\boxed{H(X|Y) \leq H(X|F)}$

(c) $H(Y) = ?$ $\quad$ $H(Y) = H(Y_1, Y_2 \ldots Y_n)\cdot\frac{1}{n} = H(Y_n|Y_1^{n-1})$

$$H(\theta, Y_1^n) = \overset{=1}{\widetilde{H(\theta)}} + H(Y_1^n|\theta) = H(\theta) + P(\theta=1)\underbrace{H(Y_1^n|\theta=1)}_{H(X_{21}^{2n})} + P(\theta=2)\cdot\underbrace{H(Y_1^n|\theta=2)}_{=H(X_{11}^{1n})}$$

$$H(\theta, Y_1^n) = 1 + \frac{1}{2}H(X_{11}^{1n}) + \frac{1}{2}H(X_{21}^{2n}) = 1 + \frac{1}{2}n\cdot\left[p_1\log\frac{1}{p_1} + (1-p_1)\log\frac{1}{1-p_1}\right. + \frac{1}{2}n\left[p_2\log\frac{1}{p_2} + (1-p_2)\log\frac{1}{1-p_2}\right] = \underline{1 + \frac{n}{2}H(p_1) + \frac{n}{2}H(p_2)}$$

$$H(Y_1^n) \geq H(Y_1^n|\theta)$$

$$\boxed{\lim_{n\to\infty}\frac{H(Y_1^n|\theta)}{n} = \frac{1}{2}H(p_1) + \frac{1}{2}H(p_2)}$$

$$\boxed{H(\theta, Y_1^n) = H(Y_1^n) + \underset{=H(Y_1^n)}{H}\overset{=0}{\widetilde{\cancel{(\theta|Y_1^n)}}}}$$

SUŠTINATA ZA PART 1 E DEKA: $\{Y_i\} = \{X_{1i}\}$ $i = 1, 2, \ldots$, ILI $\{Y_i\} = \{X_{2i}\}$ $i = 1, 2, \ldots$ OVA E ISTIOT DOKAZ KAKO VO EDITION 2 SOLUTIONS.

(a) $X_1 = \{X_{11}, X_{12}, \ldots X_{1n}\}$ $\quad$ $X_2 = \{X_{21}, X_{22}, \ldots X_{2n}\}$

$$Y = \{X_{11}X_{12}\cdots X_{1n}\} \vee \{X_{21}X_{22}\ldots X_{2n}\}$$

$$p(Y) = \left\{\frac{p_1}{2}, \frac{p_1}{2} \ldots \frac{p_1}{2}\right\} \vee \left\{\frac{p_2}{2}, \frac{p_2}{2} \ldots \frac{p_2}{n}\right\}.$$

- STATIONARY:

$x = \{x_1 x_2 - - x_n\}$ $\quad H(X) = n \cdot H(X_1) = n \cdot H(Y_1)$

$$P(x_1, x_2 \ldots x_n] = P(x_{1+l}, x_{2+l}, \ldots x_{n+l})$$

$$P(Y) = P(x_{11}\, x_{12}\, x_{13} \ldots) + P(x_{21}\, x_{22} \ldots) =$$

$$= P(x_{11+l}\, x_{12+l}\, x_{13+l} \ldots) + P(x_{21+l}, x_{22+l}, \ldots) =$$

$$= P(x_{11}) \cdot P(x_{12}) \cdot P(x_{13}) \cdots + P(x_{21}) \cdot P(x_{22}) \ldots$$

$\Rightarrow$ $\{Y_i\}$ IS STATIONARY AND $\{Y_i\}$ IS $\underbrace{\text{i.i.d PROCESS}}_{\text{VIVAT DEVA NE } \in \text{ INDEPENDENT. ZAVISI OD } \theta.}$

$$H(X_1) = H(x_{11}, x_{12} \ldots x_{1n}) = n \cdot H(x_{11}) = nH(Y_1)$$

$$H(X_2) = H(x_{21}, x_{22} \ldots x_{2n}) = nH(x_{21}) = nH(Y_2)$$

$$H(Y) = \lim_{n \to \infty} \frac{1}{n} H(Y_1, Y_2 \ldots Y_n) =$$ $\boxed{\text{RECALL ENTROPY OF TWO i.i.d}}$

$$H(Y_1, Y_2, \ldots Y_n)\Big|_{n \to \infty} = H(x_{11}, x_{12}, \ldots x_{21} x_{22} \ldots) =$$

$$= H(x_{11}, x_{12}, \ldots) + H(x_{21} x_{22} \ldots) = (n \cdot H(Y_1) + nH(Y_2))_{n \to \infty}$$

$$\cdot \lim_{n \to \infty} H(Y_1 Y_2 \ldots Y_n) = \lim_{n \to \infty} \frac{nH(Y_1) + nH(Y_2)}{n} = H(Y_1) + H(Y_2)$$

$$\frac{H(x_{11}, x_{12}, \ldots x_{21} x_{22} \ldots)}{n} = -\frac{1}{n} \sum_{x_1 x_2} P(x_{11}^{1n}, x_{21}^{2n})\, ld\, P(x_{11}^{1n} x_{21}^{2n}) =$$

$x_1 \in \{0,1\}$

$x_2 \in \{0,1\}$

$x_1 \in \{x_{11}, x_{12}\} \qquad x_2 \in \{x_{21}, x_{22}\}$

$$P(x_{11}, x_{12}) = \underbrace{P(x_{11})} \cdot P(x_{12} | x_{11}) = P(x_{11}) \cdot P(x_{12})$$

$$H(X_1) = -\sum_{x_{11} x_{12}} P(x_{11} x_{12}) \cdot ld\, P(x_{11}, x_{12}) =$$

$$= -\sum_{x_{11} x_{12}} P(x_{11}) \cdot P(x_{12})\, ld\, P(x_{11}) \cdot P(x_{12}) = -\sum_{x_{11} x_{12}} P(x_{11}) P(x_{12})\, ld\, P(x_{11})$$

$$- \sum_{x_{11} x_{12}} P(x_{11}) P(x_{12})\, ld\, P(x_{12}) = -1 \cdot \sum_{x_{11}} P(x_{11})\, ld\, P(x_{11}) - 1 \sum_{x_{12}} P(x_{12})\, ld\, P(x_{12})$$

$$= \underline{H(x_{11}) + H(x_{12}) = 2H(Y)}$$

(c) $\left.\begin{array}{l} x \in \{x_1 x_2 \dots x_n\} \\ x' \in \{x'_1 x'_2 \dots x'_n\} \end{array}\right\}$ iid $\quad P(x=x') = p_1 \cdot p_1 + \dots + p_n p_n = \sum_{i=1}^{n} p_i^2$

$p(x) = \{p_1 \; p_2 \dots p_n\}$

$p(x') = \{p_1 \; p_2 \dots p_n\}$

$\{x_{1i}\} = \{x_{11}, x_{12}, \dots x_{1n}\}$

$\{x_{2i}\} = \{x_{21}, x_{22}, \dots x_{2n}\}$

$\{\theta_i\} = \{\theta_1, \theta_2, \dots \theta_n\}$

$$\boxed{X_i = \begin{cases} x_{1i} & \text{with } p = \frac{1}{2} \\ x_{2i} & \text{with } p = \frac{1}{2} \end{cases}}$$

$= H(\theta, X_1, X_2) = H(\theta_1^n, X_{11}^{1n}, X_{21}^{2n}) =$

$$\theta_i = f(X_i) = \begin{cases} 1 & \text{IF } X_i = X_{1i} \\ 2 & \text{IF } X_i = X_{2i} \end{cases}$$

$$H(X_1^n) = H(\theta_1^n, X_1^n) = \underbrace{H(\theta_1^n)}_{n \cdot \underbrace{H(\frac{1}{2})}_{1}} + H(X_1^n / \theta_1^n) = H(X_1^n) + \underbrace{H(\theta_1^n / X_1^n)}_{0}$$

$= n + P(\theta_1 = 1, \theta_2 = 1, \dots \theta_n = 1)\, H(X_1^n|_{11\dots1}) + P(\theta_1 = 1, \theta_2 = 1, \dots \theta_n = 2) \cdot H(X_1^n|_{11\dots2}) + \dots + P(\theta_1 = 2, \theta_2 = 2, \dots \theta_n = 2) \cdot H(X_1^n|_{222\dots2})$

$= \frac{1}{2^n} H(p_1) \cdot n + \frac{1}{2^n}\left[H(p_1)\binom{n}{n-1} + H(p_2)\right] + \frac{1}{2^n}\left[H(p_1) \cdot \binom{n}{n-2} + 2H(p_2)\right] + \dots + \frac{1}{2^n} \cdot H(p_2) \cdot n =$ ~~$\frac{1}{2^n}[n + (n-1) + 2 \dots + 1] H(p_1) +$~~

~~$\frac{1}{2^n}[n + (n-1) + 2 \dots + 1] H(p_2)$~~

$n = 3 \quad \binom{n}{n-1} = \binom{3}{2} = \frac{3!}{2!\,1!} = \frac{6}{2} = 3$

112
121
211

$\binom{n}{1} = \binom{3}{1} = \frac{3!}{1!\,2!} = 3$

$\binom{n}{n} = \binom{3}{3} = \frac{3!}{3! \cdot 0!} = 1 \qquad \binom{n}{0} = \frac{n!}{0!\,n!} = 1$

$$\boxed{= n + \frac{1}{2^n}\left[\binom{n}{n}\left[H(p_1) \cdot n + H(p_2) \cdot 0\right] + \binom{n}{n-1}\left[H(p_1) \cdot (n-1) + H(p_2) \cdot 1\right] + \binom{n}{n-2}\left[(n-2) H(p_1) + 2 H(p_2)\right] + \dots + \binom{n}{6}\left[H(p_1) \cdot 0 + H(p_2) \cdot n\right]\right]}$$

$$H(Y_1^n) = n + \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{n-i} \left[(n-i) H(\gamma_1) + i\, H(\gamma_2) \right] =$$

$$= n + \frac{1}{2^n} H(\gamma_1) \underbrace{\sum_{i=0}^{n} \binom{n}{n-i} (n-i)}_{\frac{n}{2} 2^n} + \frac{1}{2^n} H(\gamma_2) \underbrace{\sum_{i=0}^{n} \binom{n}{n-i} i}_{\frac{n}{2} 2^n} =$$

MAPLE!

$$= n + \frac{1}{\cancel{2^n}} H(\gamma_1) \cdot \frac{n}{2} \cancel{2^n} + \frac{1}{\cancel{2^n}} H(\gamma_2) \cdot \frac{n}{2} \cancel{2^n} = n + \frac{n}{2} H(\gamma_1) + \frac{n}{2} H(\gamma_2)$$

$$H(Y_1^n) = n + \frac{n}{2} H(\gamma_1) + \frac{n}{2} H(\gamma_2)$$

$$\lim_{n \to \infty} \frac{H(Y_1^n)}{n} = 1 + \frac{H(\gamma_1)}{2} + \frac{H(\gamma_2)}{2}$$

$$\boxed{H(Y) = 1 + \frac{H(\gamma_1)}{2} + \frac{H(\gamma_2)}{2}} \quad \$$$

$$\boxed{\gamma + 1 - \gamma = 1}$$

(d) $P(Y) = P(x_{11} x_{12} \ldots) + P(x_{21} x_{22} \ldots) = \underset{1/2}{P}(x_{11}) \cdot \underset{1/2}{P}(x_{12}) \cdots + P(x_{21}) P(x_{2n}) \cdots$

$$-\frac{1}{n} \operatorname{ld} P(Y_1, Y_2 \ldots Y_n) = -\frac{1}{n} \operatorname{ld} \left[P(x_{11}^{n}) + P(x_{21}^{2n}) \right]$$

- RECALL ASYMPTOTIC EQUIPARTITION

$$-\frac{1}{n} \operatorname{ld} P(x_1, x_2, \ldots x_n) = -\frac{1}{n} \operatorname{ld} \prod_{i=1}^{n} P(x_i) = -\frac{1}{n} \sum_{i=1}^{n} \operatorname{ld} P(x_i)$$

$$= \sum_{i=1}^{n} \frac{1}{n} \operatorname{ld} P(x) \xrightarrow[n \to \infty]{} E\left[\operatorname{ld} \frac{1}{P(x)} \right] = \sum_{i=1}^{n} p(x) \operatorname{ld} \frac{1}{p(x)} = \underline{\underline{H(x)}}$$

LAW OF LARGE NUMBERS

$$\boxed{P(Y) = P(Y_1 Y_2 \ldots Y_n) = \sum_{i=1}^{n} \binom{n}{i} \left(\frac{\gamma_1^i q_1^{n-i}}{2} + \frac{\gamma_2^i q_2^{n-i}}{2} \right)}$$

$$\underbrace{x_1 x_2}_{2} \underbrace{x_3 x_4}_{2}$$

$$\binom{4}{0} + \binom{4}{1} + \binom{4}{2} = \binom{2}{0} + \binom{2}{1} + \binom{2}{2} =$$

$$= 1 + 2 + \frac{2!}{2!0!} = 4$$

$$\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 1 + \frac{4!}{3!} + \frac{4!}{2!2!} + \frac{4!}{3!} + 1$$

$$= 1 + \frac{24}{6} + \frac{24}{4} + \frac{24}{6} + 1 = 1 + 4 + 6 + 4 + 1 = \underline{\underline{16}} = \underline{\underline{2^4}}$$

$$P(Y) = \frac{1}{2} \underbrace{\sum_{i=1}^{n} \binom{n}{i} \gamma_1^i (1-\gamma_1)^{n-i}}_{1} + \frac{1}{2} \underbrace{\sum_{i=1}^{n} \binom{n}{i} \gamma_2^i (1-\gamma_2)^{n-i}}_{1} =$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

AKO SAMAJ DA ZA VIDIŠ VEROJATNOSTI
NA SAMO GRUPA OD SEKVENCI NAMI MENU VŽE

(d) $-\frac{1}{n} \operatorname{ld} p(Z_1 Z_2, \dots Z_n) = -\frac{1}{n} \operatorname{ld}[P(Z_1) P(Z_2) \cdots P(Z_n)] =$

$= -\frac{1}{n} \sum_{i=1}^{n} \operatorname{ld} P(Z_i) = -\sum_{i=1}^{n} \frac{1}{n} \operatorname{ld} P(Z_i) \to E[\operatorname{ld} Z] = H(Z) =$

$= 1 + \frac{H(p_1)}{2} + \frac{H(p_2)}{2}$

$-n(H(X)+\epsilon) \quad 2^{-n(H(X)+\epsilon)} \leq p(X_1 X_2 \dots X_n) \leq 2^{-n(H(X)-\epsilon)}$

$\sqrt{} \leq \operatorname{ld} p(X_1 X_2, \dots X_n) \leq -n(H(X)-\epsilon)$

$n(H(X)-\epsilon) \leq \operatorname{ld} p(X_1^n) \leq n(H(X)+\epsilon)$

EDITION 2 SOLUTIONS MANUAL

(a) $\{Z_i\}$ IS STATIONARY SINCE THE SCHEME TO CHOOSE Z_i'S DOESN'T CHANGE

(b) NO, IT IS NOT IID, SINCE THERE IS DEPENDENCE NOW – ALL Z_iS HAVE BEEN GENERATED ACCORDING TO THE SAME PARAMETER θ.

– IF THE MODELS WERE TO BE IID, THEN THE EXPRESSION $I(Z_{n+1}; Z_1^n)$ WOULD HAVE TO BE 0. HOWEVER, IF WE ARE GIVEN Z^n, THEN WE CAN ESTIMATE WHAT θ IS, WHICH IN TURN ALLOWS US TO PREDICT Z_{n+1}.

THUS, $I(Z_{n+1}; Z^n)$ IS NONZERO. $I(Z_{n+1}; Z_1^n) = H(Z_{n+1}) - H(Z_{n+1}|Z_1^n)$

MOZE ODMA DA GO PIŠEŠ DEKA 1 0, ZOŠTO $\theta = f(\underbrace{Z_1^n})$!!!

(c) $H(X) = \frac{H(p_1) + H(p_2)}{2}$

$H = \lim_{n\to\infty} \frac{1}{n} H(Z_1^n) = \lim_{n\to\infty} \frac{1}{n}\left[H(\theta) + H(Z_1^n/\theta) - H(\theta|Z_1^n)\right] =$

$/ H(\theta, Z_1^n) = H(\theta) + H(Z_1^n|\theta) = H(Z_1^n) + H(\theta|\underline{Z_1^n}) \Rightarrow$

$H(Z_1^n) = H(\theta) + H(Z_1^n/\theta) - H(\theta|Z_1^n) /$

$P(Z_1^n = X_{11}^{1n}) \cdot$

$= \lim_{n\to\infty} \frac{1}{n}\Big[1 + \frac{1}{2}\underbrace{H(Z_1^n|\theta=1)}_{H(X_{11}^{1n}) = H(p_1)\cdot n} + \frac{1}{2}\underbrace{H(Z_1^n|\theta=0)}_{H(X_{21}^{2n}) = H(p_2)\cdot n} - \sqrt{} H(\theta|\underline{Z_1^n = X_{m}^{1n}}) -$

$- \underbrace{P(Z_1^n = X_{21}^{2n})}_{1/2} \cdot H(\theta|Z_1^n = X_{21}^{2n})\Big] = \lim_{n\to\infty} \frac{1}{n}\left[1 + \frac{nH(p_1)}{2} + \frac{nH(p_2)}{2} - \frac{1}{2}\cdot 0 - \frac{1}{2}\cdot 0\right]$

$$\boxed{H = \frac{H(p_1)}{2} + \frac{H(p_2)}{2}}$$

OVA E EQVIVALENTNO NA MOETO IZVEDUVANJE NA TP.78

(d) THE PROCESS $\{Z_i\}$ IS NOT ERGODIC, SO THE AEP DOESN'T APPLY AND THE QUANTITY $-\frac{1}{n} \operatorname{ld} P(Z_1, Z_2, \dots, Z_n)$ DOES NOT CONVERGE TO THE ENTROPY RATE.

(e) WE CAN DO HUFFMAN CODDING.

⊡ PART 2: LET θ_i BE BERNULLI$\left(\frac{1}{2}\right)$. OBSERVE:

$$\boxed{Z_i = x_{\theta_i, i} \quad i = 1, 2, \dots,}$$

THUS θ IS NOT FIXED FOR ALL TIME, AS IT WAS IN FIRST PART, BUT IS CHOSEN I.I.D EACH TIME. ANSWER (a), (b), (c), (d), (e) FOR PROCESS $\{Z_i\}$, LEADING ANSWERS (a'), (b'), (c'), (d'), (e')

(a') YES $\{Z_i\}$ IS STATIONARY SINCE THE SCHEME THAT WE USE TO GENERATE Z_i's DOESN'T CHANGE WITH TIME.

(b') YES, IT IS IID, SINCE THERE'S NO DEPENDENCE NOW! EACH Z_i IS GENERATED ACCORDING TO TO INDEPENDENT PARAMETER θ_i AND

$Z_i \sim$ Bernoulli$\left(\frac{\gamma_1+\gamma_2}{2}\right)$

HMV

$$Z_i = \{1, 0\} \quad \boxed{P(Z_i) = \left\{\frac{\gamma_1+\gamma_2}{2}, 1 - \frac{\gamma_1+\gamma_2}{2}\right\}} \quad x_{1i} = \{1, 0\}$$

$$P(x_{1i}) = \{\gamma_1, 1-\gamma_1\}$$

$$x_{2i} = \{1, 0\} \quad P(x_{2i}) = \{\gamma_2, 1-\gamma_2\}$$

$$\begin{cases} P(Z_i = 1) = P(\theta=1) \cdot P(x_{1i}=1 \mid \theta=1) + P(\theta=2) \cdot \underline{P(x_{2i}=1 \mid \theta=2)} \\ \quad = \frac{1}{2} \cdot \gamma_1 + \frac{1}{2} \cdot \gamma_2 = \frac{\gamma_1+\gamma_2}{2} \\ P(Z_i = 0) = P(\theta=1) \cdot P(x_{1i}=0 \mid \theta=1) + P(\theta=2) \cdot P(x_{2i}=0 \mid \theta=2) \\ \quad = \frac{1}{2} \cdot (1-\gamma_1) + \frac{1}{2}(1-\gamma_2) = 1 - \frac{\gamma_1+\gamma_2}{2} \end{cases}$$

(c') $H(Z_i) = \frac{\gamma_1+\gamma_2}{2} \operatorname{ld} \frac{2}{\gamma_1+\gamma_2} + \left(1 - \frac{\gamma_1+\gamma_2}{2}\right) \operatorname{ld} \frac{2}{2-\gamma_1-\gamma_2} =$

$= \frac{\gamma_1+\gamma_2}{2} \operatorname{ld} \frac{2}{\gamma_1+\gamma_2} - \frac{\gamma_1+\gamma_2}{2} \operatorname{ld} \frac{2}{2-\gamma_1-\gamma_2} + \operatorname{ld} \frac{2}{2-\gamma_1-\gamma_2} =$

$= \frac{\gamma_1+\gamma_2}{2} \operatorname{ld} \frac{\cancel{2}}{\gamma_1+\gamma_2} \cdot \frac{2-\gamma_1-\gamma_2}{\cancel{2}} + \operatorname{ld} \frac{2}{2-\gamma_1-\gamma_2}$

$H(Z_i) = \frac{\gamma_1+\gamma_2}{2} \operatorname{ld} \frac{2-\gamma_1-\gamma_2}{\gamma_1+\gamma_2} + \operatorname{ld} \frac{2}{2-\gamma_1-\gamma_2}$

$H(Z) = \lim\limits_{n\to\infty} \frac{H(Z_1 Z_2 \dots Z_n)}{n} = \lim\limits_{n\to\infty} \frac{\cancel{n} H(Z_i)}{\cancel{n}}$ ⊛

$$\boxed{H(Z) = H(Z_i) = H\left(\frac{\gamma_1+\gamma_2}{2}\right) = \frac{\gamma_1+\gamma_2}{2} \operatorname{ld} \frac{2-\gamma_1-\gamma_2}{\gamma_1+\gamma_2} + \operatorname{ld} \frac{2}{2-\gamma_1-\gamma_2}}$$

ⓓ $H(Z) = 1 + \frac{H(\gamma_1)}{2} + \frac{H(\gamma_2)}{2} = 1 + \gamma_1 \operatorname{ld} \frac{1}{\gamma_1} + (1-\gamma_1) \operatorname{ld} \frac{1}{1-\gamma_1} +$

$+ \gamma_2 \operatorname{ld} \frac{1}{\gamma_2} + (1-\gamma_2) \operatorname{ld} \frac{1}{(1-\gamma_2)} = 1 + \underline{\gamma_1 \operatorname{ld} \frac{1}{\gamma_1}} + \underline{\underline{\operatorname{ld} \frac{1}{1-\gamma_1}}} - \underline{\gamma_1 \operatorname{ld} \frac{1}{1-\gamma_1}} +$

$+ \underline{\underline{\gamma_2 \operatorname{ld} \frac{1}{\gamma_2}}} + \underline{\underline{\operatorname{ld} \frac{1}{1-\gamma_2}}} - \underline{\underline{\gamma_2 \operatorname{ld} \frac{1}{1-\gamma_2}}} = 1 + \gamma_1 \operatorname{ld} \frac{1-\gamma_1}{\gamma_1} + \operatorname{ld} \frac{1}{(1-\gamma_1)(1-\gamma_2)} + \gamma_2 \operatorname{ld} \frac{1-\gamma_2}{\gamma_2}$

GI SKOCEDIV ($) , (★) VO MATICE NO NE SE [illegible]!!?

$\boxed{4.29}$ Waiting Times. LET X BE THE WAITING TIME FOR FIRST HEADS TO APPEAR IN SUCCESSIVE FLIPS OF A FAIR COIN. FOR EXAMPLE, $Pr\{X=3\} = \left(\frac{1}{2}\right)^3$. LET S_n BE THE WAITING TIME FOR THE n-th HEAD TO APPEAR. THUS, $S_0 = 0$, $S_{n+1} = S_n + X_{n+1}$ WHERE $X_1, X_2, X_3, \ldots$ ARE I.I.D ACCORDING TO DISTRIBUTION ABOVE!

(a) Is the process $\{S_n\}$ stationary?

(b) Calculate $H(S_1, S_2, \ldots S_n)$.

(c) Does the process $\{S_n\}$ have entropy rate? If so, what is it? If no, why not?

(d) WHAT IS THE EXPECTED NUMBER OF FAIR COIN FLIPS REQUIRED TO GENERATE A RANDOM VARIABLE HAVING SAME DISTRIBUTION AS S_n?

(b) $H(S_1, S_2)$

$$\boxed{\begin{aligned} S_1 &= S_0 + X_1 \\ S_2 &= S_1 + X_2 \\ S_3 &= S_2 + X_3 \end{aligned}}$$

e.g. $P_r(X_1=3) = \left(\frac{1}{2}\right)^3$

$S_1 = 3 \quad Pr\{S_1 = 3\} = \left(\frac{1}{2}\right)^3$

$S_2 = 3 + X_2$; e.g. $Pr\{X_2 = 4\} = \left(\frac{1}{2}\right)^4$

$S_2 = X_1 + X_2$

$S_2 = 3 + 4 \quad Pr\{\underline{S_2 = 7}) = \left(\frac{1}{2}\right)^{③} \cdot \left(\frac{1}{2}\right)^4 = \left(\frac{1}{2}\right)^{⑦}$ — $x_1 + x_2 = S_1 + x_2$

→ S_1

$S_{n+1} = S_n + X_{n+1} \qquad Pr.\{S_n$

$S_{n+1} = \sum_{i=1}^{n+1} X_i \qquad \sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}$

ZA: $H(S_1, S_2) \qquad S_1 = X_1 \qquad S_2 = X_1 + X_2$

$H(S_1, S_2) = H(S_1) + H(S_2|S_1)$

$H(S_1) = P(S_1) \cdot \log \frac{1}{P(S_1)} \qquad P(S_1) = \left(\frac{1}{2}\right)^{X_1}$

$$\boxed{H(S_1) = \sum_{X_1=1}^{\infty} \left(\frac{1}{2}\right)^{X_1} \log 2^{X_1}} = \sum_{X_1=1}^{\infty} X_1 \cdot \left(\frac{1}{2}\right)^{X_1} = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = \frac{\frac{1}{2}}{\frac{1}{4}} = 2$$

$$H(S_2|S_1) = \sum_{X_2=1}^{\infty} \sum_{X_1=1}^{\infty} P(S_1 = X_1) H(S_2 | S_1 = X_1) = \sum_{X_1=1}^{\infty} P(S_1 = X_1) \underbrace{\sum_{X_2=1}^{\infty} H(S_2|S_1 = X_1)}$$

$$(★) = \sum_{X_2=1}^{\infty} H(S_2|S_1 = X_1) = \sum_{X_2=1}^{\infty} \left(\frac{1}{2}\right)^{\overbrace{X_1+X_2}^{=S_2}} \overbrace{(X_1+X_2)}^{=S_2} = \left(\frac{1}{2}\right)^{X_1} \sum_{X_2=1}^{\infty} \left(\frac{1}{2}\right)^{X_2} \cdot \overset{⊕}{(X_1+X_2)}$$

$$= \left(\frac{1}{2}\right)^{X_1} \sum_{X_2=1}^{\infty} \left(\frac{1}{2}\right)^{X_2} \cdot X_1 + \left(\frac{1}{2}\right)^{X_1} \sum_{X_2=1}^{\infty} X_2 \left(\frac{1}{2}\right)^{X_2} = X_1 \left(\frac{1}{2}\right)^{X_1} \sum_{X_2=1}^{\infty} \left(\frac{1}{2}\right)^{X_2} +$$

$$+ \left(\frac{1}{2}\right)^{X_1} \cdot \frac{\frac{1}{2}}{\frac{1}{4}} = X_1 \left(\frac{1}{2}\right)^{X_1} \cdot \frac{\cancel{\frac{1}{2}}}{\cancel{\frac{1}{2}}} + \left(\frac{1}{2}\right)^{X_1 - 1} = \left(\frac{1}{2}\right)^{X_1 - 1} \left(\frac{X_1}{2} + 1\right)$$

$$① = \left(\frac{1}{2}\right)^{x_1}(x_1+2)$$

$$H(S_2|S_1) = \sum_{x_1=1}^{\infty}\left(\frac{1}{2}\right)^{x_1}\cdot\left(\frac{1}{2}\right)^{x_1}(x_1+2) = \sum_{x_1=1}^{\infty}\left(\frac{1}{4}\right)^{x_1}\cdot x_1 + 2\cdot\sum_{x_1=1}^{\infty}\left(\frac{1}{4}\right)^{x_1} =$$

$$= \frac{\frac{1}{4}}{\left(\frac{3}{4}\right)^2} + 2\cdot\frac{\frac{1}{\cancel{4}}}{\frac{3}{\cancel{4}}} = \frac{\frac{1}{\cancel{4}}}{\frac{9}{\cancel{16}\,4}} + \frac{2}{3} = \frac{4}{9} + \frac{2}{3} = \frac{4+6}{9} = \frac{10}{9}$$

$$H(S_1,S_2) = H(S_1) + H(S_2|S_1) = 2 + \frac{10}{9} = \frac{28}{9}$$

- $$H(S_1 S_2 S_3) = H(S_1 S_2) + \underbrace{H(S_3|S_1 S_2)}_{\text{NE ZAVISI OD } S_1} = H(S_1 S_2) + H(S_3|S_2)$$

$$\boxed{S_3 = S_2 + X_3}$$

$$H(S_3|S_2) = \sum_{x_3=1}^{\infty}\sum_{s_2=1}^{\infty} P(S_2=s_2)\cdot H(S_3|S_2=s_2) = \sum_{s_2=1}^{\infty}\left(\frac{1}{2}\right)^{s_2}\underbrace{\sum_{x_3=1}^{\infty} H(S_3|S_2=s_2)}_{①}$$

$$① = \sum_{x_3=1}^{\infty} H(S_3|S_2=s_2) = \sum_{x_3=1}^{\infty}\left(\frac{1}{2}\right)^{s_2+x_3}(s_2+x_3) = \left(\frac{1}{2}\right)^{s_2}(s_2+2)$$

$$H(S_3|S_2) = \sum_{s_2=1}^{\infty}\left(\frac{1}{2}\right)^{s_2}\left(\frac{1}{2}\right)^{s_2}(s_2+2) = \frac{10}{9}$$

$$H(S_1 S_2 S_3) = H(S_1 S_2) + H(S_3|S_2) = H(S_1) + H(S_2|S_1) + H(S_3|S_2)$$

$$= 2 + \underbrace{\frac{10}{9} + \frac{10}{9}}_{2=n-1} = \frac{28}{9} + \frac{10}{9} = \frac{38}{9} = 2 + \underbrace{2}_{n-1}\cdot\frac{10}{9}$$

- VO GENERALEN SLUČAJ:

$$H(S_1 S_2 \ldots S_n) = H(S_1 S_2 \ldots S_{n-1}) + H(S_n|S_{n-1}\ldots S_1) =$$

$$= H(S_1^{n-1}) + H(S_n|S_{n-1}) = 2 + (n-1)\frac{10}{9} + \frac{10}{9} = 2 + \frac{10n}{9}$$

(c) $$\boxed{H(X) = \lim_{n\to\infty}\frac{1}{n}\left(2 + \frac{10n}{9}\right) = \frac{10}{9}}$$

- EDITION 2 SOLUTIONS:

$$x_1 x_2 \ldots x_n$$

- STATIONARITY:

$$P(x_1 x_2 x_3) = P(x_2 x_3 x_4) \quad ; \quad P(x_1 x_2) = P(x_2 x_3)$$

$$\boxed{P(x_1 x_2) = P(\cancel{x_1}) + P(x_2|x_1)} \qquad \boxed{P(x_2,x_3) = P(x_2) + P(x_3|x_2)}$$

$$= P(x_2) + P(x_1|x_2)$$

- FOR THE PROCESS TO BE STATIONARY THE DISTRIBUTION MUST BE TIME INVARIANT

(a) S_0 IS ALWAYS 0. WHILE S_i, $i \neq 0$ CAN TAKE ON SEVERAL VALUES. SINCE THE MARGINALS FOR S_0 AND S_1, FOR EXAMPLE ARE NOT THE SAME, THE PROCESS CAN'T

BE STATIONARY.

(b) It is clear that the variance of S_n grows with "n" which again implies that the marginals are not time invariant.

$\{S_n\}$ is an INDEPENDENT INCREMENT PROCESS.

An independent increment process is not stationary (not even wide sence stationary) since:

$$\mathrm{var}\{S_n\} = \sum_n (s_n-\mu)^2 p(s_n) = \sum_{s_{n-1},x_n} (s_{n-1}+x_n-\mu)^2 p(s_n,x_n) =$$

$$= \sum_n (s_n-\mu)^2 p(s_n) = \sum s_n^2 p(s_n) - 2\mu \sum_n s_n p(s_n) + \mu^2 \sum_n p(s_n) =$$

$$= E[s_n^2] - 2\mu^2 + \mu^2 = \underline{E[s_n^2]} - \mu^2 \qquad \overbrace{\qquad}^{E[s_{n-1}^2]}$$

$$E[s_n^2] = \sum_{s_{n-1}x_n} (s_{n-1}+x_n)^2 p(s_{n-1}x_n) = \sum_{s_{n-1}x_n} s_{n-1}^2 p(s_{n-1},x_n)$$

$$+ 2 \underbrace{\sum_{s_{n-1}x_n} s_{n-1}x_n\, p(s_{n-1}x_n)}_{0 \text{ (STATISTIČKI NEOVISNI)}} + \underbrace{\sum_{s_{n-1}x_n} x_n^2 p(s_{n-1}x_n)}_{E[x_n^2]}$$

$$\mathrm{var}\{S_n\} = E[s_{n-1}^2] - \overbrace{E^2[s_n]}^{=\mu} + E[x_n^2]$$

$$p(x_n) = \left(\frac{1}{2}\right)^n \qquad E[x_n] = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \cdot n = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2}$$

$$\boxed{E[x_n] = 2} \qquad \mathrm{var}\{S_n\} = \mathrm{var}\{S_{n-1}\} + \mathrm{VAR}\{x_n\}$$

$$\geq \mathrm{var}\{S_{n-1}\}$$

(d) $$H(S_1S_2\dots S_n) = H(S_1) + \sum_{i=2}^{n} H(S_i \mid S_1^{i-1}) = H(S_1) + \sum_{i=2} H(S_i \mid S_{i-1})$$

$$H(S_1S_2\dots S_n) = H(X_1) + \sum_{i=2}^{n} H(S_i) = H(X_1) + (n-1)H(X_1)$$

$$H(X_1) = \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i \mathrm{ld}(2^i) = \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i \cdot i = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = 2$$

$$H(S_1^n) = n \cdot H(X_1) = n \cdot 2 = 2n$$

(e) $$H(X) = \lim_{n\to\infty} \frac{1}{n} H(S_1^n) = \lim_{n\to\infty} \frac{1}{n}(2n) = 2$$

$$X = [x_1\, x_2 \dots x_n] \qquad p_n(X) = \left[\frac{1}{n}, \frac{1}{n} \dots \frac{1}{n}\right]$$

$$E[X] = \sum_{i=1}^{n} x_i \cdot \frac{1}{n} = \frac{1}{n} \sum_{i=1}^{n} x_i \qquad x \in [0..1]$$

$n = 100$

$\boxed{\text{ZEMAT ŠO ACE } \sigma_x^2}$

$$X = [0.01, 0.02, \dots 1]$$

$$Y = [0 \dots 2\pi] \qquad P(Y) = \left[\frac{1}{|Y|}, \frac{1}{|Y|} \dots \frac{1}{|Y|}\right]$$

$$\bar{Y} = \frac{1}{|Y|} \cdot \sum Y$$

- STATIONARY PROCESS:

$X_t = \cos(t + T)$ for $t \in \mathbb{R}$ $\boxed{\{X_t\} \text{ is strictly stationary}}$

$S_n = X_n + S_{n-1}$

(f) WHAT IS THE EXPECTED NUMBER OF FAIR COIN FLIPS TO GENERATE VARIABLE THAT IS HAVING SAME DISTRIBUTION AS S_n ?

$\boxed{\text{LG OPTIMUS L3E400}}$

$\boxed{\text{ZTE KISS +}}$

$$\underbrace{X_1 X_2 X_3 X_4}_{4} \underbrace{X_5 X_6 X_7 X_8}_{4+1} X_9$$

$X_4 \downarrow$ HEAD ✓ $X_7 \downarrow$ HEAD #

PODOBRA E NOMENKLATURATA OD ~~ST.~~ 88 $\boxed{S_2=5}$ ~~$\boxed{S_4=K}$~~

$X_1 X_2 X_3 X_4 X_5$

$X_2 \downarrow$ HEAD $X_5 \downarrow$ HEAD

$$\boxed{P(X_5 = S_2)} = P(X_1=1) \cdot P(X_5=1 \mid X_1=1)$$
$$+ P(X_2=1) \cdot P(X_5=1 \mid X_2=1) + P(X_3=1) \cdot P(X_5=1 \mid X_3=1) +$$
$$+ P(X_4=1) \cdot P(X_5=1 \mid X_4=1) = \frac{1}{2} \cdot \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^2$$
$$+ \left(\frac{1}{2}\right)^4 \cdot \frac{1}{2} = 4 \cdot \left(\frac{1}{2}\right)^5 = \frac{4}{32}$$

$$\underline{P(X_n = S_n)} = \frac{1}{2} \cdot \left(\frac{1}{2}\right)^{n-1} + \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^{n-2} + \dots + \left(\frac{1}{2}\right)^{n-1} \cdot \frac{1}{2} = \underline{(n-1) \cdot \frac{1}{2^n}}$$

$$E[N] = \sum_{n=1}^{\infty} n \cdot p(n) = \sum_{n=1}^{\infty} n(n-1)\frac{1}{2^n} = \underbrace{\sum_{n=1}^{\infty} n^2 \frac{1}{2^n}}_{S_1} -$$
$$- \underbrace{\sum_{n=1}^{\infty} n \cdot \left(\frac{1}{2}\right)^n}_{S_2}$$

$$\boxed{S_1 = \sum_{n=1}^{\infty} n^2 \frac{1}{2^n}} \qquad S_2 = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = 2$$

$$\frac{dS_1}{dx} = \sum_{n=1}^{\infty} \left(n^2 x^n\right)' \qquad \int \frac{S_1 dx}{x} = \sum_{n=1}^{\infty} n^2 \int x^{n-1} dx = \sum_{n=1}^{\infty} \frac{n^2}{n} x^n$$

$$\int \frac{S_1 dx}{x} = \sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2} \qquad \frac{S_1}{x} = \left(\frac{x}{(1-x)^2}\right)'$$

$$\frac{S_1}{x} = \frac{(1-x)^2 + x \cdot 2 \cdot (1-x)}{(1-x)^4} = \frac{\cancel{(1-x)}[1-x+2x]}{(1-x)^{\cancel{4}\,3}} = \frac{1+x}{(1-x)^3}$$

$$\boxed{S_1 = \frac{x(1+x)}{(1-x)^3}} \qquad S_1 = \sum_{n=1}^{\infty} n^2 \frac{1}{2^n} = \frac{\frac{1}{2} \cdot \frac{3}{2}}{\frac{1}{2^3}} = \frac{\frac{3}{4}}{\frac{1}{\cancel{8}_2}} = 6$$

$$\boxed{E[N] = S_1 - S_2 = 6 - 2 = 4}$$

- ZA $\boxed{X_4 = S_2}$ $P(X_4 = S_2) = (n-1)\frac{1}{2^4} = \frac{3}{16}$

n S

– OVA E SAMO ZA $\boxed{n=2}$ T.e. DVA „ČEKORI"

(f) [EDITION 2 SOLUTION] THE EXPECTED NUMBER OF TRIALS REQUIRED CAN BE LOWER-BOUNDED BY $H(S_n)$ AND UPPER BOUNDED BY $H(S_n)+2$ (THEOREM 5.12.3, PG 115). S_n HAVE NEGATIVE BINOMIAL DISTRIBUTION

$$\boxed{Pr(S_n=k) = \binom{k-1}{n-1}\left(\frac{1}{2}\right)^k} \quad \text{for } k \geq n \quad ✓$$

$$\left.\begin{aligned} Pr(S_2=5) &= \binom{4}{1}\cdot\left(\frac{1}{2}\right)^5 = \frac{4}{32} \\ Pr(S_2=4) &= \binom{3}{1}\left(\frac{1}{2}\right)^4 = \frac{3}{16} \end{aligned}\right\} \rightarrow$$

SUM ZA POGODOK VEROJATNOSTA (VIDI PP. 87) (NO TOA SAMO ZA $n=2$)

- DOKAZ DEKA VAŽUVA E VEROJATNOSTA ZA $n>2$

$\boxed{n=3}$ $P(X_5=S_3)=?$

$$\boxed{\sum_{i=1}^{3} i = 5}$$

1 1 3	1 2 2
1 3 1	2 1 2
3 1 1	2 2 1

$$\binom{4}{2}=6$$

$x_1\,x_2\,x_3\,x_4\,x_5$

$\boxed{113}$ 131

$$P(X_5=S_3) = P(X_1=1, X_2=1)\cdot P(X_5=1 \mid X_1=1\,X_2=1) + P(X_1=1, X_4=1)\cdot$$
$$P(X_5=1 \mid \underset{\boxed{131}}{X_1=1}\,X_4=1) + P(X_3=1, X_4=1)\cdot\underset{\boxed{311}}{}P(X_5=1 \mid X_3=1\,X_4=1) +$$
$$+ P(X_1=1, X_3=1)\cdot P(X_5=1 \mid X_1=1\,X_3=1) + P(X_2=1, X_3=1)\cdot P(X_5=1 \mid X_2=1\,X_3=1)$$
$$+ P(X_2=1\;X_4=1)\cdot P(X_5=1 \mid X_2=1\,X_4=1)$$

$$P(X_1=1, X_2=1) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2^2}=\frac{1}{4} \qquad P(X_1=1, X_4=1) = \frac{1}{2}\cdot\underbrace{\frac{1}{2}\,\frac{1}{2}\cdot\frac{1}{2}}_{X_2=0\;X_3=0\;X_4=1} = \frac{1}{16}\,\frac{1}{2^4}\;\;{}_{X_5=1}$$

$$P(X_1=1, X_4=1) = \underbrace{\frac{1}{2}}_{X_1=0}\cdot\underbrace{\frac{1}{2}}_{X_2=0}\cdot\underbrace{\frac{1}{2}}_{X_3=1}\cdot\underbrace{\frac{1}{2}}_{X_4=1} = \frac{1}{2^4}=\frac{1}{16} \qquad \underline{P(X_5=1 \mid X_1=1\,X_2=1)} = \underbrace{\frac{1}{2}\,\frac{1}{2}}_{X_3=0\,X_4=0}\,\frac{1}{2}$$

$$P(X_5=S_3) = \frac{1}{2^2}\cdot\frac{1}{2^3} + \frac{1}{2^4}\cdot\frac{1}{2} + \frac{1}{2^4}\cdot\frac{1}{2} + \frac{1}{2^3}\cdot\frac{1}{2^2} + \frac{1}{2^3}\cdot\frac{1}{2^2} + \frac{1}{2^4}\cdot\frac{1}{2}$$

$$P(X_5=S_3) = 6\cdot\frac{1}{2^5} \quad \text{i.e} \quad P(\underset{S_n=k}{S_3=5}) = \binom{k-1}{n-1}\cdot\frac{1}{2^k} = \binom{4}{2}\frac{1}{2^5}$$

$$\binom{4}{2} = \frac{4!}{2!\,2!} = \frac{24}{4} = 6 \;/\!/$$

PROVED !!!

$\boxed{n=4}$ $\boxed{k=5}$ $x_1\,x_2\,x_3\,x_4\,x_5$

$$\left.\begin{matrix} 1\,1\,1\,2 \\ 1\,1\,2\,1 \\ 1\,2\,1\,1 \\ 2\,1\,1\,1 \end{matrix}\right\} \quad \binom{k-1}{n-1} = \binom{4}{3} = \frac{4!}{3!} = 4$$

$$\boxed{E[k] = \sum_{k=1}^{\infty} k\cdot\binom{k-1}{n-1}\left(\frac{1}{2}\right)^k = 2}$$

VIDI MAPLE MultiHopMIMOCapacity1.mw 2.11.10

$$E[S_n] = \sum_{n=1}^{N}\sum_{k=1}^{\infty} k\binom{k-1}{n-1}\left(\frac{1}{2}\right)^k = \sum_{n=1}^{N}\cdot 2 = 2N \quad \boxed{E[S_n]=2n}$$

$$E[(S_n-2n)^2] = \sum_{n=1}^{N}\cdot\underbrace{\sum_{k=1}^{\infty}(k-2n)^2\cdot\binom{k-1}{n-1}\left(\frac{1}{2}\right)^k}_{=2} = \sum_{n=1}^{N} 2 = 2n$$

• АЛТЕРНАТИВНА РЕПРЕЗЕНТАЦИЈА (TOTAL EXPECTATION)

$$E[S_n] = \sum_{S_i=1}^{\infty} P(S_i)\cdot E(S_n \mid S_n=S_i) = \sum_{S_i=1}^{\infty} P(S_i)\cdot \sum_{k=1}^{\infty} k\cdot\binom{k-1}{i-1}\cdot\left(\frac{1}{2}\right)^k$$

$\frac{1}{3}+\frac{3}{3}+\frac{5}{3}=\frac{9}{3}=3$ $\frac{1+2+3+4+5+6}{6}=\frac{21}{6}=\frac{18+3}{6}=3.5$

$$E_x(R)=E_x(R|E)\cdot P(E)+E_x(R|\bar{E})\cdot P(\bar{E}) =$$ / E -EVEN DIE, $\bar{E}$ -ODD DIE /

$$=4\cdot\frac{1}{2}+3\cdot\frac{1}{2}=\frac{7}{2}=3.5$$

• НЕМА ПОТРЕБА ОД TOTAL EXPECTATION. ВО МАКЕ ЌЕ ДОБИВ СИТЕ СРЕДНИ ВРЕДНОСТ ЗА ЗАДЕНО „n":

$$E[S_n] = 2, 4, 8, 10, 12, \dots 20, \dots \Rightarrow$$

$$\Rightarrow \boxed{E[S_n]=2n} \quad \boxed{E[S_4]=8} \quad \boxed{E[S_2]=4}$$

- ДА ГА НАЈДАМ ВЕРОЈАТНОСТА ЗА UNFAIR COIN FLIPS:

$\boxed{\begin{matrix}k=5\\n=3\end{matrix}}$ $P(X_5=S_3)=p^2\cdot(1-p)^2\cdot p\cdot + p(1-p)^2 p\cdot p + (1-p)^2\cdot p\cdot p\cdot p + p(1-p)p(1-p)p$
$+(1-p)\cdot p\cdot p\cdot(1-p)\cdot p + (1-p)\cdot p(1-p)\cdot p\cdot p = \left[p^3\cdot(1-p)^2\right]\cdot 6$

$$P(\underbrace{X_5}_{k}=\underbrace{S_3}_{n}) = \binom{4}{2} p^n\cdot(1-p)^{k-n} = \binom{4}{2} p^3(1-p)^2$$

- ВО ГЕНЕРАЛЕН СЛУЧАЈ: $\boxed{P_r(S_n=k)=\binom{k-1}{n-1}\cdot p^n(1-p)^{k-n}}$

$$E[S_n]=\left[\frac{1}{p}, \frac{2}{p}, \dots \frac{N}{p}\right] \qquad n=1,2,\dots N$$

$$\boxed{E[S_n]=\frac{n}{p} \qquad n=1,2,\dots}$$

$$\boxed{E\left[\left(S_n-\frac{n}{p}\right)^2\right]=var(S_n)=\frac{n(1-p)}{p^2}}$$

(НЕ УСПЕАВ АНАЛИТИЧКИ ДА ГО ДОКАЖАМ) → МАКЕ!!!

→ ДОКАЗ ЗА ОВА СО ПОМОШ НА ИСКОРИСТУВАЊЕТО ОД ТЕОРИЈА НА ИНФОРМАЦИИ:

$$\sum_{k=1}^{\infty}\binom{k-1}{n-1}p^n\cdot(1-p)^{k-n}=\sum_{k=1}^{\infty}\binom{k-1}{n-1}p^n\cdot q^{k-n}=\sum_{i=0}^{\infty}\binom{i}{n-1}p^n\cdot q^{i+1-n}=$$

$i=k-1$, $k=i+1$; $m=n-1$, $n=m+1$

$$=\sum_{i=0}^{\infty}\binom{i}{m}p^{m+1}\cdot q^{i-m}=p\sum_{i=0}^{\infty}\binom{i}{m}p^{m}\cdot q^{i-m}$$

- POVTORUVAME OD TEORIJA NA INFORMACII:

$$(p+q)^n = \sum_{k=0}^{n}\binom{n}{k}p^k\cdot q^{n-k} \qquad (\underset{\uparrow}{p\cdot v}+q)^n = \sum_{k=0}^{n}\binom{n}{k}p^k\cdot q^{n-k}\cdot v^k$$

POMOŠNA prom.

$$(\ )' \qquad n\cdot p(p\cdot v+q)^{n-1} = \sum_{k=0}^{n}\binom{n}{k}p^k\cdot q^{n-k}\cdot k\cdot v^{k-1} \qquad \underline{v=1}$$

$$n\cdot p\,\underbrace{(p+q)}_{1}{}^{n-1} = \sum_{k=0}^{n}k\cdot\underbrace{\binom{n}{k}p^k\cdot q^{n-k}}_{P(n,k)} = E[k]$$

$$\boxed{E[k]=n\cdot p}$$

- VO NAŠIOT SLUČAJ $P(S_n=k) = \binom{k-1}{n-1}p^n\cdot q^{k-n}$

$$\sum_{n=1}^{k}\binom{k-1}{n-1}p^n\cdot q^{k-n} = \left|\begin{matrix}n=m+1\\ m=n-1\\ n=k\\ m=k-1\end{matrix}\right| = \sum_{m=0}^{k-1}\binom{k-1}{m}p^{m-1}\cdot q^{k-m-1}$$

$$= p^{-1}\sum_{m=0}^{k-1}\binom{k-1}{m}p^m\cdot q^{k-1-m} = p^{-1}(p+q)^{k-1}$$

$$(pv+q)^{k-1} = \sum_{m=0}^{k-1}\binom{k-1}{m}p^m\cdot q^{k-1-m}v^m \qquad (\ )'$$

$$(k-1)p(pv+q)^{k-2} = \sum_{m=0}^{k-1}\binom{k-1}{m}p^m\cdot q^{k-1-m}\cdot m\,v^{m-1} \qquad v=1$$

$$(k-1)p\,\underbrace{(p+q)}_{1}{}^{k-2} = \sum_{m=0}^{k-1}\underline{m}\binom{k-1}{m}\cdot p^m\cdot q^{k-1-m} \qquad \left|\begin{matrix}n=m+1\\ m=n-1\\ m=k-1\\ n=k\end{matrix}\right|$$

$$= \sum_{n=1}^{k}(n-1)\binom{k-1}{n-1}p^{n-1}\cdot q^{k-n} = \frac{1}{p}\sum_{n=1}^{k}(n-1)\binom{k-1}{n-1}p^n\cdot q^{k-n} =$$

$$= \frac{1}{p}\sum_{n=1}^{k}n\binom{k-1}{n-1}p^n q^{k-n} - \frac{1}{p}\underbrace{\overbrace{\sum_{n=1}^{k}\binom{k-1}{n-1}p^n q^{k-n}}^{P(S_n=k)}}_{=1} \quad ?$$

$$(k-1)p = \frac{1}{p}E[n] + \frac{1}{p} \qquad (k-1)p - \frac{1}{p} = \frac{1}{p}E[n]$$

$$\boxed{E[n] = (k-1)p^2 - 1}$$

$$\binom{k}{n} = \frac{k!}{(k-n)!\cdot n!} =$$

$$= \frac{(k-1)!}{(k-1-n)!\cdot n!}\cdot\frac{k}{(k-n)} = \frac{(k-1)!}{(k-1-n+1)!\cdot n!}\cdot\frac{k(\cancel{k-1-n+1})}{(k-n)} =$$

$$= \frac{k\cancel{(k-n)}}{\cancel{(k-n)}}\cdot\frac{(k-1)!}{(k-n)!\cdot(n-1)!}\cdot n = \dots k-n \; \frac{(k-1)!}{(k-n)!\,(n-1)!} = k\cdot n\binom{k-1}{n-1}$$

$\binom{k-1}{n-1} = \frac{1}{k \cdot n} \cdot \binom{k}{n}$ $\quad S = \sum_{n=1}^{k} \binom{k-1}{n-1} \gamma^n \cdot 2^{k-n} = \sum_{n=1}^{k} \frac{1}{k \cdot n} \binom{k}{n} \varphi^n \cdot 2^{k-n}$

$S = \frac{1}{k} \sum_{n=1}^{k} \frac{1}{n} \binom{k}{n} \gamma^n \cdot 2^{k-n}$ $\quad \frac{1}{k} \sum_{n=1}^{k} \frac{1}{n} \binom{k}{n} \gamma^n \cdot 2^{k-n} \cdot \sigma^n$?

- NEMAM VIŠKA GOLMKE IZVEDIVANJA! (VIDI WIKIPEDIA ARTICLE)

- CONTINUE FROM EDITION 1 SOLUTION (Iz KONSTAT TEOREMA 5.11.3 str 139)

$p_k = Pr(S_n = k + E[S_n]) = Pr(S_n = k + 2n)$.

- LET $\phi(x)$ BE NORMAL DENSITY FUNCTION WITH MEAN ~~ZERO~~ AND VARIANCE $2n$ i.e.

