(64) $t = [0, 10, 15, 20, 32, 59, 62, 125]$; $v(t) = [0, 185, 319, 447, 742, 1325, 1445, 4151]$

(a) $v(t) = 0.00146t^3 - 0.11553t^2 + 24.98169t - 21.26872$

(b) $h = \int_0^{125} v(t)\,dt = 2.064 \cdot 10^5$ [feets]

5.5 SUBSTITUTION RULE

$$\int 2x\sqrt{1+x^2}\,dx = \left| \begin{matrix} u = 1+x^2 \\ du = 2x\,dx \end{matrix} \right| = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} = \frac{2}{3}\sqrt{(1+x^2)^3} + C$$

$$\frac{2}{3}\frac{d}{dx}\left((1+x^2)^{3/2}\right) = \frac{2}{3}\cdot\frac{3}{2}(1+x^2)^{\frac{3}{2}-1}\cdot 2x = 2x\sqrt{1+x^2}$$

$$\int f(g(x))\,g'(x)\,dx$$

Use this aproach whenever you meet such form of integral

If $F' = f$ then:

$$\int F'(g(x))\,g'(x)\,dx = \int F'(g(x))\,d(g(x)) = F(g(x)) + C$$

$$\frac{d}{dx}F(g(x)) = F'(g(x))\,g'(x) \quad ; \quad u = g(x)$$

$$\int F'(g(x))\,g'(x)\,dx = F(g(x)) + C = \int F'(u)\,du = F(u) + C$$

$F' = f$; $\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$

• Substitution rule: If $u = g(x)$ differentiable function in interval I and f is continuous on I then:

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

EXAMPLE 1:

$$I = \int x^3\cos(x^4+2)\,dx = \left| \begin{matrix} x^4+2 = u \\ 4x^3dx = du \end{matrix} \right| = \int \cos(u)\,\frac{du}{4} = \frac{1}{4}\sin(u) + C$$

$$I = \frac{1}{4}\sin(x^4+2) + C$$

EXAMPLE 2:

$$I = \int \sqrt{2x+1}\,dx = \left| \begin{matrix} u = 2x+1 \\ du = 2\,dx \end{matrix} \right| = \int \sqrt{u}\,\frac{du}{2} =$$

$$= \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} = \frac{1}{3}(2x+1)^{3/2} + C$$

$$\frac{d}{dx}(I) = \frac{1}{3}\cdot\frac{3}{2}(2x+1)^{1/2}\cdot 2 = \sqrt{2x+1}$$

• SOLUTION 2: $u = \sqrt{2x+1}$ $\quad u^2 = 2x+1$ $\quad 2u\,du = 2\,dx$

$dx = u\,du$

$$I = \int u \cdot u\,du = \int u^2\,du = \frac{1}{3}u^3 = \frac{1}{3}\sqrt{(2x+1)^3} = \frac{1}{3}(2x+1)^{3/2}$$

EXAMPLE 3 $I=\int \frac{x}{\sqrt{1-4x^2}}\,dx = \left| \begin{array}{l} u=1-4x^2 \\ du=-8x\,dx \\ x\,dx=-\frac{du}{8} \end{array} \right| = \int -\frac{du}{8\sqrt{u}} =$

$= -\frac{1}{8}\,\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = -\frac{1}{8}\,\frac{\sqrt{u}}{\frac{1}{2}} = -\frac{1}{4}\sqrt{u}+C = -\frac{1}{4}\sqrt{1-4x^2}+C$

$f(x)=\frac{x}{\sqrt{1-4x^2}}$; $g(x)' = -\frac{1}{4}\sqrt{1-4x^2} \ldots ; \; g'(x)=f(x)$

ex4 $\int e^{5x}dx = \left| \begin{array}{l} u=5x \\ du=5dx \end{array} \right| = \int e^{u}\frac{du}{5} = \frac{1}{5}e^{u} = \frac{1}{5}e^{5x}+C$

ex 5 $I=\int \sqrt{1+x^2}\,x^5\,dx = \left| \int u\,dv = u\cdot v - \int v\,du \right|$

$I = \int \underbrace{\sqrt{1+x^2}}_{u}\, \underbrace{d\left(\frac{x^6}{6}\right)}_{v} = \frac{x^6}{6}\sqrt{1+x^2} - \underbrace{\int \frac{x^6}{6}\, d\left(\sqrt{1+x^2}\right)}_{I_1}$

$\frac{d}{dx}\left(\sqrt{1+x^2}\right) = \frac{1}{2}(1+x^2)^{\frac{1}{2}-1}\cdot 2x = \frac{x}{\sqrt{1+x^2}}$

$I_1 = \int \frac{x^6\cdot x}{\sqrt{1+x^2}}\,dx = \int \frac{x^7}{\sqrt{1+x^2}}\,dx$

$\left(\frac{1}{\sqrt{1+x^2}}\right)' = -\frac{1}{2}(1+x^2)^{-\frac{1}{2}+1}\cdot 2x = x\sqrt{1+x^2}$

$\left(\frac{x}{\sqrt{1+x^2}}\right)' = \frac{x'\cdot\sqrt{1+x^2} - \frac{1}{2}(1+x^2)^{\frac{1}{2}-1}\cdot 2x\cdot x}{1+x^2} = \frac{\sqrt{1+x^2} - \frac{x^2}{\sqrt{1+x^2}}}{1+x^2}$

$= \frac{1+x^2-x^2}{\sqrt{1+x^2}\,(1+x^2)} = \frac{1}{\sqrt{(1+x^2)^3}}$

$u = \frac{1}{\sqrt{1+x^2}}$; $du = x\sqrt{1+x^2}\,dx$

$u^2 = \frac{1}{1+x^2}$; $u^2+x^2u^2 = 1$

$x^2u^2 = 1-u^2$ $\quad x^2 = \frac{1-u^2}{u^2}$ $\quad x^4 = \frac{(1-u^2)^2}{u^4}$

$x^4 = \frac{1}{u^4} - \frac{2}{u^2} + 1 = u^{-4}-2u^{-2}+1$

$I = \int \sqrt{1+x^2}\,x^5\,dx = \int x^4\left(x\sqrt{1+x^2}\,dx\right) = \int (u^{-4}-2u^{-2}+1)\,du =$

$= \frac{u^{-3}}{3} - 2\frac{u^{-1}}{1} + u = \frac{1}{3}\left(\frac{1}{\sqrt{1+x^2}}\right)^{-3} - \frac{2}{1}\left(\frac{1}{\sqrt{1+x^2}}\right)^{-1} + \frac{1}{\sqrt{1+x^2}} =$

$= \frac{1}{3}\sqrt{(1+x^2)^3} - \frac{2}{1}\sqrt{1+x^2} + \frac{1}{\sqrt{1+x^2}} = \frac{3\sqrt{(1+x^2)^4} - 10(1+x^2)+15}{15\sqrt{1+x^2}}$

$= \frac{3(1+x^2)^2 - 10(1+x^2)+15}{15\sqrt{1+x^2}} = \frac{3+6x^2+3x^4-10-10x^2+15}{15\sqrt{1+x^2}}$

$$I = \frac{8 - 4x^2 + 3x^4}{15\sqrt{1+x^2}}$$

$$(\tan x)' = \left(\frac{\sin(x)}{\cos(x)}\right)' = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$$

$$\sec(x) = \frac{1}{\cos(x)} \qquad \sinh(z) = \frac{e^z + e^{-z}}{2}$$

$$(\tan^5(x))' = 5\tan^4(x) \cdot \sec^2(x)$$

$$y = \operatorname{arctg} x \; ; \quad x = \operatorname{tg} y \; ; \quad dx = \frac{dy}{\cos^2(y)} = (1 + \operatorname{tg}^2(y))\, dy$$

$$dx = (1 + x^2)\, dy \qquad dy = \frac{dx}{1+x^2}$$

$$I = \int x^5 \sqrt{1+x^2}\, dx \; ; \qquad \int \frac{dx}{1+x^2} = \operatorname{arctg}(x) + c$$

$$u = 1 + x^2 \; ; \quad du = 2x\, dx \; ; \quad x^2 = u - 1 \; ;$$

$$I = \int x^4 \sqrt{1+x^2} \cdot x\, dx = \int (u-1)^2 \sqrt{u} \cdot \frac{du}{2} =$$

$$= \frac{1}{2} \int (u^2 - 2u + 1)\sqrt{u}\, du = \int \left(u^{5/2} - 2u^{3/2} + \sqrt{u}\right) du$$

$$= \frac{1}{2}\left(\frac{2}{7}(1+x^2)^{\frac{5}{2}+1} - \frac{4}{5}(1+x^2)^{5/2} + \frac{2}{3}(1+x^2)^{3/2}\right) =$$

$$= \frac{1}{7}(1+x^2)^{7/2} - \frac{2}{5}(1+x^2)^{5/2} + \frac{1}{3}(1+x^2)^{3/2}$$