$\phi(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\frac{x^2}{2\sigma^2}}$ $\quad \sigma^2 = 2n$ $\quad \boxed{ld\,x = \frac{\ln x}{\ln 2}}$

- THEN FOR LARGE "n" SINCE THE ENTROPY IS INVARIANT UNDER ANY CONSTANT SHIFT OF RANDOM VARIABLES, AND $\phi(x)\, ld\, \phi(x)$ IS Riemann integrable

$$H(S_n) = H(S_n - E(S_n)) = -\sum p_k\, ld\, p_k \approx -\sum \phi(k)\, ld\, \phi(k) \approx$$

$$\approx -\int \phi(x)\, ld\, \phi(x)\, dx = \frac{-1}{\ln 2} \int \phi(x) \left[\ln \frac{1}{\sigma\sqrt{2\pi}} - \frac{x^2}{2\sigma^2} \right] dx \underset{ld\,e}{=}$$

$$= \frac{1}{\ln 2} \int \phi(x) \left[\ln[\sigma\sqrt{2\pi}] + \frac{x^2}{2\sigma^2} \right] dx = ld(e) \left[\int \phi(x)\, dx\, \ln(\sigma\sqrt{2\pi}) + \int x^2 \phi(x) dx \right.$$

$ld\,x = y \quad 2^y = x \;/\, \ln() \qquad y \ln 2 = \ln x \quad y = \frac{\ln x}{\ln 2}$

$\boxed{ld\,x = \frac{\ln x}{\ln 2}}$ $\qquad \ln x = y \qquad e^y = x \;/\, ld()$

$y\, ld(e) = ld(x) \qquad y = \frac{ld(x)}{ld(e)} \qquad \ln(x) = \frac{ld(x)}{ld(e)}$

$\boxed{ld(x) = \ln(x) \cdot ld(e)}$

$$H(S_n) = ld(e) \ln(\sqrt{2\pi\sigma^2}) + \frac{ld\,e}{2\sigma^2} \underbrace{\int \frac{x^2}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx}_{\sigma^2} = ld(e) \left[\frac{1}{2} \ln 2\pi\sigma^2 + \frac{1}{2} \right]$$

$$H(S_n) = ld(e) \cdot \left[\frac{1}{2} \ln(2\pi\sigma^2) \cdot \frac{1}{2} \ln e \right] = ld(e)\, \frac{1}{2} \ln 2\pi e \sigma^2$$

$\boxed{H(S_n) = \frac{1}{2}\, ld(2\pi e \sigma^2) \quad \sigma^2 = 2 \cdot n}$ $\qquad \boxed{H(S_n) = \frac{1}{2} + \frac{1}{2}\, ld(\pi e \sigma^2)}$

- FOR EXAMPLE $\boxed{n = 100}$ $\qquad Pr(S_{100} = k) = \binom{k-1}{99} \left(\frac{1}{2}\right)^k$

- APPROXIMATION:

$H(S_{100}) \doteq \frac{1}{2}\, ld(4\pi e n) = \frac{1}{2}\, ld\, 4 + \frac{1}{2}\, ld(\pi e n) = 1 + \frac{1}{2}\, ld(\pi e n) = 5.86902$

$H(S_{100}) = -\sum_{k=100}^{\infty} \binom{k-1}{99} \frac{1}{2^k} \cdot ld\left[\binom{k-1}{99} \frac{1}{2^k}\right] = 5.86359$

$\boxed{H(S_n) \leq E[S_n] < H(S_n) + 2}$

$\boxed{4.30}$ MARKOV CHAIN TRANSITIONS

$$P = [P_{ij}] = \begin{array}{c|ccc} & 0 & 1 & 2 \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 1 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 2 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{array}$$

LET X_1 BE DISTRIBUTED UNIFORMLY OVER THE STATES $\{0,1,2\}$. LET $\{X_i\}_1^\infty$ BE A MARKOV CHAIN WITH TRANSITION MATRIX P; THUS $P(X_{n+1}=j \mid X_n=i) = P_{ij}$ $i,j \in [0,1,2]$

(a) IS $\{X_n\}$ STATIONARY ?

(b) FIND $\lim_{n\to\infty} \frac{1}{n} H(X_1, \ldots X_n)$

NOW CONSIDER DERIVED PROCESS $Z_1, Z_2, \ldots Z_n$ WHERE $Z_1 = X_1$ $Z_i = X_i - X_{i-1} \pmod 3$ $i = 2, \ldots, n$

THUS, Z^n ENCODES THE TRANSITIONS, NOT THE STATES.

(c) FIND $H(Z_1, Z_2, \ldots Z_n)$.

(d) FIND $H(Z_n)$ AND $H(X_n)$ FOR $n \geq 2$

(e) FIND $H(Z_n \mid Z_{n-1})$ FOR $n \geq 2$

(f) ARE Z_{n-1} AND Z_n INDEPENDENT FOR $n \geq 2$?

(a) $$\overbrace{[\mu_1\ \mu_2\ \mu_3]}^{M_1} = \overbrace{[\mu_1\ \mu_2\ \mu_3]}^{M_1} \cdot \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.25 & 0.5 & 0.25 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$$

$\mu_1 = 0.5\mu_1 + 0.25\mu_2 + 0.25\mu_3$
$\mu_2 = 0.25\mu_1 + 0.5\mu_2 + 0.25\mu_3$
$\mu_3 = 0.25\mu_1 + 0.25\mu_2 + 0.5\mu_3$

$$\begin{cases} -0.5\mu_1 + 0.25\mu_2 + 0.25\mu_3 = 0 \\ 0.25\mu_1 - 0.5\mu_2 + 0.25\mu_3 = 0 \\ 0.25\mu_1 + 0.25\mu_2 - 0.5\mu_3 = 0 \\ \mu_1 + \mu_2 + \mu_3 = 1 \end{cases}$$

- SUŠTINSKO ZA KREIRANJE NA STATIONARNA DISTRIBUCIJA:

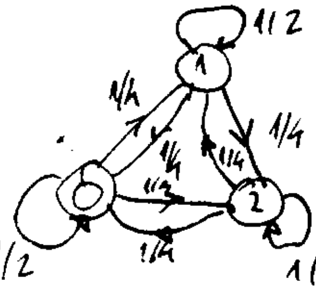
MMV $\mu_1 = \frac{1}{3}$ $\mu_2 = \frac{1}{3}$ $\mu_3 = \frac{1}{3}$

$\boxed{M_n - \text{PROBABILITY MASS FUNCTION AT TIME } n}$ MMV

TPF NE SE MENUVA SO TEK NA VREME ⇒ STATIONARNA $\boxed{M_1 = M_2 = \ldots M_n}$

(b) $$\lim_{n\to\infty} \frac{1}{n} H(X_1, \ldots X_n) = \lim_{n\to\infty} H(X_n \mid X_{n-1} \ldots X_1) = H(X_n \mid X_{n-1}) = H(X_2 \mid X_1)$$

$H(X_2 \mid X_1) = P(X_1 = 0) H(X_2 \mid X_1 = 0) + P(X_1 = 1) \cdot H(X_2 \mid X_1 = 1) + P(X_1 = 2) \cdot H(X_2 \mid X_1 = 2) = \frac{1}{3}\left[H(X_2 \mid X_1 = 0) + H(X_2 \mid X_1 = 1) + H(X_3 \mid X_1 = 2)\right]$



$H(X_2 \mid X_1 = 0) = \frac{1}{2} ld 2 + \frac{1}{4} ld 4 + \frac{1}{4} ld 4 = 3/2$

(VOLONATA OD P_{ij} KADE $i = 0$ T.E. $X_1 = 0$)

$H(X_2 \mid X_1 = 1) = \frac{1}{4} ld 4 + \frac{1}{2} ld 2 + \frac{1}{4} ld 4 = \frac{1}{4} \cdot 2 + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$

$H(X_2 \mid X_1 = 2) = \frac{1}{4} ld 4 + \frac{1}{4} ld 4 + \frac{1}{2} ld 2 = \frac{3}{2}$

$H(X_2 \mid X_1) = \frac{1}{3}\left[3 \cdot \frac{3}{2}\right] = 3/2 = 1.5$

(c) $\{X_i\} = \{\overset{X_1}{0}, \overset{X_2}{2}, \overset{X_3}{1}, \overset{X_4}{1}, 2, 2, 0 \ldots\}$

$\{Z_i\} = \{0, 2, 2, 0, 1, 0, 1 \ldots\}$

$(-1) \bmod 3 = 2$

$\boxed{\underset{x \; r}{\mathrm{mod}(-1,3)} = x - n \cdot r \qquad n = \mathrm{floor}\left(\frac{x}{r}\right)}$

$n = \left\lfloor \frac{-1}{3} \right\rfloor = \lfloor -0{,}33 \rfloor = -1 \qquad \mathrm{mod}(-1,3) = -1 + 3 = 2$

$\mathrm{mod}(-2,3) = \left/ \left\lfloor \frac{-2}{3} \right\rfloor = -1 \right/ = -2 + 3 = 1$

$$H(Z_1, Z_2, \ldots Z_n) = \sum_{i=1}^{n} H(Z_i \mid Z_1^{i-1}) = H(Z_1) + H(Z_2 \mid Z_1) + \ldots Z_n \mid Z_1^n$$

$$= H(X_1) + \underbrace{H(Z_2 \mid X_1)}_{= H(X_2)} + H(Z_3 \mid Z_2 Z_1) + H(Z_4 \mid Z_1^3) + \ldots H(Z_n \mid Z_1^n) = ⊛$$

$$H(Z_3 \mid Z_2 Z_1) = H(Z_3 \mid Z_2 X_1) = H(Z_3 \mid X_2 X_1) = H(Z_3 \mid X_2) = H(X_3)$$

$$H(Z_4 \mid Z_3 Z_2 Z_1) = H(Z_4 \mid Z_3 \underline{X_2 X_1}) = H(Z_4 \mid X_3 X_2 X_1) = H(Z_4 \mid X_3) = H(X_4)$$

- VO GENERALEN SLUČAJ

$$H(Z_i \mid Z_{i-1}, \underbrace{Z_{i-2}, \ldots Z_1}_{\text{NE ZAVISI OD OVIE ČLENOVI}})$$

$Z_{i-2} = X_{i-2} - X_{i-3}$

$Z_{i-1} = X_{i-1} - X_{i-2} \qquad Z_i = X_i - X_{i-1}$

$H(Z_i \mid Z_{i-1}) = H(Z_i \mid X_{i-1}, X_{i-2})$

Z_i NE ZAVISI OD X_{i-2}

$$\Rightarrow \underline{H(Z_i \mid Z_{i-1})} = H(Z_i \mid X_{i-1}) = \underline{H(X_i)}$$

AKO GO ZNAEŠ X_{i-1}
TOGAŠ $Z_i = X_i - X_{i-1}$
ZAVISI SAMO OD X_i.

$$⊛ = n \cdot H(X_1) = n \cdot \left(\frac{1}{3} \mathrm{ld}\, 3\right) \cdot 3 = \underline{n \cdot \mathrm{ld}\, 3}$$

? OVA VEROJATNO MI E TOČNO IZVEDUVANJE AKO JE ZEME VO RACPVID EDITION 2 SOLU.

(d) $H(Z_n) = ?$ $\qquad H(X_n) = ?$

$Z_n = (X_n - X_{n-1}) \bmod 3$

$X_{n-1} X_n$	Z_n	$P(Z_n)$
0 0	0	1/6
0 1	1	1/12
0 2	2	1/12
1 0	2	1/12
1 1	0	1/6
1 2	1	1/12
2 0	1	1/12
2 1	2	1/12
2 2	0	1/6

$P(Z_n = 1) = 3 \cdot \frac{1}{6} = \frac{1}{2}$

$P(Z_n = 2) = 1 \cdot \frac{1}{6} = \frac{1}{2}$

$P(X_n = 0) = 2 \cdot \frac{1}{6} = \frac{1}{3}$

VIDI ZA TRANSITION MATRIX P_{ij}

$$P(Z_n = 1) = P(X_{n-1} = 0)\, \boxed{P(X_n = 1 \mid X_{n-1} = 0)} + P(X_{n-1} = 1) \cdot P(X_n = 2 \mid X_{n-1} = 1) + P(X_{n-1} = 2) \cdot P(X_n = 0 \mid X_{n-1} = 2)$$

$$\underline{P(Z_n = 1)} = \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{4} = \cancel{3} \cdot \frac{1}{\cancel{3}} \cdot \frac{1}{4} = \boxed{\frac{1}{4}}$$

$$\underline{P(Z_n = 0)} = P(X_{n-1} = 1) \cdot P(X_n = 1 \mid X_{n-1} = 1) + P(X_{n-1} = 2) \cdot P(X_n = 2 \mid X_{n-1} = 2) + P(X_{n-1} = 0) \cdot P(X_n = 0 \mid X_{n-1} = 0) = \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} = \boxed{\frac{1}{2}}$$

$$\underline{P(Z_n = 2)} = P(X_{n-1} = 0) \cdot P(X_n = 2 \mid X_{n-1} = 0) + P(X_{n-1} = 1) \cdot P(X_n = 0 \mid X_{n-1} = 1) + P(X_{n-1} = 2) \cdot P(X_n = 1 \mid X_{n-1} = 2) = \frac{1}{3}\left[\frac{1}{4} + \frac{1}{4} + \frac{1}{4}\right] = \boxed{\frac{1}{4}}$$

$$H(Z_n) = 2\,\frac{1}{4}\,ld 4 + \frac{1}{2}\,ld 2 = \cancel{2}\cdot\frac{1}{\cancel{2}} + \frac{1}{2} = \frac{3}{2}$$

(*) $\boxed{\text{FOR } n\geq 2}$

- $H(X_n) = P(X_{n-1}=0)\cdot H(X_n|X_{n-1}=0) + P(X_{n-1}=1)\cdot H(X_n|X_{n-1}=1) + P(X_{n-1}=2)\cdot H(X_n|X_{n-1}=2) = \mu_1\cdot H(X_n|X_{n-1}=0) + \mu_2\, H(X_n|X_{n-1}=1) + \mu_3\, H(X_n|X_{n-1}=2)$

ANALOGNO 6 !!!

$$\boxed{H(X) = -\sum_{i=1}^{n} \mu_i \cdot \sum_{j=1}^{n} P_{ij}\, ld\, P_{ij}}$$

$$\boxed{H(X) = \mu_1\cdot\sum_{j=1}^{n} P_{1j} + \mu_2\sum_{j=1}^{n} P_{2j} + \mu_3\sum_{i=1}^{n} P_{3j}}$$

$$H(X) = \frac{1}{3}\cdot\left[\frac{1}{2}\,ld 2 + \frac{1}{4}\,ld 4 + \frac{1}{4}\,ld 4\right] + \frac{1}{3}\left[\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right] + \frac{1}{3}\cdot\frac{3}{2}$$

$$H(X) = \cancel{3}\cdot\frac{1}{\cancel{3}}\cdot\frac{3}{2} = \frac{3}{2} = 1.5$$

$$\boxed{H(X_n) = \cancel{3}\cdot\frac{1}{\cancel{3}}\,ld 3 = ld 3}$$

- VO (C) DOKAZAV DEKA $H(Z_i|Z_{i-1}) = H(X_i) \Rightarrow$

$$\underline{H(Z_n|Z_{n-1})} = H(X_n) = 1.5 \overset{(*)}{=} \underline{H(Z_n)}$$

ZNAČI SE DOBIVA $H(Z_n|Z_{n-1}) = H(Z_n) \rightarrow Z_n$ IS INDEPENDENT FROM Z_{n-1}.

$$\begin{array}{c|ccc} & & X_{k+1} & \\ X_k & 0 & 1 & 2 \\ \hline 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 1 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 2 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{array}$$

$\boxed{\text{EDITION 2 SOLUTIONS}}$

$$P(X,Y) = P(X)\cdot P(Y|X)$$

(d)

$$P(X_{k+1}-X_k|X_k) = \frac{P(X_{k+1}-X_k, X_k)}{P(X_k)}$$

$$P(X_{k+1}-X_k|X_k=0) = P(X_{k+1}) = \begin{bmatrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\end{bmatrix}$$

$$P(X_{k+1}-X_k|X_k=1) = P(X_{k+1}) = \begin{bmatrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\end{bmatrix}$$

$X_k=1$	X_{k+1}	0	1	2
	$(X_{k+1}-\cancel{X_k})\bmod 3$	2	0	1
	$P(X_{k+1}-\cancel{X_k})$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

$= P(X_{k+1})$

$X_k=2$	X_{k+1}	0	1	2
	$(X_{k+1}-X_k)\bmod 3$	1	2	0
	$P(X_{k+1}-X_k)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

$$\boxed{P(X_{k+1}-X_k|X_k=2) = \begin{bmatrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\end{bmatrix} = P(\cancel{X}_{k+1})}$$

X_k \ $X_{k+1}-X_k$	0	1	2
0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
2	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

①

$P(Z_{k+1}|Z_k) = P(Z_{k+1}|X_k, t_{k+1}) = P(Z_{k+1}|t_k) = P(\overset{①}{t_{k+1}-t_k}|t_k)$

$= P(t_{k+1}-t_k) = P(Z_{k+1})$ $\boxed{Z_{k+1} \text{ IS INDEPENDENT OF } Z_k!}$

(c) EDITION 2 SOLUTION

SINCE $(X_1, \ldots X_n)$ AND $(Z_1, Z_2 \ldots Z_n)$ ARE ONE TO ONE BY CHAIN RULE OF ENTROPY AND MARKOVITY,

$$H(Z_1, Z_2 \ldots Z_n) = H(X_1 X_2 \ldots X_n) = H(X_1) + \sum_{i=2}^{n} H(X_i|X_1^{i-1})$$

$$= H(X_1) + \sum_{i=2}^{n} H(X_i|X_{i-1}) = H(X_1) + (n-1)\left[\frac{1}{4} ld4 + \frac{1}{4} ld4 + \frac{1}{2} ld2\right]$$

$$= H(X_1) + (n-1) \cdot \left[\frac{1}{2} \cdot 2 + \frac{1}{2}\right] = H(X_1) + (n-1) \frac{3}{2}$$

- ALTERNATIVELY TAKING IN CONSIDERATION (d) & (e):

$$H(Z_1, Z_2, Z_3 \ldots Z_n) = H(Z_1) + H(Z_2) + \cdots + H(Z_n) =$$

$$= H(Z_1) + (n-1) H(Z_2) = 3\left(\frac{1}{3} ld3\right) + (n-1) \cdot \frac{3}{2} = ld3 + (n-1)\frac{3}{2}$$

(e) EDITION 2 SOLUTION

DUE TO SYMMETRY P, $P(Z_n|Z_{n-1}) = P(Z_n)$ FOR $n \geq 2$

HENCE: $\boxed{H(Z_n|Z_{n-1}) = H(Z_n) = 3/2}$

PROBLEM 4.31 MARKOV. LET $\{X_i\} \sim$ Bernoulli(p). CONSIDER ASSOCIATED MARKOV CHAIN $\{Y_i\}_{i=1}^{n}$ WHERE Y_i = (the number of ones in current run of 1's). FOR EXAMPLE, IF $X^n = 101110\ldots$, WE HAVE $Y^n = 101230\ldots$

(a) FIND ENTROPY RATE OF X^n.

(b) FIND ENTROPY RATE OF Y^n

(a) $H(X) = \lim_{n \to \infty} \frac{1}{n} H(X_1, X_2, \ldots X_n) = \lim_{n \to \infty} \frac{1}{\cancel{n}} \cancel{n} H(X_1)$

$$\boxed{H(X) = -p\, ld p - (1-p) ld(1-p) = H(p)}$$

(b) FIND THE ENTROPY RATE OF Y^n

$$H(Y) = \lim_{n \to \infty} H(Y_n|Y_1^{n-1})$$

E.G. $X^6 = 101110$ $P(X_6 = 0) = 1-p$

$Y^6 = 101230$ $P(Y_i = 1) = P(X_{i-1} = 0) \cdot p = \underline{(1-p)p}$

$P(Y_i = 2) = P(X_{i-2} = 0) \cdot P(X_{i-1} = 1) \cdot P(X_i = 2) = (1-p) \cdot p \cdot p$
$= (1-p) \cdot p^2$

$P(Y_i = n) = (1-p) \cdot p^n$

$P(Y_i = n | Y_i = n-1) = p$

$H(Y_1 Y_2) = ?$ $\boxed{p = \frac{1}{2}}$

$P(Y_2|Y_1)$

$Y_1 \backslash Y_2$	0	1	2
0	$\frac{1}{2}$	$\frac{1}{2}$	0
1	$\frac{1}{2}$	0	$\frac{1}{2}$

$P(Y_1 Y_2)$

$Y_1 \backslash Y_2$	0	1	2
0	$\frac{1}{4}$	$\frac{1}{4}$	0
1	$\frac{1}{4}$	0	$\frac{1}{4}$

$H(Y_1 Y_2) = \sum P(Y_1 Y_2) ld P(Y_1 Y_2) = \left(\frac{1}{4} \cdot 2\right) \cdot 4 = 2$

- $H(Z_1 Z_2)$

$P(Z_1 Z_2) = P(Z_1) P(Z_2|Z_1)$

$Z_1 \backslash Z_2$	0	1	2	$P(Z_1)$
0	$(1-p)^2$	$(1-p)p$	0	$(1-p)$
1	$(1-p)p$	0	p^2	p

$P(Z_2|Z_1)$

$Z_1 \backslash Z_2$	0	1	2
0	$(1-p)$	p	0
1	$(1-p)$	0	p

$$\boxed{H(Z_1 Z_2) = -(1-p)^2 \, ld(1-p)^2 - 2(1-p)p \, ld\, p(1-p) - p^2 \cdot ld\, p^2}$$

$P(Z_1, Z_2)$

$Z_1 \backslash Z_2$	0	1	2	3	4	$P(Z_1)$
0	$(1-p)^2$	$(1-p)p$	0	0	0	$1-p$
1	$p(1-p)$	0	p^2	0	0	p
2	$p^2(1-p)$	0	0	p^3	0	p^2
3	$p^3(1-p)$	0	0	0	p^4	p^3

$$(1-p)^2 + (1-p)\cdot p =$$
$$1 - 2p + \cancel{p^2} + p - \cancel{p^2} = 1-p$$

$$H(Z_1 Z_2) = -(1-p)^2 \, ld(1-p)^2 - 2p(1-p) \, ld\, p(1-p) - p^2 \, ld\, p^2 - p^2(1-p) \, ld\, p^2(1-p) - p^3 \, ld\, p^3 - p^3(1-p) \, ld\, p^3(1-p) + p^4$$

$P_{ij} = P(Z_2|Z_1)$

$Z_1 \backslash Z_2$	0	1	2	3	4
0	$(1-p)$.	p			
1	$(1-p)$	0	p	0	0
2	$(1-p)$	0	0	p	0
3	$(1-p)$	0	0	0	p

$$H(X) = -\sum_{i=1}^{n} \mu_i \sum_{j=1}^{n} P_{ij} \, ld\, P_{ij}$$

$$\Longrightarrow H(X) = \sum_{i=1}^{n} \mu_i \left[-(1-p)\, ld(1-p) - p \, ld\, p\right]$$

$$\boxed{H(X) = \sum_{i=1}^{n} H(p) \cdot \mu_i}$$

$$\boxed{H(X) = H(p) \sum_{i=1}^{n} \mu_i = H(p) \underbrace{\sum_{i=1}^{\infty} \mu_i}_{1} = H(p)}$$

- SOLUTIONS (EDITION 2): X^n AND Y^n HAVE ONE-TO-ONE MAPPING. THUS $H(Y) = H(X) = H(q)$

PROBLEM 4.32 TIME SYMMETRY. LET $\{X_n\}$ BE STATIONARY MARKOV PROCESS. WE CONDITION ON (X_0, X_1) AND LOOK INTO THE PAST AND FUTURE. FOR WHAT INDEX k IS: $H(X_{-n}|X_0 X_1) = H(X_k|X_0 X_1)$

STATIONARITY: $H(X_0 X_1 \ldots X_n) = H(X_{-n}, X_{-n+1}, \ldots X_0)$

$$H(X_0^n) = H(X_0) + \sum_{i=1}^{n} H(X_i | X_1^{i-1}) = \left| \begin{matrix}\text{SINGLE}\\ \text{ORDER}\\ \text{MC}\end{matrix} \right| = H(X_0) + H(X_1|X_0) + H(X_2|X_1) + \ldots + H(X_n|X_{n-1})$$

$$H(X_{-n}|X_0 X_1) = H(X_0|X_n X_{n+1})$$

$k = n$.. $H(X_k|X_0 X_1) = H(X_n|X_0 X_1) = H(X_0|X_n X_{-n+1})$

$k = -n$ $H(X_k|X_0 X_1) = H(X_{-n}|X_0 X_1) = H(X_0|X_n X_{n+1})$

$$I(X_0X_1;X_k) = I(X_0;X_k) + I(X_1;X_k|X_0) = H(X_k) - H(X_k|X_0X_1)$$

$$H(X_{-n}|X_0X_1) \qquad H(X_0X_1X_{-n}) = H(X_0X_1) + \underline{H(X_{-n}|X_0X_1)}$$

$$H(X_{-n}, X_{-n+1}, \dots X_0X_1) = H(X_1X_0X_{-1} \dots X_{-n}) = H(X_{-n}) +$$
$$H(X_{-n+1}|X_{-n}) + H(X_{-n+2}|X_{-n}, X_{-n+1}) + \dots + \underline{H(X_1|X_0X_{-1}\dots X_{-n})}$$

$$= H(X_{-n}) + H(X_{-n+1}|X_{-n}) + H(X_{-n+2}|X_{-n+1}) + \dots + H(X_1|X_0) =$$
$$= H(X_{-n}) + H(X_1|X_0) + H(X_1|X_0) + \dots + H(X_1|X_0) =$$
$$= H(X_{-n}) + (n+1)H(X_1|X_0)$$

$0, 1, 2 \dots 10 \Rightarrow 10 - 0 + 1 = 11$ elem

$\boxed{-n+1, -n+2, \dots 0, 1}$ $\qquad 1 + n ~~+1~~ + ~~1~~ = n+1$

$$H(X_{-n}^{1}) = H(X_{-n}) + nH(X_1|X_0) + H(X_1|X_0) = H(X_{-n}) + H(X_1|X_0) + \underline{nH(X_1|X_0)}$$

$$H(X_{-n}^{1}) = H(X_{-n}^{-1}) + H(X_0X_1|X_{-n}^{-1}) = \underline{H(X_0X_1)} + H(X_{-n}^{-1}|X_0X_1)$$

$$H(X_{-n}^{-1}|X_0X_1) = H(X_{-n}|X_0X_1) + H(X_{-n+1}|X_0X_1X_{-n}) + H(X_{-n+2}|X_0X_1X_{-n}X_{-n+1})$$
$$+ H(X_{-1}|X_0X_1X_{-n}X_{-n+1}\dots X_{-2}) = H(X_{-n}|X_0X_1) + H(X_{-n+1}|X_0X_1X_{-n})$$
$$+ H(X_{-n+2}|X_0X_1X_{-n+1}) + \dots + H(X_{-1}|X_0X_1X_{-2})$$

$$H(X_{-n}, X_{-n+1}, \dots, X_0X_1) = \underline{H(X_0X_1 \dots X_nX_{n+1})} = H(X_0) + \underline{H(X_1|X_0)(n+1)}$$

$$H(X_{-n}) + (n+1)\cancel{H(X_1|X_0)} = H(X_0) + (n+1)\cancel{H(X_1|X_0)}$$

$$\boxed{H(X_{-n}) = H(X_0)} \Rightarrow \boxed{H(X_0) = H(X_n)}$$

⊛ $H(X_0X_1X_{-n}) = H(X_0X_1) + H(X_{-n}|X_0X_1) = \underline{H(X_{-n})} + H(X_0X_1|X_{-n})$

$$H(X_0X_1) = H(X_0) + H(X_1|X_0)$$
$$\cancel{H(X_0)} + H(X_1|X_0) + H(X_{-n}|X_0X_1) = \cancel{H(X_{-n})} + \underline{H(X_0X_1|X_{-n})}$$

$$\cancel{H(X_1|X_0)} + H(X_{-n}|X_0X_1) = H(X_0X_1|X_{-n}) = H(X_0|X_{-n}) + H(X_1|X_0X_{-n})$$
$$= H(X_0) + \cancel{H(X_1|X_0)}$$

$$\boxed{H(X_{-n}|X_0X_1) = H(X_0)}$$

$$H(X_k|X_1X_0)$$

⊛ $\rightarrow$ $H(X_{-n}|X_0X_1) = \underline{H(X_n)}$ $\qquad H(X_k|X_1X_0) = |k=n| = H(X_n|X_1X_0)$
$$= \underline{H(X_n)}$$

$H(X_{-n}|X_0X_1) = H(X_k/X_1X_0)$ $\qquad \boxed{\text{FOR} \quad k = n}$

- VAŽI, ZA SEKOE $k \leq -1$ i.e $k = -1, -2, \dots -n$

ZATOA ŠTO $H(X_k|X_1X_0) = H(X_0)$ FOR $k = -1, -2, \dots -n$

EDITION 2 SOLUTION

$\underline{H(X_{-n}|X_0X_1)} = H(X_0X_1X_{-n}) - H(X_0X_1) = H(X_{-n}) + H(X_0X_1|X_{-n})$

$= H(X_{-n}) + H(X_0|X_{-n}) + H(X_1|X_{-n}X_0) - H(X_0) - H(X_1|X_0)$ $^{-H(X_0X_1)}$

$= \cancel{H(X_0)} + H(X_n|X_0) + \cancel{H(X_1|X_0)} - \cancel{H(X_0)} - \cancel{H(X_1|X_0)}$

$H(X_{-n}) = H(X_0)$ DUE TO STATIONARITY

$H(X_1|X_{-n}X_0) = H(X_1|X_0)$ DUE TO MARKOVITY

$= H(X_n|X_0) = H(X_n|X_0X_{-1}) = H(X_{n+1}|X_1X_0)$

$\boxed{K = n+1}$ I.E. VAZI ZA $\boxed{K \in (-n, n+1)}$

ISTO SO PRETHODNOTO IZVEDUVANJE POKAZAN DEKA VAZI ZA $\boxed{K = n}$ I ZA $\boxed{K \in (-n, -1)}$

PROBLEM 4.33 Chain inequality. LET $X_1 \to X_2 \to X_3 \to X_4$ FORM A MARKOV CHAIN.

SHOW THAT: $I(X_1;X_3) + I(X_2;X_4) \leq I(X_1;X_4) + I(X_2;X_3)$

$X \to Y \to Z$ DATA PROCESSING $I(X;Y) \geq I(X;Z)$

$I(X,Y;Z) = I(X;Z) + \underbrace{I(Y;Z|X)}_{\neq 0} = I(Y;Z) + \underbrace{I(X;Z|Y)}_{0}$

$\Rightarrow I(Y;Z) \geq I(X;Z)$

$I(X;Y,Z) = I(X;Y) + \underbrace{I(X;Z|Y)}_{0} = I(X;Z) + \underbrace{I(X;Y|Z)}_{\neq 0}$

$\boxed{I(X;Z) \leq I(X;Y)}$

$I(X_1X_2;X_3X_4) = I(X_1;X_3X_4) + I(X_2;X_3X_4|X_1) =$

$= I(X_1;X_3) + \underbrace{I(X_1;X_4|X_3)}_{0} + ? \underbrace{I(X_2;X_3|X_1)}_{0} + \underbrace{I(X_2;X_4|X_1X_3)}_{0}$ (OVA NE E [illegible]!!!)

$= I(X_2;X_3X_4) + I(X_1;X_3X_4|X_2) = I(X_2;X_4) + I(X_2;X_3|X_4) +$

$+ \underbrace{I(X_1;X_3|X_2)}_{0} + \underbrace{I(X_1;X_4|X_2X_3)}_{0}$

$I(X_2;X_4|X_1X_3) = H(X_3|X_1X_3) - H(X_3|X_1X_3X_4)$

$= H(X_2|X_1X_3) - H(X_2|X_1X_3X_4)$

$I(X_2;X_4|X_1X_3) = H(X_4|X_1X_3) - H(X_4|X_1X_2X_3) =$

$= H(X_4|X_3) - H(X_4|X_3) = 0$

$\boxed{I(X_1;X_3) \geq I(X_2;X_4)}$

$I(X_1X_2;X_3X_4) \equiv I(X_1;X_3X_4) + I(X_2;X_3X_4|X_1) = I(X_1;X_4) + I(X_1;X_3|X_4) +$

$+ I(X_2;X_3|X_1) + \underbrace{I(X_2;X_4|X_1X_3)}_{0} = I(X_2;X_3X_4) + I(X_1;X_3X_4|X_2) =$

$= I(X_2;X_3) + I(X_2;X_4|X_3) + I(X_1;X_3|X_2) +$

$+ \underbrace{I(x_1;x_4|x_2x_3)}_{0} \Rightarrow \boxed{I(x_2;x_3) \geq I(x_1;x_4)}$

$2\, I(x_1;x_3) \qquad \boxed{I(x_1;x_3) \geq I(x_2;x_4)}$

- ANALOGNO & DOBIVA DEKA: $I(x_1;x_3) \geq \underline{I(x_1;x_4)}$

$I(x_2;x_3) \geq I(x_2;x_4)$

$\underline{I(x_1;x_3)} = I(x_2;x_4) + I(x_2;x_3|x_4) = \underline{I(x_2;x_3x_4)}$

$I(x_2;x_3) = I(x_1;x_4) + I(x_1;x_3|x_4) + I(x_2;x_3|x_1)$

$\boxed{I(x_1;x_3) = I(x_2;x_3)}$

$\underline{I(x_1;x_3) + I(x_2;x_4)} = I(x_2;x_3) + I(x_2;x_4) \leq \underline{I(x_2;x_3) + I(x_1;x_3)}$

$I(x_2;x_4) + \underline{I(x_2;x_3|x_4)} = I(x_1;x_3) + I(x_1;x_3|x_4) + \underline{I(x_2;x_3|x_1)}$

$I(x_1;x_2x_3x_4) = \underbrace{I(x_1;x_2)} + \underbrace{I(x_1;x_3|x_2)}_{0} + \underbrace{I(x_1;x_4|x_2x_3)}$

$= I(x_1;x_3) + \underbrace{I(x_1;x_4|x_3)}_{0} + I(x_1;x_2|x_3x_4) =$

$\boxed{I(x_1;x_2) \geq I(x_1;x_3)} \qquad \boxed{I(x_1;x_2) = I(x_1;x_3) + I(x_1;x_2|x_3x_4)}$

$= I(x_1;x_4) + I(x_1;x_2|x_4) + \underbrace{I(x_1;x_3|x_4x_2)}_{0}$

$\boxed{I(x_1;x_2) \geq I(x_1;x_4)} \qquad \boxed{\begin{aligned} I(x_1;x_2) &= I(x_1;x_4) + I(x_1;x_2|x_4) \\ &= I(x_1;x_2x_4) \end{aligned}}$

$I(x_1;x_3) + I(x_2;x_4) = I(x_2;x_3x_4) + I(x_2;x_4) \leq I(x_2;x_3x_4) +$

$I(x_2;x_3) = I(x_2;x_4) + I(x_2;x_3|x_4) + I(x_2;x_3) \leq$

$I(x_1;x_3) + I(x_2;x_3|x_4) + I(x_2;x_3)$

$I(x_2;x_3|x_4) = H(x_2|x_4) - H(x_2|x_3x_4) \leq H(x_2) - H(x_2|x_3) =$

$I(x_2;x_3)$

$I(x_1;x_4) + I(x_2;x_3) = I(x_1;x_2) - I(x_1;x_2|x_4) + I(x_2;x_3) = ①$

$\boxed{I(x_1;x_3) \quad \text{Vs} \quad I(x_2;x_3)} \quad I(x_1;x_3) = I(x_2;x_3)$

$① = I(x_1;x_2) + I(x_1;x_3) - I(x_1;x_2|x_4) = I(x_1;x_3) + I(x_1;x_2) - I(x_1;x_2|x_4)$

$\geq I(x_1;x_3) + \underline{I(x_1;x_4) - I(x_1;x_2|x_4)} = I(x_1;x_3) + H(x_1) - H(x_1|x_4) -$

$- H(x_1|x_4) + H(x_1|x_2x_4)$

$I(x_1;x_3) \geq I(x_2;x_4) \quad I(x_2;x_3) \geq I(x_1;x_4)$ | $\boxed{I(x_1;x_3) \leq I(x_2;x_3)}$

$I(x_1;x_3) \geq I(x_1;x_4) \quad I(x_2;x_3) \geq I(x_2;x_4)$

$I(x_1;x_2) \geq I(x_1;x_3) \quad I(x_1;x_2) \geq I(x_1;x_4)$

$$I(x_1;x_3) + I(x_2;x_4) \leq I(x_1;x_4) + I(x_2;x_3)$$

$$I(x_1,x_2;x_3x_4) = I(x_1;x_3x_4) + I(x_2;x_3x_4|x_1) = \underline{I(x_1;x_3)} + \underbrace{I(x_1;x_4|x_3)}_{0}$$

$$+ I(x_2;x_3|x_1) + \underbrace{I(x_2;x_4|x_1x_3)}_{0} = I(x_1;x_3) + I(x_2;x_4|x_1) \cancel{+ I(x_2;x_3|x_1)}$$

$$= I(x_2;x_3x_4) + I(x_1;x_3x_4|x_2) = \underline{I(x_2;x_3)} + \underbrace{I(x_2;x_4|x_3)}_{0} +$$

$$\underbrace{I(x_1;x_3|x_2)}_{0} + \underbrace{I(x_1;x_4|x_2x_3)}_{0}$$

$$\underline{I(x_1;x_3) + I(x_2;x_3|x_1) = I(x_2;x_3)} \qquad \boxed{I(x_2;x_3) \geq I(x_1;x_3)}$$

$$\boxed{I(x_2;x_3) = I(x_1x_2;x_3)} \qquad \boxed{\text{RUKURUT}}$$

$$I(x_1;x_3) + I(x_2;x_4) \leq I(x_2;x_3) + I(x_2;x_4)$$

$$I(x_1,x_2;x_3x_4) = I(x_1;x_3x_4) + I(x_2;x_3x_4|x_1) = I(x_1;x_4) + I(x_1;x_3|x_4)$$

$$+ I(x_2;x_3|x_1) + \underbrace{I(x_2;x_4|x_3x_1)}_{0} = I(x_2;x_3x_4) + I(x_1;x_3x_4|x_2) =$$

$$= I(x_2;x_4) + I(x_2;x_3|x_4) + \underbrace{I(x_1;x_3|x_2)}_{0} + \underbrace{I(x_1;x_4|x_2x_3)}_{0}$$

$$I(x_1;x_4) + I(x_1;x_3|x_4) + I(x_2;x_3|x_1) = I(x_2;x_4) + I(x_2;x_3|x_4)$$

$$I(x_2;x_4) = I(x_1;x_4) + I(x_1;x_3|x_4) + I(x_2;x_3|x_1) - I(x_2;x_3|x_4)$$

$$\boxed{\begin{aligned} I(x_1;x_3|x_4) &= H(x_1|x_4) - H(x_1|x_3x_4) \\ I(x_2;x_3|x_4) &= H(x_2|x_4) - H(x_2|x_3x_4) \end{aligned} \;\Bigg|\; \begin{aligned} I(x_2;x_3|x_1) &= H(x_2|x_1) \\ &- H(x_2|x_1x_3) \end{aligned}}$$

$$\Rightarrow I(x_2;x_3) + I(x_2;x_4) = I(x_2;x_3) + I(x_1;x_4) + I(x_1;x_3|x_4) + I(x_2;x_3|x_1) - I(x_2;x_3|x_4)$$

$$\leq I(x_2;x_3) + I(x_1;x_4) + I(x_1;x_3|x_4) + I(x_2;x_3|x_1) \leq I(x_2;x_3) +$$

$$+ I(x_1;x_4) + H(x_1|x_4) + H(x_2|x_4)$$

$$I(x_1,x_2;x_4) = I(x_1;x_4) + I(x_2;x_4|x_1) = I(x_2;x_4) + I(x_1;x_4|x_2)$$

$$I(x_1;x_4) = I(x_2;x_4) + I(x_1;x_4|x_2) - I(x_2;x_4|x_1)$$

$$\boxed{I(x_2;x_4) = I(x_1;x_4) + I(x_2;x_4|x_1) - I(x_1;x_4|x_2)}$$

$$\Rightarrow I(x_1;x_3) + I(x_2;x_4) \leq I(x_2;x_3) + I(x_1;x_4) + I(x_2;x_4|x_1) =$$

$$= I(x_2;x_3) + I(x_1x_2;x_4) \qquad I(x_2;x_4|x_1) = \underbrace{I(x_4;x_2|x_1)}_{0}$$

$$= \underline{I(x_2;x_3) + I(x_1;x_4)}$$

$$I(x_3;x_2|x_1) = H(x_2|x_1) - H(x_2|x_1x_4) = \underline{H(x_2|x_1) - H(x_2|x_1) = 0}$$

• UŠTE POKRATOK DOKAZ:

$$I(x_1x_2;x_3x_4) = I(x_2;x_3x_4) + I(x_1;x_3x_4|x_2) = I(x_2;x_3) + \underbrace{I(x_2;x_4|x_3)}_{0} +$$

$$+ \underbrace{I(x_1;x_3|x_2)}_{0} + \underbrace{I(x_1;x_4|x_2x_3)}_{0} = I(x_1;x_3x_4) + I(x_2;x_3x_4|x_1) = I(x_1;x_3) +$$

$$\overset{0}{\cancel{I(x_1;x_4|x_3)}} + \overset{0}{\cancel{I(x_2;x_3|x_1)}} \; (= I(x_3;x_2|x_1)) + \overset{0}{\cancel{I(x_2;x_4|x_3x_1)}} \Rightarrow \boxed{I(x_1;x_3) = I(x_2;x_3)}$$

$- I(x_1x_2;x_3) = I(x_1;x_3) + \underbrace{I(x_2;x_3|x_1)}_{0} = I(x_2;x_3) + \underbrace{I(x_1;x_3|x_2)}_{0}$

$\Rightarrow \boxed{I(x_1;x_3) = I(x_2;x_3)}$ 070300585

$I(x_1,x_2;x_4) = I(x_1;x_4) + \underbrace{I(x_2;x_4|x_1)}_{0} = I(x_2;x_4) + I(x_1;x_4|x_2)$

$I(x_2;x_4) = I(x_1;x_4) - I(x_1;x_4/x_2)$ $\boxed{I(x_2;x_4) \leq I(x_1;x_4)}$

$I(x_1;x_3) + I(x_2;x_4) = I(x_2;x_3) + I(x_2;x_4) \leq I(x_2;x_3) + I(x_1;x_4)$

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IME NA POVIKUVAČA STRANA

$\boxed{I(x_1;x_3) + I(x_2;x_4) \leq I(x_2;x_3) + I(x_1;x_4)}$

070300940 L.DINOV

- EDITION 2 SOLUTIONS:

$I(x_1;x_4) + I(x_2;x_3) - I(x_1;x_3) - I(x_2;x_4) = \cancel{H(x_1)} - H(x_1|x_4) + \cancel{H(x_2)} - H(x_2|x_3)$

$- \cancel{H(x_1)} + H(x_1|x_3) - \cancel{H(x_2)} + H(x_2|x_4) = H(x_1|x_3) - H(x_1|x_4) + H(x_2|x_4) - H(x_2|x_3)$

$/ H(x_1,x_2|x_3) - H(x_2|x_1x_3) = H(x_1|x_3) + \cancel{H(x_2|x_1x_3)} - \cancel{H(x_2|x_1x_3)} /$

$= \underline{H(\cancel{x_1x_2}|x_3)} - H(x_2|x_1x_3) - \underline{H(x_1x_2|x_4)} + H(x_2|x_1x_4) + \underline{H(\cancel{x_1x_2}|x_4)} - H(x_1|x_2x_4)$

$- \underline{H(\cancel{x_1x_2}|x_3)} + H(x_1|x_2x_3) = H(x_1x_2|x_3) - H(\cancel{x_1x_2|x_3}) + H(\cancel{x_1x_2|x_4}) - H(\cancel{x_1x_2|x_4})$

$= -H(x_2|x_1x_3) + H(x_2|x_1x_4) - H(\cancel{x_1}|x_2x_4) + H(x_1|x_2x_3) = ⊛$

$-H(x_2|x_1x_3) + H(x_2|x_1x_4) - H(x_2|x_1x_4) + H(x_2|x_1x_3)) =$

$= -H(x_2|\underline{x_1}x_3x_4) + \underbrace{H(x_2|x_1x_4)} - \underbrace{H(x_2|x_1x_4)} + H(x_2|\underline{x_1}x_3x_4) = (\$)$

ZA DADENO x_1 NE E ZAVISI OD DRUGO

$\boxed{I(x_2;x_3|x_1,x_4) = H(x_2|x_1x_4) - H(x_2|x_1,x_3x_4)}$

$(\$) = 2I(x_2;x_3|x_1x_4) \geq 0 \Rightarrow \boxed{I(x_1;x_4) + I(x_2;x_3) \geq I(x_1;x_3) + I(x_2;x_4)}$

$⊛ = H(x_2|x_1x_4) - H(x_2|x_1x_3) - H(x_1|x_2x_4) + H(x_1|x_2x_3) =$

$= H(\cancel{x_2|x_1}) - H(\cancel{x_2|x_1}) - H(x_1|x_2x_4) + H(x_1|x_2x_3) =$

$= H(x_1|x_2x_3) - H(x_1|x_2x_4) \overset{?}{=} H(x_1|x_2x_3) - H(x_1|x_2x_3x_4) =$

$I(x_1;x_4|x_2x_3) \geq 0$

- USE EDIT3: $I(x_1;x_4) + I(x_2;x_3) - I(x_1;x_3) - I(x_2;x_4) = \cancel{H(x_1)} - H(x_1|x_4) + \cancel{H(x_2)} -$

$H(x_2|x_3) - \cancel{H(x_1)} + H(x_1|x_3) - \cancel{H(x_2)} + H(x_2|x_4) = H(x_1|x_3) - H(x_1|x_4) + H(x_2|x_4) - H(x_2|x_3)$

$= \cancel{H(x_1x_2|x_3)} - H(x_2|x_1x_3) - \cancel{H(x_1x_2|x_4)} + H(x_2|x_4x_1) + H(\cancel{x_1x_2}|x_4) - H(x_1|x_2x_4) +$

$- H(\cancel{x_1x_2}|x_3) + H(x_1|x_2x_3) = -H(x_2|x_1x_3) + H(x_2|x_1x_4) - H(x_1|x_2x_4) + H(x_1|x_2x_3)$

$=$ / VO EDITION 2 TVRDAT $H(x_1|x_2x_4) = H(x_1|x_2x_3)$ DUE TO MARKOVITY / $=$

$= H(x_2|x_1x_4) - H(x_2|x_1x_3) = H(x_2|x_1x_4) - \underbrace{H(x_2|x_1x_3x_4)}_{}=$

x_2 ZAVISI SAMO OD x_1 i.e. MINATOTO

$= I(x_2;x_3|x_1x_4) \geq 0$ -

$\boxed{4.34}$: BROADCAST CHANNEL Let $\boxed{X \to Y \to (Z, W)}$ form a Markov chain [i.e. $p(x,y,z,w) = p(x)\cdot p(y|x)\cdot p(z,w|y)$ for all x,y,z,w]. Show that

$$I(X;Z) + I(X;W) \leq I(X;Y) + I(Z;W)$$

$$I(X;Y) + I(Z;W) - I(X;Z) - I(X;W) = \cancel{H(X)} - \underline{H(X|Y)} + \cancel{H(Z)} - \underline{H(Z|W)}$$
$$- \cancel{H(Z)} + \underline{H(Z|X)} - \cancel{H(X)} + \underline{H(X|W)} = -H(X,Z|Y) + H(Z|XY) -$$
$$- H(Y,Z|W) + H(Y|\underline{ZW}) + H(YZ|X) - H(Y|XZ) + H(XY|W) - H(Y|WX)$$
$$= H(X,Z|Y) + H(Z|Y) - H(Y,Z|W) + H(Y|ZW) + H(YZ|X) - H(Y|X)$$
$$+ H(XY|W) - H(Y|WX)$$

$$I(X,Y;ZW) = I(Y;ZW) + \underbrace{I(X;ZW|Y)}_{0} = I(X;ZW) + I(Y;ZW|X)$$

[illegible]

088402350

Code

$$I(Y;ZW) \geq I(X;ZW)$$
$$I(Y;Z) + \underbrace{I(Y;W|Z)}_{0} \geq I(X;Z) + \underbrace{I(X;W|Z)}_{0}$$
$$I(Y;W|Z) = H(Y|Z) - \underbrace{H(W|YZ)}$$

$$X \to Y \to ZW \quad I(X;YZW) = I(X;Y) + \underbrace{I(X;ZW|Y)}_{0} = I(X;ZW) + I(X;Y|ZW)$$

$$\boxed{I(X;Y) \geq I(X;ZW)}$$ DATA PROCESSING

$$\boxed{I(X;Y) \geq I(X;Z) + I(X;W|Z)}$$

$$I(Z;W) = 0 \quad I(X;W|Z) = I(X;W) \Rightarrow$$
$$I(X;Y) + I(Z;W) \geq I(X;Z) + I(X;W)$$
$$(I(X;W|Z) = H(X|Z) - H(X|\cancel{W}Z) = H(X|Z) - H(X|Z) = 0$$

СЪМНИТЕЛНА АНАЛИЗА !!!

ако го знаеш Z го знаеш W

$$(I(X;W|Z) = H(W|Z) - H(W|XZ) = H(W|Z) - H(W|Z) = 0$$

EDITION 2 SOLUTIONS $\quad \boxed{I(X;Y) \geq I(X;ZW)}$

$$I(X;Y) + I(Z;W) - I(X;Z) - I(X;W) \geq I(X;ZW) + I(Z;W) - I(X;Z) - I(X;W)$$
$$= I(\cancel{X;Z}) + I(X;W|Z) + I(Z;W) - I(\cancel{X;Z}) - I(X;W) = H(X|Z) - H(X|WZ) + H(Z)$$
$$- H(Z|W) - H(X) + H(X|W) = H(ZW) - H(ZW|X) + \cancel{H(W)} - H(W|Z) - H(X) +$$
$$+ H(X|Z) - H(\cancel{W}) + H(W|X) = \cancel{H(ZW)} - H(XZW) + \cancel{H(X)} - H(\cancel{WZ}) + H(Z)$$
$$- H(\cancel{X}) + H(XZ) - H(X) + H(XW) - H(X) = -H(XZW) + H(Z) + H(XZ) + H(XW)$$

$$I(X;ZW) + I(Z;W) - I(X;Z) - I(X;W) = H(ZW) - H(ZW|X) + H(W) - H(W|Z) -$$
$$- H(X|Z) + H(Z|X) - H(X) + H(X|W) = H(\cancel{ZW}) - H(XZW) + \cancel{H(X)} + \cancel{H(W)} - H(\cancel{WZ}) + \cancel{H(Z)}$$
$$- \cancel{H(Z)} + H(XZ) - H(\cancel{X}) - H(X) + H(XW) - H(\cancel{W}) = \underline{-H(XZW) + H(XZ) - H(X) + H(XW)}$$
$$\cancel{H(XZW) = H(XZ) + H(W|XZ)} \qquad H(XW) = H(X) + H(W|X) =$$
$$= -\cancel{H(XZ)} - H(W|XZ) + \cancel{H(XZ)} - \cancel{H(X)} + \cancel{H(X)} + H(W|X) = H(W|X) - H(W|XZ)$$

$= I(W;Z|X) \geq 0 \Rightarrow I(X;Y) + I(Z;W) \geq I(X;Z) + I(X;W)$.

$\boxed{4.35}$ CONCAVITY OF SECOND LAW. LET $\{X_n\}_{-\infty}^{\infty}$ BE STATIONARY MARKOV PROCESS. SHOW THAT $\underline{H(X_n|X_0)}$ IS CONCAVE IN „n". SPECIFICALLY SHOW THAT:

$$H(X_n|X_0) - H(X_{n-1}|X_0) - (H(X_{n-1}|X_0) - H(X_{n-2}|X_0)) = -I(X_1;X_{n-1}|X_0X_n) \leq 0$$

THUS THE SECOND DIFFERENCE IS NEGATIVE, ESTABLISHING THAT $H(X_n|X_0)$ IS CONCAVE FUNCTION OF „n".

$$H(X_1X_2\dots X_n) = H(X_1)+H(X_2|X_1)+H(X_3|X_2X_1)+\dots+H(X_n|X_1^{n-1})$$

$$= H(X_1)+H(X_2|X_1)+H(X_3|X_2)+\dots+H(X_n|X_{n-1}) = H(X_1)+(n-1)H(X_2|X_1)$$

$$H(\mathcal{X}) = \lim_{n\to\infty}\frac{1}{n}H(X_1^n) = H(X_2|X_1)$$

$$H(\dots X_{-n}X_{-n+1}\dots X_0X_1X_2\dots X_n\dots) \overset{?}{=} H(X_{-n}|X_{-n-1})+\dots+H(X_n|X_{n-1})$$

$- (H(X_{n-1}|X_0) - H(X_{n-2}|X_0))$

$$H(X_n|X_0) - H(X_{n-1}|X_0) = H(X_0X_n) - \cancel{H(X_0)} - H(X_0X_{n-1}) + \cancel{H(X_0)} -$$

$$- H(X_0X_{n-1}) + \cancel{H(X_0)} + H(X_0X_{n-2}) - \cancel{H(X_0)} = \underline{H(X_0X_n) - H(X_0X_{n-1})}$$

$$\underline{- H(X_0X_{n-1}) + H(X_1X_{n-1})} = \boxed{H(X_0X_n) - 2H(X_0X_{n-1}) + H(X_1X_{n-1})}$$ ▼

$$I(X_1;X_{n-1}|X_0X_n) = H(X_1|X_0X_n) - H(X_1|X_0X_{n-1}X_n) =$$

$$= H(X_0X_1X_n) - H(X_0X_n) - H(X_0X_1X_{n-1}X_n) + H(X_0X_{n-1}X_n) =$$

$$= H(X_0) + H(X_1X_n|X_0) - H(X_0X_n) - \cancel{H(X_0)} - H(X_1X_{n-1}X_n|X_0) + \cancel{H(X_0)} + H(X_{n-1}X_n|X_0)$$

$$H(X_1X_n|X_0) = H(X_1|X_0) + H(X_n|X_0X_1) = H(X_n|X_0) + \underline{H(X_1|X_0X_n)}$$

$= H(X_1|X_0)$

$$= \underline{H(X_1|X_0) + H(X_n/X_0)}$$

$$= H(X_0) + H(X_1|X_0) + \cancel{H(X_n|X_0)} - H(X_0X_n) - H(X_1|X_0) - H(X_{n-1}X_n|X_0) + \cancel{H(X_{n-1}X_n|X_0)}$$

$$= H(X_0X_1) - \cancel{H(X_0X_n)} = \cancel{H(X_0)} + \underline{H(X_1|X_0)} - \cancel{H(X_0)} - \underline{H(X_n|X_0)}$$

$$= \underline{H(X_1|X_0) - H(X_n/X_0)} \geq 0 \Rightarrow \underline{H(X_n/X_0) \leq H(X_1/X_0)}$$

PECAZ:
→ KONTRADICT

- SECOND LAW OF THERMODYNAMICS:

$H(X_n|X_1)$ INCREASES WITH „n"

$H(X_n|X_1) \geq H(X_n|X_1X_2)$ / ENTROPY DECREASES WITH CONDITIONING / $= H(X_n|X_2)$ ↑ MARKOVITY

$= H(X_{n-1}|X_1)$ $\boxed{H(X_n|X_1) \geq H(X_{n-1}|X_1)}$ HMV ★

- DATA PROCESSING:

$\boxed{I(X_1;X_n) \leq I(X_1;X_{n-1})}$ $\cancel{H(X_n)} - H(X_n|X_1) \leq H(X_{n-1}) - H(X_{n-1}|X_1)$

$H(X_n) = H(X_{n-1}) \Rightarrow$ STATIONARITY $\Rightarrow \underline{H(X_n|X_1) \geq H(X_{n-1}|X_1)}$

KONSTATACIJATA ★ VO OVOJ MOMENT MI DOJDE MNOGU JAKO!!!

(#) $H(X_n|X_0) - H(X_{n-1}|X_0) - H(X_{n-1}|X_0) + H(X_{n-2}|X_0) = H(X_n|X_0) - 2H(X_{n-1}|X_0) + H(X_{n-1}|X_1) = H(X_n|X_0) - H(X_n|X_1) - H(X_{n-1}|X_0) + H(X_{n-1}|X_1) =$

$= H(X_n|X_0) - H(X_n|X_1X_0) - H(X_{n-1}|X_0) + H(X_{n-1}|X_1X_0) =$

$= \boxed{I(X_n;X_1|X_0) - I(X_{n-1};X_1|X_0)} = I(X_0X_n;X_1) - \cancel{I(X_0;X_1)} -$

$- I(X_0X_{n-1};X_1) + \cancel{I(X_0;X_1)} = I(X_0X_n;X_1) - I(X_0X_{n-1};X_1)$

BY DATA PROCESSING INEQUALITY OVA E ≤ 0

$$I(X;YZ) = I(X;Y) + \underbrace{I(X;Z|Y)}_{0} = I(X;Z) + \underbrace{I(X;Y|Z)}_{\geq 0}$$

$$\Rightarrow \boxed{I(X;Y) \geq I(X;Z)}$$

- ZNAČI DA JA DOKAŽAM DEKA ④ :

$$H(X_n|X_0) - H(X_{n-1}|X_0) - (H(X_{n-1}|X_0) - H(X_{n-2}|X_0)) = I(X_n;X_1|X_0) -$$
$$- I(X_{n-1};X_1|X_0) = I(X_1;X_n|X_0) - I(X_1;X_{n-1}|X_0) \leq 0$$

- NE ZNAM KAKO DOKAŽAT DO:

$$I(X_1;X_{n-1}|X_0X_n) = I(X_0X_1X_n;X_{n-1}) - I(X_0X_n;X_{n-1})$$
$$= I(X_0X_{n-1}X_n;X_1) - I(X_0X_n;X_1)$$

$$\left.\begin{array}{l} H(X_n|X_0) - H(X_{n-1}|X_0) \\ -(H(X_{n-1}|X_0) - H(X_{n-2}|X_0)) \end{array}\right\} = H(X_0X_n) - 2H(X_0X_{n-1}) + H(X_1X_{n-1})$$

$$H(X_{n-1}|X_0) = H(X_n|X_1) = H(X_n|X_1X_0)$$
$$H(X_{n-2}|X_0) = H(X_{n-1}|X_1) = H(X_{n-1}|X_0X_1)$$

$$H(X_n|X_0) - H(X_n|X_1X_0) - H(X_{n-1}|X_0) + H(X_{n-1}|X_0X_1) =$$
$$= \underline{I(X_1;X_n|X_0) - I(X_1;X_{n-1}|X_0)} = I(X_0X_n;X_1) - \cancel{I(X_0;X_1)} +$$
$$- I(X_0X_{n-1};X_1) + \cancel{I(X_0;X_1)} = I(X_0X_n;X_1) - I(X_0X_{n-1};X_1) =$$
$$= H(\cancel{X_1}) - H(X_1|X_0X_n) - H(\cancel{X_1}) + H(X_1|X_0X_{n-1}) = \circledast$$
$$= H(X_1|X_0X_{n-1}) - H(X_1|X_0X_n)$$

DO OVDE ŠTO SUM GO REŠIL ~~IZ~~ EDITION 2.

$$I(X_0X_{n-1}X_n;X_1) = I(X_0;X_1) + I(X_{n-1};X_1|X_0) + I(X_0;X_1) =$$
$$= I(X_0X_n;X_1) + \underline{I(X_{n-1};X_1|X_0X_n)} =$$

$$I(X_{n-1};X_1|X_0X_n) = H(X_1|X_0X_n) - H(X_1|X_0X_{n-1}X_n)$$
$$= I(X_0X_{n-1};X_1) + \underline{I(X_n;X_1|X_0X_{n-1})} = I(X_0X_{n-1};X_1) + I(X_1|X_n|X_{n-1})$$
$$I(X_1;X_n|X_0X_{n-1}) = \underline{I(X_1;X_n|X_{n-1})}$$

$$= I(X_0X_{n-1};X_1) + I(X_1;X_nX_{n-1}) - I(X_1;X_{n-1})$$

- EDITION 2 SOLUTION

$$\circledast \rightarrow I(X_0X_n;X_1) - I(X_0X_{n-1};X_1) = H(\cancel{X_1}) - H(X_1|X_0X_n) - H(\cancel{X_1}) + H(X_1|X_0X_{n-1})$$
$$= H(X_1|X_0X_{n-1}) - H(X_1|X_0X_n) = H(X_0X_1X_{n-1}) - H(X_0X_{n-1}) - H(X_1|X_nX_0)$$

$$H(X_1,X_{n-1},X_n,X_0) = H(X_0,X_1,X_{n-1},X_n) = H(X_0) + H(X_1|X_0) + H(X_{n-1}|X_1X_0)$$
$$+ H(X_n|X_{n-1}X_0X_1) = H(X_0) + H(X_1|X_0) + H(X_{n-1}|X_1) + H(X_n|X_{n-1})$$

$$H(X_1|X_0X_{n-1}) - H(X_1|X_0X_n) = H(X_1|X_0X_{n-1}X_n) - H(X_1|X_0X_n) =$$
$$= H(X_1|X_0X_nX_{n-1}) - H(X_1|X_0X_n) = -I(X_1;X_{n-1}|X_0X_n) \leq 0$$

- $\Delta_n = H(X_n|X_0) - H(X_{n-1}|X_0)$ $\qquad \Delta_n - \Delta_{n-1} \leq 0 \Rightarrow$

"$H(X_n|X_0)$ E KONKAVNA FUNKCIJA OD n"

OVA E MALKU VO KONTRADIKCIJA SO 4.4.4 KADE SE TVRDI DEKA $H(X_n|X_1)$ RASTE SO ZGOLEMUVANJE NA n.

CHAPTER 5

5.1 EXAMPLE OF CODES

A SOURCE CODE „C" FOR RANDOM VARIABLE $\mathcal{X}$ IS MAPPING FROM $\mathcal{X}$, THE RANGE OF X TO D^*, THE SET OF FINITE-LENGTH STRINGS OF SYMBOLS FROM A D-ARY ALPHABET. LET $C(x)$ DENOTE THE CODEWORD CORRESPONDING TO x AND LET $l(x)$ DENOTE LENGTH OF $C(x)$. FOR EXAMPLE $C(red)=00$, $c(blue)=11$ IS SOURCE CODE FOR $\mathcal{X}=\{red, blue\}$ WITH ALPHABET $D=\{0,1\}$

DEFINITION THE EXPECTED LENGTH $L(C)$ OF SOURCE CODE $C(x)$ FOR RANDOM VARIABLE $\mathcal{X}$ WITH PROBABILITY MASS FUNCTION $p(x)$ IS GIVEN BY

$$\boxed{L(C) = \sum_{x \in \mathcal{X}} p(x) l(x)}$$

WHERE $l(x)$ IS THE LENGTH OF THE CODEWORD ASSOCIATED WITH „x".

$\mathcal{D} = \{0, 1, 2, \dots, D-1\}$

EXAMPLE 5.1.1 LET X BE A RANDOM VARIABLE WITH FOLLOWING DISTRIBUTION AND CODEWORD ASSIGNMENT:

$\Pr(X=1)=1/2$, CODEWORD $C(1)=0$

$\Pr(X=2)=1/4$, CODEWORD $C(2)=10$

$\Pr(X=3)=1/8$, CODEWORD $C(3)=110$

$\Pr(X=4)=1/8$, CODEWORD $C(4)=111$

$$\boxed{L(C) = E[l(x)]}$$

THE ENTROPY $H(X)$ OF X IS $\frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 2 + \left(\frac{1}{8}\cdot 3\right)2 = \frac{2+2+3}{4} = \frac{7}{4} = 1.75$. $L(C) = \frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 2 + \left(\frac{1}{8}\cdot 3\right)\cdot 2 = \frac{1}{2} + \frac{2}{4} + \frac{6}{8} = \frac{7}{4} = 1.75$

ANY SEQUENCE OF BITS CAN BE UNIQUELY DECODED INTO SEQUENCE OF SYMBOLS OF $\mathcal{X}$. E.G.:

$0110111100110 = 134213$

EXAMPLE 5.1.2 CONSIDER ANOTHER SIMPLE EXAMPLE OF A CODE FOR RANDOM VARIABLE.

$\Pr(X=1) = \frac{1}{3}$ $C(1)=0$ $\Pr(X=3)=1/3$ $C(3)=11$

$\Pr(X=2) = 1/3$ $C(2)=10$

$H(X) = 3 \cdot \frac{1}{3} \operatorname{ld} 3 = \operatorname{ld} 3 = 1.58$ bits

$L(C) = E[l(x)] = \sum_{i=1}^{3} P(x_i) l(x_i) = \sum_{x=1}^{3} p(x)\cdot l(x) = \frac{1}{3}\cdot 1 + \frac{1}{3}\cdot 2 + \frac{1}{3}\cdot 2 = \frac{5}{3}$
$= 1.667$

$E[L(X)] > H(X)$

EXAMPLE 5.1.3 (Morse code)

- WE NOW DEFINE INCREASINGLY MORE STRINGENT CONDITIONS ON CODES. LET x^n DENOTE $(x_1, x_2, \dots x_n)$

DEFINITION: A CODE IS SAID TO BE NONSINGULAR IF EVERY ELEMENT OF THE RANGE OF X MAPS INTO DIFFERENT STRING IN D^*; THAT IS, $x \neq x' \Rightarrow C(x) \neq C(x')$

DEFINITION: THE EXTENSION C^* OF CODE C IS THE MAPPING FINATE-LENGHT STRINGS OF $\mathcal{X}$ TO FINATE-LENGTH STRINGS OF D, DEFINED BY:

$$C(x_1x_2 \cdots x_n) = C(x_1)C(x_2) \cdots C(x_n)$$

$\boxed{3216175}$

WHERE $C(x_1) \cdot C(x_2) \cdots C(x_n)$ INDICATES CONCATENATION OF CORRESPONDING CODEWORDS

$\boxed{\text{EXAMPLE 5.1.4}}$ IF $C(x_1)=00$ AND $C(x_2)=11$, THEN $C(x_1x_2)=0011$

DEFINITION: A CODE IS CALLED UNIQUELY DECODABLE IF ITS EXTENSION IS NONSINGULAR.

DEFINITION: A CODE IS CALLED PREFIX CODE OR AN INSTANTEOUS CODE IF NO CODEWORD IS A PREFIX OF ANY OTHER CODEWORD.

INSTANTENOUS CODE CAN BE DECODET WITHOUT REFERENCE TO FUTURE CODEWORDS SINCE THE END OF THE CODEWORD IS IMEDIATELY RECOGNIZABLE. AN INSTANTENOUS CODE IS SELF-PUNCTUATING CODE.

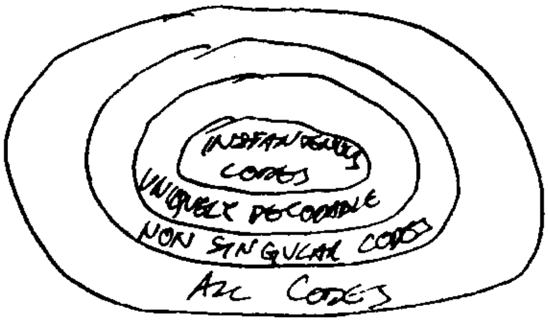


FIGURE 5.1

X	SINGULAR	NON-SINGULAR NOT-UNIQUELY	UNIQUELY NON-INST.	INSTANTENOUS
1	0	0	10	0
2	0	010	00	10
3	0	01	11	110
4	0	10	110	111

⊛

5.2 KRAFT INEQUALITY

$\boxed{\text{THEOREM 5.2.1}}$ (KRAFT INEQUALITY) FOR ANY INSTANTENOUS CODE (PREFIX CODE) OVER AN ALPHABET OF SIZE D, THE CODWORD LENGTHS $l_1, l_2, \ldots, l_m$ MUST SATISFY THE INEQUALITY

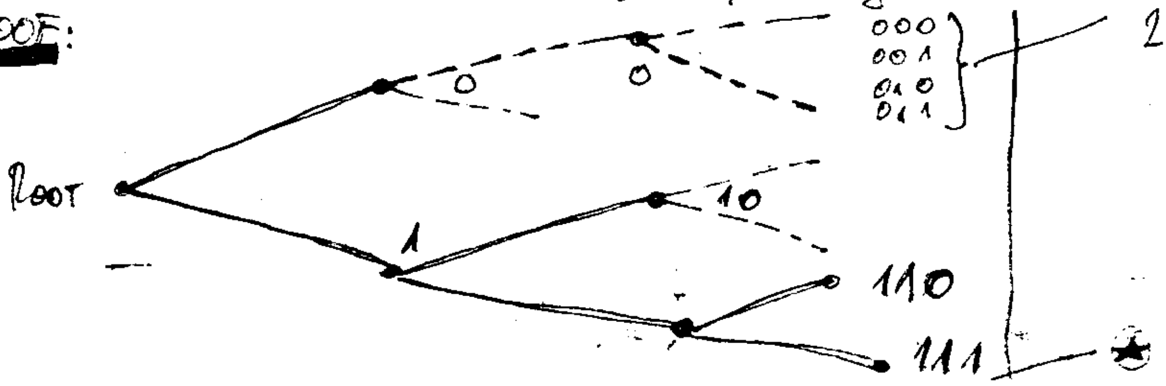
$$\boxed{\sum_i D^{-l_i} \leq 1}$$

CONVERSELY, GIVEN A SET OF CODEWORD LENGTHS THAT SATISFY THIS INEQUALITY, THERE EXIST INSTANTENEOUS CODE WITH THIS LENGTHS.

⊛ → $2^{-1} + 2^{-2} + 2^{-3} + 2^{-3} = \frac{1}{2} + \frac{1}{4} + 2 \cdot \frac{1}{8} = 1$

$2^{l_{max}-l_i} = 2^{3-1} = 2^2 = 4$

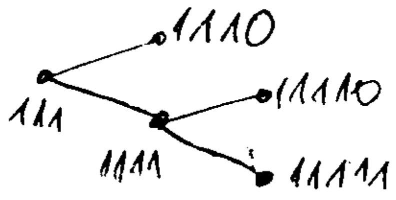
PROOF:



A CODEWORD AT LEVEL l_i HAS $D^{l_{max}-l_i}$ DESCENDANTS AT LEVEL l_{max}.

$$\sum_i D^{l_{max}-l_i} \leq D^{l_{max}} \Rightarrow \boxed{\sum_i D^{-l_i} \leq 1}$$ VIDI PROBLEM 5.2

★ ⇒ MOŽE DA PRODOLŽIŠ IOVATAM NA PRIMER!



$$2^{-1}+2^{-2}+2^{-3}+2^{-4}+2\cdot 2^{-5}=$$
$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+2\cdot\frac{1}{32}=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+2\cdot\frac{1}{16}$$
$$=\frac{1}{2}+\frac{1}{4}+\frac{1}{4}=1$$

- WE NOW SHOW THAT INFINITE PREFIX CODE ALSO SATISFIES THE CRAFT INEQUALITY.

THEOREM 5.2.2 (EXTENDED KRAFT INEQUALITY) FOR ANY COUNTABLY INFINITE SET OF CODEWORDS THAT FORM A PREFIX CODE, THE CODEWORD LENGTHS SATISFY THE EXTENDED CRAFT INEQUALITY.

$$\sum_{i=1}^{\infty} D^{-l_i} \leq 1$$

CONVERSELY, GIVEN ANY $l_1, l_2, \ldots$ SATISFYING THE EXTENDED CRAFT INEQUALITY WE CAN CONSTRUCT A PREFIX CODE WITH THESE CODEWORD LENGTHS.

PROOF: LET THE D-ARY ALPHABET BE $\{0,1,\ldots D-1\}$. CONSIDER THE i-TH CODEWORD $y_1y_2\cdots y_{l_i}$. LET $0.y_1y_2y_3\cdots y_{l_i}$ BE THE REAL NUMBER GIVEN BY THE D-ARY EXPANSION

$$\boxed{0.y_1y_2\cdots y_{l_i}=\sum_{j=1}^{l_i} y_j D^{-j}}$$

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{4+2+1}{8}=\frac{7}{8}$$

$$110 \to 0.110 = \sum_{j=1}^{3} y_j\cdot 2^{-j} = 1\cdot 2^{-1}+1\cdot 2^{-2}+0\cdot 2^{-3}=\frac{1}{2}+\frac{1}{4}=\frac{3}{4}=0.75$$

THIS CODEWORD $C(0.110)\to 0.75$ CORRESPONDS TO THE INTERVAL

$$\left[0.y_1y_2\cdots y_{l_i},\ 0.y_1y_2\cdots y_{l_i}+\frac{1}{D^{l_i}}\right] = \left[0.110,\ 0.110+\underbrace{0.001}_{2^{-3}}\right)$$

$$\boxed{[0.75,\ 0.75+0.125)=[0.75,\ 0.875)}$$

THE SET OF ALL REAL NUMBERS WHOSE D-ARY EXPANSION BEGINS WITH $0.y_1y_2\cdots y_{l_i}$. THIS IS SUBINTERVAL OF THE UNIT INTERVAL $[0,1]$. BY THE PREFIX CONDITION, THIS INTERVALS ARE DISJOINT. HENCE, THE SUM OF THEIR LENGTHS HAS TO BE LESS THEN OR EQUAL TO 1.

FOR EXAMPLE IF WE WISH TO CONSTRUCT A KRAFT CODE WITH $l_1=1$ $l_2=2$, ... WE ASIGN THE INTERVALS: $[0, \frac{1}{2})$, $[\frac{1}{2}, \frac{1}{4})$, $[\frac{1}{4}, \frac{1}{8})$ TO THE SYMBOLS WITH CORRESPONDING CODEWORDS 0, 10, ...

JAS MISLAM DEKA TREBA VAKA DA ODAT INTERVALITE.

$$[0, \tfrac{1}{2}] ; [\tfrac{1}{2}, \underbrace{\tfrac{1}{2}+\tfrac{1}{4}}_{3/4}) ; [\tfrac{3}{4}, \tfrac{3}{4}+\tfrac{1}{8}] ; [\tfrac{7}{8}, \underbrace{\tfrac{7}{8}+\tfrac{1}{16}}_{\boxed{15/16 = 0.9375}}) \cdots$$

0.75 $7/8 = 0.875$

0	10	110	111
$\downarrow$			
0.0	0.10 (0.5)	0.110 (0.75)	0.111 (0.875) ....

5.3 OPTIMAL CODES

OPTIMAL = SHORTEST EXPECTED LENGTH

LETS CONSIDER THE PROBLEM OF FINDING THE SHORTEST INSTANTANEOUS CODE. I.E. THE OPTIMAL CODE.

THIS IS EQUIVALENT TO FINDING THE SET OF LENGTHS $l_1, l_2, \ldots l_m$ SATISFYING THE KRAFT INEQUALITY AND WHOSE EXPECTED LENGTH $L=\sum p_i l_i$ IS LESS THEN EXPECTED LENGTH OF ANY OTHER PREFIX CODE. THIS IS CLASSIC OPTIMIZATION PROBLEM.

MINIMIZE: $\boxed{L = \sum p_i l_i}$

OVER ALL INTEGERS $l_1, l_2, \ldots l_m$ SATISFYING

$$\boxed{\sum D^{-l_i} \le 1}$$

$$\boxed{\begin{matrix} \log_D p_i = x & D^x = p_i \,/\, \lg \\ x \lg D = \lg p_i & x = \frac{\lg p_i}{\lg D} \end{matrix}}$$

- WE NEGLECT INTEGER CONSTRAINT ON l_i AND ASSUME EQUALITY IN THE CONSTRAINT:

$$J = \sum p_i l_i + \underline{\lambda \sum D^{-l_i}}$$

$$\frac{dJ}{dl} = \sum_i p_i + \lambda \sum \frac{d}{dl_i}\left(D^{-l_i-1}\right) = \sum_i p_i - \lambda \cdot \sum_i D^{-l_i} \cdot \ln D$$

$$(2^x)' = \left(e^{x\ln 2}\right)' = e^{x\ln 2} \cdot \underline{\ln 2} = \underline{2^x \cdot \ln 2}$$

$$(D^x)' = \left(e^{x\ln D}\right)' = e^{x\ln D} \cdot \ln D = D^x \cdot \ln D$$

$$(D^{-x})' = \left(e^{-x\ln D}\right)' = e^{-x\ln D} \cdot (-\ln D) = -\underline{D^{-x}\ln D}$$

$$\sum \left(p_i - \lambda D^{-l_i}\ln D\right) = 0 \Rightarrow p_i = \lambda D^{-l_i}\ln D$$

$$\boxed{D^{-l_i} = \frac{p_i}{\lambda \ln D}}$$

$$\sum \frac{p_i}{\lambda \ln D} = 1 \qquad \frac{1}{\lambda \ln D}\underbrace{\sum p_i}_{1} = 1$$

$$\Rightarrow \boxed{\lambda = \frac{1}{\ln D}} \longrightarrow \boxed{p_i = D^{-l_i}} \rightarrow$$

OPTIMAL CODE LENGTHS ARE:

$$\boxed{l^* = -\log_D p_i} \Rightarrow \boxed{L^* = -\sum p_i \log_D p_i = H_D(X)} \boxed{= \frac{H(X)}{\lg D}}$$

NON-INTEGER CHOICE !!!

THEOREM 5.3.1 THE EXPECTED LENGTH „L" OF ANY INSTANTENOUS D-ARY CODE FOR RANDOM VARIABLE X IS GREATER THEN OR EQUAL TO THE ENTROPY $H_D(X)$ i.e.: $\boxed{L \geq H_D(X)}$

WITH EQUALITY IF AND ONLY IF : $\boxed{p_i = D^{-l_i}}$

PROOF: ⊛

$$L - H_D(X) = \underline{\sum p_i l_i - \sum p_i \log_D\left(\frac{1}{p_i}\right)} = -\sum p_i \log_D D^{-l_i} + \sum p_i \log_D(p_i)$$

$$r_i = \frac{D^{-l_i}}{\sum_j D^{-l_j}} \qquad c = \sum D^{-l_i} \qquad r_i = \frac{D^{-l_i}}{c} \qquad D^{-l_i} = c \cdot r_i$$

$$L - H = -\sum p_i \log_D(c \cdot r_i) + \sum p_i \log_D(p_i) =$$

$$= \sum p_i \log_D\left(\frac{p_i}{r_i}\right) + \sum p_i \log \frac{1}{c} = D(p_i \| r_i) + \log_D \frac{1}{c} \geq 0$$

→ $\boxed{L \geq H_D(X)}$ PROVED !!!

$L \geq H$ IF AND ONLY IF $p_i = D^{-l_i}$ $\left(l_i = \log_D\left(\frac{1}{p_i}\right)\right)$

i.e. IF „$-\log_D(p_i)$" IS INTEGER FOR ALL „i"

DEFINITION: D-ADIC DISTRIBUTION IF EACH OF THE PROBABILITIES IS EQUAL TO D^{-n} FOR SOME „n".

5.4 BOUNDS ON THE OPTIMAL CODE LENGTH

$$H(X) \leq L \leq H(X) + 1$$

IN PREVIOUS SECTION WE PROVED THAT OPTIMAL CODEWORD LENGTHS CAN BE FOUND BY FINDING THE D-ADIC PROBABILITY DISTRIBUTION CLOSEST TO THE DISTRIBUTION OF X IN RELATIVE ENTROPY, THAT IS BY FINDING D-ADIC „r"

$\boxed{r_i = \frac{D^{-l_i}}{\sum D^{-l_j}}}$ MINIMIZE:

$$L - H_D = D(p\|r) - \log\left(\sum D^{-l_i}\right) \geq 0 \qquad \boxed{l_i = \left\lceil \log_D \frac{1}{p_i} \right\rceil}$$

SHANNON CODING

⊛ → $L = H$ FOR $l_i = \log_D\left(\frac{1}{p_i}\right)$

OVIE DOLZINI NA KODOT GO ZADOVOLUVAAT KRAFTOVOTO NEEDNAKVOST

$$\sum D^{-\lceil \log_D \frac{1}{p_i} \rceil} \leq \sum D^{-\log_D \frac{1}{p_i}} = \sum p_i = 1$$

$$\log_D \frac{1}{p_i} \leq l_i < \log_D \frac{1}{p_i} + 1 \quad / \cdot p_i \cdot \sum_{i=1}^{n}$$

$$\sum p_i \log_D \frac{1}{p_i} \leq \sum_{i=1}^{n} p_i l_i < \sum_{i=1}^{n} p_i \log_D \frac{1}{p_i} + 1 \quad (\$)$$

$$\boxed{H_D(X) \leq L \leq H_D(X) + 1}$$

THEOREM 5.4.1 LET $l_1^*, l_2^*, \ldots l_m^*$ BE OPTIMAL CODEWORD LENGHTS FOR A SOURCE DISTRIBUTION P AND A D-ARY ALPHABET, AND LET L^* BE THE ASSOCIATED EXPECTED LENGHT OF OPTIMAL CODE $\left(L^* = \sum p_i l_i^*\right)$. THEN

$$H_D(X) \leq L^* \leq H_D(X) + 1$$

PROOF: LET $l_i = \left\lceil \log_D \frac{1}{p_i} \right\rceil$ THEN l_i SATISFIES KRAFT INEQUALITY AND WE HAVE ($) :

$l_i = \lceil \log_D \frac{1}{p_i} \rceil \qquad \log_D \frac{1}{p_i} \le l_i \le \log_D \frac{1}{p_i} + 1$

$H_D(X) \le \sum l_i p_i < H_D(X) + 1 \qquad \left.\underbrace{\sum l_i^* p_i}_{L^*} \le \sum l_i p_i \right\} \Rightarrow$

$H_D(X) \le \sum l_i^* p_i < H_D(X) + 1$

- LET US CONSIDER A SYSTEM IN WHICH WE SEND SEQUENCE OF n SYMBOLS FROM $\mathcal{X}$. SYMBOLS ARE ASSUMED TO BE DRAWN I.I.D ACCORDING TO $p(x)$. WE CAN CONSIDER THESE n SYMBOLS TO BE A SUPERSYMBOL FROM ALPHABET $\mathcal{X}^n$.

(FOR THE REST OF THIS SECTION WE ASSUME THAT $D=2$ FOR SIMPLICITY)

L_n - EXPECTED CODEWORD LENGTH PER INPUT SYMBOL.

$$L_n = \frac{1}{n} \sum p(x_1 x_2 \dots x_n) \cdot l(x_1 \dots x_n) = \frac{1}{n} E[l(X_1^n)]$$

$$H(X_1, X_2 \dots X_n) \le E[l(X_1, \dots X_n)] \le H(X_1 X_2 \dots X_n) + 1$$

$$n \cdot H(X) \le E[l(X_1, \dots X_n)] \le n H(X) + 1$$

$$\boxed{H(X) \le L_n \le H(X) + \frac{1}{n}}$$

HENCE BY USING LARGE BLOCK LENGTHS WE CAN ACHIEVE AN EXPECTED CODELENGTH PER SYMBOL ARBITRARY CLOSE TO THE ENTROPY.

- WE CAN USE THE SAME ARGUMENT FOR A SEQUENCE OF SYMBOLS FROM A STOCHASTIC PROCESS THAT IS NOT NECESSARY I.I.D. IN THIS CASE WE STILL HAVE THE BOUND

$$H(X_1 \dots X_n) \le E[l(X_1 X_2 \dots X_n)] < H(X_1^n) + 1$$

L_n - EXPECTED DESCRIPTION LENGTH PER SYMBOL:

$$\frac{1}{n} H(X_1^n) \le L_n \le \frac{1}{n} H(X_1^n) + \frac{1}{n}$$

IF THE STOCHASTIC PROCESS IS STATIONARY, THEN

$$\frac{H(X_1^n)}{n} \to H(\mathcal{X}) \Rightarrow L_n \to H(\mathcal{X}) \text{ AS } n \to \infty.$$

PODSJETI SE NA PROBLEM 3.7 (HMV)

THEOREM 5.4.2 THE MINIMUM EXPECTED CODEWORD LENGTH PER SYMBOL SATISFIES:

$$\frac{H(X_1, X_2 \dots X_n)}{n} \le L_n^* < \frac{H(X_1, X_2, \dots X_n)}{n} + \frac{1}{n}$$

MOREOVER IF $X_1 X_2, \dots X_n$ IS A STATIONARY STOCHASTIC PROCESS

$$L_n^* \to H(\mathcal{X})$$

WHERE $H(\mathcal{X})$ IS ENTROPY RATE OF STOCHASTIC PROCESS. THIS THEOREM PROVIDES ANOTHER JUSTIFICATION FOR THE DEFINITION OF ENTROPY RATE - IT IS EXPECTED NUMBER OF BITS PER SYMBOL REQUIRED TO DESCRIBE THE PROCESS

- WE NOW SHOW THAT INCREASE IN EXPECTED DESCRIPTION LENGTH DUE TO INCORRECT DISTRIBUTION IS RELATIVE ENTROPY $D(p\|q)$. THUS, $D(p\|q)$ HAS CONCRETE INTERPRETATION AS INCREASE IN DESCRIPTIVE COMPLEXITY DUE TO INCORRECT INFORMATION.

THEOREM 5.4.3 (WRONG CODE) THE EXPECTED LENGTH UNDER $p(x)$ OF THE CODE ASSIGNMENT $l(x) = \lceil \log \frac{1}{q(x)} \rceil$ SATISFIES

$$H(p) + D(p\|q) \le E_p[l(X)] < H(p) + D(p\|q) + 1$$

FOR EXAMPLE, THE WRONG DISTRIBUTION MAY BE THE <u>BEST ESTIMATE</u> THAT WE CAN MAKE OF THE UNKNOWN TRUE DISTRIBUTION.

<u>PROOF:</u> $E_p[l(x)] = \sum_x p(x) \cdot l(x) = \sum_x p(x) \left\lceil ld \frac{1}{q(x)} \right\rceil <$

$$< \sum_x p(x) \left(ld \frac{1}{q(x)} + 1 \right) = \sum_x p(x) ld \left(\frac{p(x)}{q(x)} \cdot \frac{1}{p(x)} \right) + 1 =$$

$$= \sum_x p(x) \, ld \frac{p(x)}{q(x)} + \sum_x p(x) \, ld \frac{1}{p(x)} + 1 = \underbrace{D(p\|q) + H(p)}_{\text{UPPER BOUND}} + 1$$

$$E_p[l(x)] = \sum_x p(x) \left\lceil ld \frac{1}{q(x)} \right\rceil \geq \sum_x p(x) \, ld \frac{1}{q(x)} = \sum_x p(x) \, ld \frac{p(x)}{q(x)}$$

$$+ \sum_x p(x) \, ld \frac{1}{p(x)} = D(p\|q) + H(p)$$ <u>LOWER BOUND</u>

$$\boxed{H(p) + D(p\|q) \leq E_p[l(x)] < H(p) + D(p\|q) + 1}$$ PROVED!!!

THUS, BELIEVING THAT THE DISTRIBUTION IS $q(x)$ WHEN THE TRUE DISTRIBUTION IS $p(x)$ INCURS PENALTY OF $D(p\|q)$ IN THE AVERAGE DESCRIPTION LENGTH.

5.5 KRAFT INEQUALITY FOR UNIQUELY DECODABLE CODES

THEOREM 5.5.1 (McMillan) THE CODEWORD LENGTHS OF ANY UNIQUELY DECODABLE D-ARY CODE MUST SATISFY KRAFT INEQUALITY: $\sum D^{-l_i} \leq 1$

CONVERSELY, GIVEN A SET OF CODEWORD LENGTHS THAT SATISFY THIS INEQUALITY, IT IS POSSIBLE TO CONSTRUCT A UNIQUELY DECODABLE CODE WITH THESE CODEWORD LENGTHS.

- LET THE CODEWORD LENGTHS OF SYMBOL $x \in \mathcal{X}$ IS $l(x)$
- FOR THE EXTENTION CODE (C^k), THE LENGTH OF THE CODE SEQUENCE IS:

$$l(x_1 x_2 \ldots x_k) = \sum_{i=1}^{k} l(x_i)$$

C^k – k-TH EXTENTION OF THE CODE (I.E. THE CODE FORMED BY <u>CONCATENATION OF k REPETITIONS OF THE GIVEN UNIQUELY DECODABLE CODE C</u>)

WE NEED TO PROVE IS: $\sum_{x \in \mathcal{X}} D^{-l(x)} \leq 1$

$$\left(\sum_{x \in \mathcal{X}} D^{-l(x)} \right)^k = \sum_{x_1 \in \mathcal{X}} \sum_{x_2 \in \mathcal{X}} \cdots \sum_{x_k \in \mathcal{X}} D^{-l(x_1)} \cdot D^{-l(x_2)} \cdot D^{-l(x_3)} \cdots D^{-l(x_k)} =$$

$$= \sum_{x_1 x_2 \ldots x_k \in \mathcal{X}^k} D^{-l(x_1)} \cdot D^{-l(x_2)} \cdots D^{-l(x_k)} = \sum_{x^k \in \mathcal{X}^k} D^{-l(x^k)} =$$ AKO SE PODREDAT CLENOVITE PO DOLZINA NA ZBOROVITE

$$= \sum_{m=1}^{k \cdot l_{max}} a(m) \cdot D^{-m}$$

$a(m)$ IS NUMBER OF SOURCE SEQUENCES x^k MAPPING INTO CODEWORD OF LENGHT "m"

$$\left(\sum_{x\in\mathcal{X}} D^{-l(x)}\right)^k = \sum_{m=1}^{k\cdot l_{max}} a(m)\, D^{-m} \leq \sum_{m=1}^{k\cdot l_{max}} D^{m}\cdot D^{-m} = k\cdot l_{max}$$

$$\sum_{x\in\mathcal{X}} D^{-l(x)} \leq \left(k\cdot l_{max}\right)^{\frac{1}{k}}$$

SINCE THIS INEQUALITY IS TRUE FOR ALL k, IT IS TRUE IN THE LIMIT AS $k \to \infty$.

VSUŠNOST ZA $k \to \infty$ RHS E NADMAZO!!!

$$\boxed{\sum_{x\in\mathcal{X}} D^{-l(x)} \leq 1}$$

$1 \to 10 \quad 2 \to 00 \quad 3 \to 11 \quad 4 \to \underline{110}$

$k=3$

$$l(x^k) = l(x^3) = \left\{\begin{matrix} 10\,00\,11 \\ 10\,11\,00 \\ 00\,10\,11 \\ 00\,11\,10 \\ 11\,00\,10 \\ 11\,10\,00 \end{matrix}\right\} \quad 6 = a(m) = a(6) \leq 2^6 = 64$$

$\underline{D=2}$

$k=2$

$$L(x^k) = l(x^2) = \left\{\begin{matrix} 1000 \\ 0011 \\ 0010 \\ 0011 \\ 1110 \\ 1100 \end{matrix}\right\} \quad 6 = a(m) = a(4) \leq 2^6 = 64$$

COROLLARY A UNIQUELY DECODABLE CODE FOR AN INFINITE SOURCE ALPHABET $\mathcal{X}$ ALSO SATISFIES THE KRAFT INEQUALITY.

$$\sum_{i=1}^{\infty} D^{-l_i} = \lim_{N\to\infty} \sum_{i=1}^{N} D^{-l_i} \leq 1$$

5.6 HUFFMAN CODES

OPTIMAL CODE = SHORTEST EXPECTED LENGTH

ANY OTHER CODE FOR THE SAME ALPHABET CANNOT HAVE A LOWER EXPECTED LENGTH THAN THE CODE CONSTRUCTED BY THE HUFFMAN CODE ALGORITHM.

EXAMPLE 5.6.1 $\mathcal{X} = \{1, 2, 3, 4, 5\}$ $P(x) = \left\{\frac{1}{4}, \frac{1}{4}, \frac{1}{5}, 0.15, 0.15\right\}$

CODEWORD LENGTH	CODEWORD	$\mathcal{X}$	PROBABILITY			
2	01	1	0.25	0.3	0.45 1	0.55] 0 → 1
2	10	2	0.25	0.25	0.3] 0	0.45] 1
2	11	3	0.2	0.25] 0	0.25] 1	
3	000	4	0.15] 0	0.2] 1		
3	001	5	0.15] 1			

ZNAČI ODIŠ OD DESNO KON LEVO I GI MEMORIRAŠ VREDNOSTITE (0 ILI 1) ONAMU KADE ŠTO IMAŠ GRANKI. GRANKITE POSLEDOVATELNO GI OBELEŽUVAŠ SO Ø ILI 1.

$$E[l(x)] = \left(2\cdot\frac{1}{4}\right)\cdot 2 + 2\cdot 0.2 + \left(3\cdot 0.15\right)2 = 1 + 0.4 + 0.9 = 2.3\ \underline{\text{bits}}$$

EXAMPLE 5.6.2 CONSIDER THE TERNARY CODE FOR SAME RANDOM VARIABLE. NOW WE COMBINE THREE LEAST LIKELY SYMBOLS INTO ONE SUBSYMBOL AND OBTAIN THE FOLLOWING TABLE.

LENGTH	CODEWORD	X	PROBABILITY		
1	1	1	0.25	0.5	0 → 1
1	2	2	.25	0.25	1
2	00	3	0.2 0	0.25	2
2	01	4	0.15 1		
2	02	5	0.15 2		

BITI!!! NE SE STAVA POSLEDNATA „1" VO KODOT

$$E[l(x)] = 1\cdot 0.25 + 1\cdot 0.25 + 2\cdot 0{,}2 + (2\cdot 0.15)\cdot 2 = 0.5+0.4+0.6 = 1{,}5^{\text{TERNAR DGIT}}$$

$\boxed{\text{EXAMPLE 5.6.3}}$ IF $D \geq 3$, WE MAY NOT HAVE SUFFICIENT NUMBER OF SYMBOLS SO THAT WE CAN COMBINE THEM „D" AT TIME. IN SUCH A CODE WE ADD DUMMY SYMBOLS TO THE END OF THE SET OF SYMBOLS. THE DUMMY SYMBOLS HAVE PROBABILITY 0 AND ARE INSERTED TO FILL THE TREE. TOTAL NUMBER OF SYMBOL SHOLD BE $\boxed{1+k(D-1)}$, WHERE „k" IS THE NUMBER OF MERGES.

LENGTH	Codeword	X	PROBABILITY			
1	1	1	0.25	0.25	0.5	0 → 1
1	2	2	0.25	0.25	0.25	1
2	01	3	0.2	0.2 0	0.25	2
2	02	4	0.1	0.2 1		
3	000	5	0.1 0	0.1 2		
3	001	6	0.1 1			
3	002	Dummy	0 2			

$$E[l(x)] = 0.25+0.25+0.4+0.2+(0.3\cdot 2) = 0.5+0.6+0.6 = 1.7^{\text{TERNARY DIGITS}}$$

VNIMAVAJ !!! KOGA STAVAŠ D SIMBOLI VO EDEN SUPERSIMBOL, VEROJATNOSTA IZLEZE ISTA SO VEROJATNOSTA NA DRUG SYMBOL VO PODREDUVANJETO GO STAVAŠ PRVO SUPERSIMBOLOT.

5.7. SOME COMMENTS ON HUFFMAN CODES

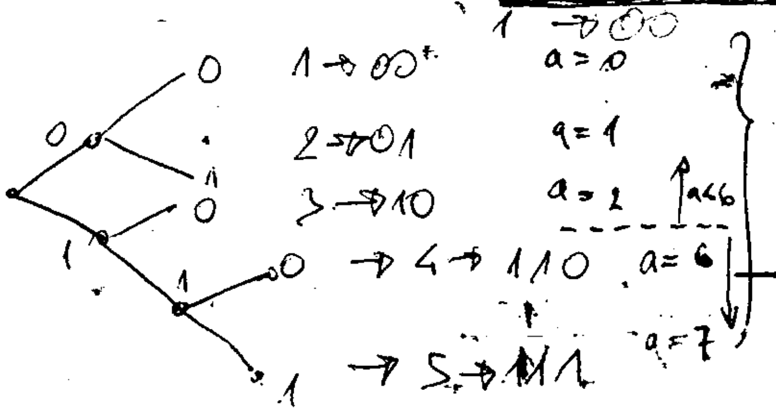
- HUFFMAN CODING FOR WEIGHTED CODEWORDS. HUFFMAN'S ALGORITHM FOR MINIMIZING $\sum w_i l_i$ CAN BE APPLIED TO ANY SET OF NUMBERS $p_i \geq 0$, REGARDLES OF $\sum w_i$. IN THIS CASE HUFFMAN CODE MINIMIZES THE SUM OF WEIGHTED CODE LENGTHS $\sum w_i l_i$ RATHER THAN AVERAGE CODE LENGHT

$\boxed{\text{EXAMPLE 5.7.1}}$

L	X	CODEWORD	WEIGHTS			
2	1	00	5	8	10 0	18
2	2	01	5	5 0	8 1	
2	3	10	4 0	5 1		
2	4	11	4 1			

MINIMUM WEIGHTED SUM: $2(5+5+4+4) = 18\cdot 2 = 36$

- HUFFMAN CODING AND SLICE QUESTIONS. (ALPHABETIC CODES)



SLICE I.E ALPHABETIC CODES BECOUSE CODES ARE ORDERED ALPHABETICALY.

$\boxed{\text{SLICE CODE}}$

EXAMPLE 5.7.3 CONSIDER RANDOM VARIABLE WITH A DISTRIBUTION $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{12}\right)$. THE HUFFMAN CODING PROCEDURE RESULTS IN CODEWORD LENGTHS - OF (2,2,2,2) OR (1,2,3,3) [DEPENDING WHERE ONE PUTS THE MERGED PROBABILITIES]. BOTH THESE CODES ACHIEVE THE SAME EXPECTED CODEWORD LENGTH. IN THE SECOND CODE, THE THIRD SYMBOL HAS LENGTH 3, WHICH IS GREATER THAN $\lceil \log \frac{1}{p_3} \rceil = 2$. THUS CODEWORD LENGTH FOR SHANNON CODE COULD BE LESS THAN OPTIMAL (HUFFMAN) CODE. THIS EXAMPLE ALSO ILUSTRATES THAT THE SET OF CODEWORD LENGTHS FOR AN OPTIMAL CODE IS NOT UNIQUE (THERE MAY BE MORE THEN ONE SET OF LENGTHS WITH THE SAME EXPECTED VALUE).

$$\begin{array}{r} 0.66 \\ 0.25 \\ \hline 0.91 \end{array} \qquad \frac{1}{4}+\frac{1}{12}=\frac{4}{12}=\frac{1}{3}$$

L(C)	C	X	PROBABILITIES		
2	00	1	0.33	0.33	0.66 0
2	01	2	0.33	0.33 0	0.33 1
2	10	3	0.25 0	0.33 1	
2	11	4	0.08 1		

$$\boxed{E[\ell(X)] = 2 \cdot 1 = 2}$$

L(C)	C	X				
1	1	1	0.33 —	0.33	0.66 0	1
2	00	2	0.33 —	0.33 0	0.33 1	
3	010	3	0.25 0	0.33 1		
3	011	4	0,08 1			

$$\boxed{E[\ell(X)] = 0.33 + 0.66 + 0.75 + 0.25 = 1 + 1 = 2}$$

(5) **FANO CODES**. FANO PROPOSED SUBOPTIMAL PROCEDURE FOR CONSTRUCTING A SOURCE CODE, WHICH IS SIMILAR TO THE ~~IDEA~~ OF SLICE CODES.

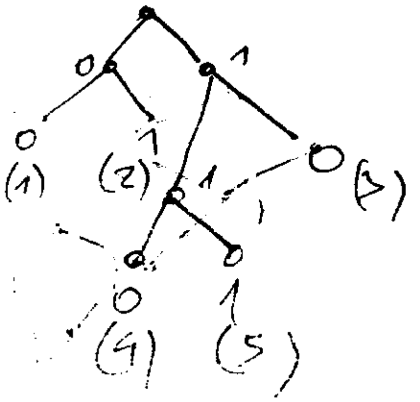
CHOSE „K" SUCH $\left| \sum_{i=1}^{k} p_i - \sum_{i=k+1}^{m} p_i \right|$ IS MINIMIZED

$\boxed{p_1 \geq p_2 \geq p_3 \cdots \geq p_m}$ — FIRST ORDER THE PROBABILITIES IN DECREASING ORDER

THIS POINT DEVIDES THE SOURCE SYMBOLS IN TWO SETS OF ALMOST EQUAL PROBABILITIES. ASSIGN 0 FOR THE FIRST BIT OF UPPER SET AND 1 FOR LOWER SET. REPEAT THIS PROCESS FOR EACH SUBSET: THIS PROCEDURE, ALTHOUGH NOT OPTIMAL ACHIEVES

$$\boxed{L(C) \leq H(X) + 2}$$

	P				
1	0.25	I (0)	I.a (0)	00	
2	0.25		I.b (1)	01	
3	0.2	II (1)	II.a (0)	10	
4	0.15		II.b (1)	II.b.1 (0)	110
5	0.15			II.b.2 (1)	111



OPTIMALITY OF HUFFMAN CODES

WITHOUT LOSS OF GENERALITY WE ASSUME THAT PROBABILITY MASSES ARE ORDERED SO THAT:

$$\boxed{p_1 \geq p_2 \geq \cdots \geq p_m}$$

THE CODE IS OPTIMAL IF $\sum p_i l_i$ IS MINIMAL.

LEMA 5.8.1 FOR ANY DISTRIBUTION, THERE EXIST AN OPTIMAL INSTANTENOUS CODE (WITH MINIMUM EXPECTED LENGTH) THAT SATISFIES THE FOLLOWING PROPERTIES:

1. THE LENGTHS ARE ORDERED INVERSELY WITH PROBABILITIES ($p_j > p_k$ THEN $l_j \leq l_k$)
2. THE TWO LONGEST CODEWORDS HAVE THE SAME LENGTH.
3. TWO OF THE LONGEST CODEWORDS DIFFER ONLY IN THE LAST BIT AND CORRESPOND TO TWO LEAST LIKELY SYMBOLS.

PROOF: THE PROOF AMOUNTS TO SWAPPING, TRIMMING AND REARANGING AS SHOWN IN FIGURE 5.3. CONSIDER AN OPTIMAL CODE C_m:

- If $p_j > p_k$ THEN $l_j \leq l_k$. HERE WE SWAP CODEWORDS. CONSIDER C'_m, WITH CODEWORDS j AND k OF C_m INTERCHANGED. THEN:

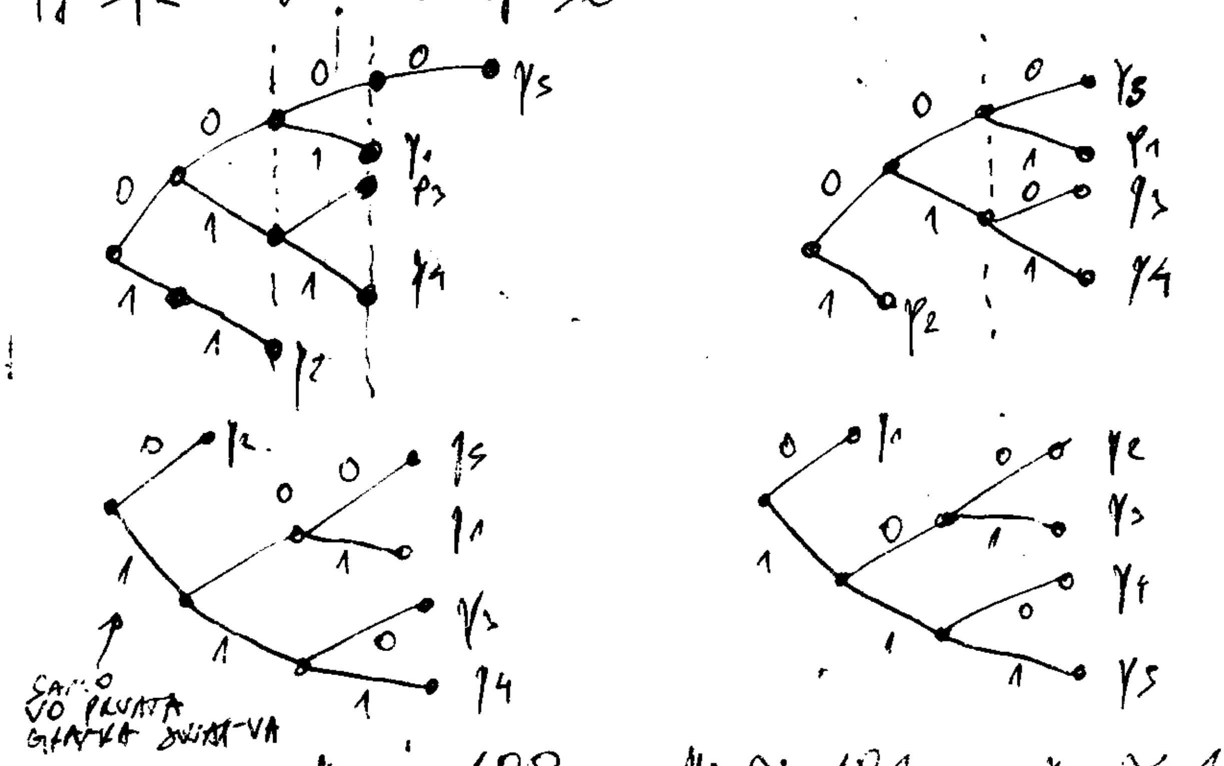
$$L(C'_m) - L(C_m) = \sum p_i l'_i - \sum p_i l_i = \underline{p_j l_k + p_k l_j} - \underline{p_j l_j - p_k l_k}$$

$$= p_k(l_j - l_k) - p_j(l_j - l_k) = (l_j - l_k)(p_k - p_j) = (p_j - p_k)(l_k - l_j)$$

$$L(C'_m) - L(C_m) = (p_j - p_k)(l_k - l_j) \geq 0 \quad \text{since } C_m \text{ is optimal}$$

$$\therefore p_j \geq p_k \Rightarrow l_k - l_j \geq 0$$

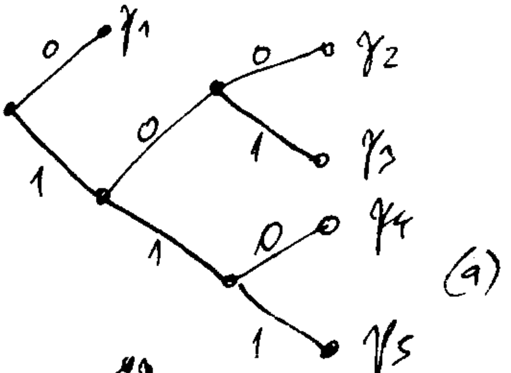
FIGURE 5.3

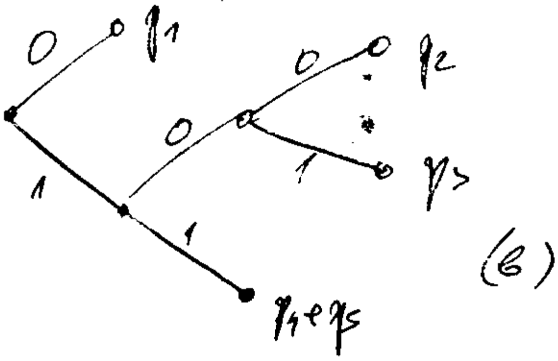


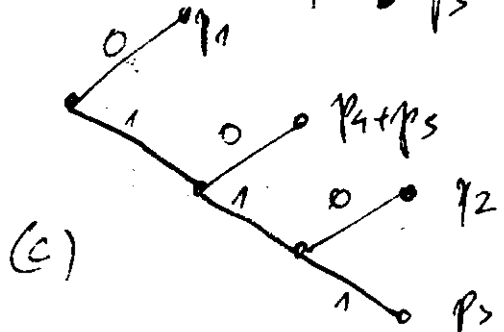
$p_1 \sim 0$ $\quad p_2 \sim 100$ $\quad p_3 \sim 101$ $\quad p_4 \sim 110$

$p_5 \sim 111$ $\quad l_1 \leq l_2 \leq \cdots l_m$ $\quad l_{m-1} = l_m$:

THE CODES THAT SATISFY $p_1 \geq p_2 \geq \cdots p_m$, $l_1 \leq l_2 \cdots l_{m-1} \leq l_m$ AND $C(x_{m-1})$ AND $C(x_m)$ DIFER ONLY IN THE LAST BIT, ARE CALLED CANNONICAL CODES.







COMBINING THE TWO LOWEST PROBABILITIES, WE OBTAIN THE CODE IN (b). REARRANGING THE PROBABILITIES IN DECREASING ORDER, WE OBTAIN CANONICAL CODE IN (c) FOR $(m-1)$ SYMBOLS.

FOR $p = (p_1, p_2, \cdots p_m)$ WITH $p_1 \geq p_2 \geq \cdots p_m$ WE DEFINE HUFFMAN REDUCTION $p' = (p_1, p_2, \cdots p_{m-2}, p_{m-1} + p_m)$ OVER THE ALPHABET SIZE $m-1$. $C^*_{m-1}(p')$ IS OPTIMAL CODE FOR p', AND $C^*_m(p)$ IS CANNONICAL OPTIMAL CODE FOR p.

OPTIMAL CODE FOR

- CONSTRUCTION OF EXTENSION CODE FOR m ELEMENTS FROM p'.