EXAMPLE 6 $I = \int \tan x\, dx \qquad (\sec(x))' = \left(\frac{1}{\cos(x)}\right)' = (\cos^{-1}(x))' =$

$$= -1 \frac{1}{\cos^2(x)} \cdot -\sin(x) = \frac{\sin(x)}{\cos^2(x)} = \operatorname{tg}(x) \cdot \sec(x)$$

$$u = \sec(x) \qquad du = \operatorname{tg}(x) \sec(x)\, dx$$

$$I = \left| \begin{matrix} u = \sec(x) \\ dx = \frac{du}{\operatorname{tg}(x)\sec(x)} \end{matrix} \right| = \int \operatorname{tg}(x) \cdot \frac{du}{\operatorname{tg}(x)\sec(x)} = \int \frac{du}{u} = \ln(u)$$

$$\boxed{I = \ln(\sec(x)) + C} \qquad \boxed{I = -\ln(\cos(x)) + C}$$

OPTIONAL: $I = \int \frac{\sin(x)}{\cos(x)} dx = \left| \begin{matrix} u = \cos x \\ du = -\sin x\, dx \end{matrix} \right| = \int -\frac{du}{u} = -\ln(u)$

$$I = -\ln|\cos(x)| = \ln|\sec(x)| + C$$

SUBSTITUTION RULE FOR DEFINITE INTEGRALS

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

F - ANTIDERIVATE f ; $F(g(x))$ is antiderivate of $f(g(x))\,g'(x)$

$$\int_a^b f(g(x))\,g'(x)\,dx = F(g(x))\Big|_a^b = F(g(b)) - F(g(a))$$

$$\int_{g(a)}^{g(b)} f(g(x))\,dg(x) = \left| \begin{matrix} u = g(x);\ du = g'(x)dx \\ x=a \quad u=g(a) \\ x=b \quad u=g(b) \end{matrix} \right| = \int_{g(a)}^{g(b)} f(u)\,du = F(g(b)) - F(g(a))$$

EXAM 7
$$\int_0^4 \sqrt{2x+1}\,dx = \left| \begin{matrix} u = 2x+1; \\ du = 2\,dx \\ x=0;\ u=1 \\ x=4;\ u=9 \end{matrix} \right| = \int_1^9 \sqrt{u}\,\frac{du}{2} = \frac{1}{2}\,\frac{u^{3/2}}{3/2}\Big|_1^9$$

$$= \frac{1}{3}\left(\sqrt[2]{9^3} - \sqrt[2]{1^3}\right) = \frac{1}{3}\left(3^3 - 1\right) = \frac{1}{3}(27-1) = \frac{26}{3}$$

EXAMP 8
$$I = \int_1^2 \frac{dx}{(3-5x)^2} = \left| \begin{matrix} u = 3-5x \\ du = -5dx \\ x=1 \quad u=-2 \\ x=2 \quad u=-7 \end{matrix} \right| = \int_{-2}^{-7} -\frac{1}{u^2}\,\frac{du}{5} = \int_{-7}^{-2} \frac{du}{5u^2}$$

$$I = \frac{1}{5}\,\frac{u^{-2+1}}{-2+1}\Big|_{-7}^{-2} = -\frac{1}{5}\,\frac{1}{u}\Big|_{-7}^{-2} = \frac{1}{5}\left(\frac{1}{-7} - \frac{1}{-2}\right) = \frac{1}{5}\left(\frac{1}{2} - \frac{1}{7}\right)$$

$$I = \frac{1}{5}\,\frac{7-2}{14} = \frac{1}{5}\,\frac{5}{14} = \frac{1}{14}$$

EXAMP 9
$$\int_1^e \frac{\ln x}{x}\,dx = \left| \begin{matrix} u = \ln x \\ du = \frac{1}{x}\,dx \\ x=1 \quad u = \ln 1 = 0 \\ x=e \quad u = \ln e = 1 \end{matrix} \right| = \int_0^1 u\,du = \frac{u^2}{2}\Big|_0^1 = \frac{1}{2}$$

SYMETRY

$f(-x) = f(x)$ even $\Rightarrow \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

$f(-x) = -f(x)$ odd $\quad \int_{-a}^{a} f(x)\,dx = 0$

$$I = \int_{-a}^{a} f(x)\,dx = \underbrace{\int_{-a}^{0} f(x)\,dx}_{I_1} + \int_0^a f(x)\,dx$$

$$I_1 = \left| \begin{matrix} x = -u \\ dx = -du \\ x=-a \quad u=a \\ x=0 \quad u=0 \end{matrix} \right| = -\int_a^0 f(-u)\,du = \int_0^a f(-u)\,du$$

$$I = \int_0^a f(-u)\,du + \int_0^a f(x)\,dx;$$ even $f(-u) = f(u)$ $\quad I = 2\int_0^a f(x)\,dx$

odd: $f(-u) = -f(u) \Rightarrow I = 0$

EXAMPLE 10 $\quad f(x) = x^6 + 1 \qquad f(-x) = (-x)^6 + 1 = x^6 + 1 = f(x)$

$$\int_{-2}^{2} (x^6+1)\,dx = 2\int_0^2 (x^6+1)\,dx = 2\left(\frac{x^7}{7} + x\right)\Big|_0^2 = 2\left(\frac{128}{7} + \frac{14}{7}\right) = \frac{142}{7}\cdot 2 = \frac{284}{7}$$

EXAMPLE 11 $f(x) = \frac{tg(x)}{1+x^2+x^4}$ odd $\int_{-\pi/2}^{\pi/2} f(x)\,dx = 0$

EXERCISES

① $\int \cos 3x\,dx = \left| \begin{matrix} u = 3x \\ du = 3dx \end{matrix} \right| = \int \cos u \frac{du}{3} = \frac{1}{3}\sin u = \frac{1}{3}\sin 3x + C$

② $\int x(4+x^2)^{10}dx = \left| \begin{matrix} u = 4+x^2 \\ du = 2x\,dx \end{matrix} \right| = \frac{1}{2}\int u^{10}du = \frac{1}{2}\frac{u^{11}}{11} = \frac{1}{22}(4+x^2)^{11} + C$

④ $\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx = \left| \begin{matrix} u = \sqrt{x} \\ du = \frac{1}{2}x^{-\frac{1}{2}}dx \\ du = \frac{dx}{2\sqrt{x}} \end{matrix} \right| = 2\int \sin u\,du = -2\cos\sqrt{x} + C$

⑥ $\int e^{\sin\theta}\cos\theta\,d\theta = \left| \begin{matrix} u = \sin\theta \\ du = \cos\theta\,d\theta \end{matrix} \right| = \int e^u du = e^u = e^{\sin\theta} + C$

⑦ $\int 2x(x^2+3)^4dx = \left| \begin{matrix} u = x^2+3 \\ du = 2x\,dx \end{matrix} \right| = \int u^4 du = \frac{1}{5}u^5 = \frac{1}{5}(x^2+3)^5 + C$

⑪ $\int \frac{1+4x}{\sqrt{1+x+2x^2}}dx = \left| \begin{matrix} u = 1+x+2x^2 \\ du = (1+4x)dx \end{matrix} \right| = \int \frac{du}{\sqrt{u}} = \frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} =$