	$C^*_{m-1}(p')$		$C_m(p)$	
p_1	w_1'	l_1'	$w_1 = w_1'$	$l_1 = l_1'$
p_2	w_2'	l_2'	$w_2 = w_2'$	$l_2 = l_2'$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
p_{m-2}	w_{m-2}'	l_{m-2}'	$w_{m-2} = w_{m-2}'$	$l_{m-2} = l_{m-2}'$
$p_{m-1} + p_m$	w_{m-1}'	l_{m-1}'	$w_{m-1} = w_{m-1}'0$ $w_m = w_{m-1}'1$	$l_{m-1} = l_{m-1}' + 1$ $l_m = l_{m-1}' + 1$

$$L(p) = \sum_{i=1}^{m} p_i l_i = \sum_{i=1}^{m-2} p_i l_i + p_{m-1} \cdot l_{m-1} + p_m \cdot l_m =$$

$$= \sum_{i=1}^{m-2} p_i l_i + p_{m-1}(l_{m-1}' + 1) + p_m(l_{m-1}' + 1) =$$

$$= \sum_{i=1}^{m-2} p_i l_i' + (p_{m-1} + p_m) l_{m-1}' + p_{m-1} + p_m =$$

$$= \sum_{i=1}^{m-1} p_i l_i' + p_{m-1} + p_m \qquad \boxed{L(p) = L^*(p') + p_{m-1} + p_m}$$

SIMILARLY FROM CANONICAL CODE FOR p, WE CONSTRUCT A CODE FOR p' BY MERGING THE CODEWORDS FOR THE TWO LOWEST-PROBABILITY SYMBOLS $m-1$ AND m WITH PROBABILITIES p_{m-1} AND p_m.

$$\underline{L(p')} = \sum_{i=1}^{m-2} p_i l_i + \overbrace{p_{m-1}(l_{m-1}-1) + p_m(l_m-1)}^{(p_{m-1}+p_m)l_{m-1}} = \left| l_m = l_{m-1} \right|$$

$$= \underbrace{\sum_{i=1}^{m-2} p_i l_i + (l_m-1)\cdot(p_{m-1}+p_m)}_{= L(p')} = \sum_{i=1}^{m-2} p_i l_i + (l_m-1)p_{m-1} +$$

$$+ (l_m-1)p_m = \sum_{i=1}^{m-2} p_i l_i + (l_{m-1}-1)p_{m-1} + (l_m-1)p_m$$

$$= \sum_{i=1}^{m-2} p_i l_i + \underbrace{l_{m-1}p_{m-1} + l_m p_m} - p_{m-1} - p_m =$$

$$= \sum_{i=1}^{m} p_i l_i - p_{m-1} - p_m = \underline{\underline{L^*(p) - p_{m-1} - p_m}}$$

$$\left.\begin{array}{l} L(p) = L^*(p') + p_{m-1} + p_m \\ L(p') = L^*(p) - p_{m-1} - p_m \end{array}\right| + \quad \rightarrow \boxed{L(p) + L(p') = L^*(p) + L^*(p')}$$

$$[L(p') - L^*(p')] + [L(p) - L^*(p)] = 0$$

$$\left.\begin{array}{l} L(p') - L^*(p') \geq 0 \qquad L(p) - L^*(p) \geq 0 \end{array}\right\} \Rightarrow \boxed{L(p) = L^*(p)}$$

I.E. THE EXTENSION OF THE OPTIMAL CODE FOR p', OPTIMAL FOR p. CONSEQUENTLY, IF WE START WITH OPTIMAL CODE FOR p' WITH $m-1$ SYMBOLS AND CONSTRUCT A CODE FOR m SYMBOLS BY EXTENDING THE CODEWORD CORRESPONDING TO $p_{m-1}+p_m$ THE NEW CODE IS ALSO OPTIMAL. STARTING WITH CODE FOR TWO ELEMENTS, IN WHICH CASE THE OPTIMAL CODE IS OBVIOUS, WE CAN BY INDUCTION EXTEND THIS RESULTS TO PROVE THE FOLLOWING THEOREM.

THEOREM 5.8.1 THE HUFFMAN CODING IS OPTIMAL; THAT IS IF C^* IS A HUFFMAN CODE AND C' IS ANY OTHER UNIQUELY DECODABLE CODE, $\boxed{L(C^*) \leq L(C')}$. MMV

SINTRA OF HUFFMAN-TO [illegible]

5.9 SHANNON-FANO-ELIAS CODING

IN THIS SECTION, WE DESCRIBE A SIMPLE CONSTRUCTIVE PROCEDURE THAT USES THE CUMULATIVE DISTRIBUTION FUNCTION TO ALLOT CODEWORDS.

$X = \{1, 2, \ldots m\}$. ASSUME THAT $p(x) > 0$ FOR ALL x

$$F(x) = \sum_{a \leq x} p(a)$$

CONSIDER THE MODIFIED CUMULATIVE DISTRIBUTION FUNCTION

$$\bar{F}(x) = \sum_{a < x} p(a) + \frac{1}{2} p(x)$$

SINCE ALL THE PROBABILITIES ARE POSITIVE, $\bar{F}(a)=\bar{F}(b)$ IF $a \neq b$, AND HENCE WE CAN DETERMINE X IF WE KNOW $\bar{F}(x)$.

ASSUME THAT WE TRUNCATE $\bar{F}(x)$ TO $l(x)$ BITS DENOTED BY $\lfloor \bar{F}(x) \rfloor_{l(x)}$. WE USE THE FIRST $l(x)$ BITS OF $\bar{F}(x)$ AS A CODE FOR X

$$\bar{F}(x) - \lfloor \bar{F}(x) \rfloor_{l(x)} < \frac{1}{2^{l(x)}}$$

EG:

$\bar{F}(x)=0.1011|001$

АКО ГО ЗАОКРУЖАМ ДО ОВДЕ НАЈГОЛЕМА ГРЕШКА ШТО МОЖЕ ДА СЕ НАПРАВИ Е:

$$\bar{F}(x) = \frac{1}{2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^7}$$

$$\lfloor \bar{F}(x) \rfloor_4 = \frac{1}{2}+\frac{1}{2^3}+\frac{1}{2^4} = 0.5+0.125+0.0625=\underline{0.6875}$$

НАЈМНОГУ ШТО МОЖЕШ ДА ЗГРЕШИШ Е: $\frac{1}{2^5}+\frac{1}{2^6}+\frac{1}{2^7}=$

$$= \frac{2^2+2+1}{2^7} \leq \frac{2^3}{2^7} = \frac{1}{2^4} \qquad \frac{1}{2^4} = \frac{2^3}{2^7} = \frac{1}{2^4}$$

VIDI NIG.152

IF: $l(x) = \left\lceil ld \frac{1}{p(x)} \right\rceil + 1$

$$2^{l(x)-1} \geq \frac{1}{p(x)}$$

$$p(x) \geq \frac{1}{2^{l(x)-1}}$$

$$\frac{p(x)}{2} \geq \frac{1}{2^{l(x)}}$$

$$\bar{F}(x) - F(x-1) = \frac{p(x)}{2} \overset{*}{>} \frac{1}{2^{l(x)}}$$

$$\boxed{l(x) \geq ld \frac{1}{p(x)} + 1}$$

$$\bar{F}(x) - F(x-1) = \frac{p(x)}{2} > \frac{1}{2^{l(x)}} \Rightarrow$$

THEREFORE $\lfloor \bar{F}(x) \rfloor_{l(x)}$ LIES WITHIN THE STEP CORRESPONDING TO X. THUS $l(x)$ BITS SUFFICE TO DESCRIBE X.

- SINCE WE USE $l(x) = \left\lceil ld \frac{1}{p(x)} \right\rceil + 1$ BITS TO REPRESENT X, THE EXPECTED LENGTH OF THIS CODE IS:

$$L = \sum_x p(x) l(x) = \sum_x p(x) \left(\left\lceil ld \frac{1}{p(x)} \right\rceil + 1 \right) \leq \sum_x p(x) \left(ld \frac{1}{p(x)} + 2 \right)$$

$$= H(X) + 2 \qquad \boxed{L \leq H(X) + 2}$$

EXAMPLE 5.9.1

X	$p(x)$	$F(x)$	$\bar{F}(x)$	$\bar{F}(x)$ BINARY	$l(x)=\left\lceil ld \frac{1}{p(x)} \right\rceil+1$	$c(x)$
1	0.25	0.25	0.125	0.001	3	001
2	0.5	0.75	0.25+0.25=0.5	0.1	2	10
3	0.125	0.875	0.25+0.5+0.0625= =0.8125	0.1101	4	1101
4	0.125	1.0	0.875+0.0625 =0.9375	0.1111	4	111

3: $\frac{1}{2^2}+\frac{1}{2^1}+\frac{1}{2^4} = 0.1101$ 4: $2^{-1}+2^{-2}+2^{-3}+2^{-4}$

$$H(X) = \frac{1}{4}\cdot 2 + \frac{1}{2} + \left(\frac{1}{8}\cdot 3\right)\cdot 2 = \frac{4+4+6}{8} = \frac{14}{8} = \frac{7}{4} = 1.75$$

$$L(C) = 3\cdot 0.25 + 2\cdot 0.5 + \left(4\cdot 0.125\right)\cdot 2 = 0.75 + 1 + 1 = \underline{\underline{2.75}}$$

EXAMPLE 5.7.2 ANOTHER EXAMPLE FOR CONSTRUCTION OF SHANNON - FANO - ELIAS CODE. IN THIS CASE SINCE THE DISTRIBUTION IS NOT DYADIC, THE REPRESENTATION OF $\bar{F}(X)$ IN BINARY MAY HAVE AN <u>INFINITE</u> NUMBER OF BITS. WE DENOTE $0.0101010101\ldots$ BY $0\overline{01}$. WE CONSTRUCT THE CODE IN THE FOLLOWING TABLE:

X	$p(X)$	$F(X)$	$\bar{F}(x)$	$\bar{F}(X)$ IN BINARY	$l(x) = \lceil \mathrm{ld}\, 1/p(x) \rceil + 1$	CODEWORD
1	0.25	0.25	0.125	0.001	3	001
2	0.25	0.5	0.375	0.01100	3	011
3	0.2	0.7	0.6	$0.1\overline{0011}$	4	1001
4	0.15	0.85	0.775	$0.1100\overline{0011}$	4	1100
5	0.15	1.0	0.925	$0.111\overline{0110}$	4	1110

- HUFFMAN CODE (Ex. 5.6.1) $L(C) = 2.3$ bits.

$$L(C') = 2(0.25\cdot 3) + 0.8 + 2(0.15\cdot 4) = 1.5 + 0.8 + 1.2 = 3.5$$

$3.5 - 2.3 = 1.2$ LONGER ON AVERAGE THEN HUFFMAN CODE.

5.10 COMPETITIVE OPTIMALITY OF SHANNON CODE

THEOREM 5.10.1 LET $l(x)$ BE THE CODEWORD LENGTHS ASSOCIATED WITH THE SHANNON CODE, AND LET $l'(x)$ BE A CODEWORD LENGTHS WITH ANY OTHER UNIQUELY DECODABLE CODE. THEN

$$\boxed{P_r\left(l(x) \geq l'(x) + c\right) \leq \frac{1}{2^{c-1}}}$$

FOR EXAMPLE, THE PROBABILITY THAT $l'(x)$ IS 5 OR MORE BITS SHORTER THAN $l(x)$ IS LESS THEN $1/16$.

<u>PROOF</u>:

$$P_r\left(l(x) \geq l'(x)+c\right) = P_r\left(\left\lceil \mathrm{ld}\frac{1}{p(x)}\right\rceil \geq l'(x)+c\right) \leq$$

$$\leq P_r\left(\mathrm{ld}\frac{1}{p(x)} \geq l'(x)+c-1\right) = P_r\left(\underline{p(x)} \leq 2^{-l'(x)-c+1}\right) =$$

$$\sum_{x:\, p(x) \leq 2^{-l'(x)-c+1}} p(x) \leq \sum_{x:\, p(x)\leq 2^{-l'(x)-c+1}} 2^{-l'(x)-c+1} \leq \sum_x 2^{-l'(x)} 2^{-c+1} =$$

$$= 2^{-c+1} \underbrace{\sum_x 2^{-l'(x)}}_{\text{KRAFT INEQUALITY}} \leq 2^{-c+1} = 2^{-(c-1)} \qquad \boxed{\sum_x 2^{-l(x)} \leq 1}$$

- $P_r\left(l(x) < l'(x+c)\right) = ?$ $\qquad P_r\left(l(x) \geq l'(x)+c\right) \leq \frac{1}{2^{c-1}}$

$$P_r\left(l(x) \geq l'(x)+c\right) = 1 - P\left(l(x) < l'(x)+c\right) \Rightarrow \boxed{P\left(l(x) < l'(x)+c\right) \geq 1 - \frac{1}{2^{c-1}}}$$

$$P_r\left(l(x) < l'(x)+c\right) \geq 1 - \frac{1}{2^{c-1}}$$

$$\left.P_r\left(l(x) < l'(x)+2\right) \geq 1 - \frac{1}{2} = \frac{1}{2}\right\} \longrightarrow \text{OVA NE GO GARANTIRA}$$

OVA ↓

<u>IN GAME THEORETIC SETTING, ONE WOULD LIKE TO ENSURE THAT $l(x) < l'(x)$, MORE OFTEN THAN $l(x) > l'(x)$...</u>

$\boxed{\text{THEOREM 5.10.2}}$ FOR A dyadic PROBABILITY MASS FUNCTION $p(x)$, LET $l(x) = ld[1/p(x)]$ BE THE WORD LENGTHS OF THE BINARY SHANNON CODE FOR THE SOURCE AND LET $l'(x)$ BE THE LENGTHS OF ANY OTHER UNIQUELY DECODABLE BINARY CODE FOR THE SOURCE. THEN:

$$\boxed{P_r\left(l(x) < l'(x)\right) \geq P_r\left(l(x) > l'(x)\right)}$$

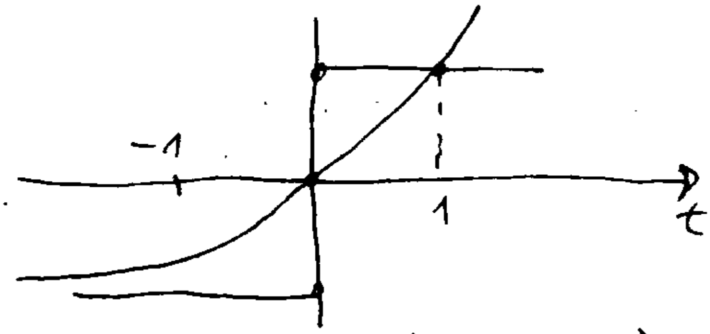
WITH AN EQUALITY IF AND ONLY IF:

$$l'(x) = l(x) \quad \text{FOR ALL } x$$

THUS, THE CODE LENGTH ASIGNMENT IS UNIQUELY COMPETITIVELY OPTIMAL.

PROOF:

$$sgn(t) = \begin{cases} 1 & \text{IF } t>0 \\ 0 & \text{IF } t=0 \\ -1 & \text{IF } t<0 \end{cases}$$



$$\boxed{sgn(t) \leq 2^t - 1} \ (\$)$$

$\boxed{\text{for } t = 0, \pm 1, \pm 2, }$

$(\$) = (a)$

$$P_r(l(x) < l'(x)) - P_r(l(x) > l'(x)) = \sum_{x: l(x) < l'(x)} p(x) - \sum_{x: l(x) > l'(x)} p(x) = \ (\star)$$

$$= \sum p(x)\, sgn(l(x) - l'(x)) = E[sgn(l(x) - l'(x))] \leq$$

$$\overset{(a)}{\leq} \sum_x p(x)\left(2^{l(x)-l'(x)} - 1\right) = \sum_x 2^{-l(x)}\left(2^{l(x)-l'(x)} - 1\right) =$$

$$= \sum_x 2^{-l'(x)} - \sum_x 2^{-l(x)} = \sum_x 2^{-l'(x)} - 1 \underset{(b)}{\leq} 1 - 1 = 0$$

(b) - KRAFT INEQUALITY $\boxed{\sum_x e^{-l(x)} \leq 1}$

(a) E "=" DOKOLKU $sgn(l(x) - l'(x)) = 2^{l(x)-l'(x)}$
TOA E TOČNO ZA $t=0$, $t=-1$ i.e

$l(x) - l'(x) = 0 \Rightarrow l(x) = l'(x)$

$l(x) - l'(x) = 1 \rightarrow l(x) = l'(x) + 1$

(b) E "=" SAMO AKO $l'(x)$ GO ZADOVOLUVA KRAFTOVOTO NERAVENSTVO SO EDNAKVOST T.E AKO $l'(x) = ld\frac{1}{p(x)}$ T.E ODNOSNO $l'(x) = l(x)$, ZA SITE x. $\boxed{E[sgn(l(x) - l'(x)] \leq 0}$

COROLARY: FOR NON-DYADIC PROBABILITY MASS FUNCTIONS

$$^{*} E[sgn(l(X) - l'(X) - 1)] \leq 0$$

KADE $l(x) = \lceil ld \frac{1}{p(x)} \rceil$ AND $l'(x)$ IS ANY OTHER CODE.

PROOF: $\Pr(l(x) < l'(x)) - \Pr(l(x) > l'(x)) = \sum p(x)\,\underline{\mathrm{sgn}(l(x)-l'(x))}$

$$\le \sum_x 2^{-l(x)+1}\mathrm{sgn}(l(x)-l'(x)) \leqslant \sum_x 2^{-l(x)+1}\cdot\left(2^{l(x)-l'(x)}-1\right)$$

$$= \left| \begin{array}{ll} \lceil ld(1/p(x)) \rceil = l(x) \Rightarrow & ld\frac{1}{p(x)}+1 \geq l(x) \Rightarrow \\ \Rightarrow \frac{1}{p(x)} \geq 2^{+l(x)-1} & p(x) \leq \underline{2^{-l(x)+1}} \end{array} \right| =$$

$$= \sum_x 2^{-\cancel{l(x)}+1+\cancel{l(x)}-l'(x)} - \sum_x 2^{-l(x)+1} = \sum 2^{-l'(x)+1} - \sum_x 2^{-l(x)+1}$$

$$\sum_x p(x)\,\mathrm{sgn}[l(x)-l'(x)] = \sum p(x)\,\mathrm{sgn}[l(x)+1-l'(x)]$$

$$\sum_x p(x)\,\mathrm{sgn}[l(x)-l'(x)] = \sum_x p(x)\,\mathrm{sgn}\left[\left\lceil ld\left(\frac{1}{p(x)}\right)\right\rceil - l'(x)\right]$$

$$ld\frac{1}{p(x)} \geqslant \left\lceil ld\frac{1}{p(x)}\right\rceil - 1$$

PROVED !!!

⊛ $E\left[\mathrm{sgn}\left(ld\frac{1}{p(x)} - l'(x)\right)\right] \le 0$

$$0 \geq E\left[\mathrm{sgn}\left(ld\frac{1}{p(x)} - l'(x)\right)\right] \geqslant E\left[\mathrm{sgn}\left(\left\lceil ld\frac{1}{p(x)}\right\rceil - 1 - l'(x)\right)\right] =$$

$$= E[l(x)-l'(x)-1] \Rightarrow \boxed{E[l(x)-l'(x)-1] \le 0}$$

- FOR DYADIC WE HAVE SHOWN:
  $E(l-l') \le 0$ i.e. $E[\mathrm{sgn}(l-l')] \le 0$
- FOR ANY FUNCTION $f(t)$ SATISFYING:
  $\boxed{f(t) \le 2^t - 1 \quad t=0,\pm 1,\pm 2\ldots} \Rightarrow E[f(l-l')] \le 0$

$$E[f(l-l')] = \sum_x p(x) f(l-l') \leqslant \sum p(x)\left(2^{l-l'}-1\right)$$

$$= \sum_x 2^{-l}\left(2^{l-l'}-1\right) = \sum 2^{-l'} - \sum_x 2^{l} = \underbrace{\sum_x 2^{-l'}}_{\text{KRAFT INEQUALITY}} - 1 \le 1-1 = 0$$

$$\Rightarrow \boxed{E[f(l-l')] \le 0}$$

5.11 GENERATION OF DISCRETE DISTRIBUTION FROM FAIR COINS

IT CAN BE ARGUED THAT ENCODED SEQUENCE IS ESSENTIALLY INCOMPRESSABLE AND THEREFORE HAS AN ENTROPY RATE CLOSE TO 1 BIT PER SYMBOL. THEREFORE THE BITS OF ENCODED SEQUENCE ARE ESSENTIALLY FAIR COIN FLIPS. ↗ VIDI PROBLEM 4.27

HOW MANY FAIR COIN FLIPS DOES IT TAKE TO GENERATE RANDOM VARIABLE X DRAWN ACCORDING TO A SPECIFIED PROBABILITY MASS FUNCTION p.

EXAMPLE 5.11.1 GIVEN A SEQUENCE OF FAIR COIN TOSSES (FAIR BITS), SUPPOSE THAT WE WISH TO GENERATE A RANDOM VARIABLE X WITH DISTRIBUTION:

$$\boxed{X = \begin{cases} a & \text{WITH PROBABILITY } 1/2 \\ b & \text{-''- \quad -''- } \quad 1/4 \\ c & \text{-''- \quad -''- } \quad 1/4 \end{cases}}$$

$0 \rightarrow X = a$ $\quad$ $11 \rightarrow X = c$

$10 \rightarrow X = b$

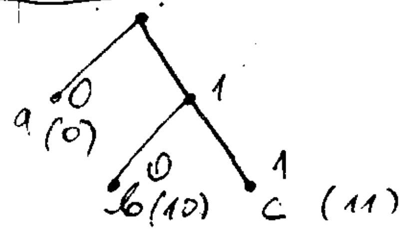
AVERAGE NUMBER OF FAIR BITS REQUIRED FOR GENERATING THE RANDOM VARIABLE IS:

$$1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} \cdot 2 = \frac{1}{2} + 1 = 1.5 \text{ bits}$$

THIS IS ALSO THE ENTROPY OF THE DISTRIBUTION. IT IS NOT UNUSUAL AS THE RESULTS OF THIS SECTION SHOW.

- GENERAL PROBLEM CAN NOW BE FORMULATED:

WE ARE GIVEN A SEQUENCE OF FAIR COIN TOSSES $Z_1, Z_2, \ldots,$ AND WE WISH TO GENERATE RANDOM VARIABLE $X \in \mathcal{X} = \{1, 2, \ldots, m\}$ WITH PROBABILITY MASS FUNCTION $p = (p_1, p_2, \ldots p_m)$. LET THE RANDOM VARIABLE T DENOTE THE NUMBER OF COIN FLIPS USED IN THE ALGORITHM.



TREE FOR GENERATION OF DISTRO $\left(\frac{1}{2}\ \frac{1}{4}\ \frac{1}{4}\right)$

THE LEAVES OF THE TREE ARE MARKED BY OUTPUT SYMBOLS X, AND PATH TO THE LEAVES IS GIVEN BY THE SEQUENCE OF BITS PRODUCED BY FAIR COIN.

THE TREE REPRESENTING THE ALGORITHM MUST SATISFY CERTAIN PROPERTIES:

① THE TREE SHOULD BE COMPLETE (i.e. EVERY NODE IS EITHER A LEAF OR HAS TWO DESCENDANTS IN THE TREE. THE TREE MAY BE INFINITE.

② THE PROBABILITY OF LEAF AT DEPTH k IS 2^{-k}. MANY LEAVES MAY BE LABELED WITH SAME OUTPUT SYMBOL - THE TOTAL PROBABILITY OF ALL THESE LEAFS SHOULD EQUAL THE DESIRED PROBABILITY OF THE OUTPUT SYMBOL.

3. THE EXPECTED NUMBER OF FAIR BITS $E[T]$ REQUIRED TO GENERATE X IS EQUAL TO THE EXPECTED DEPTH OF THE TREE.

- EXAMPLE: THE MAPPING : 00 → a 01 → b 10 → c ALSO YIELDS THE DISTRIBUTION $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$. HOWEVER THIS ALGORITHM USES 2 FAIR BITS TO GENERATE EACH SAMPLE AND IS THEREFORE NOT AS EFFICIENT AS THE MAPPING EARLIER, WHICH ONLY 1.5 BITS PER SAMPLE.

- WHAT IS THE MOST EFFICIENT ALGORITHM TO GENERATE THE GIVEN DISTRIBUTION, AND HOW IS IT RELATED TO THE ENTROPY OF THE DISTRIBUTION?

- LET $\mathcal{Y}$ DENOTE THE SET OF LEAVES OF A COMPLETE TREE. CONSIDER DISTRIBUTION ON THE LEAVES SUCH THAT THE PROBABILITY OF A LEAF AT DEPTH „k" ON THE TREE IS 2^{-k}. LET Y BE RANDOM VARIABLE WITH THIS DISTRIBUTION. THEN WE HAVE:

LEMMA 5.11.1 FOR ANY COMPLETE TREE, CONSIDER A PROBABILITY DISTRIBUTION ON THE LEAVES SUCH THAT THE PROBABILITY OF A LEAF AT DEPTH „k" IS 2^{-k}. THEN THE EXPECTED DEPTH OF THE TREE IS EQUAL TO THE ENTROPY OF THIS DISTRIBUTION.

PROOF: $$E[T] = \sum_{y \in \mathcal{Y}} k(y)\, 2^{-k(y)} \quad \Big\} \text{ EXPECTED DEPTH OF THE TREE.}$$

- ENTROPY OF DISTRIBUTION OF Y IS:

$$H(Y) = -\sum_{y \in \mathcal{Y}} \frac{1}{2^{k(y)}} \operatorname{ld} 2^{-k(y)} = \sum_{y \in \mathcal{Y}} \frac{1}{2^{k(y)}} \cdot k(y)$$

$k(y)$ - DENOTES THE DEPTH OF LEAF y. THUS

$$\boxed{H(Y) = E[T]}$$

THEOREM 5.11.1 FOR ANY ALGORITHM GENERATING X, THE EXPECTED NUMBER OF FAIR BITS USED IS GREATER THEN THE ENTROPY $H(X)$, I.E.

$$E[T] \geq H(X)$$

- Y IS DEFINED SUCH THAT FOR ANY LEAF „y" AT DEPTH „k", THE PROBABILITY THAT $Y = y$ IS 2^{-k}. BY LEMMA 5.11.1 ⇒ $E[T] = H(Y)$
- SINCE X IS FUNCTION OF Y (ONE OR MORE LEAVES MAP ONTO AN OUTPUT SYMBOL). $X = f(Y)$

$$I(X;Y) = H(X) - \underbrace{H(X|Y)}_{0} = H(Y) - H(Y|X)$$

$$\Rightarrow H(Y) = H(X) + \underbrace{H(Y|X)}_{\geq 0}$$

$$\Rightarrow H(Y) \geq H(X) \Rightarrow H(X) \leq H(Y) = E[T] \Rightarrow$$

$$\boxed{H(X) \leq E[T]}$$ PROVED !!!

- DO SUMMARY IZVEDUVANJATA SE VO MultiuserMIMOcap1.wm

SUMMARY

- KRAFT INEQUALITY: INSTANTENOUS CODES $\Leftrightarrow \sum D^{-l_i} \leq 1$
- McMILAN INEQUALITY: UNIQUELY DECODABLE CODES $\Leftrightarrow \sum D^{-l_i} \leq 1$
- CUTOFF BOUND ON DATA COMPRESSION
  $L \triangleq \sum p_i l_i \geq H_D(X)$
- SHANNON CODE
  $l_i = \left\lceil ld_D \frac{1}{p_i} \right\rceil \qquad H_D(X) \leq L \leq H_D(X)+1$

$l_i \geq ld_D \frac{1}{p_i} \qquad D^{l_i} \geq \frac{1}{p_i} \qquad p_i \geq D^{-l_i} \ / ld$

$ld\, p_i \geq -l_i \ / \cdot p_i \qquad \underline{-p_i\, ld\, p_i} \leq l_i \cdot p_i \ / \sum$

$\boxed{H_D(X) \leq E[l(X)] = L}$

- HUFFMAN CODE $\quad \boxed{L^* = \min_{\sum D^{-l_i} \leq 1} \sum p_i l_i}$
  $\boxed{H_D(X) \leq L^* \leq H_D(X)+1}$
- WRONG CODE $\quad X \sim p(x),\ l(x) = \left\lceil ld \frac{1}{q(x)} \right\rceil,\ L = \sum p(x) l(x)$
  $H(p) + D(p\|q) \leq L \leq H(p) + D(p\|q) + 1$
- STOCHASTIC PROCESSES
  $\frac{H(X_1, X_2, \ldots X_n)}{n} \leq L_n < \frac{H(X_1, X_2, \ldots X_n)}{n} + \frac{1}{n}$
- STATIONARY PROCESS
  $L_n \to H(\mathcal{X})$
- COMPETITIVE OPTIMALITY. SHANNON CODE $l(x) = \left\lceil ld \frac{1}{p(x)} \right\rceil$ VERSUS ANY OTHER CODE $l'(x)$:
  $Pr\left(l(x) \geq l'(x) + c \right) \leq \frac{1}{2^{c-1}}$

PROBLEM 5.1 UNIQUELY DECODABLE AND INSTANTENOUS CODES. LET $L = \sum_{i=1}^{m} p_i l_i^{100}$ BE THE EXPECTED VALUE OF THE 100th POWER OF THE WORD LENGTHS ASSOCIATED WITH ENCODING OF THE RANDOM VARIABLE X. LET $L_1 = \min L$ OVER ALL INSTANTENOUS CODES AND LET $L_2 = \min L$ OVER ALL UNIQUELY DECODABLE CODES. WHAT INEQUALITY RELATIONSHIP EXIST BETWEEN L_1 & L_2?

$L = \sum_{i=1}^{m} p_i l_i^{100}$ $\qquad \mathcal{X} = \{x_1, x_2 \dots x_m\}$

$H(\mathcal{X}) \leq L_1 \leq H(\mathcal{X}) + 1$ $\qquad \boxed{L_1 = \sum_{i=1}^{m} p_i l_i}$

IF! $m = 4$

$L_1 = p_1 \cdot l_1^{100} + p_2 l_2^{100} + p_3 l_3^{100} + p_4 l_4^{100} \Rightarrow$ INSTANTENOUS

$c(1) = 0$; $c(2) = 10$; $c(3) = 110$; $c(4) = 111$

$p(1) = \frac{1}{2}$; $p(2) = \frac{1}{4}$; $p(3) = 1/8$; $p(4) = 1/8$

$L_I = \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2^{100} + \left(\frac{1}{8} \cdot 3^{100}\right) \cdot 2 = \frac{1}{2} + \frac{2^{100}}{4} + \frac{3^{100}}{4}$

$L_U = ?$ $\quad c(1) = 10 \quad c(2) = 00 \quad c(3) = 11 \quad c(4) = 110.$

$\quad p(1) = \frac{1}{2} \quad p(2) = \frac{1}{4} \quad p(3) = \frac{1}{8} \quad p(4) = \frac{1}{8}$

$L_U = \frac{1}{2} \cdot 2^{100} + \left(\frac{1}{4} \cdot 2^{100}\right) + \frac{1}{8} 2^{100} + \frac{1}{8} 3^{100} = \frac{2^{100}}{2} + \frac{3 \cdot 2^{100}}{8} + \frac{3^{100}}{8} =$

$= \frac{7}{8} 2^{100} + \frac{1}{8} \cdot 3^{100}$ $\qquad \boxed{L_U < L_I}$

$L_{1I} = \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 + 2 \frac{1}{8} \cdot 3 = \frac{1}{2} + \frac{2}{4} + \frac{3}{4} = \frac{7}{4} = \underline{1.75}$

$H_i(X) = \frac{1}{2} \operatorname{ld} 2 + \frac{1}{4} \cdot 2 + \left(\frac{1}{8} \cdot 3\right) \cdot 2 = \frac{1}{2} + \frac{2}{4} + \frac{3}{4} = \frac{7}{4} = \underline{\underline{1.75}}$

$L_{1U} = \frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 2 + \frac{1}{8} \cdot 3 = \frac{8 + 4 + 5}{8} = \frac{17}{8} = \underline{\underline{2.125}}$

$\boxed{H_u(X) = H_i(X) = 1.75}$ $\qquad \boxed{L_{1U} > L_{1I}}$

$E[l^{100}(x)] = \sum_{x \in \mathcal{X}} p(x) l^{100}(x)$ $\qquad E^{100}[l(x)] = \left[\sum_{x \in \mathcal{X}} p(x) \cdot l(x)\right]^{100}$

EDITION 2 SOLUTIONS

$L = \sum_{i=1}^{m} p_i l_i^{100} \qquad L_1 = \min_{\substack{\text{INST.} \\ \text{CODES}}} \{L\} \qquad L_2 = \min_{\substack{\text{UNIQUE} \\ \text{DECO. CODES}}} \{L\}$

SINCE ALL INSTANTENOUS CODES ARE UNIQUELY DECODABLE, WE MUST HAVE $L_2 \leq L_1$. ANY SET OF CODEWORD LENGTHS WHICH ACHIEVE THE MINIMUM OF L_2 WILL SATISFY THE KRAFT INEQUALITY AND HENCE WE CAN CONSTRUCT INSTANTENOUS CODE WITH SAME CODEWORD LENGTHS, AND HENCE THE SAME L. HENCE WE HAVE $L_1 \leq L_2$. ~~FROM BOTH THIS CONDITIONS WE~~ MUST HAVE: $\boxed{L_1 = L_2}$

FROM ANY LENGTHS SATISFING KRAFT INEQUALITY INSTANTENOUS CODE CAN BE CONSTRUCTED.

INSTANTENOUS : 0, 10, 110, 111 ①

UNIQUELY D. : 00, 10, 11, 110

$\left\{\begin{matrix} 00 & 01 & 10 & 11 \\ 0 & 1 & 2 & 3 \end{matrix}\right.$

→ 0011 IS UNIQUELY DECODABLE ?

00100111 = 11

0 2 1 2

EXAMPLE (WIKIPEDIA) $\{\underline{a \to 1,\ b \to 011,\ c \to 01110,\ d \to 1110,\ e \to 10011}\}$

THE CODE IS NOT UNIQUELY DECODABLE SINCE THE STRING: 011101110011

CAN BE INTERPRETED AS:

01110 - 1110 - 011 , AND ALSO AS:

011 - 1 - 011 - 10011

$\boxed{cdb}$ $\boxed{babe}$ TWO POSSIBLE ENCODINGS

x_1 IS PREFIX OF y_1 I.E. $\boxed{x_1 w = y_1}$

IF $x_1 = 011$ & $y_1 = 01110$ $\boxed{w = 10}$

IF WE FIND:

$x_2, \dots x_p$ AND $y_2, \dots y_q$ SUCH THAT

$x_2, \dots x_p = w\, y_2 \dots y_q$ WE ARE FINISHED

THEN: $x = x_1 x_2 \dots x_p$ CAN BE DECOMPOSED AS $y_1 y_2 \dots y_q$

I.E. THE DESIRED STRING HAS AT LEAST TWO DIFFERENT DECOMPOSITIONS INTO CODEWORDS

→ $$\sum_i D^{-l_i} = 2^{-2} \cdot 4 = \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}\right) \cdot 4 = 1$$

① ⇒ $$\sum_i D^{-l_i} = 2^{-1} + 2^{-2} + 2^{-3} + 2^{-3} = \frac{1}{2} + \frac{1}{4} + 2 \cdot \frac{1}{8} = 1$$

$$L_U = 2\left(\frac{1}{2} + \frac{1}{4} + 2 \cdot \frac{1}{8}\right) = 2\left(\frac{1}{2} + \frac{1}{2}\right) = 2$$

$$L_U = 1\,\frac{1}{2} + 2 \cdot \left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} = \underline{0.75}$$

$\underbrace{0}_{y_2},\ \underbrace{10}_{x_2},\ \underbrace{11}_{x_1},\ \underbrace{111}_{y_1}$ $\qquad 2^{-1} + 2^{-2} + 2^{-2} + 2^{-1} = 1$

$\boxed{w = 1}$ $\qquad \boxed{x_2 = w\, y_2 \quad 10 = 10}$

$\underbrace{1110}_{x_1 x_2}$ $\underbrace{1110}_{y_1 y_2}$

DOKAZAV. DEKA OVOJ KOD NE E UNIQUELY ~~DE~~CODABLE.

NO ZAJOA KODOT:

0 10 11 110 NE E UNIQUELY DECODABLE

0 10 11 110
a b c d

$\underbrace{11}_{c}\underbrace{0}_{a}$ } ≙ d

0 10 01 110
a b c d

$\underbrace{01}_{c}\underbrace{10}_{b}$ = a d = $0\underbrace{110}_{a\ d}$ } UNIQUELY DECODABLE

1 01 11 110
a b c d

1101 → cb
→ da

0 01 11 110
a b c d

0110

$\underbrace{110}_{d}\underbrace{01}_{b}\underbrace{11}_{c}$ 11 $\boxed{0}\boxed{0}$ 111
c a b c

a a d c

0 01 10 111
a b c d

$\underbrace{0}_{a}\underbrace{01}_{b}\underbrace{111}_{d}\underbrace{0}_{a}$

- **SARDINAS – PETERSON** (BASIC CONCEPTS IN INFORMATION THEORY AND CODING: THE ADVENTURES OF SECRET AGENT 00111)

Seg 0: 0 10 11 110
Seg 1: ∅
Seg 2:

Seg 0: 0 10 11 111
Seg 1: 1
Seg 2: (b) 0, 1, 10

NONDECODABLE

MMV

(a) Vo Seg k pricaat gi site sufix-i na clenovi od Seg k-1 koi imaat clenovi od Seg ∅ kako prefix-i.
(b) OBRATNO!

Seg 0: 00, 10 11 110
Seg 1: 0 → NON COMPLETE WORD ⇒ UNIQUELY DECODABLE

Seg 0: 0, 01, 11, 100
Seg 1: 1
Seg 2: 1, 00
Seg 3: $\underbrace{1, 00}_{(b)}$, $\underbrace{0}_{(a)}$

NON UNIQUE DECOD

Seg 0: 0, 10, 110, 111
Seg 1: /

Seg 0: 0, 10, 11, 001
Seg 1: 01
Seg 2: 1
Seg 3: 0 NON UNIQUE DECOD

Seg 0: 0 01 11
Seg 1: 1
Seg 2: 1

THIS IS UNIQUELY DECODABLE

Seg 0: 0, 01, 001, 11
Seg 1: 0, 01
Seg 2: 1, 1
Seg 3: 1

NE E UNIQUELY DECODABLE!!!

THIS IS UNIQUELY DECODABLE CODE ((!?

a	b	c	d
0	01	11	001
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$

$1\cdot\frac{1}{2}+2\cdot\frac{1}{4}+2\cdot\frac{1}{8}+3\cdot\frac{1}{8}=\frac{4+4+2+3}{8}=\frac{13}{8}=\boxed{1.625}$

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ISTO TAKA UNIQUELY DECODABLE CODE E:

Seg0: $\boxed{0 \quad 01 \quad 001 \quad 111}$

Seg 1: 1, 01 → NE E UNIQUELY DECODABLE!!!

Seg 2: $\underbrace{1}_{a}, \underbrace{11}_{b}$

Seg 3: 11, 1

- - - - -

OVOJ KOD IMA ISTA OČEKUVANA SREDNA DOLŽINA KAKO I MOMENTALNIOT KOD: 0, 10, 110, 111

SARDINAS-PATTERSON ALGORITHM | MMV

1. LET Seg 0 BE THE COLLECTION OF ALL CODE WORDS
2. IN Seg 1 LIST THE SUFFIXES OF CODE WORDS WITH MEMBERS OF Seg 0 AS PREFIXES
3. CONSTRUCT Seg k $k>1$ USING FOLLOWING PROCEDURE
   a. IN Seg k, LIST SUFFIXES OF MEMBERS IN Seg k-1 HAVING MEMBERS OF Seg 0 AS PREFIXES
   b. IN Seg k, LIST SUFFIXES OF MEMBERS OF Seg 0, HAVING MEMBERS OF Seg k-1 AS PREFIXES.

TEST: IF FOR ANY $k>0$, Seg k CONTAINS A COMPLETE WORD, THEN THE CODE IS NOT UNIQUELY DECODABLE IN ANY SENCE

IF NO Seg k OTHER THEN Seg 0 CONTAINS A CODEW, THEN THE CODE FORMS UF DICTIONARY.

- DALI ZA KODOT ★ VAŽI KRAFTOVOTO NERAVENSTVO.

$$\left.\begin{matrix} 0 & 01 & 001 & 11 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} \end{matrix}\right\} \Rightarrow L = \frac{1}{2} + \frac{1}{4}\cdot 2 + 3\,\frac{1}{8} + 2\cdot\frac{1}{8} = = \frac{4+4+3+2}{8} = \frac{13}{8}$$

$$\sum 2^{-l} = \frac{1}{2} + 2^{-2} + 2^{-2} + 2^{-3} = \frac{1}{2} + \left(\frac{1}{4}\right) + \left(\frac{1}{4}\right) + \frac{1}{8} = \underbrace{1 + \frac{1}{8}}_{>1}$$

ZNAČI KRAFTOVOTO NERAVENSTVO NE VAŽI ZA ★ PA NORMALNO!!! ZARADI POSTOENJETO NA „01" VO Seg1 ZA ★ I $ JASNO E DEKA OVIE DVA KODA NE SE UNIQUELY DECODABLE

S0: 0 10 001 11
S1: 01
S2: 1
S3: 0 NE

S0: 0 01 100 11
S1: 1
S2: 00, 1
S3: 0, 1, 00 NE

S0: 0 10 101 111
S1: 1 NE
S2: 0

S0: 0 11 101 111
S1: 1 1 NE
S2: 1, 11

0 01 10 111 | NE
128, 0 NE

S0: 0 01 110 111
S1: 1
S2: 10, 11
S3: 0, 1 NE

S0: 1 01 010 100

S1: 00, 0

S2: 1

(NE)

S0: ⓪ 01 011 000

S1: 1, 11, 10, 00

S2: ⓪

NE

S0: 00 10 11 110

S1: 0

S2: 0

(DA)

0 01 011 0111

1 11 111 } UNIQUELY D.

ⓐ

S0: 0 10 01 11

S1: 1

S2: 0, 1

NE

$$\frac{1}{2}\cdot 2 + \frac{1}{4}\cdot 2 + \frac{1}{8}\cdot 2 + \frac{1}{8}\cdot 3 = \frac{8+4+2+3}{8} = \frac{17}{8} = \underline{\underline{2.125}}$$

$$ⓑ \rightarrow \quad \frac{1}{2} + 2\cdot\frac{1}{4} + 3\cdot\frac{1}{8} + 4\cdot\frac{1}{8} = \frac{1}{2}+\frac{1}{2}+\frac{4+3}{8} = 1+\frac{7}{8} = \frac{15}{8}$$

$$= \underline{\underline{1.875}}$$

DEFINITIVNO NE MOŽAM ~~DA~~ DA NAJDAM UNIQUELY DECODABLE CODE KOJ E RAZLIČEN OD (0, 10, 110, 111) (KOJ E INSTANTENOUS) I KOJ IMA ISTA PROSEČNA DOLŽINA i.e. 1.75.

PROBLEM 2 HOW MANY FINGERS HAS A MARTIAN? LET

$$S = \begin{pmatrix} S_1, \cdots, S_m \\ p_1, \cdots p_m \end{pmatrix}$$

THE S_i'S ARE ENCODED INTO STRINGS FROM D-SYMBOL OUTPUT ALPHABET IN UNIQUELY DECODABLE MANER. IF $m=6$ AND THE CODEWORD LENGTHS ARE $(l_1, l_2, \dots l_6) = (1, 1, 2, 3, 2, 3)$ FIND GOOD LOWER BOUND ON D. YOU MAY WISH TO EXPLAIN THE TITLE OF THE PROBLEM.

<u>SOLUTION</u>:

$$D^{-1} + D^{-1} + 2\cdot D^{-2} + 2\cdot D^{-3} \leq 1$$

$$2D^{-1} + 2D^{-2} + 2D^{-3} \leq 1 \qquad D^{-1} + D^{-2} + D^{-3} \leq \frac{1}{2}$$

$$\frac{1}{D} + \frac{1}{D^2} + \frac{1}{D^3} \leq \frac{1}{2} \qquad \frac{D^2+D+1}{D^3} \leq \frac{1}{2}$$

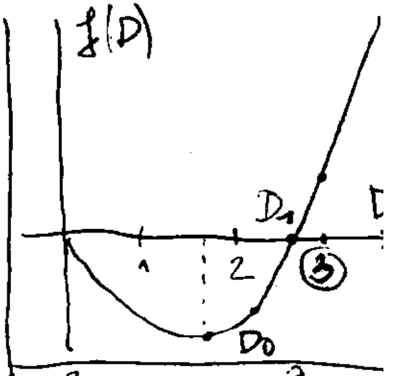
$$D^2+D+1 \leq \frac{D^3}{2} \qquad D^3 - 2D^2 - 2D - 2 \geq 0$$

$$f(D) = D^3 - 2D^2 - 2D - 2 \qquad \frac{df(D)}{dD} = 0 \qquad 3D^2 - 4D - 2 = 0$$

$$D_1^3 - 2D_1^2 - 2D_1 - 2 = 0$$

$$\boxed{D_1 = +2.91964} \doteq 3 \,//$$

$$D_0 = \frac{4 \pm \sqrt{16+24}}{6} = \frac{4 \pm 2\sqrt{10}}{6} = \frac{2}{3} \oplus \frac{\sqrt{10}}{3} \; 1.72$$



$\boxed{D=3}$

$$(l_1 l_2 l_3 l_4 l_5 l_6) = (1\ 1\ 2\ 3\ 2\ 3)$$

$$\gamma = \left(\tfrac{1}{2}\ \tfrac{1}{4}\ \tfrac{1}{8}\ \tfrac{1}{16}\ \tfrac{1}{16}\ \tfrac{1}{32}\right)$$

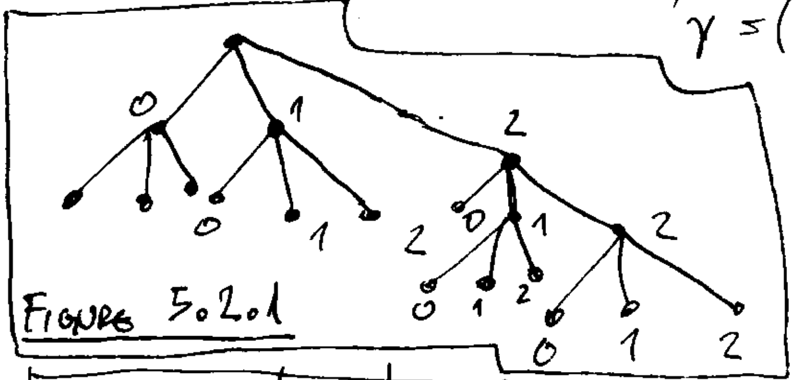


FIGURE 5.2.1

$D^{3-3} = D^0 = 1$

$D^{3-2} = 3$

$D^{3-1} = 9$

$D^3 = 27$

NODES AT EACH LEVEL!!

x_1	$\frac{1}{2}$	I			0
x_2	$\frac{1}{4}$	II			1
x_3	$\frac{1}{8}$		III		10
x_4	$\frac{1}{16}$			IV	101
x_5	$\frac{1}{32}$			IV_1	1010
x_6	$\frac{1}{32}$			IV_2	1011

Ne more so FANO da odiš !!!

x_1	1/2	1/2	1/2 0	1	1
x_2	1/4	1/4 0	1/2 1		00
x_3	1/8	1/8 1	0 2		02
x_4	1/16 0	1/8 2			010
x_5	1/32 1				011
x_6	1/32 2				012 → l_{max}

S_0: 1, 00, 02, 010, 011, 012

S_1:

INDEX CODE

$3^{③} = 27$ l_{max}

PROBLEM 5.3 SLACKNESS IN KRAFT INEQUALITY.

AN INSTANTENOUS CODE HAS WORD LENGTHS $l_1, l_2, \ldots l_m$ WHICH SATISFY THE STRICT INEQUALITY.

$$\sum_{i=1}^{m} D^{-l_i} < 1$$

THE CODE ALPHABET IS $D = \{0, 1, 2, \ldots, D-1\}$ SHOW THAT THERE EXIST ARBITRARY LONG SEQUENCE OF CODE SYMBOLS IN D^* WHICH CANNOT BE DECODED INTO SEQUENCES OF CODEWORDS. [IE NOT ALL SEQUENCES OF SYMBOLS IN D^* FORM A CODEWORD]

SOLUTION 2

LET $l_{max} = \max\{l_1, l_2, \ldots, l_m\}$

THERE ARE $D^{l_{max}}$ SEQUENCES OF LENGTH l_{max}. OF THESE SEQUENCES $D^{l_{max} - ⓛ_i}$ START WITH i-TH CODEWORD

$D^{3 - ②} = 3^{3-2} = 3$ ∴ BECOUSE OF THE PREFIX CONDITION NO TWO ~~SEQUENCES~~ CAN START WITH SAME CODEWORD. HENCE THE TOTAL NUMBER OF SEQUENCES THAT START WITH SOME CODEWORD IS:

$$\sum_{i=1}^{2} D^{n_{max}-n_i} = D^{n_{max}} \sum_{i=1}^{2} D^{-n_i} \leq D^{n_{max}}$$

(1, 00, 02, 010, 011, 012)

– DA GO RAZGLEDUVAME PRETHODNIOT PRIMER:

$$(n_1 n_2 n_3 n_4 n_5 n_6) = (1\ 1\ 2\ 3\ 2\ 3)$$

$n_1\ n_2\ n_3\ n_4\ n_5\ n_6 = n_2$

$$n_{max} = \max(1\ 1\ 2\ 3\ 2\ 3) = 3$$

$$D^{n_{max}} = 3^3 = \boxed{27}$$ — TOTAL NUMBER OF SEQUENCES WITH LENGTH n_{max}

000 010 020 100 110 120 200 210 220
001 011 021 101 111 121 201 211 221
002 012 022 102 112 122 202 212 222 } 27 NODES IN L3

$$D^{n_{max}-1} = D^{3-1} = 3^2 = 9$$

00 10 20
01 11 21
02 12 22 } 9 NODES IN L2

$$D^{n_{max}-2} = D^{3-2} = 3$$

0
1
2 } 3 NODES IN L1

VKUPNOT BROJ NA SEKVENCI ŠTO ZAPOČNUVAAT SO NEKOJ KODEN ZBOR E:

$$\sum_{i=1}^{2} D^{n_{max}-n_i} = D^{n_{max}}\left(D^{-1}+D^{-2}+D^{-3}\right) = 3^3\left(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}\right)$$

$$= 3^2+3+1 = 9+3+1 = 13$$

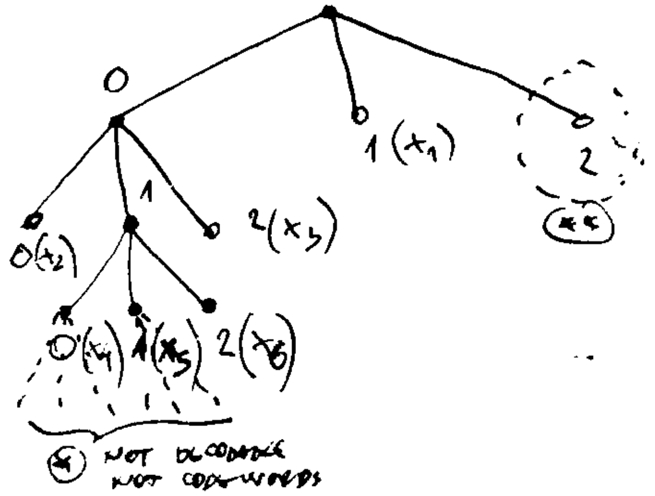
- TRINITY COLLEGE DUBLIN (soln. q1-2.pdf) MMV

$$\sum_{i=1}^{m} D^{-l_i} < 1 \ / \cdot D^{l_x}$$ l_x – MAXIMUM LENGTH

$$\underbrace{\sum_{i=1}^{m} D^{l_x-l_i}}_{} < D^{l_x}$$ → FIG 5.2.1 } NUMBER OF NODES AT LEVEL l_x (FINAL LEVEL)

FIG 5.2.1 – NUMBER OF NODES IN ALL PREVIOUS LEVELS

THE STRICT INEQUALITY MEANS THAT THERE ARE NODES AT LEVEL l_x WHICH ARE NOT CODEWORDS AND WHICH ARE NOT DESCENDANTS OF CODEWORDS. HENCE THERE ARE SEQUENCES THAT DO NOT START WITH ANY CODEWORD (**). THESE SEQUENCES AND SEQUENCES FOR WHICH SEQUENCES WITH LENGTH "l_x" ARE PREFIXES (*) CANNOT "BE DECODED.



(*) NOT DECODABLE
NOT CODEWORDS

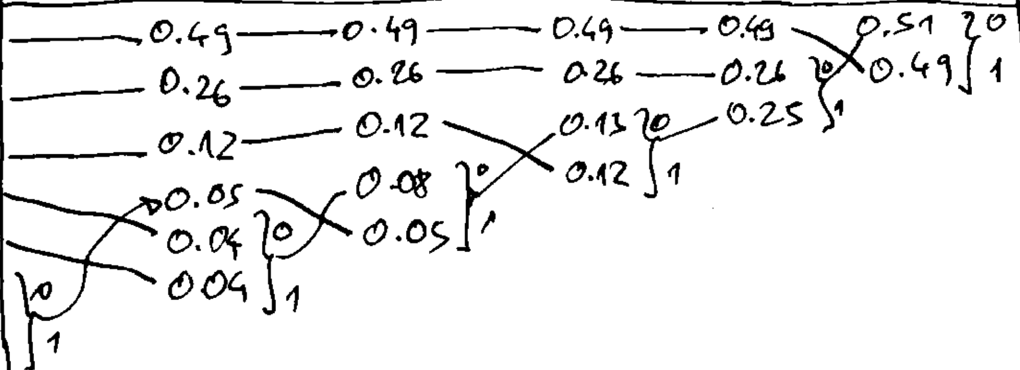
5.4 HUFFMAN CODING. CONSIDER THE RANDOM VARIABLE

$$X = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 0.49 & 0.26 & 0.12 & 0.04 & 0.04 & 0.03 & 0.02 \end{pmatrix}$$

(a) FIND BINARY HUFFMAN CODE FOR X

(b) FIND THE EXPECTED CODE LENGTH FOR THIS ENCODING

(c) FIND A TERNARY HUFFMAN CODE FOR X

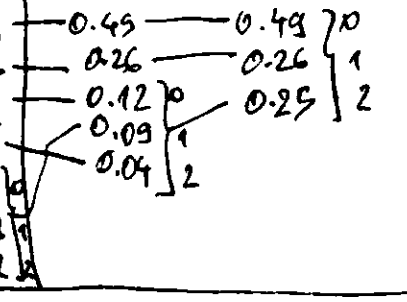


X	P	C(x)	l(x)
x_1	0.49	1	1
x_2	0.26	00	2
x_3	0.12	011	3
x_4	0.04	01000	5
x_5	0.04	01001	5
x_6	0.03	01010	5
x_7	0.02	01011	5

$$E[L(X)] = 1\cdot 0.49 + 2\cdot 0.26 + 3\cdot 0.12 + 5(0.04 + 0.04 + 0.05) =$$

$$= 0.49 + 0.52 + 0.36 + \overbrace{5\cdot 0.13}^{0.65} = 1{,}01 + 0.36 + 0.65 = 1.01 + 1{,}01 = \underline{\underline{2.02}}$$

$$H(X) = -(0.49\,ld\,0.49 + 0.26\cdot ld\,0.26 + 0.12\,ld\,0.12 + 2\cdot 0.04\,ld\,0.04 + 0.03\,ld\,0.03$$
$$+ 0.02\,ld\,0.02) = 2.01279$$



X	P	C(x)	l(x)
x_1	0.49	0	1
x_2	0.26	1	1
x_3	0.12	20	2
x_4	0.04	22	2
x_5	0.04	210	3
x_6	0.03	211	3
x_7	0.02	212	3

$$E[l(X)] = 1\cdot 0.49 + 1\cdot 0.26 + 2\cdot 0.12 + 0.08 + 0.12 + 0.09 + 0.06 =$$

$$= 0.75 + 0.24 + 0.2 + 0.15 = 0.99 + 0.35 = \underline{1.34}$$

$$H_3(X) = -(0.49\log_3(0.49) + 0.26\log_3(0.26) + \dots) = \underline{\underline{1.26993}}$$

$$ld\,x = y \qquad 2^y = x \quad / \ln = \qquad y\ln 2 = \ln(x) \Rightarrow y = \frac{\ln x}{\ln 2}$$

$$\log_3 x = y \qquad 3^y = x \quad / \,ld \qquad y\,ld\,3 = ld\,x \qquad \boxed{y = \frac{ld\,x}{ld\,3}}$$

$$H_3(X) = \frac{H(X)}{ld\,3} = \frac{2.01279}{ld\,3} = 1.26193$$

5.5 MORE HUFFMAN CODES.

FIND THE BINARY HUFFMAN CODE FOR THE SOURCE WITH PROBABILITIES $(\frac{1}{3}, \frac{1}{5}, \frac{1}{5}, \frac{2}{15}, \frac{2}{15})$ ARGUE THAT THIS CODE IS ALSO OPTIMAL FOR SOURCE WITH PROBABILITIES $(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$

X	p	p			c(x)	l(x)
x1	1/3	1/3	1/3	2/3]0	1	1
x2	1/5	4/15	2/5]0	1/3]1	000	3
x3	1/5	1/5]0	4/15 }1		001	3
x4	2/15]0	1/5]1			010	3
x5	2/15 }1		2 ČEKOR		011	3

★

$$E[l(x)] = \frac{1}{3} + \left(3\frac{1}{5}\right)2 + \frac{6}{15}\cdot 2$$
$$= \frac{1}{3} + \frac{6}{5} + \frac{12}{15} =$$
$$= \frac{5+18+12}{15} = \frac{35}{15}$$
$$= \frac{7}{3} = 2.33$$

$$H(x) = \frac{1}{3} ld 3 + \frac{2}{5} ld 5 + \frac{4}{15} ld \frac{15}{2} = 2.23$$

OVA E OPTIMALNO!!!

X	p				C(x)	l(x)
x1	1/5	2/5	2/5	3/5]0	01	2
x2	1/5	1/5	2/5]0	2/5]1	10	2
x3	1/5	1/5]0	1/5]1		11	2
x4	1/5]0	1/5 }1			000	3
x5	1/5 }1	(*)			001	3

$$E[l(x)] = 3\cdot 2\cdot\frac{1}{5} +$$
$$2\cdot 3\frac{1}{5} =$$
$$= \frac{6}{5} + \frac{6}{5} = \frac{12}{5} =$$
$$= 2.4$$ OK!!!

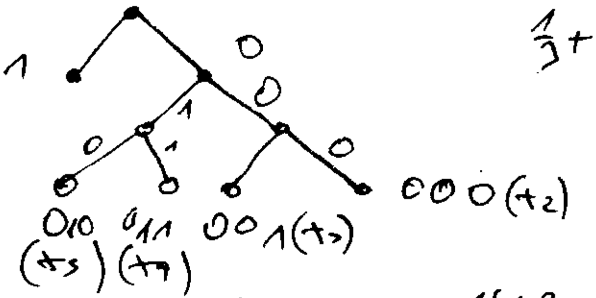
X	p					l(x)
x1	1/5	1/5	1/5	4/5]0	1	1
x2	1/5	2/5	2/5]0	1/5]1	000	3
x3	1/5	1/5]0	2/5 }1		001	3
x4	1/5]0	1/5 }1			010	3
x5	1/5 }1				011	3

$$\boxed{2.4 \leq 2.6}$$

(*) HUFFMAN CODE IS OPTIMAL!!!

$$E[l(x)] = 1\cdot\frac{1}{5} +$$
$$+ 3\cdot\frac{1}{5}\cdot 4 = \frac{1}{5} + \frac{12}{5}$$
$$= \frac{13}{5} = 2.6$$

OVA NE SMEEŠ DA GO PRAVIŠ ZOŠTO NE DOVA OPTIMALNA DOLŽINA NA CODOT!!!



$$\frac{1}{3} + \frac{1}{5} + \frac{1}{5} + \frac{2}{15} + \frac{2}{15} =$$
$$\frac{5+6}{15} + \frac{4}{15} = \frac{15}{15} = 1$$

$$\left(\frac{1}{5}\right)^2\cdot 3 + \left(\frac{1}{5}\right)^3\cdot 2 = \frac{1}{25}\cdot 3 + \frac{1}{125}\cdot 2 = \frac{15+2}{125} = \frac{17}{125}$$

$$\left(2^{-2}\right)\cdot 3 + \left(2^{-3}\right)\cdot 2 = \frac{1}{4}\cdot 3 + \frac{2}{8} = \frac{3}{4} + \frac{1}{8} = 1$$

(*) $$\frac{1}{2} + 4\cdot 2^{-3} = \frac{1}{2} + 4\cdot\frac{1}{8} = \frac{1}{2} + \frac{1}{2} = 1$$

$$L = E[l(x)] = \sum_{i=1}^{n} l_i p(x_i) = \sum_{x\in X} l(x) p(x)$$

★ MOV ZGREŠENO VO VTOROT ČEKOR!!!

$\frac{2}{5} > \frac{1}{3}$ TREBA DA ODI NAGORE!!! ZATOA OVOJ DAVA:

$$\boxed{2{,}26 > 2{,}33}$$

x	p				C(x)	l(x)
x1	1/3	1/3	2/5	3/5]0	00	2
x2	1/5	4/15	1/3]0	2/5]1	10	2
x3	1/5	1/5]0	4/15 }1		11	2
x4	2/15]0	1/5]1			010	3
x5	2/15 }1				011	3

$$l(x) = \frac{1}{3}\cdot 2 + \left(\frac{1}{5}\cdot 2\right)2 +$$
$$+ 2\left(3\frac{2}{15}\right) = \frac{2}{3} + \frac{4}{5} +$$
$$+ \frac{12}{15} = \frac{2}{3} + \frac{4}{5} + \frac{4}{5} >$$
$$= \frac{10+24}{15} = \frac{34}{15} = 2.26$$ 2.33

3220392 - OTVORENA SLUŠALKA

$$\boxed{E[L(*)] = \frac{12}{5} = 2,4 \text{ bits}}$$

$H(X) = 5 \cdot \frac{1}{5} \cdot ld5 = ld5 = 2.32$

$E[L(\text{ANY CODE})] = \sum_{i=1}^{5} \frac{l_i}{5} = \frac{(k)}{5}$ bits

NOTE LOWER POSIBLE VALUE OF $E[L]$ IS: $\frac{11}{5}$

$\frac{11}{5} = 2.2$ $\quad E[L] \geq H(X) \quad E[L] \geq 2.32$

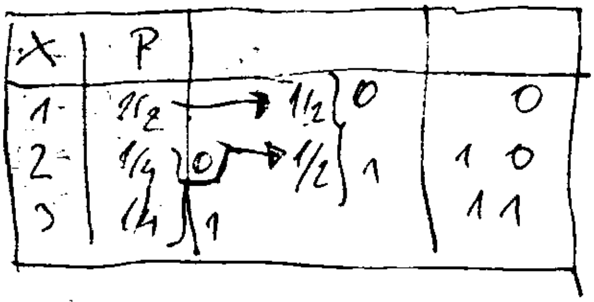
NE MOZE DA BIDE POMALA OČEKUVANA DOLŽINA OD H(X) IA SLEDUVA:

$E[L(*)] = \frac{12}{5}$ } E NAJMALATA SLEDNA DOLŽINA i.e (*) CODOT E OPTIMALEN

ALTERNATIVELY: EACH HUFFMAN CODE PRODUCED BY HUFFMAN ENCODING IS OPTIMAL, AND THEY ALL HAVE THE SAME AVERAGE LENGTH ' !!!

PROBLEM 5.6 BAD CODES WHICH OF THESE CODES CANNOT BE HUFFMAN CODES FOR ANY PROBABILITY ASSIGNMENT?

(a) {0, 10, 11} (b) {00, 01, 10, 110} (c) {01, 10}



(a) HUFFMAN CODE

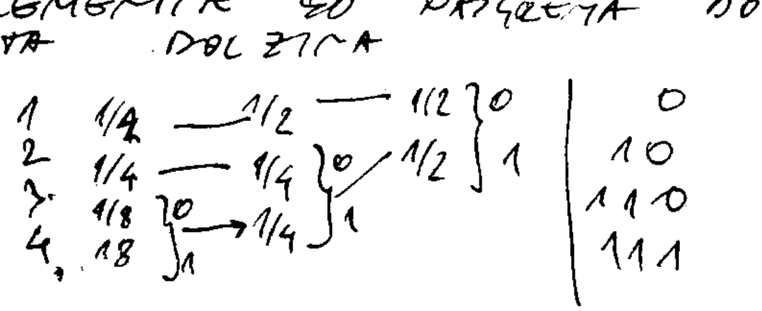
(b) H(X) IS MAXIMAL FOR UNIFORM DISTRIBUTION.

$p = \left[\frac{1}{2} \ \frac{1}{4} \ \frac{1}{8} \ \frac{1}{8}\right]$

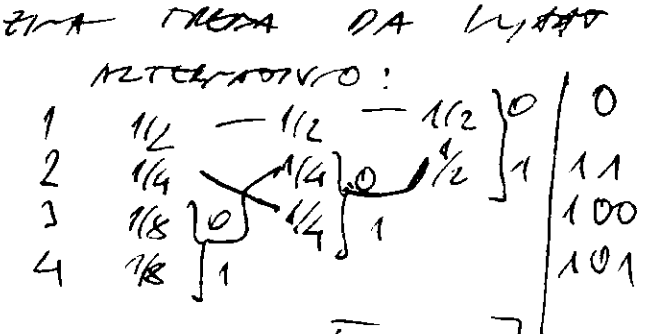
$H(X) = \frac{1}{2} ld2 + \frac{1}{4} \cdot 2 + \left(\frac{1}{8} \cdot 3\right) \cdot 2 = \frac{1}{2} + \frac{1}{2} + \frac{3}{4} = \frac{2+2+3}{4} = 1.75$

$L(X) = 2 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 2 \cdot \frac{1}{8} + 3 \frac{1}{8} = 1 + \frac{1}{2} + \frac{5}{8} = \frac{8+4+5}{8} = \frac{17}{8} = \underline{2.125}$

(b) NE E HUFFMANOV code ~~ZATOA~~ ŠTO NE E ISPOLNETA LEMMA 5.8.1 KOJA VELI ~~DEKA~~ ELEMENTITE SO NAJGOLEMA DOLŽINA TREBA DA IMAAT ISTA DOLŽINA



ALTERNATIVNO:



KAKO I DA VRTIŠ OPTIMALNITE DOLŽINI $\in \{1, 2, 3\}$

$E[l_0(X)] = \frac{1}{2} + 2\frac{1}{4} + 2\left(3\frac{1}{8}\right) = \frac{2}{4} + \frac{2}{4} + \frac{3}{4} = \frac{7}{4} = 1.75$

$E[l_b(X)] = \left(2 \cdot \frac{1}{2}\right) + 2 \cdot \frac{1}{4} + 2 \cdot \frac{1}{8} + 3 \cdot \frac{1}{8} = \frac{4}{4} + \frac{2}{4} + \frac{1}{4} + \frac{3}{8} = \frac{7}{4} + \frac{3}{8} = \frac{17}{8} = 2.125$

JAŠKA NE NI OPTIMALEN

(c) NE E HUFFMAN-OV ZATOA ŠTO NE E SO MINIMALNA SREDNA DOLŽINA

$X = \{x_1, x_2\}$; $p = [p, q]$

$E[l_c(x)] = 2 \cdot p + 2 \cdot q = 2$ } POŠTOI KOD SO SREDNA DOLŽINA 1 $\Rightarrow$

$\underline{\{0, 1\}}$

- IN OTHER WORDS (b) CAN BE SHORTENED TO: $\{00, 01, 10, 11\}$
- AND (c) CAN BE SHORTENED TO $\{0, 1\}$

PROBLEM 5.7 HUFFMAN 20 QUESTIONS. CONSIDER A SET OF "n" OBJECTS. LET $X_i = 1$ OR 0 ACCORDINGLY AS THE i-th OBJECT IS GOOD OR DEFECTIVE. LET $X_1, X_2, \ldots, X_n$ BE INDEPENDENT WITH $Pr\{X_i = 1\} = p_i$; AND $p_1 > p_2 > \ldots > p_n > \frac{1}{2}$. WE ARE ASKED TO DETERMINE THE SET OF DEFECTIVE OBJECTS. ANY YES-NO QUESTION YOU CAN THINK OF IS ADMISSIBLE.

(a) GIVE A GOOD LOWER BOUND ON THE MINIMUM AVERAGE NUMBER OF QUESTIONS REQUIRED.

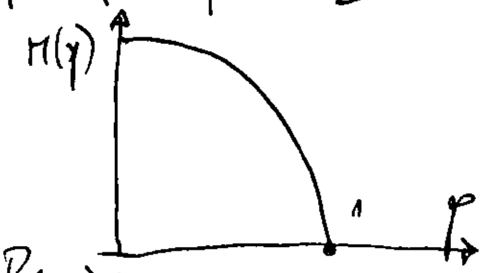
(b) IF THE LONGEST SEQUENCE OF QUESTIONS IS REQUIRED BY "NATURE'S" ANSWERS TO OUR QUESTIONS, WHAT (IN WORDS) IS THE LAST QUESTION WE SHOULD ASK? ASSUME COMPACT (MINIMUM AVERAGE LENGTH) SEQUENCE OF QUESTIONS. WHAT TWO SETS ARE WE DISTINGUISH WITH THIS QUESTION.

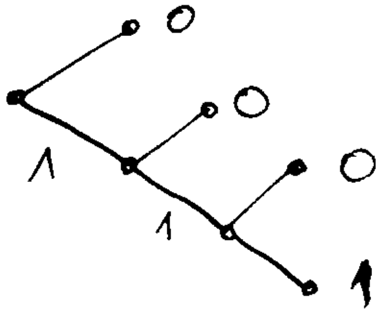
(c) GIVE AN UPPER BOUND (WITHIN ONE QUESTION) ON THE MINIMUM AVERAGE NUMBER OF QUESTIONS REQUIRED.

(a) $\boxed{n=3}$ $X = \{X_1, X_2, X_3\}$ $X_1 = \{0, 1\}$ $p = \{1-p_1, p_1\}$

$X_2 = \{0, 1\}$ $p = \{1-p_2, p_2\}$; $X_3 = \{0, 1\}$, $p = (1-p_3, p_3)$

$p_1 > p_2 > p_3 > \frac{1}{2}$





c(x)	
0	X_1 IS DEFECT.
10	X_2 IS DEFECT.
110	X_3 IS DEFECT.

- PRAŠANJETO E DALI MOŽE SITE TIE DA SE NEISPRAVNI ILI MOŽE SAMO EDEN. MOŽE TOGAŠ KODOT ĆE BIDE: 010110. AKO E SAMO X_2 KODOT E: 10 I TN.

$P(x_i=1) = \gamma_i \qquad \gamma_1 > \gamma_2 > \dots > \gamma_n > \frac{1}{2}$

$\boxed{H(X) \leq E[Q_n] < H(X)+1}$ $\quad Q_n \rightarrow$ EXPECTED AVERAGE LENGTH PER SYMBOL

$$H(X) = \lim_{n\to\infty} \frac{1}{n} H(x_1, x_2, \dots x_n) = \frac{1}{n}\left(H(x_1)+H(x_2)+\dots H(x_n)\right)$$

$$0 < H(x_1) < H(x_2) < H(x_3) \dots < \underline{\underline{H(x_n)}} < 1$$

$$H(X) = \frac{1}{n}\left(H(x_1)+H(x_2)\dots+H(x_n)\right) < \frac{1}{n} n \cdot H(x_n) < \left(\frac{1}{2}\operatorname{ld} 2\right)\cdot 2 = 1$$

$\bigcirc\!\!\!\!\!\!\!\!\!\!\quad H(X) < 1 \qquad E[Q_n] > H(X) < 1 \quad ?$

$$H(X) = \frac{1}{n}\left(H(x_1)+H(x_2)+\dots H(x_n)\right) > \frac{1}{n} n \cdot H(x_n) > (1\cdot \operatorname{ld} 1)\cdot 2 = 0$$

$H(X) > 0 \qquad \boxed{0 < H(X) < 1}$

$0 < E(Q_n) < H(X) + 1/n < 1 \qquad \boxed{0 < E[Q_n] < 1}$

SOLUTION FOR (a) & (c)

(b) <u>IS THE OBJECT DEFECTIVE?</u>

$$H(x_1 x_2 \dots x_n) \leqslant E[Q] \leqslant H(x_1 \dots x_n) + 1$$

$$H(x_1, x_2 \dots x_n) = \left(H(x_1)+H(x_2)\dots+H(x_n)\right) < n \cdot H(x_n) < n$$

$$H(x_1, x_2 \dots x_n) = H(x_1)+H(x_2)+\dots+H(x_n) > n\, H(x_1) > n \cdot 0 > 0$$

$$\boxed{0 \leqslant E[Q] \leqslant n+1}$$

<u>EDITION 2 SOLUTIONS:</u> (a) MOST LIKELY SEQUENCE IS ALL 1'S, WITH PROBABILITY OF $\prod_{i=1}^{n} \gamma_i$, AND LEAST LIKELY SEQUENCE IS ALL 0'S WITH PROBABILITY $\prod_{i=1}^{n} (1-\gamma_i)$.

$$E[Q] \geq H(X_1, X_2, \dots X_n) \overset{*}{=} \sum H(X_i) = \sum H(\gamma_i)$$

(b) THE LAST BIT IN THE HUFFMAN CODE DISTINGUISHES BETWEEN THE LEAST LIKELY SOURCE SYMBOLS.

THE TWO LEAST LIKELY SEQUENCES ARE:

$$\underset{0\,0\,0\,0\,\dots\,0}{(1-\gamma_1)(1-\gamma_2)\dots(1-\gamma_n)} \quad \& \quad \underset{0\quad 0\ \dots\ 0\ 1}{(1-\gamma_1)(1-\gamma_2)\dots(1-\gamma_{n-1})\gamma_n}$$

THUS THE THE LAST QUESTION WILL ASK "IS $X_n = 1$"

I.E. IS THE LAST ITEM DEFECTIVE?

(c) UPPER BOUND ON THE MINIMUM AVERAGE NUMBER OF QUESTIONS IS UPPER BOUND ON THE AVERAGE LENGTH OF A HUFFMAN CODE, I.E.

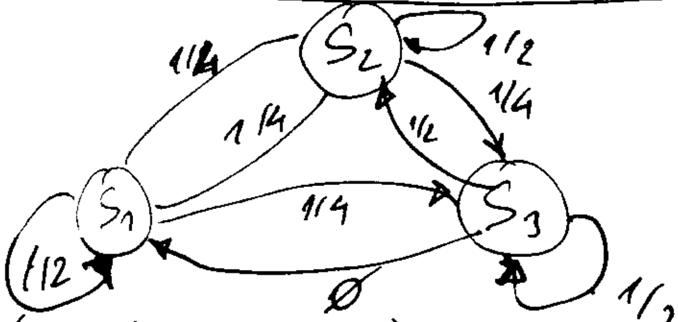
$$H(X_1, X_2, \dots X_n) + 1 = \sum_{i=1}^{n} H(\gamma_i) + 1$$

5.8 SIMPLE OPTIMUM COMPRESSION OF A MARKOV SOURCE. CONSIDER THE THREE-STATE MARKOV PROCESS $U_1, U_2, \dots$ HAVING TRANSITION MATRIX

$P(U_n \mid U_{n-1})$	$U_{n-1} \backslash U_n$	S_1	S_2	S_3
	S_1	1/2	1/4	1/4
	S_2	1/4	1/2	1/4
	S_3	0	1/2	1/2

THUS, THE PROBABILITY THAT S_1 FOLLOWS S_3 IS EQUAL TO ZERO. DESIGN THREE CODES C_1, C_2, C_3 (ONE FOR EACH STATE 1, 2 AND 3) EACH CODE MAPPING ELEMENTS OF THE SET OF S_i'S INTO SEQUENCES OF 0'S AND 1'S, SUCH THAT THIS MARKOV PROCESS CAN BE SENT WITH MAXIMUM COMPRESSION BY THE FOLLOWING SCHEME:

(a) NOTE THE PRESENT SYMBOL $Z_n = i$.
(b) SELECT CODE C_i.
(c) NOTE THE NEXT SYMBOL $Z_{n+1} = j$ AND SEND THE CODEWORD IN C_i CORRESPONDING TO j.
(d) REPEAT FOR THE NEXT SYMBOL. WHAT IS THE AVERAGE MESSAGE LENGTH OF THE NEXT SYMBOL CONDITIONED ON THE PREVIOUS STATE $Z_n = i$ USING THIS CODING SCHEME? WHAT IS THE UNCONDITIONAL AVERAGE NUMBER OF BITS PER SOURCE SYMBOL? RELATE THIS TO ENTROPY RATE $H(U)$ OF THE MARKOV CHAIN.



$$H(U) = \lim_{n \to \infty} H\left(U_n \mid U_{n-1}^1\right) = H(U_n \mid U_{n-1}) \underset{\text{STATIONARITY}}{=} \underline{\underline{H(U_2 \mid U_1)}}$$

$$H(U_2|U_1) = P(U_1 = S_1) \cdot H(U_2 | U_1 = S_1) + P(U_1 = S_2) \cdot H(U_2 | U_1 = S_2) + $$
$$+ P(U_1 = S_3) \cdot H(U_2 | U_1 = S_3) = \sum_{i=1}^{3} \mu_i \cdot H(U_2 | U_1 = S_i)$$

$$\boxed{H(X) = -\sum_{i=1}^{n} \sum_{j=1}^{n} \mu_i \cdot P_{ij} \, \mathrm{ld} \, P_{ij}}$$

$$[\mu_1 \ \mu_2 \ \mu_3] = [\mu_1 \ \mu_2 \ \mu_3] \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$\boxed{\begin{aligned} \mu_1 &= \tfrac{1}{2}\mu_1 + \tfrac{1}{4}\mu_2 \\ \mu_2 &= \tfrac{1}{4}\mu_1 + \tfrac{1}{2}\mu_2 + \tfrac{1}{2}\mu_3 \end{aligned} \quad \mu_3 = \tfrac{1}{4}\mu_1 + \tfrac{1}{4}\mu_2 + \tfrac{1}{2}\mu_3}$$

$-\frac{1}{2}\mu_1 + \frac{1}{4}\mu_2 = 0$; $\frac{1}{4}\mu_1 - \frac{1}{2}\mu_2 + \frac{1}{2}\mu_3 = 0$;

$\frac{1}{4}\mu_1 + \frac{1}{4}\mu_2 - \frac{1}{2}\mu_3 = 0$; $\mu_1 + \mu_2 + \mu_3 = 1$;

$$\boxed{\mu_1 = \frac{2}{9} ; \quad \mu_2 = \frac{4}{9} ; \quad \mu_3 = \frac{1}{3}}$$

$$H(\mathcal{U}) = \mu_1 \cdot H(U_2|U_1=s_1) + \mu_2 \cdot H(U_2|U_1=s_2) + \mu_3 \cdot H(U_3|U_1=s_3)$$

$$= \frac{2}{9}\left[\frac{1}{2}\log_2 2 + \frac{1}{4}\cdot 2 + \frac{1}{4}\,2\right] + \frac{4}{9}\left[\frac{1}{4}\cdot 2 + \frac{1}{2} + \frac{1}{4}\cdot 2\right] + \frac{1}{3}\left[\frac{1}{2} + \frac{1}{2}\right]$$

$$= \frac{\cancel{2}}{9}\cdot\frac{3}{\cancel{2}} + \frac{\cancel{4}}{\cancel{9}_3}\cdot\frac{3}{\cancel{2}} + \frac{1}{3} = \frac{1}{3} + \frac{2}{3} + \frac{1}{3} = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} = \underline{\underline{1.33}}$$

- $\boxed{U_{n-1} = s_1}$ $P(U_n|U_{n-1}=s_1) = \left[\frac{1}{2} \quad 1/4 \quad 1/4\right]$

$U_n = [s_1 \quad s_2 \quad s_3]$

U_n	$P(U_n\|U_{n-1}=s_1)$		$C_1(U_n)$	$\ell(U_n)$
s_1	1/2 → 1/2]0		0	1
s_2	1/4]0 → 1/2]1		1 0	2
s_3	1/4]1		1 1	2

$\boxed{*1}$

$$\boxed{E[\ell_1] = 1\cdot\frac{1}{2} + 2\cdot\frac{1}{4} + 2\cdot\frac{1}{4} = \frac{3}{2}}$$

$E[\ell_1] = E[\ell|c_1]$

- $\boxed{U_{n-1} = s_2}$

U_n	$P(U_n\|U_{n-1}=s_2)$		$C_2(U_n)$	$\ell(U_n)$
s_2	1/2	1/2]0	0	1
s_1	1/4]0 →	1/2]1	1 0	2
s_3	1/4]1		1 1	2

$\boxed{*2}$

$$\boxed{E[\ell_2] = 1\cdot\frac{1}{2} + \left(\ell\cdot\frac{1}{4}\right)\cdot 2 = \frac{3}{2}}$$

$E[\ell_2] = E[\ell|c_2]$

- $\boxed{U_{n-1} = s_3}$

U_n	$P(U_n\|U_{n-1}=s_3)$	$C_3(U_n)$	$\ell(U_n)$
s_2	1/2	0	1
s_3	1/2	1	1
s_1	0		

$\boxed{*3}$

$$\boxed{E[\ell_3] = \frac{1}{2} + \frac{1}{2} = 1}$$

$E[\ell_3] = E[\ell|c_3]$

$$E[\ell] = \mu_1 \cdot E[\ell_1] + \mu_2 E[\ell_2] + \mu_3 \cdot E[\ell_3] = \sum_{i=1}^{3} \mu_i \cdot E[\ell|c_i]$$

$$= \frac{2}{9}\cdot\frac{3}{2} + \frac{\cancel{4}^2}{\cancel{9}_3}\cdot\frac{3}{\cancel{2}} + \frac{1}{3}\cdot 1 = \frac{1}{3} + \frac{2}{3} + \frac{1}{3} = \frac{4}{3} = 1.33$$

$$\boxed{E[\ell] = H(\mathcal{U}) = 1.333}$$

μ_i	U_{n-1}	U_n	C	$\ell(U_n)$	$\mu_i \cdot P_i$
2/9	s_1	s_1	1	1	$\frac{2}{9}\cdot\frac{1}{2} = 1/9$
		s_2	1 0	2	$\frac{2}{9}\cdot\frac{1}{4} = 1/18$
		s_3	1 1	2	$2/9 \cdot 1/4 = 1/18$
4/9	s_2	s_1	1 0	2	$4/9 \cdot 1/4 = 1/9$
		s_2	0	1	$4/9 \cdot 1/2 = 2/9$
		s_3	1 1	2	$4/9 \cdot 1/4 = 1/9$
1/3	s_3	s_1	~~1~~	2 0	0 0
		s_2	0	1	1/6 1/6
		s_3	1	1	1/6 1/6

$E[\ell] = 1\cdot 1/9 + 2\cdot 1/18 + 2\cdot 1/18 +$
$+ 2\cdot 1/9 + 1\cdot 2/9 + 2\cdot 1/9 + 2\cdot 1/6$
$= 2/9 + 1/9 + 2/9 + 4/9 + 2/6 =$
$= \frac{8}{9} + \frac{1}{9} + \frac{1}{3} = 1 + \frac{1}{3} = \frac{4}{3} =$
$= 1.33$

(*1, *2, *3) THE AVERAGE MESSAGE LENGTH OF THE NEXT SYMBOL CONDITIONED ON THE PREVIOUS STATE BEING S_i.

$$H(U_n|U_{n-1}=S_1) = \frac{1}{2} + \frac{1}{4}\cdot ld4 + \frac{1}{4}\cdot ld4 = 3/2 \quad bit/symbol$$

$$H(U_n|U_{n-1}=S_2) = 1/2 + (1/4 \cdot 2)\cdot 2 = 3/2 \quad bit/symbol$$

$$H(U_n|U_{n-1}=S_3) = \frac{1}{2} + 1/2 = 1 \quad bit/symbol$$

NOTE: $E[L|C_i] = H(U_n|U_n=S_i)$

$\Rightarrow$ ENTROPY ~~OF~~ THE STATE AFTER STATE i, $H(X_2|X_1=S_i)$ IS EQUAL TO EXPECTED LENGTH OF EACH CODE C_i.)

$\boxed{5.9}$ OPTIMAL CODE LENGTHS THAT REQUIRE ONE BIT ABOVE ENTROPY. THE SOURCE CODING THEOREM SHOWS THAT THE OPTIMAL CODE FOR RANDOM VARIABLE X HAS AN EXPECTED LENGTH LESS THAN $H(X)+1$. GIVE AN EXAMPLE OF A RANDOM VARIABLE FOR WHICH THE EXPECTED LENGTH OF THE OPTIMAL CODE IS CLOSE TO $H(X)+1$. [I.E. FOR ANY $\epsilon>0$, CONSTRUCT A DISTRIBUTION FOR WHICH THE OPTIMAL CODE HAS $L > H(X)+1-\epsilon$]

$$\boxed{\sum_{i=1}^{n} D^{-l_i} \leq 1} \qquad H(X) \leq \sum_{i=1}^{n} l_i p_i < H(X)+1$$

$$l_i = \left\lceil ld \frac{1}{p_i} \right\rceil \qquad H(X) \leq \sum_{x\in X} l(x)p(x) < \underline{\underline{H(X)}}+1$$

$$L = \sum_{x\in X} l(x)\cdot p(x) \qquad s.t \quad \sum_{x\in X} D^{-l(x)} \leq 1$$

$$\frac{d}{dl(x)}\left[\sum_{x\in X} l(x)\cdot p(x) + \lambda\left(\sum_{x\in X} D^{-l(x)} - 1\right)\right] = 0$$

$$\sum_{x\in X} p(x) + \lambda \sum_{x\in X} -\ln D\, D^{-l(x)} = 0$$

$$\frac{d}{dx}\left(D^x\right) = \frac{d}{dx} e^{+x\ln D} \longleftarrow \begin{cases} y = D^x \;/\ln \\ \ln y = \ln D^x \\ y = e^{x\ln D} \end{cases}$$

$$= e^{x\ln D}\cdot \ln D \;// = \underline{\underline{\ln D\cdot D^x}}$$

$$\boxed{\frac{d}{dx}\left(D^{-x}\right) = \underline{\underline{-\ln D\, D^{-x}}}}$$

$$\sum_{x\in X}\left(p(x) - \lambda\cdot\ln D\cdot D^{-l(x)}\right) = 0 \Rightarrow$$

$$p(x) = \lambda\cdot\ln D\cdot D^{-l(x)} \qquad D^{-l(x)} = \frac{p(x)}{\lambda\cdot\ln D}$$

$$\sum_{x\in X} \frac{p(x)}{\lambda\cdot \ln D} = 1 \Rightarrow \frac{1}{\lambda\cdot\ln D} = 1 \quad \lambda = \frac{1}{\ln D}$$

$$p(x) = \frac{1}{\ln D}\cdot\ln D\cdot D^{-l(x)} \Rightarrow \boxed{p(x) = D^{-l(x)}}$$

$p(x) = D^{-l(x)} \Rightarrow -l^*(x) = \log_D p(x)$ $\quad l^*(x) = \log_D\left(\frac{1}{p(x)}\right)$

$L_1 = \sum_{x \in \mathcal{X}} l(x) p(x) \qquad l(x) = \left\lceil ld \frac{1}{q(x)} \right\rceil$

$$\sum_{x \in \mathcal{X}} \left\lceil ld \frac{1}{q(x)} \right\rceil p(x) \leq \sum_{x \in \mathcal{X}} \left[ld \frac{1}{q(x)} + 1 \right] p(x) =$$

$$= \sum_{x \in \mathcal{X}} p(x)\, ld \frac{1}{q(x)} + \sum_{x \in \mathcal{X}} p(x) = \sum_{x \in \mathcal{X}} p(x)\, ld \frac{p(x)}{q(x) \cdot p(x)} + 1$$

$$= \sum_{x \in \mathcal{X}} p(x)\, ld \frac{1}{p(x)} + \sum_{x \in \mathcal{X}} p(x)\, ld \frac{p(x)}{q(x)} + 1$$

$$= H(X) + D(p \| q) + 1$$

$$\boxed{l(x) = ld \frac{1}{p(x)} + 1} \Rightarrow L = \sum_{x \in \mathcal{X}} p(x)\, ld \frac{1}{p(x)} + \sum_{x \in \mathcal{X}} p(x) = H(X) + 1$$

$$\to ld \frac{1}{p(x)} = l(x) - 1 \qquad \boxed{p(x) = 2^{-l(x)+1}} \; ★$$

- $l(x) = [2, 3, 4, 4]$

$x \in [1, 2, 3, 4]$

$p(x) = [2^{-1}, 2^{-2}, 2^{-3}, 2^{-3}] = \left[\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right]$

$H(X) = \frac{1}{2} + \frac{1}{4} \cdot 2 + \left(\frac{1}{8} \cdot 3\right) \cdot 2 = 1 + \frac{3}{4} = 1{,}75$

$E[l(x)] = 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{4} + \left(4 \cdot \frac{1}{8}\right) \cdot 2 = 1 + \frac{3}{4} + 1 = \underline{2.75}$

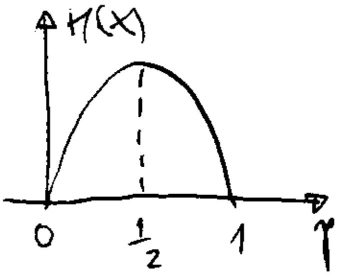
$\boxed{E[l(x)] = H(X) + 1} \leftarrow$

- $l(x) = [2, 3, 4, 5, 5] \qquad p(x) = \left[\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}\right]$

$\boxed{H(X) = 1.875 \qquad E[l(x)] = 2{,}875}$

- <u>EDITION 2 SOLUTION:</u>

$x \in [0, 1] \quad p(x) = [\varepsilon, 1-\varepsilon]$



$p = P_r(X = 1) \to 1 \Rightarrow H(X) \to 0$

$E[l(x)] = 1 \cdot 1 + 1 \cdot 0 = 1$

$\Rightarrow E[l(x)] = H(X) + 1 = 0 + 1 = 1$

IZRAZOT ★ E POGENERALIZIRANO REŠENIE KOE GO OPFAĆA I TRIVIJALNOTO REŠENIE GORE T.E.

$l(x) = 1 \to p(x) = 2^{-l(x)+1} = 2^{-1+1} = 2^0 = 1$

$\boxed{5.10}$ TERNARY CODES THAT ACHIEVE THE ENTROPY BOUND

A RANDOM VARIABLE X TAKES m = VALUES AND HAS ENTROPY $H(X)$. AN INSTANTENOUS TERNARY CODE IS FOUND FOR THIS SOURCE WITH AVERAGE LENGTH :

$$L = \frac{H(X)}{ld3} = H_3(X)$$

(a) SHOW THAT EACH SYMBOL OF X HAS PROBABILITY OF THE FORM 3^{-i} FOR SOME i.

(b) SHOW THAT m IS ODD.

(a)

$$\sum_{x\in X} D^{-l(x)} \leq 1 \qquad \sum_{x\in X} 3^{-l(x)} \leq 1$$

$$E[l(x)] = \sum_{x\in X} p(x)\cdot l(x) = \underbrace{\sum_{x\in X} p(x)\cdot \log_3(x)}_{H_3(x)} = \frac{H(x)}{ld3}$$

$$\log_3(x) = y \qquad 3^y = x \;/\; ld \qquad y\, ld3 = ld x$$

$$y = \frac{ld x}{ld3}$$

(b) $x \in [1,2,3] \qquad p(x) = [\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$

X	P	P	C(X)	L(X)
x_1	0.25	0.5 0	1	1
x_2	0.25	0.25 1	2	1
x_3	0.2 0	0.25 2	00	2
x_4	0.15 1		01	2
x_5	0.15 2		02	2

$E[l(x)] = 1.5$

$H_3(x) = 1.44197$

X	P	P	C(X)	L(X)
x_1	1/3	1/3 0	1	1
x_2	1/3	1/3 1	2	1
x_3	1/9 0	1/3 2	00	2
x_4	1/9 1		01	2
x_5	1/9 2		02	2

$E[l(x)] = 1.333$

$H_3(x) = 1.333$

(a) $L - H_3(x) = \sum_{x\in X} p(x)\cdot l(x) - \sum_{x\in X} p(x) \log_3 \frac{1}{p(x)} \geq$

$\geq \; / \; l(x) = \lceil \log_D \frac{1}{p(x)} \rceil \geq \log_D \frac{1}{p(x)} \; / \geq \sum_x p(x) \log_3 \frac{1}{p(x)} - \sum p(x) \log_3 \frac{1}{p(x)}$

$= 0 \quad \Rightarrow \quad L - H_3(x) \geq 0$

EQUALITY IF : $l(x) = \log_3 \frac{1}{p(x)}$ i.e. $\circledast \; \boxed{p(x) = 3^{-l(x)}}$

ALTERN: $p(x) = \frac{D^{-l_i}}{\sum D^{-l_i}} \qquad c = \sum D^{-l_i} \qquad r_i = \frac{D^{-l_i}}{c}$

$L - H_D(x) = -\sum_x p(x) \log_D D^{-l(x)} - \sum_x p(x) \log_D \frac{1}{p(x)} = -\sum_x p(x) \log_D (r\,c) - \sum p\, ld \frac{1}{p}$

$\sum p\, ld \frac{p}{r} + \sum p\, ld \frac{1}{c}$

⊛ $p(x) = 3^{-l(x)}$ $\boxed{l(x) = -\log_3(p(x)) = \log_3\left(\frac{1}{p(x)}\right)}$

$\log_3 \frac{1}{p(x)}$ is integer !!

(б) $x = [x_1, x_2 \dots x_n]$ "n" is odd

– ДА ПРЕТПОСТАВИМЕ ДЕКА $\underline{n = 2k}$

$p(x) = [\ p(x_1), p(x_2) \dots p(x_n)]$

$\sum_{x \in X} p(x) = 1$ $\sum_{x \in X} D^{-l(x)} \le 1$

e.g. $n = 2$ $x \in [x_1\ x_2]$

$p(x) \in [\frac{1}{3}\ \ \frac{2}{3}]$ $\left(\frac{2}{3}\right)$ НЕ СЕ ДОБИВА ОД 3^{-i}

e.g. $n = 4$ $x \in [x_1\ x_2\ x_3\ x_4]$

$p(x) \in [\frac{1}{3}\ \ \frac{1}{3}\ \ \underbrace{\frac{2}{9}\ \ \frac{1}{9}}_{1/3}]$ $\left(\frac{2}{9}\right)$ НЕ СЕ ДОБИВА ОД 3^{-i}

$\sum_{i=1}^{n} 3^{-l(i)} = 1$ $\frac{1}{3^{l_1}} + \frac{1}{3^{l_2}} \cdots \frac{1}{3^{l_n}} = 1$

$$\left(\sum_{x \in X} D^{-l(x)}\right)^k = \sum_{x_1 \in X} \sum_{x_2 \in X} \cdots \sum_{x_k \in X} D^{-l(x_1)} \cdot D^{-l(x_2)} \cdots D^{-l(x_k)}$$

$$= \sum_{x \in X^k} D^{l(x_1, x_2 \dots x_k)} = \sum_{x \in X^k} D^{l(x^k)} = \sum_{m=1}^{k \cdot l_{max}} a(m) D^{-m}$$

$$= |a(m) \le D^m| \le \sum_{m=1}^{k \cdot l_{max}} \cancel{D^m} \cdot \cancel{D^{-m}} = k \cdot l_{max} \Rightarrow$$

$\sum_{x \in X} D^{-l(x)} \le (k \cdot l_{max})^{1/k}$ $\boxed{\sum_{x \in X} D^{-l(x)} \le 1}$

$\sum_{i=1}^{n} q^i = q \cdot \frac{1-q^n}{1-q}$ $S = q + q^2 + \dots q^n$

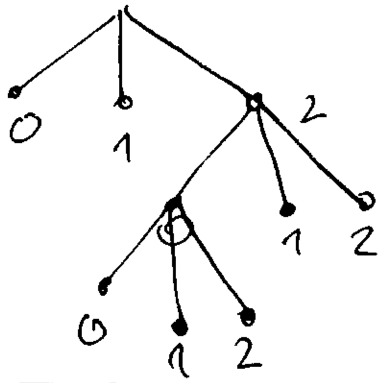
$qS = q^{n+1} + q^n + \dots q^2$

$S(1-q) = q - q^{n+1}$ $S = q\frac{1-q^n}{1-q}$

$S = \frac{1}{3} \frac{1-(\frac{1}{3})^n}{1-\frac{1}{3}} = \frac{1}{\cancel{3}} \frac{1-\frac{1}{3^n}}{\frac{2}{\cancel{3}}} = \frac{1}{2}\left(1 - \frac{1}{3^{4}}\right)$

$3 + 9 + 9 + 27 = 12 + 9 + 27 = 17 + 27 = 44$

SOLUTIONS 2 EDITION (6) WE KNOW FROM THEOREM 5.2.1, THAT GIVEN THE SET OF LENGTHS, l_i, WE CAN CONSTRUCT A TERNARY TREE WITH NODES AT THE DEPTHS l_i. NOW SINCE $\sum 3^{-l_i} = 1$, THE TREE MUST BE COMPLETE. A COMPLETE TERNARY TREE HAS AN ODD NUMBER OF LEAVES. THUS THE NUMBER OF SOURCE SYMBOLS IS ODD.



ALTERNATIVELY: (FROM DADIC NUMBER THEORY)

$$\sum_i 3^{-l_i} = 1 = 3^{-l_{max}} \sum_i 3^{l_{max}-l_i} = 1$$

$$\sum_i 3^{l_{max}-l_i} = 3^{l_{max}}$$

EACH OF THE TERMS IN THE SUM IS ODD, AND SINCE THEIR SUM IS ODD NUMBER OF TERMS IN THE SUM HAS TO BE ODD. (THE SUM OF AN EVEN NUMBER OF ODD TERMS IS EVEN) THUS THERE ARE AN ODD NUMBER OF SOURCE SYMBOLS FOR ANY CODE THAT MEETS THE TERNARY ENTROPY BOUND.

$$\underbrace{\frac{1}{3}+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^2}+\frac{1}{3^3}\cdot 3}_{\text{7 TERMS (EACH ODD)}} = \frac{2}{3}+\frac{2}{9}+\frac{1}{9} = \frac{3+6}{9} = 1$$

5.11 SUFFIX CONDITION. CONSIDER CODES THAT SATISFY THE SUFFIX CONDITION, WHICH SAYS THAT NO CODEWORD IS A SUFFIX OF ANY OTHER CODEWORD. SHOW THAT SUFFIX CONDITION CODE IS UNIQUELY DECODABLE, AND SHOW THAT THE MINIMUM AVERAGE LENGTH OVER ALL CODES SATISFYING THE SUFFIX CONDITION IS THE SAME AS THE AVERAGE LENGTH OF THE HUFFMAN CODE FOR THAT RANDOM VARIABLE

X				C(x)	X	P			C(x)
x_1	1/2	1/2	1/2 } 0	1	x_1	1/2	1/2	1/2 } 0	0
x_2	1/4	1/4 } 0	1/2 } 1	01	x_2	1/4	1/4 } 0	1/2 } 1	10
x_3	1/8 } 0	1/4 } 1		000	x_3	1/8 } 0	1/4 } 1		110
x_4	1/8 } 1			001	x_4	1/8 } 1			111

AKO GI ČITAŠ OD LEVO KON DESNO ĆE DOBIEŠ SUFIX CODE!!! } $\underbrace{0, 01, 011, 111}_{\text{SUFIX CODE}}$

$$E[l(T)] = 1\cdot\frac{1}{2} + 2\cdot\frac{1}{4} + 2\cdot 3\cdot\frac{1}{8} = \frac{2}{4}+\frac{2}{4}+\frac{5}{4} = 7/4 = 1.75$$

↳ SAME AS FOR HUFFMAN CODE !!!

→ INVERTING ALL THE BITS OR EXCHANGING TWO CODEWORDS OF SAME LENGTH WILL GIVE ANOTHER OPTIMAL CODE.

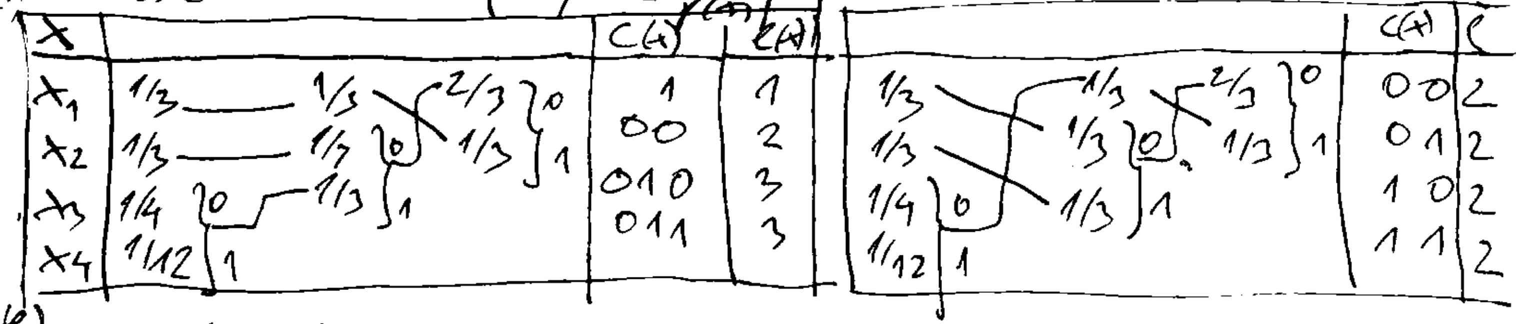
5.12 SHANNON CODES AND HUFFMAN CODES. CONSIDER RANDOM VARIABLE X THAT TAKES ON FOUR VALUES WITH PROBABILITIES $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{12}\right)$

(a) CONSTRUCT A HUFFMAN CODE FOR THIS RV

(b) SHOW THAT THERE EXIST TWO DIFFERENT SETS OF OPTIMAL LENGTHS FOR CODEWORDS; NAMELY, SHOW THAT CODEWORD LENGTH ASSIGNMENTS (1, 2, 3, 3) AND (2, 2, 2, 2) ARE BOTH OPTIMAL

(c) CONCLUDE THAT THERE ARE OPTIMAL CODES WITH CODEWORD LENGTHS FOR SOME SYMBOLS THAT EXCEED THE SHANNON CODE LENGTH ($\lceil ld \frac{1}{p(x)} \rceil$)

(a)



X	C(x)	l(x)	C(x)	l
x_1	1	1	00	2
x_2	00	2	01	2
x_3	010	3	10	2
x_4	011	3	11	2

(b)

$$H(X) = \left(\frac{1}{3} ld 3\right) \cdot 2 + \frac{1}{4} \cdot 2 + \frac{1}{12} ld 12 = 1.32707$$

$$l_1(x) = \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 2 + \frac{1}{4} \cdot 3 + \frac{1}{12} \cdot 3 = 1 + \frac{3}{4} + \frac{1}{4} = 2$$

$$l_2(x) = \left(\frac{1}{3} \cdot 2\right) \cdot 2 + \left(\frac{1}{4} + \frac{1}{12}\right) \cdot 2 = \frac{4}{3} + \frac{4}{12} \cdot 2 = \frac{4}{3} + \frac{2}{3} = 6/3 = 2$$

(c) $\lceil ld 3 \rceil = \lceil 1.585 \rceil = 2$ $\quad$ $\boxed{ld 4 = 2}$ $\quad$ $\lceil ld 12 \rceil = \lceil 3.585 \rceil = 4$

$L_1(x_3) = 3 > 2 \leftarrow$ SHANNON CODE

ON AVERAGE SHANNON CODE CANNOT BE SHORTER THAN HUFFMAN CODE.

5.13 <u>TWENTY QUESTIONS</u>. A PLAYER A CHOSES SOME OBJECT IN UNIVERSE, AND PLAYER B ATTEMPTS TO IDENTIFY THE OBJECTS WITH A SERIES OF YES-NO QUESTIONS. SUPPOSE THAT PLAYER B IS CLEVER ENOUGH TO USE THE CODE ACHIEVING THE MINIMAL EXPECTED LENGTH WITH RESPECT TO PLAYER A'S DISTRIBUTION. WE OBSERVE THAT PLAYER B REQUIRES AN AVERAGE 38.5 QUESTIONS TO DETERMINE THE OBJECT. FIND A ROUGH LOWER BOUND TO THE NUMBER OF OBJECTS IN THE UNIVERSE.

$H(X) \leq E[l(x)] < H(X) + 1$ $\qquad$ $\boxed{H(X) = 38.5}$

$P(A) = [p_1, p_2, \dots\dots p_n]$ $\qquad$ $H(X) = \sum_i p_i \, ld \frac{1}{p_i}$

$$H_{max}(X) = \sum \frac{1}{n} ld n = n \cdot \frac{1}{n} ld n = ld n \geq H(X)$$

$38.5 \geq H(X) \leq ld n$ $\qquad$ $38.5 \geq ld n$ $\qquad$ $n = 2^{38.5} = ~~0.389 \cdot 10^{12}~~$

- EDITION 2 SOLUTION: $\qquad$ $H(X) + 1 > E[l(X)]$ $\qquad$ $H(X) > E[l(X)] - 1$

$37.5 = E[l(X)] - 1 < H(X) \leq ld n$ $\quad$ $\Rightarrow$ $\quad$ $\boxed{n \geq 2^{37.5} = 1.944 \cdot 10^{11} \text{ OBJECTS}}$

5.14 HUFFMAN CODE. FIND THE (a) BINARY AND (b) TERNARY HUFFMAN CODES FOR RANDOM VARIABLE X WITH PROBABILITIES.

$$X = (1, 2, 3, 4, 5, 6)$$

$$p = \left(\frac{6}{21}, \frac{5}{21}, \frac{4}{21}, \frac{3}{21}, \frac{2}{21}, \frac{1}{21}\right)$$

X	p(x)					C(x)	l(x)
1	6/21	6/21	6/21	9/21	12/21 0	01	2
2	5/21	5/21	6/21	6/21 0	9/21 1	10	2
3	4/21	4/21	5/21 0	6/21 1		11	2
4	3/21	3/21 0	4/21 1			000	3
5	2/21 0	3/21 1				0010	4
6	1/21 1					0011	4

$$H(x) = \sum_{i=1}^{6} p_i \operatorname{ld} \frac{1}{p_i} = 2.398 \qquad E[l(x)] = \sum_{i=1}^{6} l_i\, p(x_i) = \underline{\underline{2{,}43}}$$

X				C(x)	l(x)
~~1~~	~~6/21~~	~~6/21~~	~~15/21 0~~	~~00~~	~~2~~
~~2~~	~~5/21~~	~~6/21 0~~	~~6/21 1~~	~~01~~	~~2~~
~~3~~	~~4/21~~	~~5/21 1~~	~~2~~	~~02~~	~~2~~
~~4~~	~~3/21 0~~	~~4/21 2~~		~~10~~	~~2~~
~~5~~	~~2/21 1~~			~~11~~	~~2~~
~~6~~	~~1/21 2~~			~~12~~	~~2~~

~~$E[l(x)] = 2$~~

~~$H_3(x) = \frac{2{,}398}{\operatorname{ld} 3}$~~

~~$H_3(x) = 1.51$~~

X				C(x)	l(x)
1	6/21	6/21	10/21 0	1	1
2	5/21	5/21	6/21 1	2	1
3	4/21	4/21 0	5/21 2	00	2
4	3/21	3/21 1		01	2
5	2/21 0	3/21 2		020	3
6	1/21 1			021	3
7	0 2			022	3

$E[l(x)] = 1{,}62$

00	0.3	0.3	0.4	0.6 0
01	0.3	0.3	0.3 0	0.4 1
11	0.2	0.2 0	0.3 1	
100	0.1 0	0.2 1		
101	0.1 1			

5.15 CONSTRUCT BINARY HUFFMAN CODE FOR THE FOLLOWING DISTRIBUTION ON FIVE SYMBOLS: $p = (0.3; 0.3; 0.2, 0.1, 0.1)$

(a) WHAT IS THE AVERAGE LENGTH OF THIS CODE?