$= 2\sqrt{1+x+2x^2}$

⑱ $\int y^3\sqrt{2y^4-1}\,du = \left| \begin{matrix} 2y^4-1 = u \\ 8y^3dy = du \end{matrix} \right| = \int \sqrt{u}\frac{du}{8} = \frac{1}{12}(2y^4-1)^{3/2} + C$

㉑ $\int \frac{\ln^2 x}{x}dx = \left| \begin{matrix} u = \ln x \\ du = \frac{dx}{x} \end{matrix} \right| = \int u^2 du = \frac{1}{3}u^3 = \frac{1}{3}\ln^3 x + C$

㉗ $\int e^x\sqrt{1+e^x}dx = \left| \begin{matrix} u = 1+e^x \\ du = e^x dx \end{matrix} \right| = \int \sqrt{u}\,du = \frac{2}{3}(1+e^x)^{3/2} + C$

㉚ $\int \frac{ax+b}{\sqrt{ax^2+2bx+c}}dx = \left| \begin{matrix} u = ax^2+2bx+c \\ du = 2ax+2b\,dx \\ \frac{du}{2} = (ax+b)dx \end{matrix} \right| = \frac{1}{2}\int \frac{du}{\sqrt{u}} =$

$= \frac{1}{2}\frac{(ax^2+2bx+c)^{1/2}}{\frac{1}{2}} = \sqrt{ax^2+2bx+c} + C$

㉝ $\int \sqrt{\cot x}\sec^2 x\,dx = \left| \begin{matrix} u = tg(x) \\ du = \frac{1}{\cos^2 x}dx \\ du = \sec^2 x\,dx \end{matrix} \right| = \int \frac{du}{\sqrt{u}} = \frac{u^{\frac{1}{2}}}{\frac{1}{2}}$

$= 2\sqrt{u} = 2\sqrt{tg(x)}$

$I = \int \sqrt{ctg x}\csc^2 x\,dx = \left| \begin{matrix} u = ctg x \\ du = \frac{-\sin x\cdot\sin x - \cos x\cdot\cos x}{\sin^2 x}dx \end{matrix} \right|$ $du = -\frac{1}{\sin^2 x}dx$, $du = -\csc^2 x\,dx$

$I = -\int \sqrt{u}\,du = -\frac{2}{3}u^{3/2} = -\frac{2}{3}\sqrt{ctg^3 x} + C$

㉞ $\int \frac{\cos(\pi/x)}{x^2}dx = \left| \begin{matrix} u = \frac{\pi}{x} \\ du = -\frac{\pi}{x^2}dx \end{matrix} \right| = -\int \frac{1}{\pi}\cos u\,du = \frac{-1}{\pi}\sin\left(\frac{\pi}{x}\right) + C$

(35) $I = \int \cot(x)\,dx$ $\qquad \left(\frac{1}{\sin x}\right)' = (-1)\frac{1}{\sin^2 x} \cdot \cos(x) = -\cot(x)\cdot\csc(x)$

$u = \frac{1}{\sin x}$; $du = -ctg(x)\cdot\frac{dx}{\sin x}$ $\quad du = -u\cdot\cot(x)\,dx$

$\cot(x)dx = -\frac{du}{u}$

$I = -\int\frac{du}{u} = -\ln(u) = -\ln\left|\frac{1}{\sin x}\right| = +\ln|\sin x| + C =$
$= -\ln|\csc(x)| + C$

(36) $\int\frac{\sin x}{1+\cos^2 x}dx = \left| \begin{array}{l} u = 1+\cos^2 x \\ du = -2\cos x\sin x\,dx \end{array} \right|$

$u = \cos x$; $du = -\sin x\,dx$

$-\int\frac{du}{1+u^2} = -\arctan(u) = -\arctan(\cos x) + C$

(37) $I = \int\sec^3 x\tan x\,dx$ $\qquad \left(\frac{1}{\cos x}\right)' = \frac{\sin x}{\cos^2 x} = tg x\sec(x)$

$u = \frac{1}{\cos x}$; $\quad du = tg x\sec(x)dx$

$I = \int u^2\,du = \frac{1}{3}u^3 = \frac{1}{3}\sec^3(x) + C$

(38) $I = \int(x^3+1)^{\frac{1}{3}}x^5\,dx = \int\sqrt[3]{x^3(x^3\pm1)}\;x^4\,dx = \int\sqrt[3]{x^6(x^3+1)}\;x^3\,dx$

$u = (x^3+1)^{\frac{1}{3}}$ $\quad du = \frac{1}{3}(x^3+1)^{-\frac{2}{3}}\cdot 2x^2\,dx$

$u = x^3+1$ $\quad du = 3x^2\,dx$

$I = \int\sqrt{x^9(x^3+1)}\;x^2\,dx$; $\qquad u = x^3$; $\quad du = 3x^2dx$
$\frac{du}{3} = x^2dx$

$I = \frac{1}{3}\int\sqrt[3]{u^3(u+1)}\,du = \frac{1}{3}\int u\sqrt[3]{(u+1)}\,du$

$u = x^3+1$ $\quad du = 3x^2\,dx$ $\qquad \boxed{x^3 = u-1}$

$I = \int\sqrt[3]{u}\;\underbrace{x^3}_{u-1}\;\underbrace{x^2dx}_{\frac{du}{3}} = \frac{1}{3}\int\sqrt[3]{u}\,(u-1)\,du =$

$= \frac{1}{3}\int u^{4/3}du - \frac{1}{3}\int u^{\frac{1}{3}}du = \frac{1}{3}\;\frac{u^{\frac{7}{3}}}{\frac{7}{3}} - \frac{1}{3}\;\frac{u^{\frac{4}{3}}}{\frac{4}{3}}$

$I = \frac{1}{7}(x^3+1)^{7/3} - \frac{1}{4}(x^3+1)^{4/3} + C$

(39) $\int x^a\sqrt{b+cx^{a+1}}\,dx = \left| \begin{array}{l} u = b+cx^{a+1} \\ du = c(a+1)x^a dx \\ x^a dx = \frac{du}{c(a+1)} \end{array} \right| =$

$= \frac{1}{c(a+1)}\int\sqrt{u}\,du = \frac{1}{c(a+1)}\,\frac{2}{3}\left(b+cx^{a+1}\right)^{3/2} + C$

(40) $\int \sin t \sec^2(\cos t)\,dt = \left| \begin{matrix} u=\cos t \\ du = -\sin t\,dt\end{matrix}\right| = -\int \sec^2(u)\,du$

$= -tg\,u = -tg(\cos(t))$

(41) $I=\int \frac{1+x}{1+x^2}\,dx = \int\frac{dx}{1+x^2} + \int \frac{x\,dx}{1+x^2} = I_1+I_2$

$y=arctg\,x \quad x = tg\,y \quad dx = (1+tg^2 y)\,dy \quad dy = \frac{dx}{1+x^2}$

$I_1 = \int dy = y + C = arctg\,x + C$

$I_2 = \left| \begin{matrix} u = x^2+1 \\ du = 2x\,dx \\ x\,dx = \frac{du}{2}\end{matrix}\right| = \frac12\int\frac{du}{u} = \frac12 \ln u = \frac12\ln(x^2+1)$

$I = I_1+I_2 = arctg(x) + \frac12 \ln(x^2+1)+C$

(42) $\int \frac{x}{1+x^4}\,dx = I \qquad 1+x^4=u^4 \quad 4x^3dx = 4u^3du \quad x^3dx = u^3du$

$x=\sqrt[4]{1+u^4}$

$= \left|x^2=u \quad 2x\,dx = du\right| = \frac12\int\frac{du}{1+u^2} = \frac12\, arctg(u) = \frac12\, arctg(x^2)+C$

(43) $I=\int\frac{x}{\sqrt[4]{x+2}}\,dx = \left| \begin{matrix} u=x+2 \\ du=dx \\ x=u-2\end{matrix}\right| = \int\frac{u-2}{\sqrt[4]{u}}\,du = \int u^{1-\frac14}du - 2\int u^{-\frac14}du$