(b) CONSTRUCT A PROBABILITY DISTRIBUTION p' ON FIVE SYMBOLS FOR WHICH THE CODE THAT YOU CONSTRUCTED IN PART (a) HAS AVERAGE LENGTH (UNDER p') EQUAL TO ITS ENTROPY $H(p')$

X					C(x)	l(x)
x_1	0.3	0.3	0.4	0.6 0	00	2
x_2	0.3	0.3	0.3 0	0.4 1	01	2
x_3	0.2	0.2 0	0.3 1		10	2
x_4	0.1 0	0.2 1			110	3
x_5	0.1 1				111	3

$$E[l(x)] = \sum_{i=1} p(x_i)\cdot l_i$$

$$E[l(x)] = 2.2$$

$$H(X) = 2.171$$

WRONG CODE: NE ZNAMO „p" TEKO „q"

$$E[l(x)] = \sum_{x \in X} p(x) \cdot \left\lceil ld \frac{1}{q(x)} \right\rceil \le \sum_{x \in X} p(x) \left[ld \frac{1}{q(x)} + 1 \right] =$$

$$= \sum_{x} p(x) \cdot ld \frac{p(x)}{q(x) p(x)} + 1 = \sum_{x} p(x)\, ld \frac{p}{q} + \sum_{x} p\, ld \frac{1}{p} + 1$$

$$= D(p \| q) + H(X) + 1$$

$$\boxed{H(X) + D(p\|q) \le E[l(x)] < H(X) + D(p\|q) + 1}$$

$$l(x) = [2, 2, 2, 3, 3]$$

$$p'(x) = \left[\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right]$$

$$\boxed{p'(x) = 2^{-l(x)}} \quad \boxed{l(x) = ld \frac{1}{p(x)}}$$

$$L'(x) = \sum_{x} l(x) \cdot p'(x) = \left(2 \cdot \frac{1}{4}\right) \cdot 3 + \left(3 \cdot \frac{1}{8}\right) 2 =$$

$$= \frac{3}{2} + \frac{3}{4} = \frac{9}{4} = \underline{2,25}$$

$$H'(x) = \sum p'(x) \cdot ld \frac{1}{p'(x)} = \left(\frac{1}{4} \cdot 2\right) \cdot 3 + \left(\frac{1}{8} \cdot 3\right) \cdot 2 = \frac{3}{2} + \frac{3}{4} = \frac{9}{4}$$

$\boxed{5.16}$ HUFFMAN CODES. CONSIDER VARIABLE X THAT TAKES 6 VALUES $\{A, B, C, D, E, F\}$ WITH PROBABILITIES $\{0.5, 0.25, 0.1, 0.05, 0.05, 0.05\}$

(a) CONSTRUCT BINARY HUFFMAN CODE FOR THIS RV. WHAT IS AVERAGE LENGTH?

(b) CONSTRUCT QUATERNARY HUFFMAN CODE FOR THIS RV. [CODE SYMBOLS a, b, c, d]. WHAT IS $E[l(X)]$?

(c) CONVERT QUATERNARY CODE IN BINARY USING:
$a \to 00$; $b \to 01$; $c \to 10$; $d \to 11$
WHAT IS THE LENGTH OF BINARY CODE?

(d) FOR ANY RANDOM VARIABLE X, let L_H BE THE AVERAGE LENGTH OF THE BINARY HUFFMAN CODE FOR THE RV, AND LET L_{QB} BE THE AVERAGE LENGTH CODE CONSTRUCTED BY FIRST BUILDING A QUATERNARY HUFFMAN CODE AND CONVERTING IT TO BINARY. SHOW THAT: $\boxed{L_H \le L_{QB} < L_H + 2}$

(e) THE LOWER BOUND IN THE EXAMPLE IS TIGHT. GIVE AN EXAMPLE WHERE THE CODE CONSTRUCTED BY CONVERTING AN OPTIMAL QUATERNARY CODE IS ALSO OPTIMAL BINARY CODE

(f) THE UPPER BOUND (i.e. $L_{QB} < L_H + 2$) IS NOT TIGHT. IN FACT BETTER BOUND IS $L_{QB} \le L_H + 1$. PROVE THIS BOUND, AND PROVIDE AN EXAMPLE WHERE THIS BOUND IS TIGHT.

X						c(x)	ℓ(x)
A	0.5	0.5	0.5	0.5	0.5] 0	0	1
B	0.25	0.25	0.25	0.25] 0	0.5] 1	10	2
C	0.1	0.1	0.15] 0	0.25] 1		111	3
D	0.05	0.1] 0	0.1] 1			1101	4
E	0.05] 0	0.05] 1				11000	5
F	0.06] 1					11001	5

$$\boxed{L_H = E[\ell(x)] = 2}$$

$H(X) = 1.981$

X			c(x)	$\ell_q(x)$	$c_B(x)$
A	0.5	0.5] a	a	1	00
B	0.25	0.25] b	b	1	01
C	0.1	0.15] c	d	1	11
D	0.05] a	0.1] d	ca	2	1000
E	0.05] b		cb	2	1001
F	0.05] c		cc	2	1010
G	0] d		cd	2	1011

$E[\ell(x)] = 1.15$ quaternary digits

$$\boxed{E[\ell_B(x)] = 2.3 \text{ bits}}$$

$$\boxed{\ell_B(x) = \ell_q(x) \cdot 2}$$

$\ell_q(x) = \ell_B(x)/2$

(d) $L_Q = \sum_{x \in X} \ell_q(x) p(x) = \sum_{x \in X} \cdot \frac{1}{2} \ell_B(x) p(x) = L_{BQ}/2$

$$\boxed{L_{BQ} = \sum_{x \in X} L_B \cdot p(x)}$$

$$\boxed{H_Q(X) \leq L_Q(X) < H_Q(X) + 1}$$

$\frac{H(X)}{\mathrm{ld}\,4} \leq L_Q(X) < \frac{H(X)}{\mathrm{ld}\,4} + 1$

$H(X) \leq 2 L_Q(X) < H(X) + 2$

$L_Q = \frac{L_{BQ}}{2} \Rightarrow H(X) \leq 2 \cdot \frac{L_{BQ}}{2} < H(X) + 2$

$$\boxed{H(X) \leq L_{BQ} < H(X) + 2}$$

$H(X) \leq L_H < H(X) + 1$

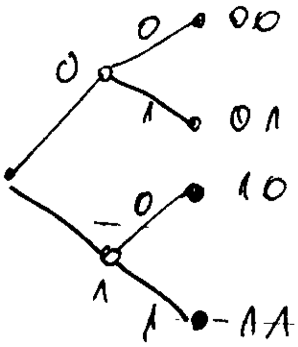
$\boxed{H(X) > L_H - 1}$

$H(X) \leq L_H$

$$\boxed{L_H - 1 \leq L_{BQ} < L_H + 2}$$

$L_H - 1 \leq L_{BQ} \leq L_H + 1$

(e)



X				
1	1/2	1/2	3/4] 0	1
2	1/4	1/2] 0	1/2] 1	00
3	1/8] 0	1/4] 1	?	010
4	1/8] 1			011

$\frac{1}{2} + \frac{1}{2} + 2\ 3 \cdot \frac{1}{8} = 1 + \frac{3}{4} = \underline{1.75}$

X			
$\frac{1}{2}$	$\frac{1}{2}$	1/2] 0	0
1/4	1/4] 0	1/2] 1	10
1/4] 0	1/4] 1		110
0] 1			

$X=\{1,2,3,4\}$ $p(x)=\left[\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right]$

x				$C_H(x)$
1	1/4	1/2	1/2 0	00
2	1/4	1/4 0	1/2 1	01
3	1/4 0	1/4 1		10
4	1/4 1			11

$$L_H(x) = 4\cdot\left(\frac{1}{4}\cdot 2\right) = 2$$

X			C_Q	C_{BQ}
1	0.25	a	a	00
2	0.25	b	b	01
3	0.28	c	c	10
4	0.25	d	d	11

$$L_{BQ} = \left(2\cdot\frac{1}{4}\right)\cdot 4 = 2$$

$$\boxed{L_Q = \frac{1}{4}\cdot 4 = 1}$$

$$\boxed{L_H = L_{QD}}$$

RV WITH EQUIPROBABLE VALUES.

(P.) $H(X) \le L(X) < H(X)+1$ $H(X) > L(X)-1$

$H(X) \le L_{BQ} < H(X)+2$

$H(X) \le L_{BQ} < L(X)-1+2$

$H(X) \le L_{BQ} < L(X)+1$

MY SOLUTION

X			$C(L)$	$C(L_{BQ})$	$C(L_Q)$
1	1/2	1/2 1/2 0	0	$0\boxed{0}$	a
2	1/4	1/4 0 1/2 1	10	10	c
3	1/8 0	1/4 1	110	$110\boxed{0}$	da
4	1/8 1		111	$111\boxed{0}$	dc

DODAVAŠ 0-e NA KODOVITE SO NEPAREN BROJ NA BITI.

$$L_Q = \frac{1}{2}E[\ell_Q(X)] = \frac{1}{2}\left\{\sum_{x:\ell(x)\text{ even}} \ell(x)\cdot p(x) + \sum_{x:\ell(x)\text{ is odd}}[\ell(x)+1]\,p(x)\right\} =$$

$$=\frac{1}{2}\Big(\sum_{x:\ell(x)\text{ is even}}\ell(x)p(x) + \sum_{x:\ell(x)\text{ is odd}}\ell(x)p(x) + \sum_{x:\ell(x)\text{ is odd}} p(x)\Big] = \frac{1}{2}\Big[\sum_{x\in X}\ell(x)p(x) + \sum_{x:\ell(x)\text{ is odd}} p(x)\Big]$$

$$=\frac{1}{2}\Big[L_H + \sum_{x:\ell(x)\text{ is odd}} p(x)\Big] < \frac{1}{2}\left[L_H+1\right]$$

$2L_Q < L_H+1$ $L_{BQ}=2L_Q$

$$\boxed{L_{BQ} < L_H+1}$$

AN EXAMPLE WHERE THIS UPPER BOUND IS TIGHT IS THE CASE WHEN WE HAVE ONLY TWO POSSIBLE SYMBOLS.
THEN $L_H=1$ & $L_{QB}=2$

X	$p(x)$	$C(x)$	$C_{BQ}(x)$	$C_Q(x)$
x_1	1/2	0	00	a
x_2	1/2	1	10	c

$$L_H = \left(1\cdot\frac{1}{2}\right)\cdot 2 = 1$$

$$L_{BQ} = \left(2\cdot\frac{1}{2}\right)\cdot 2 = 2$$

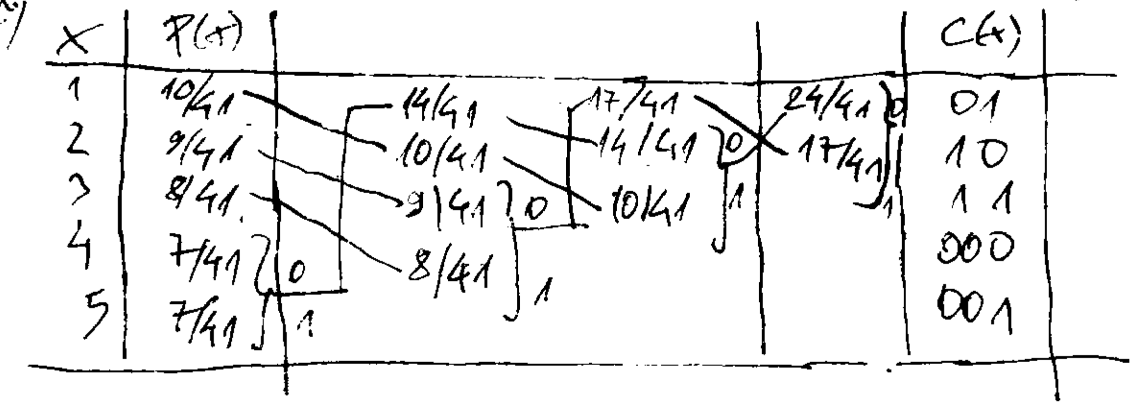
$$\boxed{L_{BQ} = 1+1 = 2}$$

$\boxed{5.17}$ DATA COMPRESSION. FIND AN OPTIMAL SET OF BINARY CODEWORD LENGTHS $l_1, l_2, \ldots$ (MINIMIZING $\sum p_i l_i$) FOR AN INSTANTANEOUS CODE FOR EACH OF FOLLOWING PROBABILITY MASS FUNCTIONS:

(a) $p = \left(\frac{10}{41}, \frac{9}{41}, \frac{8}{41}, \frac{7}{41}, \frac{7}{41}\right)$

(b) $p = \left[\frac{9}{10}, \frac{9}{10}\frac{1}{10}, \frac{9}{10}\frac{1}{10^2}, \frac{9}{10}\frac{1}{10^3}, \ldots\right]$

(a.)



X	P(x)				C(x)
1	10/41	14/41	17/41	24/41	01
2	9/41	10/41	14/41	17/41	10
3	8/41	9/41	10/41		11
4	7/41	8/41			000
5	7/41				001

(b)

x	p(x)
1	9/10
2	9/10²
3	9/10³
4	9/10⁴
5	1/10⁴

$$\frac{9 10^3 + 9 10^2 + 9\cdot 10 + 18}{10^4} =$$

$$= \frac{\qquad}{10^4}$$

$$\frac{9}{10}\sum_{i=0}^{\infty}\frac{1}{10^i} = \frac{\cancel{9}}{\cancel{10}}\,\frac{1}{\frac{\cancel{9}}{\cancel{10}}} = 1$$

$$\frac{9}{10}\sum_{i=0}^{4}\left(\frac{1}{10}\right)^i = \frac{1-\frac{1}{10^{⑤}}}{1-\frac{1}{10}}\cdot\frac{9}{10} = \frac{\frac{10^5-1}{10^5}}{\frac{\cancel{10-1}}{10}}\cdot\frac{\cancel{9}}{10} = \frac{\cancel{10}(10^5-1)}{10^5}\frac{1}{\cancel{10}}$$

N+1

$$= \frac{10^5-1}{10^5} = 1-\frac{1}{10^5}$$

$$\frac{9}{10}\sum_{i=0}^{3}\left(\frac{1}{10}\right)^i = \frac{1-\frac{1}{10^4}}{1-\frac{1}{10}}\cdot\frac{9}{10} = \frac{\frac{10^4-1}{10^4}}{\frac{\cancel{9}}{\cancel{10}}}\cdot\frac{\cancel{9}}{\cancel{10}} = \frac{10^4-1}{10^4} =$$

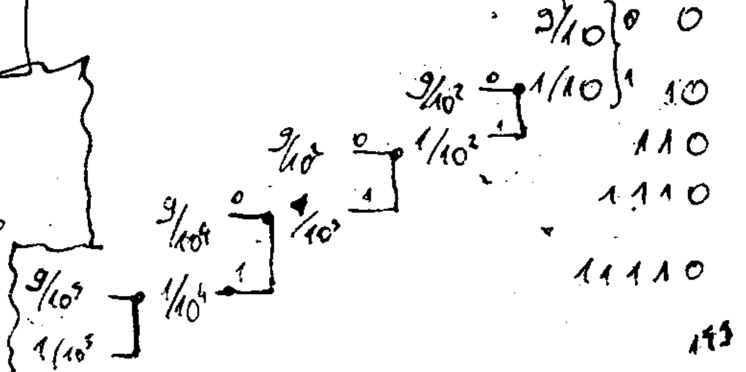
$$= 1-\frac{1}{10^4}$$

$$\boxed{\frac{1}{10^4} = \frac{9}{10^5}+\frac{1}{10^5} = \frac{10}{10^5} = \frac{1}{10^4}}$$

$$l(x) = \sum_{i=1}^{\infty} i\cdot\frac{9}{10^i} = 9\sum_{i=1}^{\infty} i\cdot\frac{1}{10^i} =$$

$$= 9\sum_{i=0}^{\infty} i\cdot\frac{1}{10^i} = 9\cdot\frac{\frac{1}{10}}{\left(1-\frac{1}{10}\right)^2} = \frac{9}{\cancel{10}\,\frac{81}{10\cancel{0}}} = \frac{90}{81}$$

$$= 10/9$$



$$l_1, l_2, \dots l_n = 0, 10, 110, 1110, \dots \underbrace{1111\dots10}_{n-1}$$

- EDITION 2 SOLUTION. CUMULATIVE PROBABILITY OF ALL SYMBOLS IS:

$\{x : x > i\}$

	X	P(X)
	1	0.9
	2	0.9/10
	3	$0.9/10^2$
i =	4	$0.9/10^3$
	5	$0.9/10^4$
	6	⋮

SLIKA 1

$$\sum_{j>i} 0.9\cdot(0.1)^{j-1} = 0.9 \sum_{j>i}(0.1)^{j-1} = 0.9\left(1-\sum_{j=1}^{i}(0.1)^{j-1}\right)$$

$$=0.9\left[1-\sum_{j=0}^{i-1}(0.1)^{j}\right] = 0.9\left[1-\frac{1-(0.1)^{i}}{1-0.1}\right] =$$

$$=0.9\left[\frac{0.9-1+(0.1)^{i}}{0.9}\right] = -0.1+(0.1)^{i} = (0.1)^{i-1}$$

$$(0.1)^{i-1} = (0.1)^3 = \frac{1}{10^3}$$

$$(0.1)^{i-1} < 0.9\cdot(0.1)^{i-1}$$

$$(0.1)^3 < 0.9\cdot(0.1)^3$$

THE CUMULATIVE SUM OF ALL THE REMAINING TERMS IS LESS THAN THE LAST TERM USED. THUS THE HUFFMAN CODE WILL ALWAYS MERGE THE LAST TWO TERMS. THEREFORE IN THIS CASE HUFFMAN CODE IS OF FORM: 1, 01, 001, 0001, ... OR 0, 10, 110, 1110...

5.18 CLASSES OF CODES. CONSIDER CODE $\{0, 01\}$

(a) Is it instantenous?

(b) Is it UNIQUELY DECODABLE?

(c) Is it NONSINGULAR

(a) NO „0" IS PREFIX OF „01"

(b) 001, 010, 0|01|00, 0101

YES! IT IS UNIQUELY DECODABLE

c) YES! SINCE IT IS UNIQUELY DECODABLE

5.19 GAME OF HI-LO.

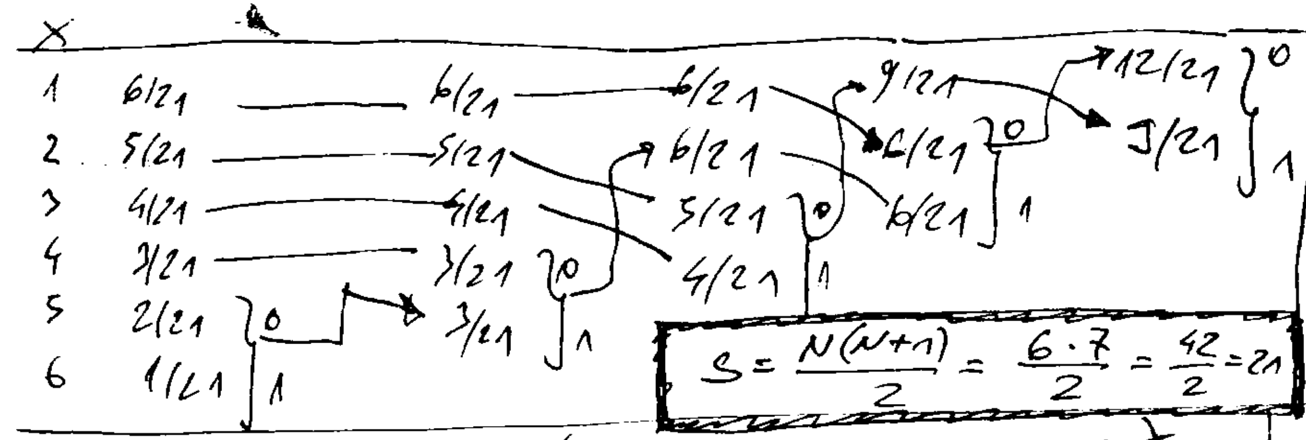
(a) A COMPUTER GENERATES A NUMBER X ACCORDING TO KNOWN PROBABILITY MASS FUNCTION $p(x)$ $x \in \{1, 2, .. 100\}$ THE PLAYER ASKS A QUESTION, "IS $x = i$?" AND IS TOLD "YES", "YOUR TOO HIGH," OR "YOUR TOO LOW". HE CONTINUES FOR A TOTAL OF 6 QUESTIONS. IF HE IS RIGHT (I.E. HE RECEIVES THE ANSWER „YES") DURING THIS SEQUENCE, HE RECEIVES A PRIZE OF VALUE $v(x)$. HOW SHOULD THE PLAYER PROCEED TO MAXIMIZE THE EXPECTED WINNINGS?

(b) PART (a) DOESN'T HAVE MUCH TO DO WITH INFORMATION THEORY. CONSIDER THE FOLLOWING VARIATION: $X \sim p(x)$ PRIZE $= v(x)$, $p(x)$ KNOWN AS BEFORE. BUT ARBITRARY

YES-NO QUESTIONS ARE ASKED SEQUENTIALLY UNTIL X IS DETERMINED. (DETERMINED DOESN'T MEAN THAT YES ANSWER IS RECEIVED) QUESTIONS COST 1 UNIT EACH. HOW SHOULD THE PLAYER PROCEED? WHAT IS EXPECTED PAYOFF?

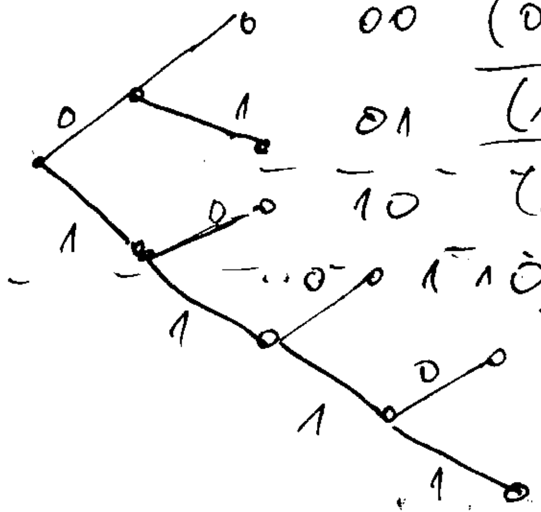
(c) CONTINUING PART (b), WHAT IF $v(x)$ IS FIXED BUT $p(x)$ CAN BE CHOSEN BY THE COMPUTER (AND THEN ANNOUNCED TO THE PLAYER)? THE COMPUTER WISHES TO MINIMIZE THE PLAYER EXPECTED RETURN. WHAT SHOULD $p(x)$ BE? WHAT IS THE EXPECTED RETURN TO THE PLAYER?

(a) - Huff Code



$$S = \frac{N(N+1)}{2} = \frac{6 \cdot 7}{2} = \frac{42}{2} = 21$$

l(x)	c(x)
2	00
2	10
2	11
3	010
4	0110
4	0111



		x
00	(0)	1
01	(1)	2
10	(2)	3
110	(6)	4
1110	(14)	5
1111	(15)	6

↑ x ↓

- IZBIRAM $x = 6$

1° DAZI $x = 2$! LOW (1)
2° DAZI $x = 3$? LOW (1)
3° DAZI $x = 4$? LOW (1)
4° DAZI $x = 5$! LOW (1)

LOW ≙ "1"
HIGH ≙ "0"

- IZBIRAM $x = 1$

1° DAZI $x = 2$ HIGH (0) → $x = 1$

- IZBIRAM $x = 2$

DAZI $x = 3$ → HIGH (0)
DAZI $x = 1$ → LOW (1)
} $x = 2$; ($c(x) = 01$)

RECALL: FANO - ELIAS CODING

$$F(x) = \sum_{a \leq x} p(a) \qquad \bar{F}(x) = \sum_{a<x} p(a) + \frac{1}{2} p(x)$$

$$\bar{F}(x) - \lfloor \bar{F}(x) \rfloor_{\ell(x)} < \frac{1}{2^{\ell(x)}}$$

$$0.010 1 1 \overset{\ell(x)}{\downarrow} 0 1$$

$$\lfloor \bar{F}(x) \rfloor_{5} = 2^{-2} + 2^{-4} + 2^{-5} = \frac{1}{4} + \frac{1}{16} + \frac{1}{32}$$

$$= \frac{8+2+1}{32} = \frac{11}{32} \qquad \bar{F}(x) \boxed{\frac{11}{32} + \frac{1}{128}}$$

$$\bar{F}(x) - \lfloor \bar{F}(x) \rfloor_{\ell(x)} = \cancel{\frac{11}{32}} + \frac{1}{128} - \cancel{\frac{11}{32}} = \frac{1}{128} < \frac{1}{32}$$

$$0.000011111\cdots = \frac{1}{64} + \frac{1}{128} + \cdots$$

$$\frac{1}{2^6} \sum_{i=0}^{\infty} \left(\frac{1}{2^i}\right) = \frac{1}{2^6} \cdot \frac{1}{1-\frac{1}{2}} = \frac{1}{2^5} = \frac{1}{32} = \frac{1}{2^{\ell(x)}}$$

$$s = 1 + \frac{1}{2} + \frac{1}{2^2} \cdots \frac{1}{2^n} \qquad s\left(1 - \frac{1}{2}\right) = 1 - \frac{1}{2^{n+1}}$$

$$\frac{s}{2} = \frac{1}{2} + \frac{1}{2^2} \cdots \frac{1}{2^{n+1}} \qquad s = \frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}}$$

$$n \to \infty \qquad \boxed{s = \frac{1}{1 - \frac{1}{2}}}$$

$$\text{IF} \quad \ell(x) = \left\lceil \log \frac{1}{p(x)} \right\rceil + 1 \Rightarrow \log \frac{1}{p(x)} \leq \ell(x) - 1$$

$$p(x)^{-1} \leq 2^{\ell(x)-1} \qquad p(x) \geq \frac{1}{2^{\ell(x)-1}} = \frac{2}{2^{\ell(x)}}$$

$$\frac{p(x)}{2} \geq \frac{1}{2^{\ell(x)}} \qquad \frac{1}{2^{\ell(x)}} \leq \frac{p(x)}{2} = \bar{F}(x) - \lfloor \bar{F}(x) \rfloor_{\ell(x)} =$$

$$= \bar{F}(x) - F(x-1)$$

(1) $x \in [1,2,3,4,5,6] \quad p(x) = [p_1\ p_2\ p_3\ p_4\ p_5\ p_6]$

ALGORITHM
G:
HI → 0
LO → 1
YES → TOGGLE
PREVIOUS BIT.

$\boxed{1°}$

1
2
③
4
5
6

IS $x = 4$ → hi (0)
IS $x = 2$ → Low (1)

$\boxed{x=3} \Rightarrow C(x=3) = 01$

$\boxed{2°}$ ① 2 3 ¦ 4 5 6

IS $x=4$ → HI (0)
IS $x=2$ → HI (0)

$\Rightarrow$ $x = ①$; $C(x=1) = 00$

$\boxed{3°}$ 1 ② 3 ¦ 4 5 6

IS $x=4$ → hi (0)
IS IS $x=2$? YES (1)

$x = 2$; $C(x=2) = 01$

$\boxed{4°}$ 1 2 3 ¦ 4 ⑤ 6

IS $x=4$ → LOW (1)
IS $x=5$ → YES (0)

$C(x=5) = 10$

$\boxed{5°}$ 1 2 3 ¦ 4 5 ⑥

IS $x=4$ (LOW), IS $x=5$ (LOW) $\quad C(x=6) = 11$

- ZNAČI ZA 6 ELEMENTI NAOĐA ODGOVOR SO DVE PRAŠANJA. SEKOGAŠ SE POTREBNI 2 PRAŠANJA.

- $X = [1, ②, 3, 4, 5, 6, 7, 8, 9, 10]$ $\boxed{C(x=2) = 001}$

1° $x=6$ Hi ; $x=3$ Hi ; $x=2$ Yes } TRI PRAŠANJA

2° IZBRAN e 7: $x=6$ LOW ; $x=8$ HIGH ; $C(x=7)=10$

3° IZBRAN e 9: $\underbrace{x=6 \text{ LOW}; x=8 \text{ LOW}; x=10 \text{ HIGH}}_{3 \text{ PRAŠANJA}}$

POSLEDNOTO NE MORA DA POSTAVI (SE PODRAZBIRA.)

OČIGLEDNO SO $\boxed{ld|X|}$ GO DOBIVAT MAKSIMALNIOT BROJ NA POTREBNI PRAŠANJA.

$$D(q\|2) = \sum q\, ld \frac{1}{2} = \left| 2 = \frac{1}{|X|} \right| = -\sum q\, ld\frac{1}{q} + \sum q\, ld\frac{1}{|X|^{-1}}$$

$$= -H_q(X) + ld|X| \sum q = -H_q(X) + ld|X| \geq 0$$

- $\boxed{H_q(X) \leq ld|X|}$ $\boxed{H_{max}(X) = ld|X|}$

- $\boxed{H(X) \leq E[L(X)] < H(X)+1}$

- $X = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]$

① IZBRAN e ⑮: $x=8$ LOW ; $x=12$ LOW ; $x=14$ LOW ; $x=16$ HIGH ILI $x=15$ YES

VKUPNO 4 PRAŠANJA

- $X = [1, 2, 3, \ldots 50 \ldots 75 \ldots 87 \ldots 93 \ldots 96 \ldots 98\ 100]$

IZBRAN e 99: $x=50$ LOW ; $x=75$ LOW ; $x=87$ LOW ; $x=93$ LOW ; $x=96$ LOW ; $x=98$ LOW ; $x=100$ HIGH ILI $x=99$ YES

ZAOKRUŽUVAM NA POMALIOT $\frac{75+100}{2} = \frac{175}{2} = |87.5| = 87$

IZBRAN e 99: $x=50$ LOW ; $x=75$ LOW ; $x=88$ LOW ; 94 LOW ; 97 LOW ; 99 YES ; 6 PRAŠANJA $\boxed{C(X=99) = 111112}$

AKO ZAOKRUŽUVAT NA POGOLEMIOT TOČNO 6 PRAŠANJA!!

p	X	QUESTIONS		l(x)	Hi-0 ; Lo-1 ; Yes-2	l(x)
1/10	1	6 Hi ; 4 Hi ; 3 Hi ; 2 Hi ; Yes	0000	4	00002	5
1/10	2	6 Hi ; 4 Hi ; 3 Hi ; 2 Yes	0001	4	0002	4
1/10	3	6 Hi ; 4 Hi ; 3 Yes ;	001	3	002	3
1/10	4	6 Hi ; 4 Yes	01	2	02	2
1/10	5	6 Hi ; 4 Lo ; Yes	01	2	012	3
1/10	6	6 Yes	1	1	2	1
1/10	7	6 Lo ; 8 Hi ; Yes	10	2	102	3
1/10	8	6 Lo ; 8 Yes	11	2	1	1
1/10	9	6 Lo ; 8 Lo ; 9 Yes	110	3	112	3
1/10	10	6 Lo ; 8 Lo ; 9 Lo ; Yes	111	3	1112	4

$\boxed{E[l(x)] = \sum q l = 2.9}$

TERNARY DIGITS.

$q = 1/55$

$L = 2.8$

$H = 2$ 153

x		C(x)	l(x)	p
1	4 Hi 3 Hi 2 Hi Yes	0002	4	6/21
2	4 Hi 3 Hi Yes	002	3	5/21
3	4 Hi Yes	02	2	4/21
4	Yes	2	1	3/21
5	4 Lo Yes	12	2	2/21
6	4 Lo 5 Lo Yes	112	3	1/21

$$S = \sum_{i=1}^{n} i = \frac{n\cdot(n+1)}{2}$$

$n = 10$

$\frac{10\cdot 11}{2} = 55$

$E[l(x)] = 2.28 \doteq 2.3$ TERNARY DIGITS

$\boxed{H_3(X) = 1.513}$ ← $p = \frac{i}{21}$

$p = 1/6 \quad i = 1 \ldots 6 \Rightarrow E[l(x)] = 2.5$

$\boxed{H_3(X) = 1.63}$ → $\boxed{H_{max}}$

- NEZGODNO E ZAOKRUŽUVANJETO METODI TREBA OVOJ ALGORITAM: AKO DOBIEŠ „LO" ZAOKRUŽI NA POGOLEMIOT A AKO DOBIEŠ „HI" ZAOKRUŽI NA POMALIOT: ★

X		C(x)	l(x)	P_1	P_2
1	4 Hi 2 Hi Yes	002	3	1/6	6/21
2	4 Hi Yes	02	2	1/6	5/21
3	4 Hi 2 Lo Yes	012	3	1/6	4/21
4	Yes	2	1	1/6	3/21
5	Lo Yes	12	2	1/6	2/21
6	Lo Lo Yes	112	3	1/6	1/21

$E_2[l(x)] = 2.28 \doteq 2.3$

$E_1[l(x)] = 2.33$

$$H_3(X) \leq E[Q] < H_3(X) + 1$$

$\boxed{1.5 \leq E[Q] < 2.6}$ T.D.

- $X = [1, 2 \ldots 10]$

$p_1 = [1/10 \ldots 1/10]$ $\qquad$ $p_2 = [10/55, 9/55 \ldots 1/55]$

$\boxed{H_2(X) = 1.95818}$ $\qquad$ $\boxed{H_1(X) = 2.096}$

$E_1[l(x)] = 2.9$

$E_2[l(x)] = 2.673$

$\boxed{E[l(x)] < 3.9}$

$5\sigma = 50 \quad \sigma = \frac{50}{3} =$

- $X = [1, 2, \ldots 100]$ $\qquad$ $p_1 = [1/100, 1/100, \ldots 1/100]$

$p_2 = [1/5050, 2/5050, 3/5050, \ldots 100/5050]$

$H_1(X) = 6.64 \qquad H_2(X) = 6.37 \qquad H_1 = H_{max}$

$6.37 \leq E[l(x)] < 7.64$

WITHOUT FINAL „YES"

ALTERNATIVNA METODA, SOGLASNO ★

x		C(x)	l(x)		
1	6 Hi 3 Hi 2 Hi Yes	0002	4	③	000
2	6 Hi 3 Hi Yes	002	3	3	002
3	6 Hi Yes	02	2	2	02
4	6 Hi 3 Lo Yes	012	3	3	012
5	6 Hi 3 Lo 4 Lo Yes	0112	4	③	011
6	Yes	2	1	1	2
7	6 Lo 8 Hi Yes	102	3	②	10
8	6 Lo Yes	12	2	2	12
9	6 Lo 8 Lo Yes	112	3	②	11
10	6 Lo 8 Lo 9 Lo Yes	1112	4	③	111

$E_1[l(x)] = 2.9$

$E_2[l(x)] = 2.872$

$

(6) DOVOLKU SE RAZLEGUVA (4) NO NE SE MNOZAT NEVOTIEMITE (ZES) ODGOVOR OČEKUVANATA DOLZINA NA IZRAČANJA IZRESUVA:

$E[l_1] = \sum l(x) p_1(x) = 2.4$ $\quad E[l_2] = \sum l(x) p_2(x) \doteq 2.3$ } POMALI ZA (0.5) VO SPOREDBA SO (4)

- PRETPOSTAVKA DEKA NA POLA OD KODOVITE ČE IM SE SKRATI DOLZINATA ZA „1" ⇒

$$\sum_{x\in X} l'(x)p(x) = \sum_{x\in X/2}(l(x)-1)p(x) + \sum_{x\in X/2} l(x)p(x)$$

$$\boxed{HMV} = \sum_{x\in X/2} l(x)p(x) + \sum l(x)p(x) - \sum_{x\in X/2} p(x) = E[l(x)] - 1/2$$

POMA

ZA DOVOLNO GOLEM BROJ NA ELEMENTI TEŽI KON 0.5

$E[X] = \bar{X} = \frac{1+2+3+4+5+6}{6} = \frac{21}{6} = \underline{3.5}$ } — ZA KOCKA ZA ZAMAD

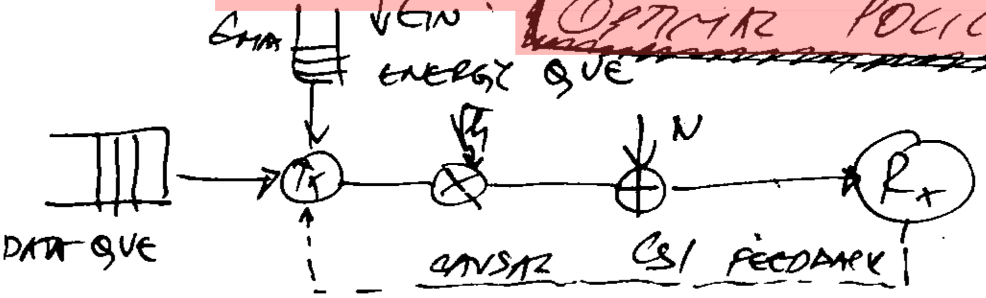
SAMPLE AVERAGE $= \sum_{i=1}^{4} x\,p(x)$

$\sum x\,p(x) = \bar{x} \xrightarrow[n\to\infty]{}$ SAMPLE AVERAGE

$$\frac{N(1)}{N} = \frac{N(2)}{N} = \frac{N(3)}{N} = \frac{N(4)}{N} = \frac{N(5)}{N} = \frac{N(6)}{N} = \frac{1}{6}$$

CONTINUE ON P.165

TRANSMISSION WITH ENERGY HARVESTING NODES IN FADING WIRELESS CHANNELS: OPTIMAL POLICIES



$\boxed{y = \sqrt{h}\,x + n}$

h → SQUARED FADING

n → GAUSSIAN RANDOM NOISE

$\frac{1}{2}\, ld(1+h\cdot p)$ - BITS OF DATA SERVED

p - TRANSMIT POWER

$p = x^2(t)$

$r(t) = \frac{1}{2}\, ld(1+h(t)p(t))$ } INSTANTENOUS RATE

- FADING LEVEL „h" AND ENERGY ARRIVALS ARE STOCHASTIC PROCESSES" IN TIME THAT ARE MARKED BY POISSON COUNTING PROCESSES WITH RATES λ_e & λ_e

$$d(t) = \max\{t_i^e : t_i^e \le t\}$$

$$\sum_{j=0}^{d(t)} E_j - \int_0^t p(u)\,du \le E_{max} \qquad \forall t \in [0,T]$$

- MAXIMIZING THROGHPUT IN STATIC CHANNEL

$$\sum_{i=1}^{l} L_i \gamma_i \le \sum_{i=0}^{l-1} E_i \qquad l = 1, \dots N+1$$

- BATTERY CAPACITY CONSTRAINT

$$\sum_{i=0}^{l} E_i - \sum_{i=1}^{l} L_i \gamma_i \le E_{max} \qquad l = 1, 2, \dots N$$

$$\boxed{\begin{aligned} &\max_{\gamma_i \ge 0} \sum_{i=1}^{N+1} \frac{L_i}{2} ld(1+\gamma_i) \\ &\text{S.T.} \quad \sum_{i=1}^{l} L_i \gamma_i \le \sum_{i=0}^{l-1} E_i \qquad l = 1, \dots N+1 \\ &\qquad\quad \sum_{i=0}^{l} E_i - \sum_{i=1}^{l} L_i \gamma_i \le E_{max} \qquad l = 1, \dots N \end{aligned}}$$

$$\mathcal{L} = \sum_{i=1}^{N+1} \frac{L_i}{2} ld(1+\gamma_i) - \sum_{i=1}^{N+1} \lambda_j \left(\sum_{i=1}^{j} L_i \gamma_i - \sum_{i=0}^{j-1} E_i \right)$$
$$- \sum_{j=1}^{N} \mu_j \left(\sum_{i=0}^{j} E_i - \sum_{i=1}^{j} L_i \gamma_i - E_{max} \right)$$

- ADDITIONAL COMPLEMENTARY SLACKNESS CONDITIONS:

$$\lambda_j \left(\sum_{i=1}^{j} L_i \gamma_i - \sum_{i=0}^{j-1} E_i \right) = 0 \qquad j = 1, 2, \dots N$$

$$\mu_j \left(\sum_{i=0}^{j} E_i - \sum_{i=1}^{j} L_i \gamma_i - E_{max} \right) = 0 \qquad j = 1, 2, \dots N$$

- KARUSH-KUHN-TUCKER CONDITIONS

$$P_\xi(k;\lambda) = \frac{\lambda^k}{k!} e^{-\lambda} \qquad \bar{\xi} = \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda} =$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=1}^{\infty} k \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} = \left| \begin{matrix} u=k-1 \\ k=1 \\ u=0 \end{matrix} \right|$$

$$= e^{-\lambda} \sum_{u=0}^{\infty} \frac{\lambda^{u+1}}{u!} = \lambda \cdot e^{-\lambda} \cdot \sum_{u=0}^{\infty} \frac{\lambda^u}{u!} = \lambda \cdot e^{-\lambda} \cdot e^{\lambda} = \underline{\underline{\lambda}}$$

$$p(r) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \qquad x = r^2$$

$$p(x) = \frac{p(r)}{\frac{dx}{dr}}\Bigg|_{r_{1,2}=\pm\sqrt{x}} = \frac{\frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}}{2r}\Bigg|_{r=\pm\sqrt{x}} =$$

NE POSTOI ZA r≤0

$$= \frac{1}{2\sigma^2} \cdot e^{-\frac{r^2}{2\sigma^2}}\Big|_{r=+\sqrt{x}} = \frac{1}{2\sigma^2} \cdot e^{-\frac{x}{2\sigma^2}} = \frac{1}{2\sigma^2} e^{-\frac{x}{2\sigma^2}}$$

$$p(x) = \frac{1}{2\sigma^2} \cdot e^{-\frac{x}{2\sigma^2}} = p_{exp}(x) \qquad \boxed{\lambda = \frac{1}{2\sigma^2}}$$

$$\text{mean}(p_{exp}(x)) = \lambda^{-1} = 2\sigma^2$$

$$\text{mean}(p(x)) = 2 \cdot \sigma^2 = /\sigma = 1/ = 2$$

- VO CLANAKOT BARAAT $\lambda = \frac{1}{\textcircled{2}}$

ZATOA ŠTO OVA VAKA GO IMAM IZBERANO VO POISSON-TA RASPREDELBA.

ZA DA DOBIJAM TAKVA EKSPONENCIJALNA TREBA DA IZBERAM VO NORMALNATA OD KOJA GENERIRAM ~~ONA~~ $\sigma = \frac{1}{2}$

$$\Rightarrow \text{mean}(p(x) = 2 \cdot \sigma^2 = 2 \cdot \frac{1}{4} = \frac{1}{2}$$

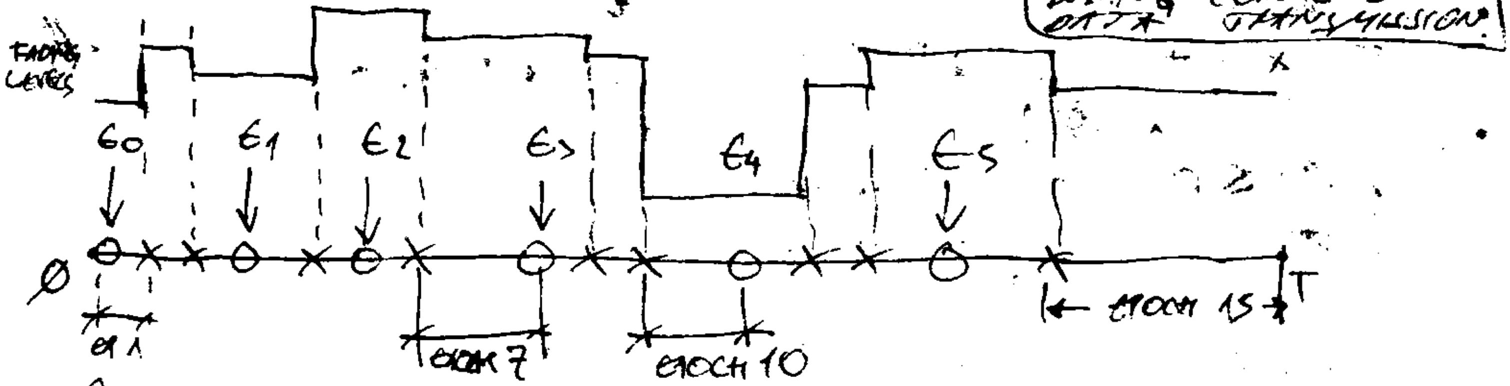
THE FADING LEVEL $[0, t_1^f)$ IS l_1, IN $[t_1^f, t_2^f)$ IS l_2, AND SO ON.

$$0 \; t_1^f \; t_2^f \; \ldots \; t_n^f \qquad 0 \; t_1^e \; t_2^e \; \ldots \; t_n^e$$

$$l_1 \; l_2 \qquad l_n \qquad\qquad E_0 \; E_1 \; E_2 \; \ldots \; E_n$$

$$\{(t_i^e, E_i)\}_{i=0}^{\infty} \qquad \{(t_i^f, l_i)\}_{i=1}^{\infty}$$

COMPLETELY DEFINE THE EVENTS THAT TAKE PLACE DURING COURSE OF DATA TRANSMISSION.



- ENERGY THAT HAS NOT ARRIVED YET CANNOT BE USED. AT THE CURRENT TIME THERE IS CAUSALITY CONSTRAINT ON THE POWER MANAGEMENT POLICY AS:

$$\int_0^{t_i^e} p(u)\,du \leq \sum_{j=0}^{i-1} E_j \quad \forall i$$

$\{t_1, t_2, \dots t_N\}$ $\quad$ $\{E_0, E_2, \dots E_N\}$

EPOCH LENGTHS ARE: $L_i = t_i - t_{i-1}$ $\quad i = 1, \dots N$
$L_{N+1} = T - t_N$ $\quad t_0 = 0$
$N+1$ - TOTAL NUMBER OF EPOCHS

- DUE TO FINITE BATTERY STORAGE CAPACITY, WE NEED TO MAKE SURE THAT ENERGY LEVEL IN THE BATTERY NEVER EXCEEDS E_{max}.

Let $d(t) = \max\{t_i^e : t_i^e \le t\}$

$$\sum_{j=0}^{d(t)} E_j - \int_0^t p(u)\,du \le E_{max} \quad \forall t \in [0,T]$$

KARUSH - KUHN - TUCKER CONDITIONS

- NON LINEAR OPTIMIZATION PROBLEM:

MINIMIZE $f(x)$
SUBJECT TO: $g_i(x) \le 0$ $\quad h_j(x) = 0$
$g_i (i = 1, \dots m)$ $\quad h_j (j = 1, \dots l)$

- NECESSARY CONDITIONS: SUPPOSE THAT THE OBJECTIVE FUNCTION $f: R^n \to R$ AND CONSTRAINT FUNCTIONS: $g_i: R^n \to R$ AND $h_j: R^n \to R$ ARE CONTINUOUSLY DIFFERENTIABLE AT A POINT x^*. If x^* IS LOCAL MINIMUM THAT SATISFIES SOME REGULARITY CONDITIONS (SEE BELOW), THEN THERE EXIST CONSTANTS $\mu_i (i = 1, \dots m)$ $\lambda_j (j = 1, \dots l)$ CALLED KKT MULTIPLIERS SUCH THAT:

- STATIONARITY:

$$\boxed{\nabla f(x^*) + \sum_{i=1}^{m} \mu_i \nabla g_i(x^*) + \sum_{j=1}^{l} \lambda_j \nabla h_j(x^*) = 0}$$

- PRIMAL FEASIBILITY:

$g_i(x^*) \le 0 \quad \forall i = 1, \dots m$
$h_j(x^*) = 0 \quad \forall j = 1, \dots l$

- DUAL FEASIBILITY

$\mu_i \ge 0$, for all $i = 1, \dots m$

- COMPLEMENTARY SLACKNESS -

$\mu_i g_i(x^*) = 0$, for all $i = 1, \dots m$

$$+\sum_{i=1}^{N+1}\frac{L_i}{2}\,ld(1+\gamma_i)-\sum_{j=1}^{N+1}\lambda_j\left(\sum_{i=1}^{j}L_i\gamma_i-\sum_{i=0}^{j-1}\epsilon_i\right)-\sum_{j=1}^{N}\mu_j\left(\sum_{i=0}^{j}\epsilon_i-\sum_{i=1}^{j}L_i\gamma_i-\epsilon_{max}\right) \leftarrow \text{(\$)}$$

$$\nabla\left[\sum_{i=1}^{N+1}\frac{L_i}{2}\,\overbrace{ld(1+\gamma_i)}^{(1)}\right]-\nabla\overbrace{\left[\sum_{j=1}^{N+1}\lambda_j\left(\sum_{i=1}^{j}L_i\gamma_i-\sum_{i=0}^{j-1}\epsilon_i\right)\right]}^{(2)}+$$

$$-\nabla\left[\sum_{j=1}\mu_j\left(\sum_{i=0}^{j}\epsilon_i-\underbrace{\sum_{i=1}^{j}L_i\gamma_i-\epsilon_{max}}_{(3)}\right)\right]=0$$

$$(1)\quad \frac{d}{d\gamma}\left[\frac{L_i}{2}\,ld(1+\gamma)\right]=\frac{L_i}{2\cdot\ln 2}\,\frac{d}{d\gamma}\left[\ln(1+\gamma)\right]=$$

$$=\frac{L_i}{2\ln 2}\,\frac{1}{(1+\gamma)}\cdot 1 \qquad \frac{L_i/2\ln 2}{1+\gamma_i}\longrightarrow$$

$$(2)\quad \sum_{j=1}^{N+1}\lambda_j\sum_{i=1}^{j}L_i\frac{d\gamma_i}{d\gamma_i}-0=\sum_{j=i}^{N+1}\lambda_j\cdot L_i\ \Big\}$$

ZA SEKOE i = IMA PO EDEN VAKOV REZULTAT OD IZVODOT!!!

$$(3)\quad -\sum_{j=i}^{N}\mu_j\;(L_i)$$

IZVODOT OD CLENOVITE L_i SEKOGAŠ E 0 !!!

– ZA KONKRETNOTO γ_i = γ_{N+1} (\$) SE SVEDUVA NA:

$$\frac{L_i/2\ln 2}{1+\gamma_i^*}-\sum_{j=1}^{N+1}\lambda_i\cdot L_i+\sum_{j=1}\mu_j L_i=0$$

$$\frac{L_i}{2(\ln 2)(1+\gamma_i^*)}=\left(\sum_{j=1}^{N+1}\lambda_i-\sum_{j=1}^{N+1}\mu_i\right)\cancel{L_i}$$

$$1+\gamma_i^*=\frac{1}{2(\ln 2)\left(\sum_{j=i}^{N+1}\lambda_i-\sum_{j=i}^{N}\mu_i\right)};\qquad \boxed{\gamma_i^*=\frac{1/2\ln 2}{\sum_{j=i}^{N+1}\lambda_j-\sum_{j=i}^{N}\mu_i}-1}$$

$$\gamma_{N+1}^*=\frac{1/(2\ln 2)}{\lambda_{N+1}-\underset{\downarrow\, 0}{\mu_{N+1}}}-1 \qquad \boxed{\gamma_{N+1}^*=\frac{1/(2\ln 2)}{\lambda_{N+1}}-1}$$

E.G. $\boxed{N=4}$ $\boxed{\mathring{i}=3}$

$$\frac{d}{d\gamma_3}\left[\sum_{i=1}^{5}\frac{L_i}{2}\,ld(1+\gamma_i)-\sum_{j=1}^{5}\lambda_j\left(\sum_{i=1}^{j}L_i\gamma_i-\sum_{i=0}^{j-1}\epsilon_i\right)-\sum_{j=1}^{4}\mu_j\left[\sum_{i=0}^{j}\epsilon_i-\sum_{i=1}^{j}L_i\gamma_i-\epsilon_{max}\right]\right]$$

$$\frac{d}{d\gamma_3}\left[\frac{L_3}{2}\,ld(1+\gamma_3)\right]-\frac{d}{d\gamma_3}\left[\sum_{j=3}^{5}\lambda_j\left(\sum_{i=1}^{j}L_i\gamma_i-\sum_{i=1}^{j-1}\epsilon_i\right)\right]-\frac{d}{d\gamma_3}\sum_{j=3}^{4}\mu_j\left[\sum_{i=0}^{j}\epsilon_i-\sum_{i=1}^{j}L_i\gamma_i-\epsilon\right.$$

$$= \frac{L_3}{2}\frac{1/\ln(2)}{1+\gamma_3} - \frac{d}{d\gamma_3}\left[\lambda_3\sum_{i=1}^{3}L_i\gamma_i + \lambda_4\sum_{i=1}^{4}L_i\gamma_i + \lambda_5\sum_{i=1}^{5}L_i\gamma_i\right] + \frac{d}{d\gamma}\left[\mu_3\sum_{i=1}^{3}L_i\gamma_i + \mu_4\sum_{i=1}^{4}L_i\gamma_i\right] = \frac{L_3}{2}\frac{1/\ln(2)}{1+\gamma_3} - \left[\lambda_3L_3+\lambda_4L_3+\lambda_5L_3\right] + \left[\mu_3L_3+\mu_4L_3+\mu_5L_3\right] = \frac{L_3}{2\ln(2)[1+\gamma_3]} - L_3\sum_{j=3}^{5}\lambda_j + L_3\sum_{j=3}^{4}\mu_j$$

(■) ↙ VO GENERALEN SLUČAJ ZA ∀i, ∀N:

$$(\#) \rightarrow \frac{L_i}{2\ln(2)\,[1+\gamma]} - L_i\sum_{j=i}^{N+1}\lambda_j + L_i\sum_{j=i}^{N}\mu_j = 0$$

$$\frac{1}{2\ln 2\,[1+\gamma_i]} = \sum_{j=i}^{N+1}\lambda_j - \sum_{j=i}^{N}\mu_j \Rightarrow \boxed{\gamma_i^* = \frac{1}{2\ln(2)\left[\sum_{j=i}^{N+1}\lambda_j - \sum_{j=i}^{N}\mu_j\right]} - 1 \quad i=1,2,\dots N} \quad (\bigstar)$$

$$\max_{\gamma_i\geq 0}\sum_{i=1}^{N+1}\frac{L_i}{2}\,\mathrm{ld}(1+\gamma_i) \qquad s.t. \sum_{i=1}^{l}L_i\gamma_i \leq \sum_{i=0}^{l-1}\epsilon_i \quad l=1,\dots N+1 \quad (7)$$

$$s.t. \sum_{i=0}^{l}\epsilon_i - \sum_{i=1}^{l}L_i\gamma_i \leq \epsilon_{max} \quad l=1,2..N \quad (8)$$

THEOREM 1 WHEN $\epsilon_{max}=\infty$, THE OPTIMAL POWER LEVELS IS A MONOTONICALY INCREASING SEQUENCE: $\gamma_{i+1}^* \geq \gamma_i^*$. MOREOVER IF FOR SOME l, $\sum_{i=1}^{l}L_i\gamma_i^* < \sum_{i=0}^{l-1}\epsilon_i$ THEN $\underline{\gamma_l^* = \gamma_{l+1}^*}$

PROOF: SINCE $\epsilon_{max}=\infty$, CONSTRAINTS:

$$\sum_{i=0}^{l}\epsilon_i - \sum_{i=1}^{l}L_i\gamma_i \leq \epsilon_{max} \qquad l=1,\dots N+1$$

ARE SATISFIED WITHOUT EQUALITY AND $\mu_i=0$ FOR ALL i = BY SLACKNESS CONDITIONS IN:

$$\mu_j\left(\sum_{i=0}^{j}\epsilon_i - \sum_{i=1}^{j}L_i\gamma_i - \epsilon_{max}\right)=0 \quad \Big\} \quad \epsilon_{max}\rightarrow\infty \Rightarrow \mu_j=0$$

FROM (★) SINCE $\lambda_i\geq 0$ OPTIMUM γ_i^* ARE MONOTONICALY INCREASING: $\gamma_{i+1}^* \geq \gamma_i^*$.

$$\gamma_{i+1}^* = \frac{1}{2\ln(2)\sum_{j=i+1}^{N+1}\lambda_j} \geq \gamma_i = \frac{1}{2\ln(2)\sum_{j=i}^{N+1}\lambda_j}$$

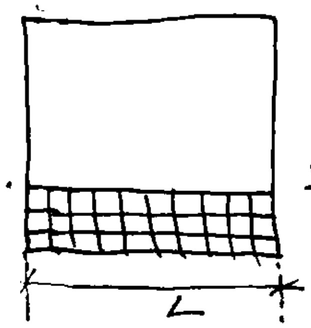
MOREOVER, IF FOR SOME l: $\sum_{i=1}^{l}L_i\gamma_i^* < \sum_{i=0}^{l-1}\epsilon_i$ THEN $\lambda_l=0$, WHICH MEANS $\gamma_{i+1}^* = \gamma_i^*$

$$\lambda_l\Bigg(\underbrace{\sum_{i=1}^{l}L_i\gamma_i - \sum_{i=0}^{l-1}\epsilon_i}_{<0}\Bigg)=0 \Rightarrow \underline{\lambda_l=0} \Rightarrow \gamma_l^* = \frac{1}{2\ln(2)\sum_{j=l+1}^{N+1}\lambda_j} = \gamma_{l+1}^*$$

A. DIRECTIONAL WATER FILLING ALGORITHM

XPRESS
OSL
XA
OML
FortMP
LINDO
PCx
LPSOLVE

AGENT SERV. 9089
HTTPS : 443
HTTP : 80

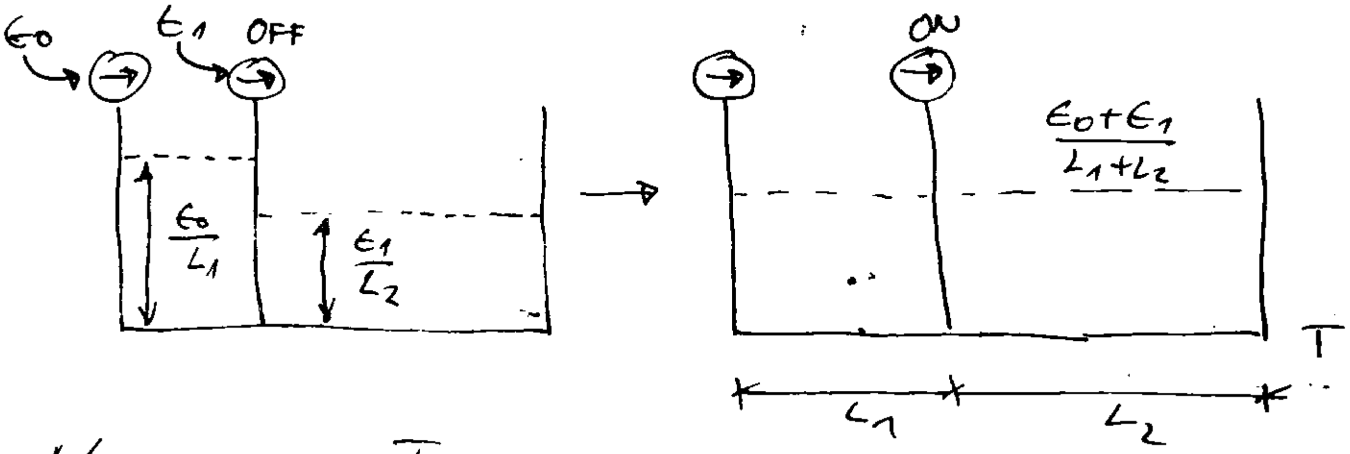


$\epsilon = 30$ UNITS $\quad L = 10 \Rightarrow$

WATER LEVEL $= \frac{\epsilon}{L} = \frac{30}{10} = 3$

$\frac{\epsilon}{L} = 3 \Rightarrow$ WATER LEVEL!!!

- RIGHT PERMEABLE TAP



■ MAXIMIZING THROUGHPUT IN A FADING CHANNEL

THE CHANNEL CHANGES M TIMES AND ENERGY ARRIVES N TIMES IN THE DURATION $[0,T)$. HENCE WE HAVE $M+N+1$ EPOCHS.

- TRANSMIT POWER IN EPOCH "i":
  $p_i \quad i = 1, \ldots, M+N+1$

$E_{in}(i)$ ENERGY WHICH ARRIVES AT EPOCH "i":

$E_{in}(i) = E_j$ FOR SOME "j" IF "i" IS ENERGY ARRIVAL

$E_{in}(i) = 0$ IF EVENT "i" IS FADE LEVEL CHANGE

$E_{in}(1) = E_0$

- OPTIMIZATION PROBLEM IS:

$$\max_{p_i \geq 0} \sum_{i=1}^{M+N+1} \frac{L_i}{2} \operatorname{ld}(1 + h_i p_i) \qquad (13)$$

$$\text{s.t.} \sum_{i=1}^{\ell} L_i p_i \leq \sum_{i=1}^{\ell} E_{in}(i), \quad \forall \ell \qquad (14)$$

$$\sum_{i=0}^{\ell} E_{in}(i) - \sum_{i=1}^{\ell} L_i p_i \leq E_{max}, \quad \forall \ell \qquad (15)$$

$$L = \sum_{i=1}^{M+N+1} \frac{L_i}{2} \operatorname{ld}(1 + h_i p_i) - \sum_{j=1}^{M+N+1} \lambda_j \left[\sum_{i=1}^{j} L_i p_i - \sum_{i=1}^{j} E_{in}(i) \right]$$

$$- \sum_{j=1}^{M+N+1} \mu_j \left(\sum_{i=1}^{j} E_{in}(i) - \sum_{i=1}^{j} L_i p_i - E_{max} \right) + \sum_{i=1}^{M+N+1} \eta_i p_i$$

- COMPLEMENTARY SLACKNESS CONDITIONS:

$$\lambda_i \left[\sum_{j=1}^{j} L_i \gamma_i - \sum_{i=1}^{j} E_{in}(i) \right] = 0 \quad \forall j \qquad (17)$$

$$\mu_j \left(\sum_{i=1}^{j} E_{in}(i) - \sum_{i=1}^{j} L_i \gamma_i - E_{max} \right) = 0 \quad \forall j \qquad (18)$$

$$\boxed{\eta_j \gamma_j = 0 \quad \forall j} \qquad (19)$$

① PP.160 ⇒ $\frac{L_3 \cdot h_3}{2\ln(2)[1+\gamma_3 h_3]} - L_3 \sum_{j=3}^{N+M+1} \lambda_j + L_3 \sum_{j=3}^{N+M} \mu_j + \eta_3$

- ZA BILO KOE i =

$$\frac{L_i h_i}{2\ln(2)[1+\gamma_i h_i]} - L_i \sum_{j=i}^{N+M+1} \lambda_j + L_i \sum_{j=i}^{N+M} \mu_j + \eta_i = 0$$

$$\frac{L_i h_i}{2\ln(2)[1+\gamma_i h_i]} = L_i \left[\sum_{j=i}^{N+M+1} \lambda_j - \sum_{j=i}^{N+M} \mu_j \right] - \eta_i$$

$$\frac{1}{[1+\gamma_i h_i]} = \frac{2\ln 2}{h_i} \left[\sum_{j=i}^{N+M+1} \lambda_j - \sum_{j=i}^{N+M} \mu_j \right] - \frac{2\ln(2) \cdot \eta_i}{L_i h_i}$$

$$1+\gamma_i h_i = \frac{1}{\frac{2\ln 2}{h_i}\left[\sum \lambda_j - \sum \mu_i\right] - \frac{2\ln(2)\eta_i}{L_i h_i}}$$

$$\boxed{\gamma_i^* = \frac{1}{2\ln(2)} \frac{1}{\sum_{j=i}^{N+M+1} \lambda_j - \sum_{j=i}^{N+M} \mu_j - \frac{\eta_i}{L_i}} - \frac{1}{h_i}}$$

IF $\gamma_i^* \neq 0 \Rightarrow \boxed{\eta_i = 0} \Rightarrow$

$$\gamma_i^* = \frac{1}{2\ln 2} \frac{1}{\sum_{j=i}^{N+M+1} \lambda_j - \sum_{j=i}^{N+M} \mu_j} - \frac{1}{h_i} = v_i - \frac{1}{h_i}$$

$$\boxed{v_i = \frac{1}{2\ln 2\left[\sum_{j=i}^{N+M+1} \lambda_j - \sum_{j=i}^{N+M} \mu_j\right]}} \quad (21) \qquad \boxed{\gamma_i^* = \left[v_i - \frac{1}{h_i}\right]} \quad (20)$$

- PADE APROXIMATION OF $\ln(1+x)$

$$\ln(1+x) = \frac{x}{4} + \frac{9}{8} - \frac{27}{8(3+2x)}$$

$$\max_{\gamma_i \geq 0} \sum_{i=1}^{N+1} \frac{L_i}{2} ld(v_i) \quad \text{st.}$$

$v_i = 1+\gamma_i$

$\gamma_i = v_i - 1$

$$\sum_{i=1}^{\ell} L_i (v_i - 1) \leq \sum_{i=0}^{\ell-1} E_i$$

$$\sum_{i=0}^{\ell} E_i - \sum_{i=1}^{\ell} L_i(v_i - 1) \leq E_{max}$$

997　　　9, 8, 9, 6, 8, 9, 8, 6, 0　　　010925

$\epsilon = [\epsilon_0, \epsilon_1, \epsilon_2, ---\ \epsilon_N]$　　$t = [t_1, t_2, \dots t_N]$

$L = [\ L_1, L_2, ---\ L_N, L_{N+1}]$　　$L_i = t_i - t_{i-1}$

$\gamma = [\ \gamma_1, \gamma_2, ---\ \gamma_N, \gamma_{N+1}]$　　$\boxed{L_{N+1} = T - t_N}$

$$\max_{\gamma_i \geq 0} \sum_{i=1}^{N+1} \frac{L_i}{2} ld(1+\gamma_i)$$

$$st.\ \sum_{i=1}^{l} L_i \gamma_i \leq \sum_{i=0}^{l-1} \epsilon_i \quad l=1, \dots N+1$$

$$st.\ \sum_{\boxed{i=0}}^{l} \epsilon_i - \sum_{i=1}^{l} L_i \gamma_i \leq \epsilon_{max} \quad l=1,2 \dots N$$

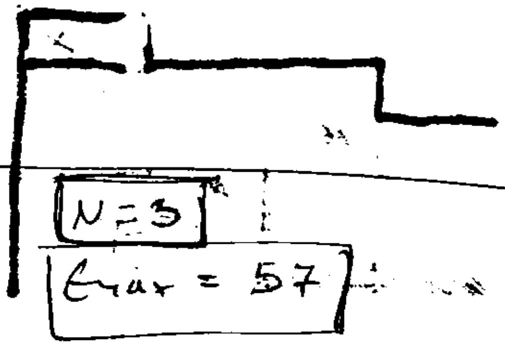
ALTERNATIVE:

$\epsilon = [\epsilon_1, \epsilon_2, \epsilon_3, ---\ \epsilon_{N+1}]$

$L = [L_1\ L_2, L_3, ---\ \epsilon_{N+1}]$

$\gamma = [\gamma_1, \gamma_2, \gamma_3, ---\ \gamma_{N+1}]$

$$\max_{\gamma_i \geq 0} \sum_{i=1}^{N} \frac{L_i}{2} ld(1+\gamma_i)$$



EXAMPLE:

$\epsilon = [3\ \ 1\ ;2\ ;1]$
$L = [0.2; 0.1; 0.3; 0.2]$
$\gamma = [\gamma_1; \gamma_2; \gamma_3; \gamma_4]$
$t = [0\ ;1\ ;2\ ;3\ \ 4]$

$\boxed{N = 3}$

$\boxed{\epsilon_{max} = 57}$

$$\max \quad \frac{0.2}{2} ld(1+\gamma_1) + \frac{0.1}{2} ld(1+\gamma_2) + \frac{0.3}{2} ld(1+\gamma_3) + \frac{0.2}{2} ld(1+\gamma_4)$$

CONST 1:

$L_1 \cdot \gamma_1 \leq \epsilon_0$　　$0.2\gamma_1 \leq 3$

$L_1\gamma_1 + L_2\gamma_2 \leq \epsilon_0 + \epsilon_1$　　$0.2\gamma_1 + 0.1\gamma_2 \leq 4$

$L_1\gamma_1 + L_2\gamma_2 + L_3\gamma_3 \leq \epsilon_0 + \epsilon_1 + \epsilon_2$　　$0.2\gamma_1 + 0.1\gamma_2 + 0.3\gamma_3 \leq 6$

$L_1\gamma_1 + L_2\gamma_2 + L_3\gamma_3 + L_4\gamma_4 \leq \epsilon_0 + \epsilon_1 + \epsilon_2 + \epsilon_3$　　$0.2\gamma_1 + 0.1\gamma_2 + 0.3\gamma_3 + 0.4\gamma_4 \leq 7$

CONST 2

$\epsilon_0 + \epsilon_1 - L_1\gamma_1 \leq \epsilon_{max}$　　$4 - 0.1\gamma_1 \leq 57$

$\epsilon_0 + \epsilon_1 + \epsilon_2 - L_1\gamma_1 - L_2\gamma_2 \leq \epsilon_{max}$　　$6 - 0.2\gamma_1 - 0.1\gamma_2 \leq 57$

$\epsilon_0 + \epsilon_1 + \epsilon_2 + \epsilon_3 - L_1\gamma_1 - L_2\gamma_2 - L_3\gamma_3 \leq \epsilon_{max}$　　$7 - 0.2\gamma_1 - 0.1\gamma_2 - 0.3\gamma_3 \leq 57$

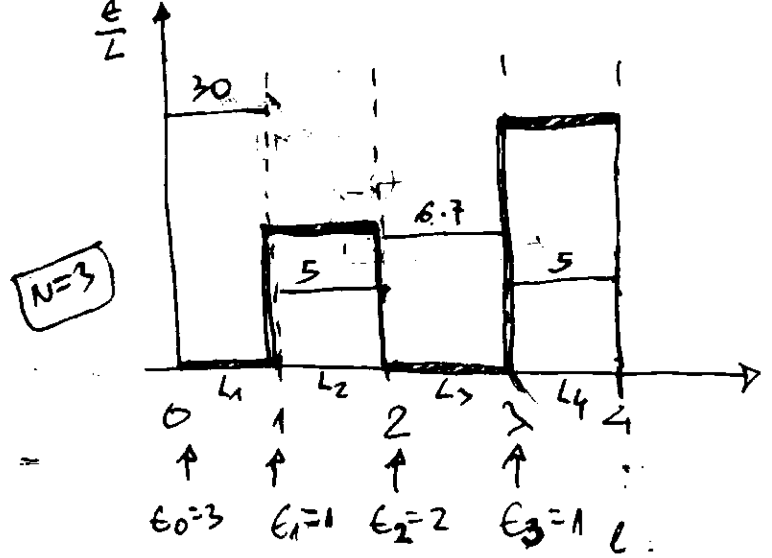
CONST 1 (CHECK)

$\gamma = [1, 1, 1, 1]$

1°　$0.2 \leq 3$　SLACK: 2.8
2°　$0.3 \leq 4$　SLACK: 3.7
3°　$0.6 \leq 6$　-"-: 5.4
4°　$0.8 \leq 7$　-"-: 6.2

CONST 2: (CHECK)

1°　$4 - 0.1 \leq 57$　SLACK: $57 - 3.9 = 53.1$
2°　$6 - 0.3 \leq 57$　-"-: $57 - 5.7 = 51.3$
3°　$7 - 0.6 \leq 57$　-"-: $57 - 6.4 = 50.6$



$$\overset{\epsilon_0\ \epsilon_1\ \epsilon_2\ \epsilon_3}{\epsilon = [3, 1, 2, 1]}$$

$$\epsilon_{max} = 5$$

$$\boxed{\Sigma L\gamma = [0; 2; 2; 7]}$$

$$\Sigma\epsilon = [3, 4; 6; 7]$$

$$\sum_0^l \epsilon - \sum_1^l L\gamma = [4; 4; 5]$$

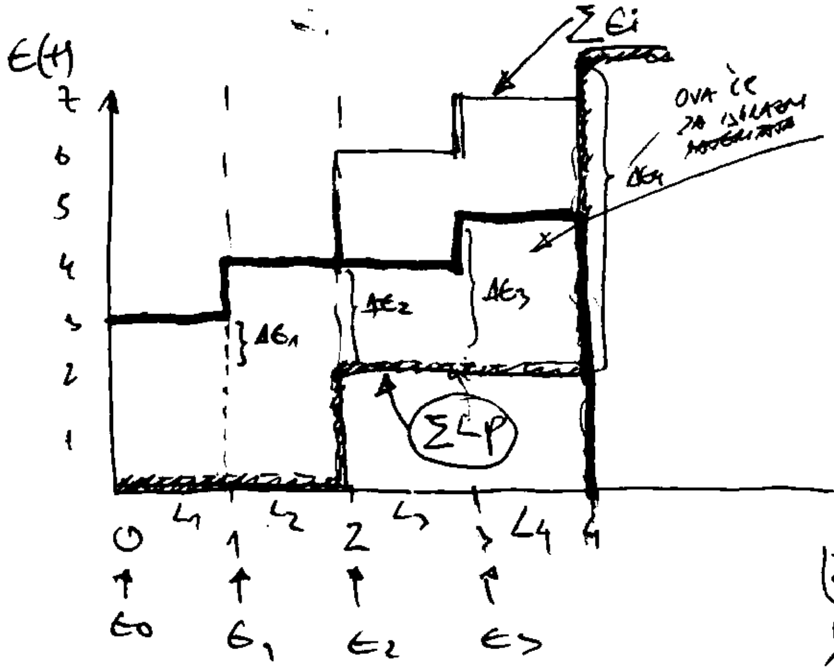
$$L = [0.1; 0.2; 0.3, 0.2]$$

$$\frac{\epsilon}{L} = [30; 5; 6.7; 5]$$

$$\frac{\epsilon_{max}}{L} = (50; 25; 16.7; 25]$$

$$\gamma = [0, 19, 0, 28]$$

[illegible] Е НИВОТО НА ЕНЕРГИЈА ВО БАТЕРИЈАТА. НЕ МОЖЕ ДА ОТИДЕ НАД $\epsilon_{max}=5$



$$\boxed{CST\,1}\quad \sum_{i=1}^{l} L_i\gamma_i \le \sum_{i=0}^{l-1}\epsilon_i \qquad l=1\ldots N$$

$$\boxed{CST\,2}\quad \sum_{i=0}^{l}\epsilon_i - \sum_{i=1}^{l} L_i\gamma_i \le \epsilon_{max} \qquad l=1\ldots N$$

CST 1 КАЖУВА ДЕКА ВКУПНО ПОТРОШЕНАТА СНАГА НЕ МОЖЕ ДА БИДЕ ПОГОЛЕМА ОД АКУМУЛИРАНАТА СНАГА

CST 2 КАЖУВА ДЕКА АКУМУЛИРАНАТА СНАГА ВО ДАДЕН МОМЕНТ НЕ МОЖЕ ДА БИДЕ ПОГОЛЕМА ОД ϵ_{max} ШТО Е МАКСИМАЛЕН КАПАЦИТЕТ НА БАТЕРИЈАТА.

OBJECTIVE FUNCTION: $\max_{\gamma_i \ge 0} \sum_{i=1}^{N} \frac{L_i}{2}\, ld(1+\gamma_i)$

$A\cdot x \le b \qquad L\cdot\gamma \le \epsilon$ (МАТРИЦИ)

$$\begin{bmatrix} L_1 & 0 & 0 & 0 \\ L_1 & L_2 & 0 & 0 \\ L_1 & L_2 \cdot L_3 & 0 \\ L_1 & L_2 & L_3 & L_4 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{bmatrix} \le \begin{bmatrix} \epsilon_0 \\ \epsilon_0+\epsilon_1 \\ \epsilon_0+\epsilon_1+\epsilon_2 \\ \epsilon_0+\epsilon_1+\epsilon_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \\ 7 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0.1 & 0.2 & 0 & 0 \\ 0.1 & 0.2 & 0.3 & 0 \\ 0.1 & 0.2 & 0.3 & 0.2 \end{bmatrix}}_{A_1} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{bmatrix} \le \underbrace{\begin{bmatrix} 3 \\ 4 \\ 6 \\ 7 \end{bmatrix}}_{b_1}$$

$$-\sum_{i=1}^{\ell} L_i \gamma_i \le E_{max} - \sum_{i=0}^{\ell} c_i \qquad \ell = 1 \ldots N \qquad \boxed{E_{max} = 10}$$

$$\underbrace{\begin{bmatrix} -L_1 & 0 & 0 \\ -L_1 & -L_2 & 0 \\ -L_1 & -L_2 & -L_3 \end{bmatrix}}_{A_2} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} \le 10 - \begin{bmatrix} 4 \\ 6 \\ 7 \end{bmatrix} \doteq \underbrace{\begin{bmatrix} 6 \\ 4 \\ 3 \end{bmatrix}}_{b_2}$$

$$\begin{bmatrix} L_1 & 0 & 0 & 0 \\ L_1 & L_2 & 0 & 0 \\ L_1 & L_2 & L_3 & 0 \\ L_1 & L_2 & L_3 & L_4 \\ -L_1 & 0 & 0 & 0 \\ -L_1 & -L_2 & 0 & 0 \\ -L_1 & -L_2 & -L_3 & 0 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \\ 7 \\ 4 \\ 6 \\ 7 \end{bmatrix}$$

arr_inst = [1 0 0 1 1 1 0 0 1 . . . . 0 ① 0]

no_en_arrivals = sum(arr_inst) = 11

CONTINUE FROM P.155

(6) ОТКРИВАЊЕТО НА ВИСТИНСКИОТ БРОЈ ШТО ГО ЗАМИСЛИ КОМПЈУТЕРОТ /ИГРАЧОТ ЌЕ ГО ЧИНИ (ВИДИ PP.155) НЕ ПОВЕЌЕ ОД:

PP.156 →

$$\boxed{\sum_{x\in X} l(x)\cdot p(x) - 0.5} \doteq E[l(x)] - 0.5$$

$$H_3(X) \le E[l(X)] \le H_3(X) + 1 \qquad H_3(X) = \sum_{x\in X} p(x) \log_3(x)$$

RECALL

$$\sum_{x\in X} D^{-l(x)} \le 1 \qquad l(x) = \log_D\left(\frac{1}{p(x)}\right) \qquad \boxed{p(x) = D^{-l(x)}} \Rightarrow$$

$$\sum_{x\in X} D^{-l(x)} = 1 \qquad \sum_{x\in X} l(x)\, p(x) = \sum_{x\in X} \log_D\left(\frac{1}{p(x)}\right) p(x) = H_D(X)$$

$$E(l(x)) \le \sum_{x} p(x) \left\lceil ld\left(\frac{1}{p(x)}\right)\right\rceil \le \sum_{x\in X} p(x)\left(ld\frac{1}{p(x)} + 1\right) = H(X) + 1$$

$$E[l(x)] \ge \sum_{x\in X} p(x) \left\lfloor ld\left(\frac{1}{p(x)}\right)\right\rfloor \ge \sum_{x\in X} p(x)\left[ld\frac{1}{p(x)}\right] = H(X) \Rightarrow$$

$$H(X) \le E[l(x)] = \sum_{x} l(x)\, p(x) \le H(X) + 1$$

– ZNAČI IGRAČOT NEMA DA PLATI POVEĆE OD:

$$\sum_x l(x)p(x) - 0.5 = E(l(x)) - 0.5 \leq H_3(X)+1-0.5 = \underline{H_3(X)+0.5}$$

THE PLAYER SHOULD FOLLOW SLICE CODING ALGORITHM. PAYOFF WILL BE:

$$V(X) - \lceil H_3(X)+0.5 \rceil = \underbrace{V(X)}_{\text{PRIZE}} - \underbrace{H_3(X)-0.5}_{\text{COST}}.$$

(c) $p(x_1) > p(x_2) \geq p(x_3) - - - \geq p(x_n)$.

NE E VAKA!!! VIDI □ PP. 167

VO VAKOV SLUČAJ AKO GO SLEDI HUFFMAN-OT ALGORITAM ĆE IMA MAKSIMALEN PROFIT BIDEJKI HUFFMANOVIOT ALGORITAM E OPTIMALEN, I DOBIENATA KODNA DOLZINA ĆE BIDE MINIMALNA T.E POMALA OD ONAA SO SLICE CODING.

PRIMER! ZA ($)

$E[l_1] = 2,4$ (SLICE CODE) UNIFORM p =

$E[l_2] = 2.3$ (SLICE CODE) $p_n \geq p_{n-1} \geq$ - - -

$11 \to 9 \to 7 \to 5 \to 3$

X	p(X)					c(X)	l(X)
1	10/55	10/55	10/55	18/55	27/55 0	2	1
2	9/55	9/55	10/55	10/55	18/55 1	01	2
3	8/55	8/55	9/55	10/55 0	10/55 2	02	2
4	7/55	7/55	8/55	9/55 1		10	2
5	6/55	6/55	7/55 0	8/55 2	▲	11	2
6	5/55	5/55	6/55 1			12	2
7	4/55	4/55 0	5/55 2			000	3
8	3/55	3/55 1				001	3
9	2/55 0	3/55 2				0020	4
10	1/55 1					0021	4
0	0 2					–	

$H(X) = 1.95818$.

$E[l(X)] = 2.0546$

1° PAZI $X = ①$ → L_0

2° PAZI $X \in 2, ③$ → L_0

3° PAZI $X \in 4, 5, ⑥$ → L_0

4° PAZI $X \in 7, 8, 9, 10$

④

"0" → 2, 3, 7, 8, 9, 10

"1" →

- SEKAKO BARANJETO VO (C) E KOMPJUTEROT KOJ JA ZADAVA RASPREDELBATA NA $\{p(x_1), p(x_2) \ldots p(x_n)\}$ ZA DA GO MINIMIZIRA PROFITOT NA IGRAČOT.

$H(X)$ E MAKSIMALNO ZA UNIFORMNA RASPREDELBA T.E.

$\boxed{p(x_1)=p(x_2) \ldots p(x_n) = p}$ $\boxed{H_{max} = \sum_{i=1} p\log_2(\frac{1}{p})}$

$= n \cdot \frac{1}{n} ld\, n = ld\, n = \log_2|X|$

OČEKUVANIOT PROFIT NA IGRAČOT E: $v(X) - H_n(X) - 0.5 =$
$= v(X) - ld|X| - 0.5$

EDITION 2 SOLUTIONS (a) SE BROI 1 YES !!!

- 1 QUESTION → 1 VALUE OF X
- 2 QUESTIONS

$2^5 = 32$

$\boxed{2^6 = 64}$

△1Q

2Q		
1	2Hi Yes	VIDI OPIS VO SOLUTION OF 2 EDITION
②	Yes	
3	2Lo Yes	

3Q	
1	2Hi Yes
2	Yes
3	2Lo 3Yes
4	2Lo 3Lo Yes

△2Q

1	3Hi 1Yes
2	3Hi 1Lo Yes
3	Yes
4	3Lo 4Yes
5	3Lo 4Lo 1Yes

2Q

△2Q

1	3Hi 1Yes
2	3Hi 1Lo 2Yes
3	Yes
4	3Lo 5Hi 4Yes
5	3Lo 5Yes
6	3Lo 5Lo 6Yes

3Q

△2Q

1	4Hi 2Hi 1Yes
2	4Hi 2Yes
3	4Hi 2Lo 3Yes
④	Yes
5	4Lo 5Yes
6	4Lo 5Lo 6Yes
7	4Lo 5Lo 6Lo ⑦Yes

$\boxed{3Q}$ → SO „Yes"

△3Q → BEZ „Yes"

- PRETPOSTAVKA ZA „K" PRAŠANJA ĆE POGODIŠ 2^K-1 BROJ OD „X" T.E. AKO $x \in [1, 2, \ldots 2^K-1]$ SO „K" PRAŠANJA ĆE OTKRIEŠ KOJ BROJ GO IZBRAL KOMPJUTEROT.
- $n = K+1$

$\left.\begin{matrix} 1 \\ 2 \\ \vdots \\ 2^K-1 \\ 2^{K+1}-1 \end{matrix}\right\}$ SO UŠTE EDNO PRAŠANJE ĆE POGODIŠ $2^{K+1}-1$

DODATE NA KOCKATA !!!

SO 3-TO PRAŠANJE SEKVENCATA JA DELIŠ NA DVE SEKVENCI SO DOLŽINA 2^K.

PRIMER: $\boxed{K=2}$

$\left.\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}\right\}$ 2Q ⇒ $2^2-1 = 3$

4

$\left.\begin{matrix} 5 \\ 6 \\ 7 \end{matrix}\right\}$ 1Q

$2^{2+1}-1 = 7$

3Q

NAVISTINA ĆE GO NAJDEŠ ZA 3Q.

- ZA „6" PRAŠANJA $2^6-1 = 64-1 = 63$

SO OVOJ ALGORITAM SO „6" PRAŠANJA MOŽEŠ DA POGODIŠ 63 BROJKI.

$\boxed{K=n}$ INDUKCIJATA VAŽI ZA BILO KOE „K" ZATOA ŠTO AKO VAŽI DEKA SEKVENCIJA OD 2^K-1 MOŽE DA SE POGODI SO „K" PRAŠANJA POKAŽI SEKVENCA OD $2^{K+1}-1$ BROJA ŠE POGODUVA SO K+1. ZA DELIŠ
PRAŠANJA ZATOA ŠTO SO POSLEDNOTO PRAŠANJE SEKVENCATA JA DELIŠ NA POLA (2^K-1)

- CONVEX OPTIMIZATION PROBLEM

$$\sum_{i=1}^{3} \frac{L_i}{2} ld\,(1+p_i) = (1+p_1)^{\frac{L_1}{2}}(1+p_2)^{\frac{L_2}{2}}(1+p_3)^{\frac{L_3}{2}}$$

$\boxed{L_1 = L_2 = L_3 = 2}$ $\quad (1+p_1)(1+p_2)(1+p_3)$

MINIMIZE: $\frac{1}{2} x^T Q x + c^T x$ $\qquad 0.000191027$

(b) EDITION 2 SOLUTION — EXPECTED RETURN

$$\textcircled{R} = \sum_{x \in X} \gamma(x)\left[v(x) - l(x)\right] = \sum_{x \in X} \gamma(x) v(x) - \overbrace{\underbrace{\sum_{x \in X} l(x)\gamma(x)}_{\le H+1}}^{\le H(X)}$$

$$\sum_{x \in X} \gamma(x) v(x) - H - 1 \le R \le \sum_{x \in X} \gamma(x) v(x) - H$$

(c) STANDARD MINIMIZATION PROBLEM WITH CONSTRAINTS.

$$J(\gamma) = \underbrace{\sum \gamma_i v_i + \sum \gamma_i \log \gamma_i}_{= \sum_{x \in X} \gamma(x) v(x) - H} + \lambda\left(\sum \gamma_i - 1\right)$$

$$\sum_{i=1}^{N} v_i + \sum_{i=1}^{N}\left(ld\,\gamma_i + \gamma_i \cdot \frac{1}{\gamma_i}\right) + N \cdot \lambda + \underbrace{\cancel{\sum \gamma_i}^{1} - 1}_{0} =$$

$$= \sum_{i=1}^{N}\left(v_i + ld\,\gamma_i + 1\right) + \sum_{i=1}^{N} \lambda = \sum_{i=1}^{N}\left(v_i + ld\,\gamma_i + 1 + \lambda\right) = 0$$

$\boxed{v_i + ld\,\gamma_i + 1 + \lambda = 0}$ $\qquad ld\,\gamma_i = -v_i - 1 - \lambda$

$$\gamma_i = 2^{-v_i - 1 - \lambda} = \frac{2^{-v_i}}{2^{1+\lambda}} = \frac{2^{-v_i}}{\sum_i 2^{-v_i}} \;??$$

$$r_i = \frac{2^{-v_i}}{\sum_j 2^{-v_j}} \qquad 2^{-v_i} = r_i \sum_j 2^{-v_j}$$

$$\sum \gamma_i v_i + \sum \gamma_i\, ld\,\gamma_i = \sum \gamma_i v_i + \sum \gamma_i\, ld\, 2^{-v_i} = \sum \gamma_i v_i + \sum \gamma_i\, ld\, r_i + \sum \gamma_i\, ld \sum_j 2^{-v_j}$$

$$= \sum \gamma_i v_i \underset{?}{\oplus} \sum \gamma_i\, ld\, r_i \underset{?}{\oplus} ld \sum_j 2^{-v_j} = D(\gamma_i \| r_i) - ld \sum_j 2^{-v_j}$$

$$\sum \gamma_i v_i + \sum \gamma_i\, ld\, r_i = \sum \gamma_i v_i + \sum \gamma_i\, ld\, 2^{-v_i} - \sum \gamma_i\, ld \sum 2^{-v_i}$$

$$\sum_{\forall i} \frac{2^{-v_i}}{2^{1+\lambda}} = 1 \qquad 2^{1+\lambda} = \sum 2^{-v_i}$$

$\sum_{x \in X} D^{-l(x)} \leq 1$ $\quad l(x) = \log_D \frac{1}{p(x)} = \quad \sum_{x \in X} D^{\log_D p(x)} = \underline{\sum p(x) = 1}$

$ld\, p_i = -v_i - \lambda - 1$ $\quad$ $\boxed{ld \frac{1}{p_i}} = v_i + \lambda + 1$

$2^{-v_i} \cdot 2^{-\lambda-1} = p_i$ $\quad$ $\boxed{2^{-v_i} = \frac{p_i}{2^{-\lambda-1}}}$ $\quad p_i = 2^{-(v_i+\lambda+1)}$

$\sum 2^{-v_i} = 1$ $\quad \sum \frac{p_i}{2^{-\lambda-1}} = 1$ $\quad \rightarrow \boxed{\lambda = -1}$

PROBLEM 5.20 Huffman Codes with cost. WORDS LIKE Run! Help! and Fire! ARE SHORT, NOT BECAUSE THEY ARE FREQUENTLY USED, BUT PERHAPS BECAUSE TIME IS PRECIOUS IN SITUATIONS IN WHICH THIS WORDS ARE REQUIRED. SUPPOSE THAT $X = i$ WITH PROBABILITY p_i, $i = 1, 2, \ldots, m$. LET l_i be the NUMBER OF BINARY SYMBOLS IN THE CODEWORD ASSOCIATED WITH $X = i$, AND LET c_i DENOTE THE COST PER LETTER OF THE CODEWORD WHEN $X = i$. THUS THE AVERAGE COST C OF THE DESCRIPTION OF X IS $C = \sum_{i=1}^{m} p_i c_i l_i$

(a) MINIMIZE C OVER ALL $l_1, l_2, \ldots l_m$ SUCH THAT $\sum 2^{-l_i} \leq 1$. IGNORE ANY IMPLIED INTEGER CONSTRAINTS ON l_i. EXHIBIT THE MINIMIZING $l_1^*, l_2^*, \ldots, l_m^*$ AND THE ASSOCIATED MINIMUM VALUE C^*.

(b) HOW WOULD YOU USE THE HUFFMAN CODE PROCEDURE TO MINIMIZE C OVER ALL UNIQUELY DECODABLE CODES? LET C_{HUFF} DENOTE THIS MINIMUM

(c) CAN YOU SHOW THAT

$$\boxed{C^* \leq C_{HUFF} \leq C^* + \sum_{i=1}^{m} p_i c_i}$$

RECALL (GENERATION OF DISCRETE DISTRIBUTIONS FROM FAIR COIN FLIPS)

$$X = \begin{cases} a & \text{with } p = 1/2 \\ b & \text{with } p = 1/4 \\ c & \text{with } p = 1/4 \end{cases}$$

0
10
11

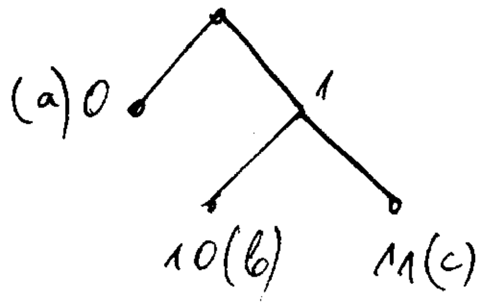
$\frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 + \frac{1}{4} \cdot 2 =$
$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$
$= 1.5.$

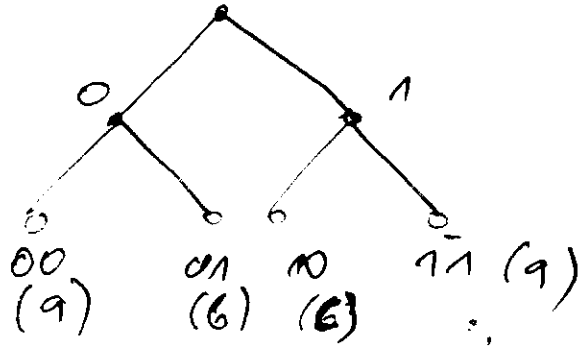
$H(X) = \frac{1}{2} ld\, 2 + \left(\frac{1}{4} ld\, 4\right) \cdot 2 = \frac{1}{2} + \frac{1}{2} \cdot 2 = 3/2$ bits

COIN TOSSES: $Z_1, Z_2, \ldots$

$X = \{1, 2, \ldots, m\}$ $p = \{p_1, p_2, \ldots p_m\}$

T – NUMBER OF COIN FLIPS USED IN ALGORITHM.





$$E[T] = \sum_{y \in Y} k(y) 2^{-k(y)}$$

$$H(Y) = \sum_{y \in Y} 2^{-k(y)} \operatorname{ld} 2^{k(y)} = \sum_{y \in Y} k(y) 2^{-k(y)}$$

k(y) – DEPTH OF THE LEAF "y"; ⇒ $E[T] = H(Y)$

T.5.11.1 $E[T] \geq H(X)$ $\boxed{X = f(Y)}$

$$H(X,Y) = H(X) + \underbrace{H(Y|X)}_{\neq 0} = H(Y) + \underline{H(X|Y)}_{0}$$

$$H(X) = H(Y) - H(Y|X) \Rightarrow \boxed{H(X) \leq H(Y)}$$

$$E[T] = H(Y) \geq H(X) \quad \boxed{H(X) \leq E[T]}$$

$\boxed{\text{T.5.11.2}}$ · $\mathcal{X}$ HAVE D-ADIC DISTRIBUTION THEN:

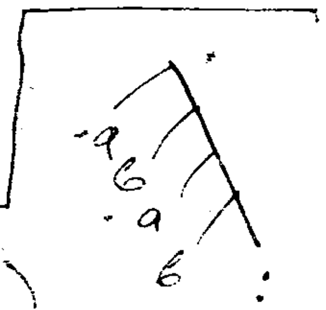
$$E[T] = H(\mathcal{X}) \qquad \frac{1}{2} + \frac{1}{8} + \frac{1}{16} = \frac{4+2+1}{16} = \frac{7}{16}$$

$$p_i = \sum_{j \geq 1} p_i^{(j)} \qquad p_i^{(j)} = 2^{-j} \text{ or } 0$$

$$\{p_i^{(j)} : i = 1, 2, \ldots m, \; j \geq 1\} \qquad = \sum_i p_i = 1$$

$\boxed{\text{Ex. 5.11.2}}$ $\mathcal{X} = \begin{cases} a & p(a) = 2/3 \\ b & p(b) = 1/3 \end{cases}$

$$\left(\frac{2}{3}\right)_{10} = (0.1010101)_2 \qquad \left(\frac{1}{3}\right)_{10} = (0.01010)_2$$



ATOMS ARE: $\frac{2}{3} = 0.10101010\ldots$

$\frac{1}{3} = 0.0101010\ldots$

$$\frac{2}{3} \to \left(\frac{1}{2}, \frac{1}{8}, \frac{1}{32}, \ldots\right) \qquad \frac{1}{3} \to \left(\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \ldots\right)$$

$$\boxed{H(X) \le E[T] \le H(X) + 2}$$

$(p_1, p_2, \dots p_n)$ $\qquad p_1 \to (p_1^{(1)}, p_1^{(2)}, \dots),$

$E[T] = H(Y)$ DYADIC DISTRIBUTION OF ATOMS

$Y \to (p_1^{(1)}, p_1^{(2)}, \dots, p_2^{(1)}, p_2^{(2)}, \dots, p_n^{(1)}, p_n^{(2)} \dots)$

$H(Y) = H(X) + H(Y|X) \qquad H(Y|X) < 2$

$$H(Y) = -\sum_{i=1}^{n} \sum_{j \ge 1} p_i^{(j)} \operatorname{ld} p_i^{(j)} = \sum_{i=1}^{n} \sum_{j: p_i^{(j)} \ge 0} j \cdot 2^{-j}$$

$$\boxed{T_i = \sum_{j: p_i^{(j)} \ge 0} j 2^{-j}}$$

$$2^{-(n-1)} > p_i \ge 2^{-n}$$

$$(n-1) < -\operatorname{ld} p_i \le n$$

$p_i^{(j)} > 0$ IF $j \ge n$

$$T_i = \sum_{j: j \ge n\ p_i^{(j)} > 0} j 2^{-j} \qquad p_i = \sum_{j: j > n,\ p_i^{(j)} > 0} 2^{-j}$$

$$T_i < -p_i \operatorname{ld} p_i + 2 p_i$$

$$\underline{T_i + p_i \operatorname{ld} p_i - 2 p_i} < T_i - p_i(n-1) - 2p_i = T_i - p_i(n-1+2)$$

$$= T_i - p_i(n+1) = \sum_{j: j \ge n\ p_i^{(j)} > 0} j 2^{-j} - (n+1) \sum_{\substack{j: j \ge n \\ p_i^{(j)} > 0}} 2^{-j} = \sum_{\substack{j: j \ge n \\ p_i^{(j)} > 0}} (j-n-1) 2^{-j}$$

$$= -2^{-n} - 0 + \sum_{\substack{j: j \ge n+2 \\ p_i^{(j)} \ge 0}} (j-n-1) 2^{-j} = \left| \begin{matrix} k = j-n-1 & j = k+n+1 \\ j = n+2 & k = 1 \end{matrix} \right| =$$

$$= -2^{-n} - 0 + \sum_{k: k \ge 1,\ p_i^{(k+n+1)} > 0} k\, 2^{-(k+n+1)} \le -2^{-n} + \sum_{k: k \ge 1} k \cdot 2^{-(k+n+1)}$$

$$= -2^{-n} + \left(\sum_{k \ge 1} k \cdot 2^{-k}\right) \cdot 2^{-(n+1)} = \qquad \sum_{n=1}^{\infty} n \cdot x^n = \frac{x}{(1-x)^2}$$

$$= -2^{-n} + 2^{-(n+1)} \cdot \frac{\frac{1}{2}}{\frac{1}{\cancel{4}\,2}} = -2^{-n} + 2^{-(n+1)} 2 = -2^{-n} + 2^{-n} = 0$$

HENCE: $T_i < -p_i \operatorname{ld} p_i + 2 p_i \qquad E[T] = \sum_i T_i$

$$E[T] < -\sum_i p_i \operatorname{ld} p_i + 2 = H(X) + 2$$

– ON AVERAGE $H(X) + 2$ COIN FLIPS SUFFICE TO SIMULATE RANDOM VARIABLE X?

$$\boxed{\Pr[l(x) \ge l^*(x) + c] \le \frac{1}{2^{c-1}}}$$

$Pr[l(x) \geq l'(x)+c] = Pr[\lceil ld \frac{1}{p(x)} \rceil \geq l'(x)+c] \leq$

$Pr[ld \frac{1}{p(x)} \geq l'(x)+c-1] = Pr[p(x) \leq 2^{-l'(x)-c+1}] =$

$= \sum_{x:\, p(x) \geq 2^{-l'(x)-c+1}} p(x) \leq \sum_{x:\, p(x) \geq 2^{-l'(x)-c+1}} 2^{-l'(x)-c+1} = 2^{-c+1} \underbrace{\sum_x 2^{-l'(x)}}_{\leq 1 \text{ KRAFT}}$

$\leq 2^{-c+1}$ // $\Rightarrow$ $\boxed{Pr[l(x) \geq l'(x)+c] \leq 2^{-c+1}}$ $\frac{1}{2^{c-1}}$

$Pr[l(x) \geq l'(x)+2] \leq \frac{1}{2}$ $\quad Pr[l(x) < l'(x)+2] =$

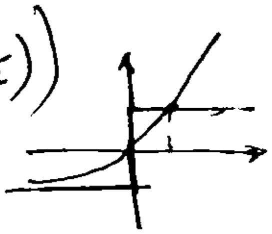
$= 1 - Pr[l(x) \geq l'(x)+2] \geq \frac{1}{2}$

$\boxed{7.5.10.2}$ $\quad Pr(l(x) < l'(x)) \geq Pr(l(x) > l'(x))$

$$\boxed{sgn(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}}$$

$sgn(t) \leq 2^t - 1$

$t = 0, \pm 1, \pm 2, \pm 3$



$Pr(l'(x) < l(x)) - Pr(l'(x) > l(x)) = \sum_{x:\, l(x) < l'(x)} p(x) - \sum_{x:\, l(x) > l'(x)} p(x) =$

$= \sum_x p(x)\, sgn[l(x) - l'(x)] \leq \sum_x 2^{-l(x)} [2^{l(x)-l'(x)} - 1] =$ (6)

$= \sum_x 2^{-l'(x)} - \sum_x 2^{-l(x)} = \sum_x 2^{-l'(x)} - 1 \leq \underbrace{1}_{\text{KRAFT}} - 1 = 0$ NEQ

$\Rightarrow Pr(l'(x) < l(x)) \leq Pr(l'(x) > l(x))$

EQUALITY FOR: $sgn(t) = 2^t - 1$ FOR $t = 0$ & $t = 1$ i.e.

$l(x) - l'(x) = 0 \quad l(x) = l'(x)$ OR

$l(x) - l'(x) = 1 \quad l(x) = l'(x) + 1$

(6) $\Rightarrow \sum 2^{-l'(x)} = 1 \Rightarrow \boxed{l(x) = l'(x)}$

CORROLARY (FOR NONDYADIC) $\quad E[sgn(l(x) - l'(x) - 1)] \leq 0$

$l(x) = \lceil ld \frac{1}{p(x)} \rceil$

$Pr(l(x) < l'(x)+1) > Pr(l(x) > l'(x)+1)$

$Pr(l'(x) < l(x)-1) - Pr(l'(x) > l(x)-1) = \sum_{x:\, l'(x) < l(x)-1} p(x) - \sum_{x:\, l'(x) > l(x)-1} p(x) =$

$= -\sum p(x)\, sgn(l'(x) - l(x) + 1) = -\sum p(x)\, sgn(l'(x) + 1 - l(x)) =$

$= \sum p(x)\, sgn(l(x) - l'(x) - 1) \leq \sum p(x) (2^{l(x)-l'(x)-1} - 1) \leq$ (*)

$l(x) = \left\lceil ld \frac{1}{p(x)} \right\rceil \qquad l(x) \leq ld \frac{1}{p(x)} + 1$

$ld \frac{1}{p(x)} \geq l(x) - 1 \qquad ld\, p(x) \leq -l(x) + 1 \quad p(x) \leq 2^{-l(x)+1}$

$$\circledast = \sum_x 2^{-l(x)+1} \cdot \left(2^{l(x) - l'(x) - 1} - 1\right) = \sum_x \cancel{2^{-l'(x)}} - \sum 2^{-l(x)+1}$$

$$= \underbrace{\sum_x 2^{-l(x)}}_{\leq 1} - \sum 2^{-l'(x)+1} \leq 1 - \underbrace{\sum 2^{-l(x)+1}}_{\geq p(x)} \leq 1 - \underbrace{\sum_x p(x)}_{1} = 0$$

$Pr\left(l'(x) < l(x) - 1\right) \leq Pr\left(l'(x) \geq l(x) - 1\right]$ i.e

$Pr\left(l(x) > l'(x) + 1\right) \leq Pr\left(l(x) < l'(x) + 1\right]$

$X = i \qquad p_i, \; i = 1, 2, \dots m$

l_i - NUMBER OF BINARY SYMBOLS IN CODEWORD ASSOCIATED WITH $X = i$

c_i - COST PER LETTER

- AVERAGE COST OF DESCRIPTION OF X IS:

$$C = \sum_{i=1}^{m} p_i c_i l_i$$

$$\min \sum_{i=1}^{m} p_i c_i l_i \qquad l_i - l_1, l_2 \dots l_m$$

$$\text{s.t.} \quad \sum_{i=1}^{m} 2^{-l_i} \leq 1$$

$$\frac{d}{dl_i} \left\{ \sum_{i=1}^{m} p_i c_i l_i + \lambda \left[\sum_{i=1}^{m} 2^{-l_i} - 1 \right] \right\} = 0 \qquad l_i = l_1, l_2 \dots l_m$$

$$\sum_{i=1}^{m} p_i c_i + \lambda \sum_{i=1}^{m} -\ln 2 \cdot 2^{-l_i} = 0$$

$$\boxed{\frac{d}{dx}\left(2^{-x}\right) = \frac{d}{dx}\left(e^{-x \ln 2}\right) = e^{-x \ln 2}(-\ln 2) = -\ln 2 \, e^{-x \ln 2}}$$

$$\sum_{i=1}^{m} \left(p_i c_i - \lambda \cdot \ln 2 \cdot 2^{-l_i}\right) = 0 \qquad p_i c_i = \lambda \ln 2 \cdot 2^{-l_i}$$

$$2^{-l_i} = \frac{p_i c_i}{\lambda \ln 2} \qquad \sum_{i=1}^{m} 2^{-l_i} = 1 \quad \sum_{i=1}^{m} \frac{p_i c_i}{\lambda \ln 2} = 1$$

$$\frac{c_i}{\lambda \ln 2} = 1 \qquad \boxed{\lambda = \frac{c_i}{\ln 2}} \qquad p_i \cancel{c_i} = \cancel{c_i}\, 2^{-l_i} \rightarrow$$

$$\boxed{p_i = 2^{-l_i}} \rightarrow l_i^* = ld\left(p_i^{-1}\right) \qquad L^* = \sum p_i l_i = \sum_{i=1}^{m} p_i \, ld \frac{1}{p_i} = H(X)$$

$$\boxed{C^* = \sum_{i=1}^{n} p_i c_i \,\mathrm{ld}\frac{1}{p_i}}$$

Solution 2 SOLUTION

$$C = \sum_{i=1}^{n} p_i c_i n_i \qquad \sum 2^{-n_i} < 1$$

ASSUME EQUALITY IN CONSTRAINT: $r_i = 2^{-n_i}$

$$\boxed{q_i = \frac{p_i c_i}{Q}; \quad Q = \sum p_i c_i;} \qquad C = \sum_{i=1}^{n} p_i c_i n_i = Q\sum_{i=1}^{n} q_i n_i$$

$$n_i = \mathrm{ld}\frac{1}{r_i}; \quad r_i = 2^{-n_i}; \qquad C = Q\sum_{i=1}^{n} q_i \,\mathrm{ld}\frac{1}{r_i} =$$

$$= Q\left(\sum_{i=1}^{n} q_i \,\mathrm{ld}\frac{q_i}{r_i} + \sum_{i=1}^{n} q_i \,\mathrm{ld}\frac{1}{q_i}\right) = \underline{Q\, D(q\|r) + Q H(q)}$$

$$\boxed{r_i = q_i} \Rightarrow \quad 2^{-n_i} = \frac{p_i c_i}{\sum_{i=1}^{n} p_i c_i} \qquad \boxed{n_i^* = -\mathrm{ld}\frac{p_i c_i}{\sum_{i=1}^{n} p_i c_i}}$$

- MINIMUM COST FOR THIS ASSIGNMENT IS:

$$\boxed{C^* = Q H(q)}$$

◦ PROBLEM 19 REVISITED **MMV**

$$v_i + \mathrm{ld}\, p_i + 1 + \lambda = 0 \qquad \boxed{p_i = 2^{-v_i - \lambda - 1}}$$

$$p_i = \frac{2^{-v_i}}{2^{\lambda+1}} \qquad \sum p_i = 1 \qquad \sum \frac{2^{v_i}}{2^{\lambda+1}} = 1$$

$$\boxed{2^{\lambda+1} = \sum_{j=1}^{n} 2^{v_j}}$$

MMV

$$\boxed{p_i = \frac{2^{-v_i}}{\sum_{i=1}^{n} 2^{v_i}}} \qquad \text{i.e.} \quad r_i = \frac{2^{-v_i}}{\sum_{i=1}^{n} 2^{v_i}}$$

$$J(p) = \sum p_i v_i + \sum p_i \,\mathrm{ld}\, p_i + \lambda\left(\sum_{i=1}^{n} p_i - 1\right) = \sum p_i v_i + \sum p_i \,\mathrm{ld}\, p_i +$$

$$+ \lambda\left(\sum_{i=1}^{n} \frac{2^{-v_i}}{\sum_{j=1}^{n} 2^{v_j}} - 1\right) = \sum p_i v_i + \sum p_i \,\mathrm{ld}\, p_i + \underbrace{\lambda(1-1)}_{0} =$$

$$= \sum p_i v_i + \sum p_i \,\mathrm{ld}\, p_i = \sum p_i v_i + \sum p_i \,\mathrm{ld}\, \underline{\sum_{j=1}^{n} 2^{v_j}} =$$

$$= \left| 2^{-v_i} = r_i \sum_{i=1}^{n} 2^{v_j} \qquad \boxed{-v_i = \mathrm{ld}\, r_i + \mathrm{ld}\sum_{j=1}^{n} 2^{v_j}} \right| =$$

$$= -\sum_{i=1}^{n} p_i \,\mathrm{ld}\, r_i - \underbrace{\sum r_i}_{1}\left(\mathrm{ld}\sum_{i=1}^{n} 2^{v_i}\right) + \sum p_i \,\mathrm{ld}\, p_i =$$

$$= \sum_{i=1}^{m} p_i \, ld \, p_i - \sum p_i \, ld \, r_i - ld \sum_{i=1}^{m} 2^{v_i}$$

$$\boxed{J(p) = D(p\|r) - ld \sum_{i=1}^{m} 2^{v_i}}$$

RETURN WILL BE MINIMIZED BY CHOSING $\boxed{p_i = r_i}$

(b) IF WE USE $q_i = \frac{c_i p_i}{Q}$ INSTEAD OF p_i FOR THE HUFFMAN PROCEDURE, WE OBTAIN CODE WHICH MINIMIZE THE EXPECTED COST.