$I = \int u^{3/4}du - 2\int u^{-1/4}du = \frac{u^{\frac34+1}}{\frac34+1} - 2\,\frac{u^{-\frac14+1}}{-\frac14+1} = \frac47 u^{7/4} - 2\,\frac43 u^{3/4}$

$I = \frac47(x+2)^{7/4} - \frac83(x+2)^{3/4}$

(44) $I = \int\frac{x^2}{\sqrt{1-x}}\,dx = \left| \begin{matrix} u=1-x \\ du=-dx \\ x=1-u\end{matrix}\right| = \int\frac{(1-u)^2\,du}{\sqrt u} = \int\frac{du}{\sqrt u} - \int\frac{2u\,du}{\sqrt u} + \int\frac{u^2\,du}{\sqrt u}$

$I = \frac{u^{-\frac12+1}}{-\frac12+1} - 2\int\sqrt u\,du + \int u^{2-\frac12}du = 2\sqrt u - 2\,\frac{u^{3/2}}{3/2} + \frac{u^{\frac32+1}}{\frac32+1}$

$I = 2\sqrt u - \frac43 u^{3/2} + \frac25 u^{5/2} = 2\sqrt{1-x} - \frac43(1-x)^{3/2} + \frac25(1-x)^{5/2}+C$

(45) $I = \int\frac{3x-1}{(3x^2-2x+1)^4}\,dx = \left| \begin{matrix} u=3x^2-2x+1 \\ du=(6x-2)dx \\ du/2 = (3x-1)dx\end{matrix}\right| = \frac12\int\frac{du}{u^4} = \frac12\,\frac{u^{-4+1}}{-4+1}$

$I = -\frac16 u^{-3} = -\frac16\,\frac{1}{(3x^2-2x+1)^3}$

(46) $I=\int\frac{x}{\sqrt{x^2+1}}\,dx = \left| \begin{matrix} u=x^2+1 \\ du = 2x\,dx\end{matrix}\right| = \frac12\int\frac{du}{\sqrt u} = \frac12\,\frac{u^{-\frac12+1}}{\frac12} = \sqrt u$

$I = \sqrt{x^2+1}+C \qquad G(x)=\sqrt{x^2+1} \qquad (\sqrt{x^2+1})' = \frac12(x^2+1)^{\frac12-1}\,2x$

$g(x) = \frac{x}{\sqrt{x^2+1}} \qquad = \frac{x}{\sqrt{x^2+1}}$

(47) $I = \int \sin^3 x \cos x \, dx = \left| \begin{array}{l} \cancel{u = \cos x} \\ \cancel{du = -\sin x \, dx} \end{array} \quad \begin{array}{l} u = \sin x \\ du = \cos x \, dx \end{array} \right|$

$I = \int u^3 \, du = \frac{1}{4} u^4 = \frac{1}{4} \sin^4(x) + C$

(48) $I = \int tg^2\theta \sec^2\theta \, d\theta = \left| \begin{array}{l} u = tg\,\theta \\ du = \sec^2\theta \, d\theta \end{array} \right| = \int u^2 \, du = \frac{1}{3} u^3$

$I = \frac{1}{3} tg^3\theta + C$

(53) $I = \int_0^{\pi} \sec^2(t/4) \, dt = \left| \begin{array}{l} u = t/4 \\ 4du = dt \\ t=0 \quad u=0 \\ t=\pi \quad u=\pi/4 \end{array} \right| = 4\int_0^{\pi/4} \sec^2(u) \, du = 4 \, tg\, u \Big|_0^{\pi/4}$

$I = 4\left(tg \frac{\pi}{4} - tg\, 0\right) = 4(1-0) = 4$

(54) $I = \int_{1/6}^{1/2} \csc(\pi t) \cot(\pi t) \, dt = \int_{1/6}^{1/2} \frac{1}{\sin(\pi t)} \frac{\cos(\pi t)}{\sin(\pi t)} \, dt = \left| \begin{array}{l} u = \sin(\pi t) \\ du = \pi \cos(\pi t) \, dt \\ \frac{du}{\pi} = \cos(\pi t) \, dt \\ t = 1/6 \quad u = \sin(\pi/6) = \frac{1}{2} \\ t = 1/2 \quad u = \sin(\pi/2) = 1 \end{array} \right.$

$I = \frac{1}{\pi} \int_{0.5}^{1} \frac{du}{u^2} = \frac{1}{\pi} \frac{u^{-2+1}}{-2+1} \Big|_{0.5}^{1} = -\frac{1}{\pi} \frac{1}{u} \Big|_{0.5}^{1}$

$I = -\frac{1}{\pi}(1-2) = \frac{1}{\pi}$

(55) $I = \int_{-\pi/6}^{\pi/6} tg^3\theta \, d\theta = \int_{-\pi/6}^{\pi/6} \frac{\sin^3\theta}{\cos^3\theta} \, d\theta = \left| \begin{array}{l} u = \cos\theta \\ du = -\sin\theta \, d\theta \\ \theta = -\frac{\pi}{6} \; ; \; u = -\frac{\sqrt{3}}{2} \end{array} \quad \begin{array}{l} \theta = \frac{\pi}{6} \\ u = \frac{\sqrt{3}}{2} \end{array} \right| =$

$= -\int_{-\frac{\sqrt{3}}{2}}^{\sqrt{3}/2} \frac{\sin^2\theta}{u^3} \, du = -\int_{-\sqrt{3}/2}^{\sqrt{3}/2} \frac{1-u^2}{u^3} du = \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \left(\frac{1}{u} - \frac{1}{u^3}\right) du = \left(\ln(u) - \frac{u^{-3+1}}{-3+1}\right)\Big|_{-\sqrt{3}/2}^{\frac{\sqrt{3}}{2}}$

$= \left(\ln|u| + \frac{1}{2u^2}\right)\Big|_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} = \ln\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2 \cdot \frac{3}{4}} - \ln\left(\frac{\sqrt{3}}{2}\right) - \frac{1}{2 \cdot \frac{3}{4}} = 0$

$\frac{1}{2} tg^2(x) - \frac{1}{2} \ln\left(\frac{\cos^2 x + \sin^2 x}{\cos^2 x}\right) = \frac{1}{2} \frac{\sin^2 x}{\cos^2 x} - \frac{1}{2}\left(\ln 1 - 2\ln(\cos x)\right)$ ($\ln 1 \to 0$)

$= \frac{1}{2}\left(\frac{1-\cos^2 x}{\cos^2 x}\right) + \ln(\cos x) = \frac{1}{2} \frac{1}{\cos^2 x} - \frac{1}{2} + \ln(\cos x)$

$\ln\left(\frac{1}{\cos^2 x}\right) = -2\ln(\cos(x)) = \left| x = \frac{\pi}{6} \right| = -2 \ln \frac{\sqrt{3}}{2} =$

$= \ln\left(\frac{\sqrt{3}}{2}\right)^{-2} = \ln\left(\frac{2}{\sqrt{3}}\right)^2 = \ln \frac{4}{3}$

(57) $\int_1^2 \frac{e^{1/x}}{x^2} dx = \left| \begin{array}{l} u = \frac{1}{x} \\ du = (-1)x^{-1-1} dx \\ du = -\frac{1}{x^2} dx \end{array} \quad \begin{array}{l} x=1 \quad u=1 \\ x=2 \quad u=1/2 \end{array} \right| = -\int_1^{1/2} e^u \, du = \int_{1/2}^{1} e^u \, du$

$= e - \sqrt{e}$

(58) $\int_0^1 x e^{-x^2} dx = \left| \begin{matrix} u = -x^2 & x=0 \; u=0 \\ du = -2x\,dx & x=1 \; u=-1 \\ -\frac{du}{2} = x\,dx & \end{matrix} \right| = -\frac{1}{2}\int_0^{-1} e^{+u} du = \frac{1}{2}\int_{-1}^{0} e^u du = \frac{1}{2}\left(1-\frac{1}{e}\right)$

(60) $I = \int_{-\pi/2}^{\pi/2} \frac{x^2 \sin x}{1+x^6} dx$ $y = \operatorname{arctg} x \quad y' = \frac{1}{1+x^2}$