(c)

$$C^* \le C_{HUFFMAN} \le C^* + \sum_{i=1}^{m} p_i c_i$$

$$n_i = \left\lceil ld \frac{1}{q_i} \right\rceil \qquad ld \frac{1}{q_i} \le n_i \le ld \frac{1}{q_i} + 1 \quad / \cdot p_i c_i \ \sum$$

$$\sum p_i c_i \, ld \frac{1}{q_i} \le \underbrace{\sum p_i c_i n_i}_{C_{HUFFMAN}} \le \sum p_i c_i \, ld \frac{1}{q_i} + \sum p_i c_i$$

$$Q \sum \frac{p_i c_i}{Q} ld \frac{1}{q_i} \le C_{HUFF} \le Q \sum \frac{p_i c_i}{Q} ld \frac{1}{q_i} + \sum \frac{p_i c_i}{Q}$$

$$Q \sum q_i \, ld \frac{1}{q_i} \le C_{HUFF} \le Q \sum q_i \, ld \frac{1}{q_i} + Q$$

$$Q \cdot H(q) \le C_{HUFF} \le Q \, H(q) + Q$$

$$\boxed{C^* \le C_{HUFF} \le C^* + Q}$$

PROVED !!!

$\boxed{5.21}$ CONDITIONS FOR UNIQUE DECODABILITY PROVE THAT CODE "C" IS UNIQUELY DECODABLE IF AND ONLY IF THE EXTENSION

$$C^k(x_1, x_2, \dots, x_k) = C(x_1) C(x_2) \cdots C(x_k)$$

IS ONE TO ONE MAPPING FROM X^k TO D^* FOR EVERY $k \ge 1$. (THE "ONLY IF PART" IS OBVIOUS)

SECOND EDITION SOLUTIONS: IF C^k IS NOT ONE-TO-ONE FOR SOME "k", THEN C IS NOT UD, SINCE THERE EXIST TWO DISTINCT SEQUENCES, $(x_1, \dots, x_k)$ AND $(x_1', \dots x_k')$ SUCH THAT

$$C^k(x_1, \dots x_k) = C(x_1) C(x_2) \cdots C(x_k) = C(x_1') C(x_2') \cdots C(x_k')$$
$$= C^k(x_1', x_2', \dots x_k')$$

CONVERSELY, IF C IS NOT UD THEN BY DEFINITION THERE EXIST DISTINCT SEQUENCES OF SOURCE SYMBOLS, $(x_1 \dots x_k)$ AND $(y_1 \dots y_k)$ SUCH THAT:

$$c(x_1)c(x_2)\cdots c(x_i) = c(y_1)\cdot c(y_2)\cdots c(y_j)$$

- CONCATENATING INPUT SEQUENCES WE OBTAIN:

$$c(x_1)\cdots c(x_i)\cdot c(y_1)\cdots c(y_j) = c(y_1)\cdots c(y_j)c(x_1)\cdots c(x_i)$$

$\Rightarrow C^k$ IS NOT ONE TO ONE FOR $k = i + j$.

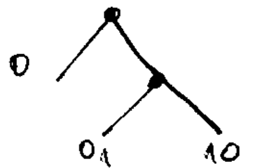
$\boxed{5.22}$ AVERAGE LENGTH OF AN OPTIMAL CODE. PROVE THAT $L(p_1, \dots p_m)$, THE AVERAGE CODEWORD LENGTH FOR AN OPTIMAL D-ARY PREFIX CODE FOR PROBABILITIES $\{p_1, \dots p_m\}$, IS CONTINUOUS FUNCTION OF $p_1, p_2, \dots p_m$. THIS IS TRUE EVEN THOUGH THE OPTIMAL CODE CHANGES DISCONTINUOUSLY AS PROBABILITIES VARY.

- $$L = \sum_{i=1}^{m} l_i \cdot p_i \qquad l_i = ld \frac{1}{p_i}$$

$$L = \sum_{i=1}^{m} p_i \, ld \frac{1}{p_i} = H(p)$$

$$\lim_{p \to 0} \left(p \, ld \, p^{-1} \right) = \lim_{p \to 0} \left(p \, ld \frac{1}{p} \right) = \lim \frac{ld \frac{1}{p}}{\frac{1}{p}} =$$

$$= \lim_{p \to 0} \frac{\left(ld \frac{1}{p}\right)'}{\left(\frac{1}{p}\right)'} = \frac{1}{\ln 2} \lim_{p \to 0} \frac{\frac{1}{\left(\frac{1}{p}\right)} \cdot \cancel{\frac{-1}{p^2}}}{\cancel{-\frac{1}{p^2}}} = \frac{1}{\ln 2} \lim_{p \to 0} p = 0$$



$\boxed{\text{PROBLEM 5.23}}$ (VIDI APPENDIX)

GO DOKAZAV ZA $D = 2$, $X = [x_1, x_2, x_3]$

SOLUTIONS 2

$$\sum_{i=1}^{m} D^{-l_i} = 1$$

IF THE CODE DOES NOT SATISFY THE PREFIX CONDITION, THEN AT LEAST ONE CODEWORD, SAY $c(x_1)$, IS A PREFIX OF ANOTHER, SAY $c(x_m)$
⇒ THEN THE PROBABILITY THAT RANDOM GENERATED SEQUENCE BEGINS WITH THE CODEWORD IS AT MOST:

$$\sum_{i=1}^{m-1} D^{-l_i} \le 1 - D^{-l_m} < 1$$

$c(x_1)$ ILI $c(x_1)c(x_2)$ ILI $c(x_1)c(x_2)\cdots c(x_{m-1})$ ⓘ?

(b) $\{0, 01, 11\}$

$01111 \cdots 1110$

EVEN: $0, 11, 11, \cdots 11, 0$

ODD: $01, 11, 11, \cdots, 11, 0$

MORA DA ZA-ČEKA CELA PORUKA
⇒ DECODING DELAY → ∞

5.24 OPTIMAL CODES FOR UNIFORM DISTRIBUTIONS.

CONSIDER A RANDOM VARIABLE WITH „m" EQUIPROBABLE OUTCOMES. $H(X) = ld\ m$ bits

(a) DESCRIBE THE OPTIMAL INSTANTENOUS CODE FOR THIS SOURCE AND COMPUTE AVERAGE CODEWORD LENGTH.

(b) FOR WHAT VALUES OF „m" DOES THE AVERAGE CODEWORD LENGTH L_m EQUAL THE ENTROPY $H = ld(m)$?

(c) WE KNOW THAT $L < H+1$ FOR ANY PROBABILITY DISTRIBUTION. THE REDUNDANCY OF A VARIABLE-LENGTH CODE IS DEFINED TO BE:

$$\rho = L - H$$

FOR WHAT VALUE(S) OF m, WHERE $2^k \leq m \leq 2^{k+1}$ IS THE REDUNDANCY OF THE CODE MAXIMIZED? WHAT IS THE LIMITING VALUE OF THIS WORST-CASE REDUNDANCY AS $m \to \infty$.

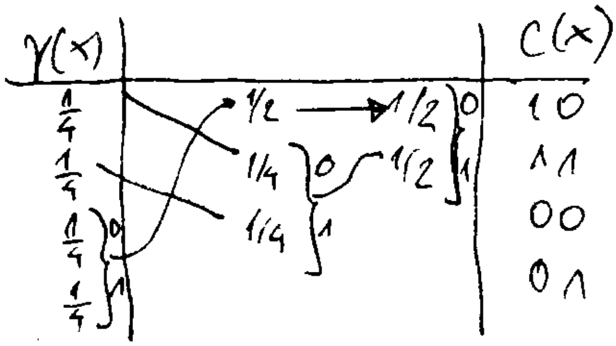
Alt + Shift + H

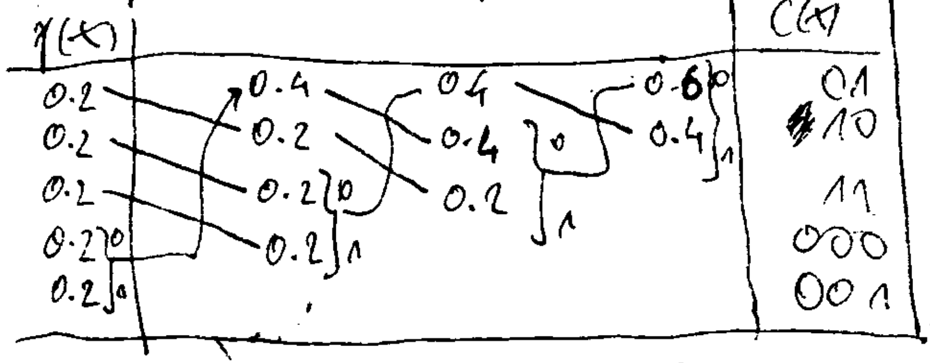
U₀ 27400012

Ap.V 4.6.0.207

PIN : 21824897

ENGINEERING SCREEN





$$E[l''(x)] = \frac{1}{5}\sum_x l(x) =$$

$$\frac{1}{5}(3\cdot 2 + 2\cdot 3) = \frac{12}{5} = 2.4$$

$$E[l'(x)] = \frac{1}{4}\sum_x l(x) =$$

$$= \frac{1}{4}\cdot 4\cdot 2 = 2$$

(a) $E[l(x)] = \sum_x p(x)\,l(x) = \frac{1}{m}\sum_x l(x)$

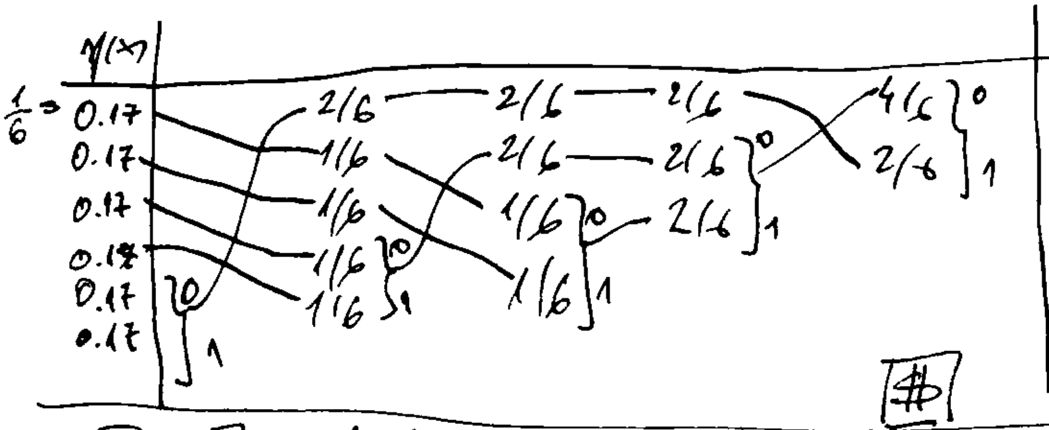
$$H'(x) = \left(\frac{1}{4}\cdot ld4\right)\cdot 4 = ld4 = 2$$

$$H''(x) = \left(\frac{1}{5}\,ld5\right)\cdot 5 = ld5 = 2.32193$$

(b) $H(x) = \sum \frac{1}{m}\,ld\,m = E[l(x)] = \sum \frac{1}{m}\,l(x)$

$ld\,m = l(x)$ $\quad m = 2^{l(x)}\ \forall x$

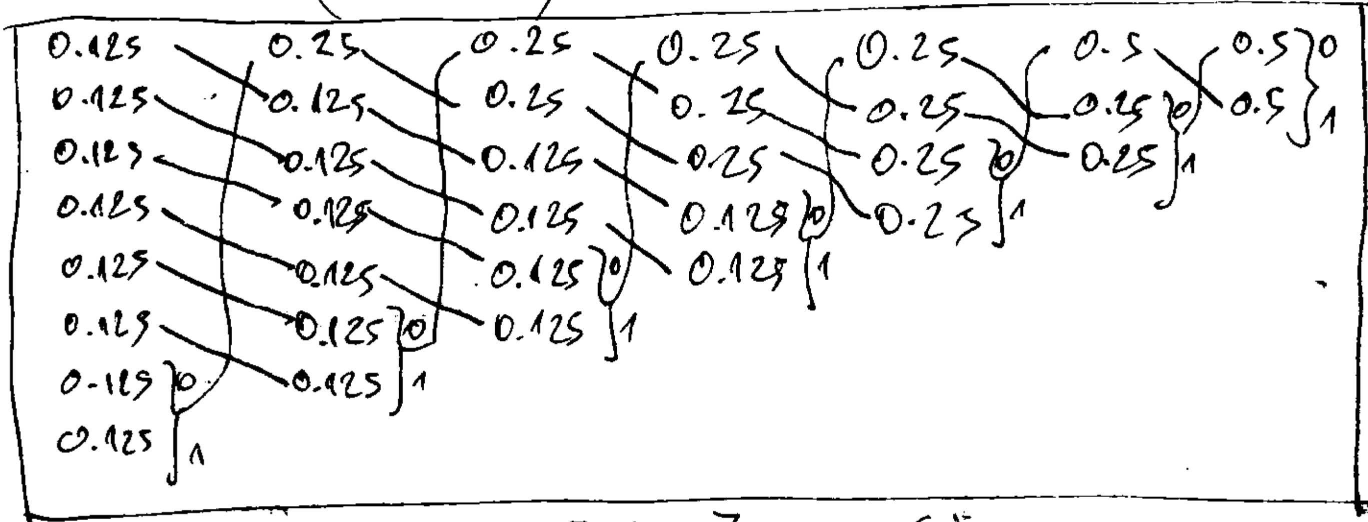
$\mathcal{X} = \{x_1 x_2 \cdots x_m\}$ $\quad m = 4$ $\quad \boxed{l(x) = 2}$



0 1 0	3
0 1 1	3
0 0 0	3
0 0 1	3
1 0	2
1 1	2

$$E[l(x)] = \frac{1}{6}(4\cdot 3+2\cdot 2) = \frac{16}{6} = 2.666$$

$$H(x) = \left(\frac{1}{6}\operatorname{ld}6\right)6 = \operatorname{ld}6 = 2.585$$



000
001
010
011
100
101
110
111

- Дефинитивно $E[l(x)] = H(x)$ ако:

$l(x) = \operatorname{ld} n$ за секое $x \in [x_1, x_2 \, - - \, x_n]$

т.е $n = 2^{l(x)} \; \forall x$

(c) $\rho = L - H = \frac{1}{n}\sum l(x) - \operatorname{ld} n$

$\frac{d\rho(n)}{dn} = 0 \qquad -\frac{1}{n^2}\sum l(x) - \frac{1}{n}\cdot\frac{1}{\ln(2)} = 0$

$-\sum\frac{l(x)}{n^2} - \frac{1}{n}\sum\frac{1}{n\ln 2} = 0 \qquad -\sum\left(\frac{l(x)}{n^2} + \frac{1}{n^2\ln 2}\right) = 0$

$l(x) + \frac{1}{\ln 2} = 0 \qquad l(x) = -\frac{1}{\ln 2}$

$\max \rho$

st $2^k \leq n \leq 2^{k+1}$ $\qquad \boxed{k \leq \operatorname{ld} n \leq k+1}$

$\rho = \sum\left(\frac{l(x)}{n} - \frac{\operatorname{ld} n}{n}\right) \qquad \rho = \rho_{max}$ if $\operatorname{ld} n = k$

$\lim_{n\to\infty}\rho_{max} = \lim_{n\to\infty}\left(\frac{1}{n}\sum l(x) - k\right) = -k$

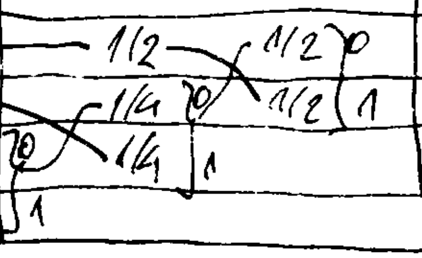
EDITION 2 SOLUTIONS

UNIFORMLY DISTRIBUTED CODEWORDS, THERE EXISTS AN OPTIMAL BINARY VARIABLE LENGTH PREFIX CODE SUCH THAT THE LONGEST AND SHORTEST CODEWORDS DIFFER BY AT MOST ONE BIT.

$m_S - c_S$ MESSAGE AND CODEWORD FOR THE SHORTER CODEWORD

$m_L - c_L$ MESSAGE AND CODEWORD FOR THE LONGER CODE.

$c_s' = c_s 0$ $c_l' = c_s 1$ (A)



$p(x)$	$C(x)$	$C'(x)$
1/2	1	10
1/4	01	010
1/8	000	011
1/8	001	11

$c_s = 1$
$c_s' = 10$
$c_l' = 11$

(B) $c_s = 01$ $c_s' = 010$
$c_l' = 011$

$$E(l(x)) = \frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 2 + \left(\frac{1}{8}\cdot 3\right)2 = \frac{1}{2} + \frac{1}{2} + \frac{3}{4} = 1 + \frac{3}{4} = \frac{7}{4}$$

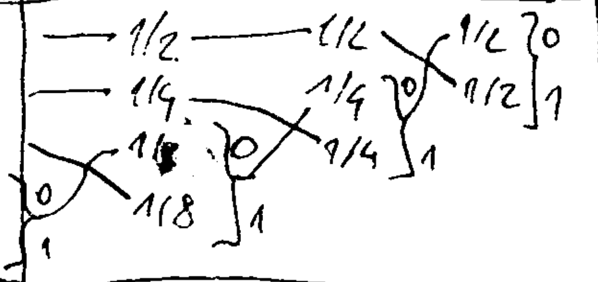
$$E(l'(x)) = \frac{1}{2}\cdot 2 + \left(\frac{1}{4}\cdot 3\right)\cdot 2 + \frac{1}{8}\cdot 2 = 1 + \frac{3}{2} + \frac{1}{4} = \frac{4+6+1}{4} = \underline{11/4}$$

FOR SOURCE WITH "n" MESSAGES.

$l(m_s) = \lfloor ld\, n \rfloor$ $l(m_l) = \lceil ld\, n \rceil$

$\boxed{d = n - 2^{\lfloor ld\, n \rfloor}}$ $n = 4$ $d = 4 - 2^2 = 0$

MMV: OVA E LOGIČNO!! AKO $n \sim 2^x$ TOGAŠ $d = 0$



$p(x)$	$C(x)$	$C'(x)$
1/2	1	10
1/4	01	010
1/8	001	001
1/16	0000	011
1/16	0001	11

$$E[l(x)] = \frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 2 + \frac{1}{8}\cdot 3 + \frac{1}{16}\cdot 4\cdot 2 = \frac{1}{2} + \frac{1}{2} + \frac{3}{8} + \frac{1}{2} = \frac{8+4+3}{8} = \frac{15}{8} = 1.87$$

$E[l'(x)] = 2.43$

$\boxed{n=5}$ $d = 5 - 2^{\lfloor ld\, 5 \rfloor} = 5 - 2^2 = 5 - 4 = 1$

$l(m_s) = \lfloor ld\, n \rfloor = \lfloor ld\, 5 \rfloor = 2$ //

$l(m_l) = \lceil ld\, n \rceil = \lceil ld\, 5 \rceil = 3$ //

THEN THE OPTIMAL CODE HAS $2d$ CODEWORDS OF lenght $\lceil ld\, n \rceil$ AND $n-2d$ CODEWORDS OF LENGTH $\lfloor ld\, n \rfloor$

e.g. $\boxed{\#}$ $n=6$ $d = n - 2^{\lfloor ld 6 \rfloor} = 6 - 2^2 = 2$

$2 \cdot 2 = 4$ $\lceil ld\, n \rceil = 3$. 4 CODEWORDS LENGHT 3

$6-4=2$ $\lfloor ld 6 \rfloor = 2$ 2 CODEWORDS " " 2

$$L = \frac{1}{n}\left[2d \cdot \lceil ld\, n \rceil + (n-2d)\lfloor ld\, n \rfloor \right)\Big] = \frac{1}{n}\Big[2d\left(\lceil ld\, n \rceil - \lfloor ld\, n \rfloor\right)$$

$$+ n \lfloor ld\, n \rfloor \Big] = \frac{1}{n}\left[2d + n\lfloor ld\, n \rfloor\right] = \frac{2d}{n} + \lfloor ld\, n \rfloor$$

(b) $H(X) = ld\, n = \frac{2d}{n} + \lfloor ld\, n \rfloor$

$$ld\, n - \lfloor ld\, n \rfloor = \frac{2d}{n}$$

$$L = \sum_i p_i l_i = -\sum_i p_i \, ld\, 2^{-l_i} = -\sum_i p_i \, ld\left(\frac{2^{-l_i}}{p_i} \cdot p_i\right) =$$

$$= -\sum_i p_i \, ld(p_i) - \sum_i p_i \, ld \frac{2^{-l_i}}{p_i} =$$

$$= \sum_i p_i \, ld \frac{1}{p_i} + \sum_i p_i \, ld \frac{p_i}{2^{-l_i}} = H(X) + D(p \| q)$$

$\boxed{q = 2^{-l_i}}$ $\Rightarrow$ IF $p = q = 2^{-l_i}$ $L = H(X)$

WHEN "n" IS POWER OF 2

$ld\, 2^x = \frac{2d}{n} + \lfloor ld\, 2^x \rfloor$ $x = \frac{2d}{n} + x$ $\Rightarrow$

$\Rightarrow \frac{2d}{n} = 0 \Rightarrow \boxed{d = 0} \rightarrow$ ALL CODEWORDS ARE WITH EQUAL LENGTH !!

ISTORO SUM GO KONSTANTINOS NA Π. 178.

(c) $\boxed{n = d + 2^m}$ $r = L - H$

$$r = \frac{2d}{n} + \lfloor ld\, n \rfloor - ld\, n = \frac{2d}{n} + \underbrace{\lfloor ld(d+2^m) \rfloor}_{m =} - ld(d+2^m)$$

$$= \frac{2d}{n} + m - ld(d+2^m) = \frac{2d}{d+2^m} + m - \frac{ln(d+2^m)}{ln 2}$$

$$\frac{\partial r}{\partial d} = \frac{2(d+2^m) - 2d}{(d+2^m)^2} + 0 - \frac{1}{ln 2} \cdot \frac{1}{d+2^m} = 0 .$$

$$ln 2 \left[2(d+2^m) - 2d\right] - (d+2^m) = 0 \Rightarrow$$

$$2 ln 2 (d+2^m) - 2d\, ln 2 - (d+2^m) = 0$$

$$2 \cdot \ln 2 \cdot 2^n - (d + 2^n) = 0 \qquad \boxed{d^* = 2^n(2\ln 2 - 1)}$$

$$\boxed{d^* = 0.38629 \cdot 2^n}$$

$$r^* = \frac{2 \cdot 0.3862 \cdot 2^n}{0.3862 \cdot 2^n + 2^n} + n - \frac{\ln(0.3862 \cdot 2^n + 2^n)}{\ln 2} =$$

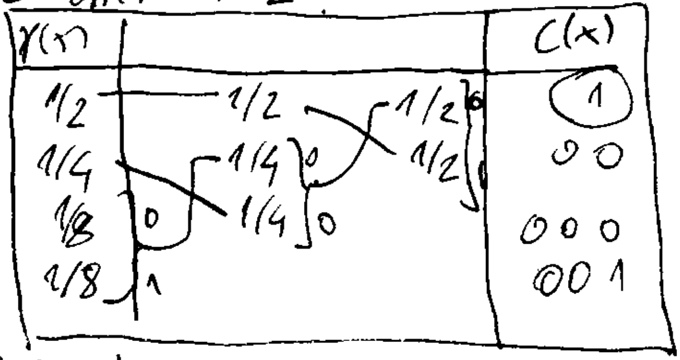
$$= \frac{2 \cdot 0.3862}{0.3862 + 1} + \cancel{n} - \cancel{ld\, 2^n} - ld(1.3862) =$$

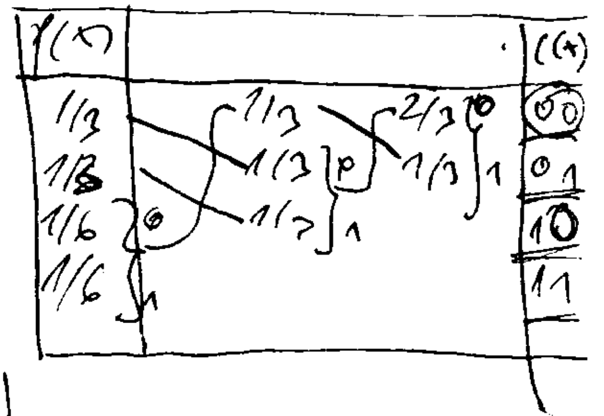
$$= \frac{2 \cdot 0.7862}{0.3862 + 1} - ld(1.3862) = 0.08607$$

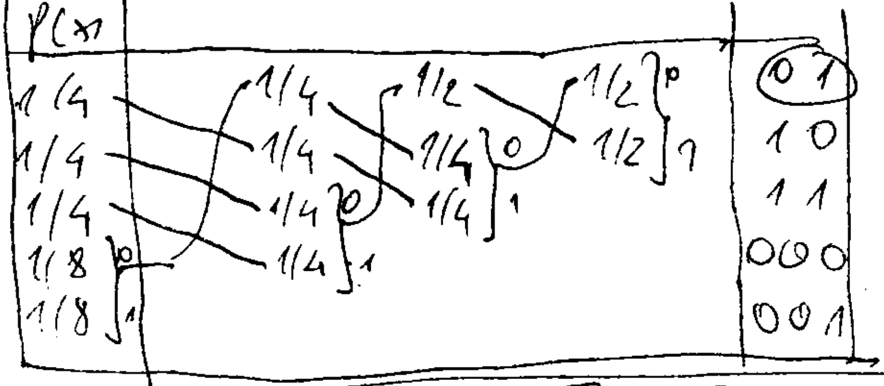
PROBLEM 5.25 OPTIMAL CODEWORD LENGHTS. ALTHOUGH THE CODEWORD LENGHTS OF AN OPTIMAL VARIABLE-LENGTH CODE ARE COMPLICATED FUNCTIONS OF THE MESSAGE PROBABILITIES $\{p_1, p_2, \dots p_m\}$, IT CAN BE SAID THAT LESS PROBABLE SYMBOLS ARE ENCODED INTO LONGER CODEWORDS. SUPPOSE THAT THE MESSAGE PROBABILITIES ARE GIVEN IN DEACRESING ORDER: $p_1 > p_2 > \dots p_m$

(a) PROVE THAT FOR ANY BINARY HUFFMAN CODE, IF THE MOST PROBABLE MESSAGE SYMBOL HAS $p_1 > \frac{2}{5}$ THAT SYMBOL MUST BE ASSIGNED CODEWORD LENGHT 1.

(b) PROVE THAT FOR ANY BINARY HUFFMAN CODE, IF THE MOST PROBABLE MESSAGE SYMBOL HAS PROBABILITY $p_1 < \frac{1}{3}$ THAT SYMBOL MUST BE ASSIGNED A CODEWORD LENGHT ≥ 2.







$p_1 > 0.4$ 1 bit

$p_1 < 0.33$ 2 bit

$$\boxed{l(x) = \lceil ld\, 3 \rceil = \lceil 1.584 \rceil = 2}$$

$$\boxed{l(x) = \left\lceil ld \frac{1}{p(x)} \right\rceil = \left\lceil ld \frac{5}{2} \right\rceil = \lceil 1.32 \rceil = 2}$$

$L(X) = \sum_{x} p(x) l(x) = p_1 l_1 + \sum_{i=2}^{n} p_i l_i > \frac{2 l_1}{5} +$

$\sum_{x > x_1} p(x) l(x)$ $\qquad L(X) \leq H(X) + 1$

$H(X) + 1 \geq L(X) > \frac{2 l_1}{5} + \sum_{x > x_1} p(x) l(x)$

$l_1 \cdot \frac{2}{5} < H(X) + 1 - \sum_{x > x_1} p(x) l(x) + p_1 l_1 - p_1 l_1 =$

$= H(X) + 1 - \underbrace{\sum_{x} p(x) l(x)}_{E[l(X)]} + p_1 l_1 \leq \cancel{H(X)} + 1 - \cancel{H(X)} + p_1 l_1$

$| E[l(X)] \geq H(X) |$

$\frac{2 l_1}{5} \leq 1 + p_1 l_1 \leq 1 + l_1 \qquad \frac{2}{5} l_1 - l_1 \leq 1$

- $\left(\frac{2}{5} - \frac{5}{5}\right) l_1 \leq 1 \qquad -\frac{3}{5} l_1 \leq 1 \qquad l_1 \geq -\frac{5}{3}$ $\boxed{??}$

- DA ZEMAM $\boxed{p_1 = \frac{1}{2}}$

$L(X) = \frac{1}{2} l_1 + \sum_{i \geq 2} p_i l_i = p_1 l_1 + p_2 l_2 + \dots p_n l_n$

$p_1 \geq p_2 \geq p_3 \dots p_n \qquad l_1 \geq l_2 \geq l_3 \dots \geq l_n$.

$L(X) \geq p_1 l_1 + p_2 l_1 + \dots + p_n l_1$ $\qquad \boxed{l_1 \leq L(X)}$

$L(X) \leq p_1 l_1 + p_1 l_2 + \dots + p_1 l_n$

$L(X) \leq \frac{2}{5} (l_1 + l_2 + \dots l_n)$

$\boxed{l_1 \leq H(X) + 1}$

$\frac{2}{5} (l_1 + l_2 + \dots l_n) \geq H(X)$ $\qquad \boxed{l_1 + l_2 + \dots + l_n \geq \frac{5}{2} H(X)}$

- LEMA 5.8.1 (EXTENSION OF OPTIMAL CODE)

$L(X) = p_1 l_1 + (p_2 + p_3 + \dots p_n) \cdot l_2$ $\underbrace{3/5}$

$p_1 = \frac{2}{5} \Rightarrow L(X) = \frac{2}{5} l_1 + \left(1 - \frac{2}{5}\right) l_2$

$H(X) = \frac{2}{5} ld \frac{5}{2} + \frac{3}{5} ld \frac{5}{3} = 0.97$

$L(X) \geq H(X) \qquad \frac{2}{5} l_1 + \frac{3}{5} l_2 \geq 0.97$

$l_1 = l_2 \Rightarrow \quad \frac{2}{5} l_1 + \frac{3}{5} l_1 \geq 0.97 \quad l_1 \geq 0.97$

- $p_1 = 1/3 \qquad p_2 + \dots p_m = 2/3$

$L(X) = \frac{1}{3} l_1 + \frac{2}{3} l_2 \qquad H(X) = \frac{1}{3} \cdot ld 3 + \frac{2}{3} ld \frac{3}{2}$
$= 0.11830$

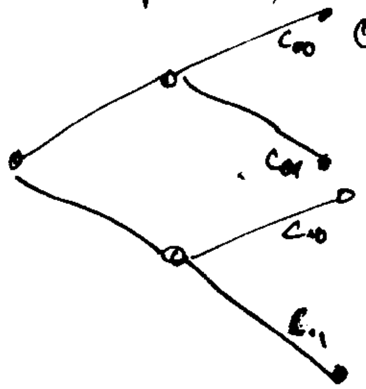
EDITION 2 SOLUTION

p(x)		C
0.49	0.51 0	1
0.49 0	0.49 1	00
0,02 1		01

(a) SUPPOSE, FOR THE SAKE OF CONTRADICTION, THAT $l_1 \geq 2$. THEN THERE ARE NO CODEWORDS OF LENGHT 1; OTHERWISE c_1 WOULD NOT BE SHORTEST CODEWORD. WITHOUT LOSS OF GENERALITY, WE CAN ASSUME THAT c_1 BEGINS WITH 00. FOR $x, y \in \{0,1\}$ LET C_{xy} DENOTE THE SET OF CODEWORDS BEGINNING WITH xy.

$$\boxed{p(C_{01}) + p(C_{10}) + p(C_{11}) = 1 - p_1 < 3/5}$$

e.g. $p(C_{10}) + p(C_{11}) < 2/5$



$00 \rightarrow 1$

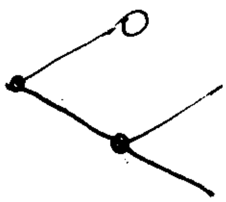
$01 \rightarrow 01$

$C_{10} \rightarrow 000$

$C_{11} \rightarrow 001$.

THIS IMPROVEMENT CONTRADICTS THE ASSUMPTION THAT $l_1 \geq 2$ AND SO $l_1 = 1$.

(b) FOR THE SAKE OF CONTRADICTION, ASSUME $l_1 = 1$.
$c_1 = 0$, $p(C_{10}) + p(C_{11}) = 1 - p_1 > 2/3$
SO AT LEAST ONE OF THESE SETS (WITHOUT LOSS OF GENERALITY C_{10}) HAS PROBABILITY GREATER THAN 2/3. WE CAN NOW OBTAIN BETTER CODE BY INTERCHANGING THE SUBTREE OF THE DECODING TREE BEGINNING WITH 0 WITH SUBTREE BEGINNING WITH 10;



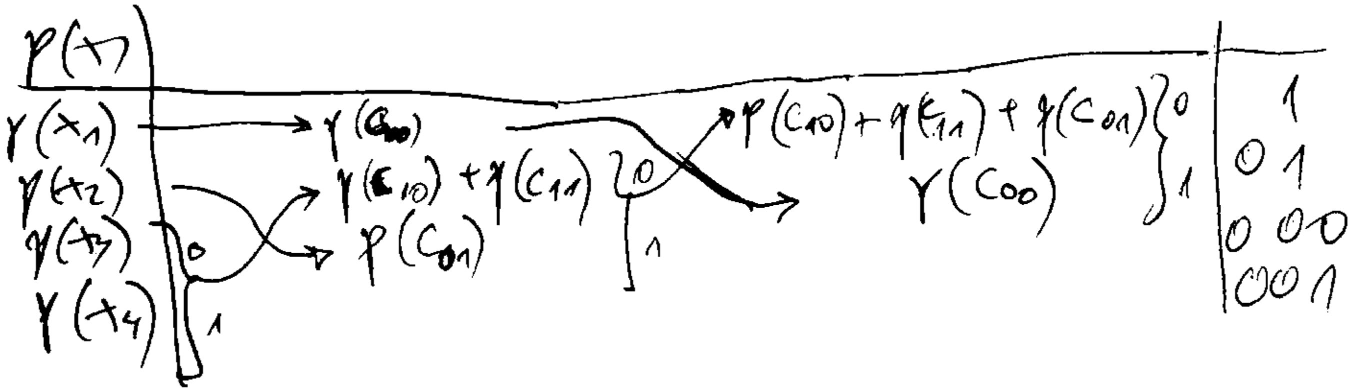
$0 \longrightarrow 10$

$01 \longrightarrow 01$

$10 \longrightarrow 0$

$\Rightarrow \boxed{l_1 \geq 2}$

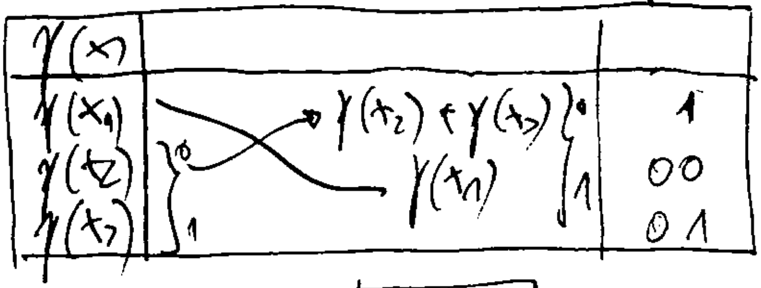
CONTRADICTS THE ASSUMPTION.



$$\gamma(x_3)+\gamma(x_4)=\gamma(c_{10})+\gamma(c_{11})<2/5$$

$$\gamma(x_1)>2/5$$

$$\gamma(x_2)+\gamma(x_3)+\gamma(x_4)=\gamma(c_{01})+\gamma(c_{10})+\gamma(c_{11})<3/5$$



$$\boxed{\frac{1}{3}+\gamma_2+\gamma_3+\gamma_4=1}$$

PROBLEM 26 MERGES. COMPANIES WITH VALUES $W_1, W_2, \dots W_m$ ARE MERGED AS FOLLOWS. THE TWO LEAST VALUABLE COMPANIES ARE MERGED. THE VALUE OF THE MERGE IS THE SUM OF THE VALUES OF THE TWO MERGED COMPANIES. THIS CONTINUES UNTILL ONE SUPERCOMPANY REMAINS. LET V EQUAL THE SUM OF VALUES OF THE MERGES. THUS V REPRESENTS THE TOTAL REQUIRED READ VOLUME OF THE MERGES. FOR EXAMPLE IF $W=(3322)$ THE MERGES YIELD: $(3322)\to(4,3,3)\to(6,4)\to 10$, AND $V=4+6+10=20$

(a) ARGUE THAT V IS MINIMUM VOLUME ACHIEVABLE BY SEQUENCES OF PAIR-WISE MERGES TERMINATING IN ONE SUPERCOMPANY.

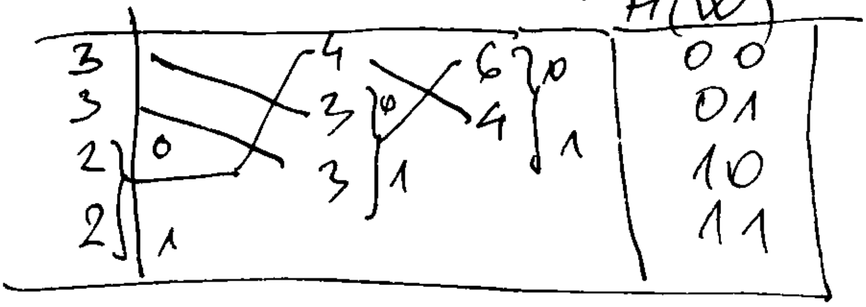
(b) LET $W=\sum W_i$, $\tilde{W}_i=W_i/W$ AND SHOW THAT MINIMUM MERGE VOLUME V. SATISFIES:

$$W\,H(\tilde{W})\leq V\leq W\,H(\tilde{W})+W$$

(c) $l_i=\left\lceil ld\frac{1}{\tilde{W}_i}\right\rceil \leq ld\frac{1}{\tilde{W}_i}+1$ [illegible]

$l_i \leqslant ld\frac{1}{\tilde{W}_i} + 1 \;/ \tilde{W}_i ; \sum_{\tilde{W}_i}$

$\sum_{\tilde{W}_i} \tilde{W}_i l_i \leqslant \underbrace{\sum_{\tilde{W}_i} \tilde{W}_i \, ld\frac{1}{\tilde{W}_i}}_{H(\tilde{W})} + \underbrace{\sum_{\tilde{W}_i} \tilde{W}_i}_{1}$?



MINIMUM WEIGHT SUM:

$E[l(x)] = 2\cdot 2 + 2\cdot 2 + 3\cdot 2 + 3\cdot 2 = 2(10) = 20$

$V = E[l(x)]$ $\quad \boxed{V = \sum l(x)\cdot W_i = \sum_i l_i W_i}$

$l_i = \left\lceil ld\frac{1}{\tilde{W}_i} \right\rceil$ $\quad l_i \leqslant ld\frac{1}{\tilde{W}_i} + 1 \;/ W_i$

$W_i l_i \leqslant W_i \, ld\frac{1}{\tilde{W}_i} + W_i \;/ \sum_i$

$\sum_i W_i l_i \leqslant \frac{W}{W}\sum W_i \, ld\frac{1}{\tilde{W}_i} + \underbrace{\sum W_i}_{W}$

$V \leqslant W \sum \frac{W_i}{W} ld\frac{1}{\tilde{W}_i} + W = \underline{W H(\tilde{W}) + W}$

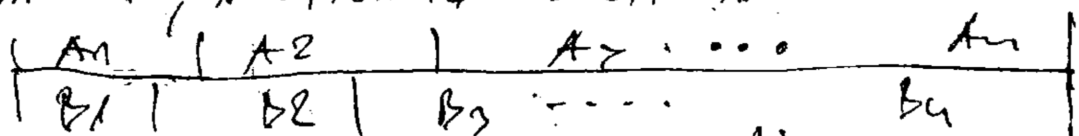
- FROM OTHER SIDE:

$l_i \geqslant ld\frac{1}{\tilde{W}_i} \;/ \cdot W_i ; \sum_i$ $\quad V \geqslant W\sum_i \frac{W_i}{W} ld\frac{1}{\tilde{W}_i}$

$\boxed{V \geqslant W \sum_i \tilde{W}_i \, ld\frac{1}{\tilde{W}_i}} \Rightarrow V \geqslant W\cdot H(\tilde{W})$

$\boxed{W H(\tilde{W}) \leqslant V \leqslant W H(\tilde{W}) + W}$

5.27 SARDINAS-PETTERSON TEST FOR UNIQUE DECODABILITY. A CODE IS NOT UNIQUELY DECODABLE IF AND ONLY IF THERE EXIST A FINITE SEQUENCE OF CODE SYMBOLS WHICH CAN BE RESOLVED IN TWO DIFFERENT WAYS INTO A SEQUENCES OF CODEWORDS. THAT IS, A SITUATION SUCH AS:



MUST OCCUR WHERE EACH A_i AND EACH B_i IS A CODEWORD. NOTE THAT B_1 MUST BE A PREFIX OF A1

A1 WITH SOME RESULTING „DANGLING SUFIX". EACH DANGLING SUFIX MUST IN TURN BE ~~EITHER~~ A PREFIX OF A CODEWORD OR HAVE ANOTHER CODEWORD AS ITS PREFIX, RESULTING IN ANOTHER DANGLING SUFFIX. FINALY THE LAST DANGLING SUFFIX IN THE SEQUENCE MUST ALSO BE A CODEWORD. THUS ONE CAN SET UP A TEST FOR UNIQUE DECODABILITY, (WHICH IS ESSENTIALY SARDINAS-PETERSON TEST) IN THE FOLLOWING WAY: CONSTRUCT SET OF ALL POSIBLE DANGLING SUFFIXES. THE CODE IS UNIQUELY DECODABLE IF AND ONLY IF S CONTAINS NO CODEWORDS.

(a) State the PRECISE RULES FOR BUILDING THE SET S.

(b) SUPPOSE THE CODEWORD LENGHTS ARE l_i, $i = 1, 2 \ldots m$. FIND A GOOD UPPER BOUND ON THE NUMBER OF ELEMENTS IN THE SET S

✓ (c) DETERMINE WHICH OF FOLLOWING CODES ARE UNIQUELY DECODABLE:

i. $\{0, 10, 11\}$
II. $\{0, 01, 11\}$
III. $\{0, 01, 10\}$
IV. $\{0, 01\}$
V. $\{00, 01, 10, 11\}$
VI. $\{110, 11, 10\}$
VII. $\{110, 11, 100, 00, 10\}$

✓ (d) FOR EACH UNIQUELY DECODABLE CODE IN PART (c), CONSTRUCT, IF POSSIBLE AN INFINITE ENCODED SEQUENCE WITH A KNOWN STARTING POINT, SUCH THAT IT CAN BE RESOLVED INTO CODEWORDS IN TWO DIFFERENT WAYS. (UNIQUE DECODABILITY DOESN'T IMPLY FINITE DECODABILITY).

(c) $\boxed{I}$ S_0: 0, 10, 11 } UNIQUELY DECODABLE (PREFIX CODE)
S_1: ×

$\boxed{II}$ S_0: 0, 01, 11 } UNIQUELY DECODABLE (SUFIX CODE)
S_1: 1
S_2: $1 = S_3 = S_4 = S_5 \ldots$

(III) S_0: 0, 01, 10
S_1: 1 NOT
S_2 0

STRIG 010 HAS TWO VALID PARSINGS

(IV) S_0: $\{0, 01\}$
S_1: 1 YES (SUFIX CODE)

(V) S_0: 00, 01, 10, 11 } PREFIX CODE
S_1: ×

(VI) S_0: 110, 11, 10 } $\boxed{\text{YES}}$
S_1: 0
S_2: ∅

$\boxed{VII.}$ S_0: 110, 11, 100, 00, 10 } YES
S_1: 0, 0 $S_1 = \{0\}$
S_2: 0, 0 $S_2 = \{0\} = S_1$

(d) $\boxed{VII}$ $\underline{1101100}\ \underline{11111}00000\ldots 0$

ALTERNATIVI $10000\ldots 00$
$10,00,\ldots 00$
$100,00,00,\ldots 00$

$11,11,110,00,00\ldots 00,\ldots$
$11,11,11,00,00,00,\ldots 00,\ldots$ } DVA NAČINI NA DECOD.

$\boxed{II}$ $0011111\ldots 1\ldots$

$0,0,11,11,11,\ldots 11,\ldots$
$0,01,11,\ldots$ } 2 NAČINI NA DECODIRANJE

$\boxed{I}$ $0111111,\ldots 1,1,1$
$0,11,11,11,\ldots 11,11,\ldots$ } DVA E ZA II

ZA PREFIX CODE NE MOŽE SEKVENCATA DA ZA DECODIRA NA DVA RAZLIČNI NAČINI!!!

- SAMO ZA ②, VII MOŽE DA SE FORMIRA BESKONEČNA SEKVENCA KOJA MOŽE DA SE DECODIRA NA DVA RAZLIČNI NAČINI.

(b) UPPER BOUND OF ELEMENTS IN „S"?

$l_i \quad i=1,2,\ldots n$

$L=\max\{l_i\}$ e.g. $i=n \Rightarrow \boxed{l_n=L}$

- PRETPOSTAVKA E DEKA SITE ~~PRE~~PREFIX-I NA l_n MOŽE DA ~~SE~~ ZNAT KAKO l_i KADE $i\neq n$

$x_n = \underbrace{011011\ldots 10}_{l_n-1}$

$S = 1+2+\cdots 2^n$
$2S = 2+\cdots 2^{n+1}$

MAKSIMALNIOT BROJ NA PREFIXI ĆE BIDE:

$$\sum_{i=1}^{l_n-1} 2^i = 2+2^2+2^3+\cdots 2^{l_n-1} = 2^{l_n}\left(\frac{1}{2^{l_n-1}}+\frac{1}{2^{l_n-2}}+\cdots\frac{1}{2}\right)$$

$$= 2^{l_n}\sum_{i=1}^{l_n-1}\frac{1}{2^i} = 2^{l_n-1}\sum_{i=0}^{l_n-2}\frac{1}{2^i} = 2^{l_n-1}\frac{1-\left(\frac{1}{2}\right)^{l_n-1}}{1-\frac{1}{2}} =$$

$$= 2^{l_n}\left(1-\frac{1}{2^{l_n-1}}\right) = 2^{l_n}-2 = \underline{2\left(2^{l_n-1}-1\right)}$$

$U_B\{S\} = 2^{l_n}-2$ KADE $l_n \to$ NAJDOLGIOT KODEN ZBOR.

E.G. $l_n=3$ GORNATA GRANICA E:

$U_B\{S\} = 2^3-2 = 8-2=6$

EDITION 2 SOLUTIONS

$$x_1x_2x_3x_4x_5x_6x_7x_8 = x_1x_2,x_3x_4x_5,x_6x_7x_8 = \underline{x_1x_2}x_3x_4,x_5x_6x_7x_8$$

$\downarrow$ NOT UNIQUE DECODABLE.

S_2: x_3x_4 $S_4 = \underbrace{x_6x_7x_8}_{\text{CODE WORD}}$
S_3: x_5

PROBLEM 28 SHANNON CODE. CONSIDER FOLLOWING METHOD FOR GENERATING CODE FOR RANDOM VARIABLE X WHICH TAKES ON m VALUES $\{1, 2, \dots m\}$ WITH PROBABILITIES $p_1 p_2, \dots p_m$. ASUME THAT PROBABILITIES ARE ORDERED: $p_1 > p_2 > \dots p_m$.

DEFINE: $F_i = \sum_{k=1}^{i-1} p_k$ THE SUM OF PROBABILITIES OF ALL SYMBOLS LESS THEN i. THEN THE CODEWORD FOR i = IS THE NUMBER $F_i \in [0,1]$ ROUNDED OF TO l_i BITS, WHERE $l_i = \lceil ld \frac{1}{p_i} \rceil$.

(a) SHOW THAT THE CODE CONSTRUCTED BY THIS PROCESS IS PREFIX FREE AND THE AVERAGE LENGTH SATISFIES: $H(X) \leq L \leq H(X)+1$

(b) CONSTRUCT THE CODE FOR THE PROBABILITY DISTRIBUTION $(0.5, 0.25, 0.125, 0.125)$

(a)

$$E[l(x)] = \sum_{i=1}^{4} l_i p_i = \sum_{i=1}^{4} \left\lceil ld \frac{1}{p_i} \right\rceil p_i \leq$$

$$\leq \sum_{i=1}^{4} \left(ld \frac{1}{p_i} + 1 \right) p_i = H(X) + 1$$

$$E[l(x)] = \sum_{i=1}^{4} \left\lceil ld \frac{1}{p_i} \right\rceil p_i \geq \sum_{i=1}^{4} ld \frac{1}{p_i} p_i = H(X)$$

(b)

i	$p(x)$	$F_i(x)$	$C(x)$	$ld \frac{1}{p_i}$	$bin(F_i(x))$
1	0.5	0	0	1	0
2	0.25	0.5	10	2	0.100
3	0.125	0.75	110	3	0.110
4	0.125	0.875	111	3	0.111

PREFIX CODE!!! °°

$\boxed{E[l(x)] = 1.75}$

$\boxed{H(X) = 1.75}$

$$0.\overset{1}{1}\overset{2}{1}\overset{3}{1}\overset{4}{1} \qquad 2^{-4} = \frac{1}{16}$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{8+4+2+1}{16} = \frac{15}{16} + \frac{1}{16} = \frac{16}{16} = 1$$

EDITION 2 Solution:

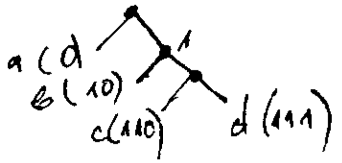
(a.) $ld \frac{1}{p_i} \leq \left\lceil ld \frac{1}{p_i} \right\rceil = l_i \leq ld \frac{1}{p_i} + 1 \quad / \cdot p(i) / \sum_i$

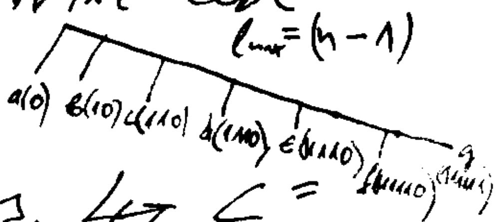
$$\sum_i p_i \, ld\left(\frac{1}{p_i}\right) \leq \sum p_i l_i \leq \sum_i p_i \, ld \frac{1}{p_i} + 1$$

$$H(X) \leq E[l(X)] \leq H(X) + 1$$

PROBLEM 5.22 AVERAGE LENGTH OF AN OPTIMAL CODE

A.1 $\boxed{L(C) = \sum_{\lambda=1}^{4} \gamma_i l_i}$





PROBLEM 5.23 UNUSED CODE SEQUENCES. LET C = BE A VARIABLE-LEGTH CODE THAT SATISFIES THE KRAFT NEQUALITY WITH EQUALITY BUT DOES NOT SATISFY THE PREFIX CONDITION.

(a) PROVE THAT SOME FINITE SEQUENCE OF CODE ALPHABET SYMBOLS IS NOT PREFIX OF ANY SEQUENCE OF CODEWORDS.

(b) (Optional) PROVE OR DISAPROVE: C HAS INFINITE DECODING DELAY.

$$x \in [x_1^1, x_2^2, x_3^3]$$
$$C(x) \in [0, 01, 10]$$
$$\gamma(x) \in \left[\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right]$$
$$l(x) = [1, 2, 2]$$

$$\sum_{i=1}^{3} D^{-l(x)} = \sum_{x} 2^{-l(x)} =$$
$$= 2^{-1} + 2^{-2} + 2^{-2} = \frac{1}{2} + 2\cdot\frac{1}{\cancel{4}_2} = 1$$

1,1,1	0|0|0
1 1 2	0|0|01
1 1 3	0|0|10
1 2 1	0,01,0
1 2 2	0,01,01
1 2 3	0,01,10
1 3 1	0,10,0
1 3 2	0,10,01
1 3 3	0,10,10
2 1 1	01,0,0
2 1 2	01,0,01
2 1 3	01,0,10
2 2 1	01,01,0
2 2 2	01,01,01
2 2 3	01,01,10
2 3 1	01,10,0
2 3 2	01,10,01
2 3 3	01,10,10
3 1 1	10,0,0
3 1 2	10,0,01
3 1 3	10,0,10
3 2 1	10,01,0
3 2 2	10,01,01
3 2 3	10,01,10
3 3 1	10,10,0
3 3 2	10,10,01
3 3 3	10,10,10

- VO BILO KOLKAVA SEKVENCA OD OVIE KODNI ZBOROVI NE MOŽE DA SE JAVI SEKVENCA KOJA ČE SODRŽI POVEČE OD DVE POSLEDOVATELNI EDINICI (E.G 111,1111...)

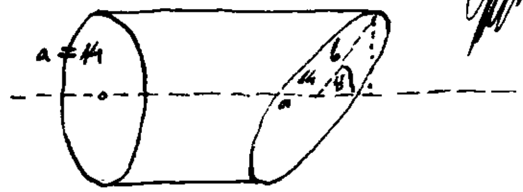
078 262121

078282848

77500
46500
31000

[redacted]

46500
31000
77500



$\sin\alpha = \frac{b}{a}$

$\frac{a}{b} = \sin 45 = \frac{\sqrt{2}}{2}$

129
12
117

$b = \frac{2}{\sqrt{2}} \cdot a = \sqrt{2} \cdot 14 = \underline{\underline{14\sqrt{2}}}$

$= 19.8$

$2b = 39{,}6$ mm

За:
. $a = 14.5$ $b = 20.51$ $2b = 41.01$

3069444

071227493

15505

My Pictures 6.82 GB (7.332 687.769)
My Pict (Отвлот) 6924 items
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