$y = \arcsin x \quad x = \sin y \quad dx = \cos y\, dy \;; \; dx = \sqrt{1-\sin^2 y}\, dy \quad dy = \frac{dx}{\sqrt{1-x^2}}$

$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin(x) \qquad x = \sin y^2 \;; \; dx = +\cos(y^2)\cdot 2y \cdot dy$

$x = \cos^2(y) \qquad dx = 2\cos y \cdot (-\sin y)\, dy = -2\sqrt{x}\,\sqrt{1-x}\; dy$

$\cos y = \sqrt{x}$ $\qquad dy = \frac{(-1)}{2}\frac{dx}{\sqrt{x-x^2}}$

$1-\sin^2 y = x \quad \sin^2 y = 1-x$

$\sin y = \sqrt{1-x}$

$\to y = \arccos(\sqrt{x})$

$x^2 = \cos u \qquad 2x\,dx = -\sin u\, du$

$x^3 = \cos u \qquad 3x^2 dx = -\sin u\, du \;; \quad x^2 = \cos u = \sqrt{1-\sin^2 u}$

$x^2 = \sqrt{1-\sin^2(u)} \qquad x^4 = 1-\sin^2(u) \qquad \sin^2 u = 1-x^4 \qquad \sin u = \sqrt{1-x^4}$

$y_1 = \frac{x^2}{1+x^6}$ → even function $\qquad y_2(x) = \sin(x)$ – odd function

$y = y_1(x)\cdot y_2(x)$ → odd function ; $\int_{-\pi/2}^{\pi/2} y(x)\,dx = 0$

(61) $\int_0^{11} \frac{dx}{\sqrt[3]{(1+2x)^2}} = \left| \begin{matrix} u = 1+2x \\ du = 2dx \\ x=0 \quad u=1 \\ x=11 \quad u=23 \end{matrix} \right| = \frac{1}{2}\int_1^{23} \frac{du}{u^{2/3}} = \frac{1}{2} \left.\frac{u^{-\frac{2}{3}+1}}{-\frac{2}{3}+1}\right|_1^{23} = \frac{3}{2}\sqrt[3]{u}\Big|_1^{23}$

$I = \frac{3}{2}\left(\sqrt[3]{23}-1\right) = 2.7658$

(63) $I = \int_1^2 x\sqrt{(x-1)}\,dx = \left| \begin{matrix} u = x-1 \\ du = dx \\ x = u+1 \\ x=1 \quad u=0 \\ x=2 \quad u=1 \end{matrix} \right| = \int_0^1 (u+1)\sqrt{u}\,du = \int_0^1 \left(u^{3/2}+u^{1/2}\right) du$

$I = \left(\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right)\Big|_0^1 = \frac{2}{5}+\frac{2}{3} = \frac{6+10}{15} = \frac{16}{15}$

(65) $I = \int_e^{e^4} \frac{dx}{x\sqrt{\ln x}} = \left| \begin{matrix} u = \ln x & x=e \;; \; u=1 \\ du = \frac{dx}{x} & x=e^4 ;\; u=4 \end{matrix} \right| = \int_1^4 \frac{du}{\sqrt{u}} = 2u^{1/2}\Big|_1^4$

$I = 2\left(\sqrt{4}-1\right) = 2\cdot 1 = 2$

(66) $I = \int_0^{1/2} \frac{\arcsin(x)}{\sqrt{1-x^2}}\, dx$; $u = \arcsin(x)$. $dx = +\cos u \, du$

$x = \sin u$. $dx = \sqrt{1-\sin^2 u}\, du$

$dx = \sqrt{1-x^2}\, du$ $\quad du = \frac{dx}{\sqrt{1-x^2}}$; $x=0$; $u=0$

$x=\frac{1}{2}$; $u=\frac{\pi}{6}$

$$I = \int_0^{\pi/6} u\, du = \frac{1}{2} u^2 \Big|_0^{\pi/6} = \frac{1}{2}\left(\frac{\pi^2}{36} - 0\right) = \frac{\pi^2}{72}$$

(68) $\int_0^a x\sqrt{a^2-x^2}\, dx = \left| \begin{array}{l} u = a^2-x^2 \\ du = -2x\,dx \\ x\,dx = -du/2 \end{array} \quad \begin{array}{ll} x=0 & u=a^2 \\ x=a & u=0 \end{array} \right| = \frac{1}{2}\int_0^{a^2} \sqrt{u}\, du$

$I = \frac{1}{2} \frac{u^{3/2}}{3/2} = \frac{1}{3} a^{2 \cdot \frac{3}{2}} = \frac{1}{3} a^3$

(71) $y = \sqrt{2x+1}$ $\quad 0 \le x \le 1$ $\quad P_1 \approx \frac{1\cdot(y(1)-1)}{2} = \frac{1}{2}\sqrt{2+1} = \frac{\sqrt{3}-1}{2}$

$P_2 = 1\cdot 1 = 1$ $\quad P = P_1 + P_2 = \frac{1}{2} + \frac{\sqrt{3}}{2} \doteq 1.4$

(72) $y = 2\cdot\sin(x) - \sin(2x)$; $\quad 0 \le x \le \pi$

$\int_0^\pi \sin(x)\, dx = 2$ $\quad P = 2\cdot 2 - 0 = 4$

(73) $\int_{-2}^{2} (x+3)\sqrt{4-x^2}\, dx = \int_{-2}^{2} x\sqrt{4-x^2}\, dx + \int_{-2}^{2} 3\sqrt{4-x^2}\, dx$

$\qquad\qquad I_1 \qquad\qquad I_2$

$I_1 = \left| \begin{array}{l} 4-x^2 = u \\ du = -2x\,dx \\ x=-2 \quad u=0 \\ x=2 \quad u=0 \end{array} \right| = -\frac{1}{2}\int \sqrt{u}\, du = -\frac{1}{2}\frac{u^{3/2}}{3/2} = -\frac{1}{3}u^{3/2}$

$I_1 = -\frac{1}{3}\sqrt{(4-x^2)^3}\Big|_{-2}^{2} = -\frac{1}{3}\left(\sqrt{(4-2^2)^3} - \sqrt{(4-2^2)^3}\right) = 0$

$I_2 = 3\int_{-2}^{2}\sqrt{4-x^2}\, dx = \left| \begin{array}{l} u = 4-x^2 \\ du = -2x\,dx \\ dx = \frac{du}{-2x} \end{array} \quad \begin{array}{l} dx = -\frac{du}{x} ; \\ x^2 = 4-u \\ x = \sqrt{4-u} \end{array} \right|$

$I_2 = -3\int_{0}^{0} \sqrt{u}\, \frac{du}{\sqrt{4-u}} = -3\int_0^0 \sqrt{\frac{u}{4-u}}\, du = 0$

$I_2 = 3\int_{-2}^{2}\sqrt{4-x^2}\, dx = 3\cdot 2\int_0^2 \sqrt{(2-x)(2+x)}\, dx = \left| \begin{array}{l} u = 2+x \\ du = dx \\ x = u-2 \\ x=0 \quad u=2 \\ x=2 \quad u=4 \end{array} \right| :$

$= 6\int_0^4 \sqrt{(2-u+2)u}\, du = \int_0^4 \sqrt{(4-u)u}\, du$

$$I_2 = \int \sqrt{4-x^2}\,dx = \left| \begin{matrix} x = 2\sin u ; \\ dx = 2\cos u\,du ; \end{matrix} \right| = \int \left(\sqrt{4-4\sin^2 u}\right) 2\cos u\,du$$

$$I_2 = \int \sqrt{4(1-\sin^2 u)}\; 2\cdot\cos u\,du = \int 2\cdot\cos u \cdot 2\cos u\,du =$$

$$= 4\int \cos^2 u\,du = \sout{2\int \sqrt{1-\sin^2 u}\,\cos u\,du}$$

$$I_2 = \left| \begin{matrix} x=-2 & -2 = 2\sin u ; & u = -\frac{\pi}{2} \\ x = 2 & 2 = 2\sin u ; & u = \frac{\pi}{2} \end{matrix} \right| = 4\int_{-\pi/2}^{\pi/2} \cos^2 u\,du = 8\int_0^{\pi/2} \cos^2 u\,du$$

$$I_2 = 8\int_0^{\pi/2} \cos u\; d(\sin u) = \left| \begin{matrix} v = \sin u \\ dv = +\cos u\,du \\ u = 0 \quad v = 0 \\ u = \pi/2 \quad v = 1 \end{matrix} \right| = 8\int_0^1 \sqrt{1-v^2}\,dv$$

$$4 - x^2 = 4\sin^2 u ; \qquad -2x\,dx = \sqrt{\cos u}\,du$$

$$x^2 = 4 - 4\sin^2 u \qquad dx = -\frac{4\sin u\cos u}{\sqrt{4-4\sin^2 u}}\,du$$

$$x = \sqrt{4-4\sin^2 u}$$

$$I_2 = -\int 2\sin u\,\frac{4\sin u\cos u}{2\sqrt{1-\sin^2 u}}\,du = -4\int \frac{\sin^2 u}{\sqrt{1-\sin^2 u}}\,d\sin u$$

$$I_2 = -4\int \frac{z^2}{\sqrt{1-z^2}}\,dz = \left| \begin{matrix} v = \arcsin z & dv = \frac{dz}{\sqrt{1-z^2}} \\ z = \sin v & \\ dz = \cos v\,dv & \end{matrix} \right|$$

$$= -4\int \sin^2 v\,dv$$

$$y^2 = 4 - x^2 \qquad x^2 + y^2 = 4 \qquad \boxed{R = 2}$$

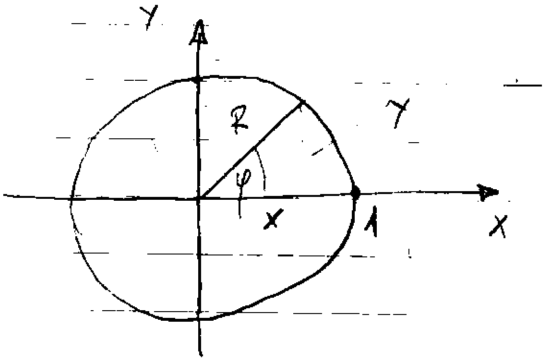
$$y = \sqrt{4-x^2}$$

$$I_2 = 3\int_{-2}^{2} \sqrt{4-x^2}\,dx = 3\cdot\left(\frac{R^2\pi}{2}\right) = 3\cdot\frac{4\cdot\pi}{2} = 6\pi$$

---

(74) $$\int_0^1 x\sqrt{1-x^4}\,dx = \left| \begin{matrix} u = x^2 \\ du = 2x\,dx \\ x = 0 \quad u = 0 \\ x = 1 \quad u = 1 \end{matrix} \right| = \frac{1}{2}\int_0^1 \sqrt{1-u^2}\,du$$

$$= \frac{1}{2}\left(\frac{R^2\pi}{4}\right) = \frac{\pi}{8}$$



$$x^2 + y^2 = 1$$
$$y = \sqrt{1 - x^2}$$
$$P = 2\int_{-1}^{1} \sqrt{1 - x^2}\, dx$$

$R = 1 \qquad \sin\varphi = y \;; \quad \cos\varphi = x \;; \qquad dx = -\sin\varphi\, d\varphi$

$$P = -2 \int_{-1}^{1} \sqrt{1 - \cos^2\varphi}\, \sin\varphi\, d\varphi$$

$P = R^2\pi = \pi \qquad \frac{y}{R} = \sin\varphi \qquad y = R\sin\varphi$

$$\frac{P}{2} = \int_0^{\pi} R\sin\varphi\, d\varphi = -R\cdot\cos\varphi \Big|_0^{\pi} = -R\left(\cos\pi - \cos 0\right) = 2R$$

$\sin\varphi = \frac{y}{R}, \qquad y = \sqrt{R^2 - x^2} \;; \quad \sin\varphi = \frac{1}{R}\sqrt{R^2 - x^2}$

$$I = \boxed{\int \sqrt{R^2 - x^2}\, dx} \qquad \cos\varphi\, d\varphi = \frac{1}{2}\frac{1}{R}\left(R^2 - x^2\right)^{\frac{1}{2} - 1}\cdot(-2x)\, dx$$

$\cos\varphi\, d\varphi = \frac{1}{R}\frac{x}{\sqrt{R^2 - x^2}}\, dx \;; \quad y = R\cdot\sin\varphi$

$\boxed{R\sin\varphi = \sqrt{R^2 - x^2}} \qquad R\cos\varphi\, d\varphi = \frac{x}{\sqrt{R^2 - x^2}}\, dx$

$R\cos\varphi\, d\varphi = \frac{x}{R\sin\varphi}\, dx \;; \quad x^2 = -R^2\sin^2\varphi + R^2 \Rightarrow$

$x = \sqrt{R^2(1 - \sin^2\varphi)}$

$R^2\cos\varphi\cdot\sin\varphi\, d\varphi = R\sqrt{1 - \sin^2\varphi}\, dx$

$$dx = \frac{R^2\cos\varphi\sin\varphi}{R\sqrt{1 - \sin^2\varphi}}\, d\varphi$$

$$I = \int R\sin\varphi \frac{R^2\cos\varphi\sin\varphi}{R\sqrt{1 - \sin^2\varphi}}\, d\varphi = R^2 \int \frac{\sin^2\varphi\cos\varphi}{\sqrt{1 - \sin^2\varphi}}\, d\varphi$$

$$I = \left| \begin{array}{l} u = \sin\varphi \\ du = \cos\varphi\, d\varphi \end{array} \right| = R^2 \int \frac{u^2}{\sqrt{1 - u^2}}\, du = \left| \begin{array}{l} v = \arcsin(u) \\ \sin v = u \\ dv = \frac{du}{\sqrt{1 - u^2}} \end{array} \right|$$

$$\boxed{I = R^2 \int \sin^2 v\, dv}$$

$\cos(v + v) = \cos v\cdot\cos v - \sin v\cdot\sin v$

$\cos(2v) = \cos^2 v - \sin^2 v =$

$= 1 - \sin^2 v - \sin^2 v = 1 - 2\sin^2 v$

$$\cos(2v) = 1 - 2\sin^2 v \qquad \sin^2 v = \frac{1}{2}(1 - \cos 2v).$$

$$I = R^2 \int \sin^2 v \, dv = R^2 \int \frac{1}{2}(1 - \cos 2v)\, dv = \frac{R^2}{2}\Big(v - \underbrace{\int \cos(2v)\, dv}_{I^*}\Big)$$

$$I^* = \int \cos(2v)\, dv = \left| \begin{matrix} w = 2v \\ dw = 2\,dv \end{matrix} \right| = \frac{1}{2}\int \cos(w)\, dw = \frac{1}{2}\cos w$$

$$I^* = \frac{1}{2}\cos(2v); \qquad I = \frac{R^2}{2}\left[v - \frac{1}{2}\cos(2v)\right]$$

$$\boxed{I = \int \sqrt{R^2 - x^2}\, dx}, \qquad x = R \cdot \sin(u); \quad \sin(u) = \frac{x}{R}$$

$$u = \arcsin\left(\frac{x}{R}\right); \qquad dx = R \cdot \cos u \, du;$$

$$I = \int \sqrt{R^2 - R^2 \cdot \sin^2 u} \cdot R \cdot \cos u \, du = R^2 \int \cos^2 u \, du$$

$$\cos(2u) = \cos(u+u) = \cos^2 u - \sin^2 u = 2\cos^2 u - 1$$

$$\cos^2 u = \frac{1}{2}(1 + \cos(2u))$$

$$I = R^2 \int \frac{1}{2}(1 + \cos(2u))\, du = \frac{R^2}{2}\left[u + \frac{1}{2}\sin(2u)\right]$$

$$\boxed{I = \frac{R^2}{2}\left[\arcsin\left(\frac{x}{R}\right) + \frac{1}{2}\underbrace{\sin\left(2 \cdot \arcsin\frac{x}{R}\right)}_{(*)}\right]}$$

~~ex: $\int \sqrt{4 - x^2}\, dx$~~

$$(*) = \sin\left(2 \cdot \arcsin\frac{x}{R}\right) = \sin(2u) =$$ ~~cos…~~

$$= \sin u \cdot \cos u + \sin u \cdot \cos u = 2\sin u \cos u = 2 \cdot \frac{x}{R} \cdot \sqrt{1 - \sin^2 u}$$

$$= 2 \cdot \frac{x}{R}\sqrt{1 - \frac{x^2}{R^2}} = \frac{2x}{R^2}\sqrt{R^2 - x^2}$$

$$I = \frac{R^2}{2}\left[\arcsin\left(\frac{x}{R}\right) + \frac{1}{2} \cdot \frac{2x}{R^2}\sqrt{R^2 - x^2}\right]$$

$$\boxed{I = \frac{x}{2}\sqrt{R^2 - x^2} + \frac{R^2}{2}\arcsin\left(\frac{x}{R}\right)}$$

ex: $$\int \sqrt{4 - x^2}\, dx = \left| R = 2 \right| = \frac{x}{2}\sqrt{4 - x^2} + 2\arcsin\left(\frac{x}{2}\right)$$

$$\int_{-2}^{2} \sqrt{4 - x^2}\, dx = \frac{2}{2}\underbrace{\sqrt{4-4}}_{0} + 2\arcsin(1) + \frac{2}{2}\underbrace{\sqrt{4-4}}_{0} - 2\arcsin(-1)$$

$$= 2 \cdot \frac{\pi}{2} - 2\left(-\frac{\pi}{2}\right) = \pi + \pi = 2\pi$$

(75) $y = e^{\sqrt{x}}$; $x = 0 \div 1$ (a) ; $y = 2x e^x$ $x = 0 \div 1$ (b) ; $y = e^{\sin x} \sin 2x$; $x = 0 \ldots \pi/2$ (c) ;

(a) $I_A = \int_0^1 e^{\sqrt{x}} dx = \left| \begin{matrix} u = \sqrt{x} & x = 0 \quad u = 0 \\ du = +\frac{1}{2} \frac{1}{\sqrt{x}} dx & x = 1 \quad u = 1 \\ dx = +2u\,du & \end{matrix} \right| = +2 \int_0^1 e^u \cdot u\,du$

$\int v\,dw = w \cdot v - \int w\,dv$

$\int e^u \cdot dw = \left| w = \frac{u^2}{2} \right| = \int e^u\, d\left(\frac{u^2}{2}\right) = \frac{u^2}{2} \cdot e^u - \int \frac{u^2}{2}\, d(e^u)$

$\int u\, d(e^u) = u \cdot e^u - \int e^u du = u \cdot e^u - e^u$

$I_A = +2\left(u e^u \Big|_0^1 - e^u \Big|_0^1\right) = +2(e - 0 - e + 1) = +2$

(b) $y = 2x e^x$ $x = 0 \ldots 1$

$I_B = \int_0^1 2x e^x\, dx = \left| \begin{matrix} \int v\,du = uv - \int u\,dv \\ u = e^x \end{matrix} \right| = \int_0^1 2x\, du =$

$= e^x \cdot 2x \Big|_0^1 - \int_0^1 e^x\, d(2x) = e^x 2x \Big|_0^1 - 2e^x \Big|_0^1 = e2 - 2e^1 + 2 = 2$

(c) $y = e^{\sin x} \sin(2x)$

$I_C = \int e^{\sin x} \sin(2x)\, dx$; $I_C = \int e^{\sin x}\, 2 \cdot \sin x \cos x\, dx$

$\sin(2x) = 2 \sin x \cos x$; $I_C = 2 \int e^{\sin x} \sin x\, d(\sin x) = 2 \int e^y y\, dy$

$I_C = 2 \int y e^y dy = 2 \int y\, d(e^y) = 2\left(y e^y - \int e^y dy\right) = 2y e^y - 2e^y$

$I_C = 2\left[\sin x\, e^{\sin x} \Big|_0^{\pi/2} - e^{\sin x} \Big|_0^{\pi/2}\right] = 2(1e - e + 1) = 2$ //

(76) $r(t) = 450.268\, e^{1.125677t}$ GROW RATE OF BACTERIA PER HOUR

$r(0) = 450.268$

$N_3 = 400 + \int_0^3 450.268\, e^{1.125677t} dt = 400 + 450.268\left(e^{1.12567 \cdot 3} - 1\right) \cdot \frac{1}{1.12567}$

$=$

(77) 0.5 L/S - MAXIMUM RATE OF AIR FLOW INTO THE LUNGS

$f(t) = \frac{1}{2}\sin(2\pi t/5)$ - RATE OF AIR FLOW

$$V(t) = \int_0^t \frac{1}{2}\sin(2\pi t/5)\,dt = \frac{1}{2}\int_0^t \sin(2\pi t/5)\,dt = \left| \begin{matrix} u = 2\pi t/5 \\ du = \frac{2\pi}{5}dt \end{matrix} \right| =$$

$$V(t) = \frac{1}{2}\frac{5}{2\pi}\int_0^t \sin(u)\,du = -\frac{1}{2}\frac{5}{2\pi}\cos 2\pi t/5 \Big|_0^t = -\frac{5}{4\pi}\left(\cos 2\pi t/5 - 1\right)$$

$$V(t) = \frac{5}{4\pi}\left(1 - \cos(2\pi t/5)\right) \text{[liters]}$$

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(78) • RATE OF PRODUCTION CALCULATORS

$$\frac{dx}{dt} = 5000\left(1 - \frac{100}{(t+10)^2}\right) \text{ calculators/week}$$

$$x = \int_3^4 5000\left(1 - \frac{100}{(t+10)^2}\right)dt = \int_3^4 5000\,dt - \int_3^4 5000\frac{100}{(t+10)^2}\,dt =$$

$$= 5000 - 5\cdot 10^5 \int_3^4 \frac{dt}{(t+10)^2} = \left| \begin{matrix} u = t+10 \\ du = dt \\ t=3 \quad u=13 \\ t=4 \quad u=14 \end{matrix} \right| = 5000 - 5\cdot 10^5 \int_{13}^{14} \frac{du}{u^2} =$$

$$= 5000 - 5\cdot 10^5 \frac{u^{-2+1}}{-2+1}\Big|_{13}^{14} = 5000 + 5\cdot 10^5\left(\frac{1}{14} - \frac{1}{13}\right) = 5000 + 5\cdot 10^5 \frac{13-14}{14\cdot 13}$$

$$= 5000 - \frac{5\cdot 10^5}{182} = \frac{205\,000}{91} = 2252.7473$$

(79) $f(x)$ is continuous $\int_0^4 f(x)\,dx = 10$ $\int_0^2 f(2x)\,dx = ?$

$$\int_0^2 f(x)\,dx + \int_2^4 f(x)\,dx = 10$$

$F(4) - F(0) = 10$ $F(0) = -10 + F(4)$

$F(2) - F(0) = ?$ $F(2) - (F(4) - 10) = 10 - F(4) + F(2)$

$$I = \int_0^2 f(2x)\,dx = \left| \begin{matrix} u = 2x \quad du = 2dx \\ x = u/2 \\ x=0 \quad u=0 \\ x=2 \quad u=4 \end{matrix} \right| = \int_0^4 f(u)\,d\left(\frac{u}{2}\right) = \frac{1}{2}\int_0^4 f(u)\,du$$

10

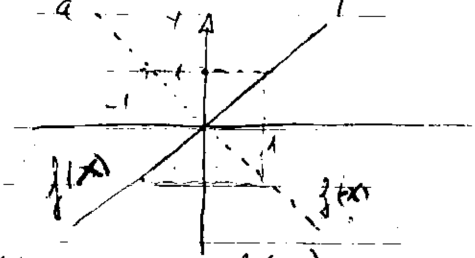
$I = 10/2 = 5$

(80) $\int_0^9 f(x)\,dx = 4 \quad I = \int_0^3 x f(x^2)\,dx = ?$

$u = x^2 \quad du = 2x\,dx \;;\quad x\,dx = \frac{du}{2} \;;\quad x = 0\; u = 0 \;;\; x = 3\; u = 9$

$I = \int_0^3 f(x^2)\,x\,dx = \frac{1}{2}\int_0^9 f(u)\,du = \frac{1}{2}\cdot 4 = 2$

(81) $\int_a^b f(-x)\,dx = \left| \begin{matrix} u = -x & x = a & u = -a \\ du = -dx & x = b & u = -b \end{matrix} \right| = -\int_{-a}^{-b} f(u)\,du = \int_{-b}^{-a} f(u)\,du$



$\int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$

$\int_{-1}^0 x\,dx = \frac{x^2}{2}\Big|_{-1}^0 = 0 - \frac{1}{2} = -\frac{1}{2}$

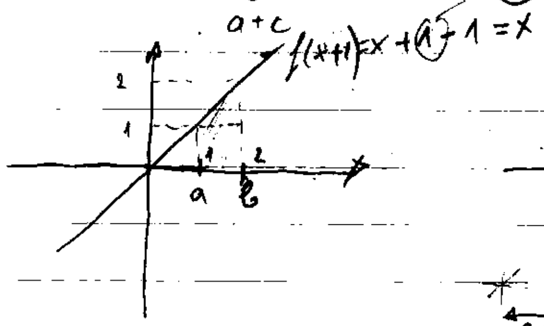
$\int_0^1 -x\,dx = -\frac{x^2}{2} = -\frac{1}{2}$

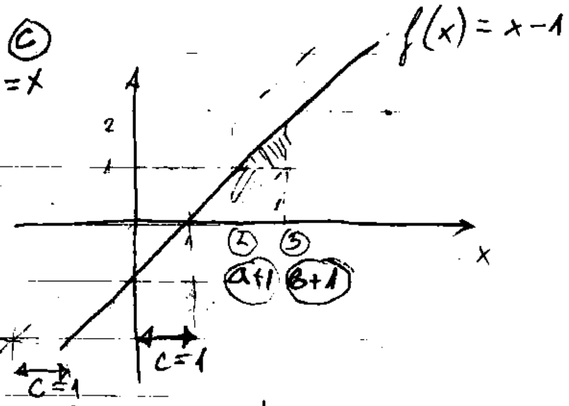
$f(x) = x \qquad f(-x) = -x$

$\int_{-1}^0 f(x)\,dx = \int_{-1}^0 x\,dx = \frac{x^2}{2}\Big|_{-1}^0 = 0 - \frac{1}{2} = -\frac{1}{2}$

(82) $\int_a^b f(x+c)\,dx = \int^{b+c} f(x)\,dx \qquad \begin{matrix} u = x + c & x = a & u = a + c \\ du = dx & x = b & u = b + c \end{matrix}$

$\int_a^b f(x+c)\,dx = \int_{a+c}^{b+c} f(u)\,du$





(83) $a, b \geq 0 \quad \int_0^1 x^a (1-x)^b\,dx = \int_0^1 x^b (1-x)^a\,dx$

$\int_0^1 x^a (1-x)^b\,dx = \left| \begin{matrix} u = 1 - x \quad du = -dx \\ x = (1 - u) \\ x = 0 \quad u = 1 \;;\; x = 1 \quad u = 0 \end{matrix} \right| = -\int_1^0 (1-u)^a \cdot u^b\,du = \int_0^1 (1-u)^a u^b\,du$

(84) $u = \pi - x \qquad \boxed{I = \int_0^\pi x f(\sin(x))\,dx = \frac{\pi}{2}\int_0^\pi f(\sin(x))\,dx}$

$x = \pi - u$

$du = -dx$

$x = 0 \;;\; u = \pi$

$x = \pi \;;\; u = 0$

$I = -\int_\pi^0 (\pi - u) f(\sin(\pi - u))\,du = \int_0^\pi (\pi - u) f(+\sin(u))\,du$

$I = \pi\int_0^\pi f(\sin(u))\,du - \underbrace{\int_0^\pi u f(\sin(u))\,du}_{I} \;;\quad 2I = \pi\int_0^\pi f(\sin(u))\,du$

$I = \frac{\pi}{2}\int_0^\pi f(\sin(x))\,dx$

(85) $u=\pi-x$ $\quad I=\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}\,dx=\left|\begin{matrix} x=\pi-u \\ x=0 \quad u=\pi \\ x=\pi \quad u=0 \\ du=-dx\end{matrix}\right|=-\int_{\pi}^{0}\frac{(\pi-u)\sin u}{1+\cos^2(\pi-u)}\,du$

$$I=\int_0^{\pi}\frac{\pi\sin u}{1+\cos^2 u}\,du-\underbrace{\int_0^{\pi}\frac{u\sin u}{1+\cos^2 u}\,du}_{I}$$

$$2I=\pi\int_0^{\pi}\frac{\sin u}{1+\cos^2 u}\,du \qquad I=\frac{\pi}{2}\int_0^{\pi}\frac{-d(\cos u)}{1+\cos^2 u}=-\frac{\pi}{2}\int\frac{dz}{1+z^2}$$

$z=\cos u \quad dz=-\sin u\,du$
$u=0 \quad z=\cos(0)=1$
$u=\pi \quad z=\cos(\pi)=-1$

$$I=-\frac{\pi}{2}\int_{+1}^{-1}\frac{dz}{1+z^2}=\frac{\pi}{2}\int_{-1}^{1}\frac{dz}{1+z^2}$$

$$I=\frac{\pi}{2}\operatorname{arctg} z\Big|_{-1}^{1}=\frac{\pi}{2}\left(\operatorname{arctg}1-\operatorname{arctg}(-1)\right)=\frac{\pi}{2}\left(\frac{\pi}{4}+\frac{\pi}{4}\right)=\frac{\pi}{2}\,\frac{\pi}{2}$$

$$\boxed{I=\frac{\pi^2}{4}}$$

MOŽE DA SE REŠI SO KORISTENJE NA (84)

$$\int_0^{\pi}x f(\sin(x))\,dx=\frac{\pi}{2}\int_0^{\pi}f(\sin(x))\,dx$$

$$I=\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}\,dx=\left|\ f(\sin(x))=\frac{\sin x}{1+1-\sin^2 x}=\frac{\sin x}{2-\sin^2 x}=\frac{t}{2-t^2}\ \right|=$$

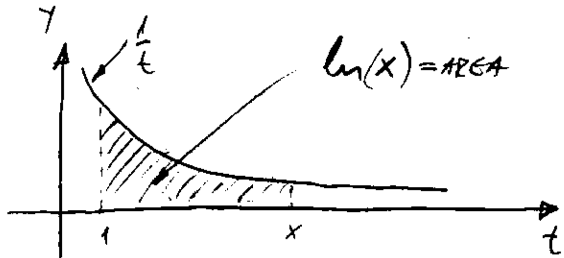
$$=\int_0^{\pi}x f(\sin x)\,dx=\frac{\pi}{2}\int_0^{\pi}\frac{\sin x\,dx}{1+\cos^2 x}=\left|\begin{matrix} u=\cos x \\ du=-\sin x\,dx \\ x=0 \quad u=1 \\ x=\pi \quad u=-1\end{matrix}\right|=$$

$$=-\frac{\pi}{2}\int_1^{-1}\frac{du}{1+u^2}=+\frac{\pi}{2}\int_{-1}^{1}\frac{du}{1+u^2}=\frac{\pi}{2}\left[\operatorname{arctg}(1)-\operatorname{arctg}(-1)\right]=\frac{\pi}{2}\,\frac{\pi}{2}=\frac{\pi^2}{4}$$

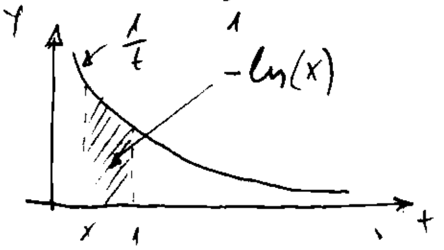
5.6 LOGARITHM DEFINED AS AN INTEGRAL

$y=a^x \qquad x=\log_x a$

- NATURAL LOGARITHM

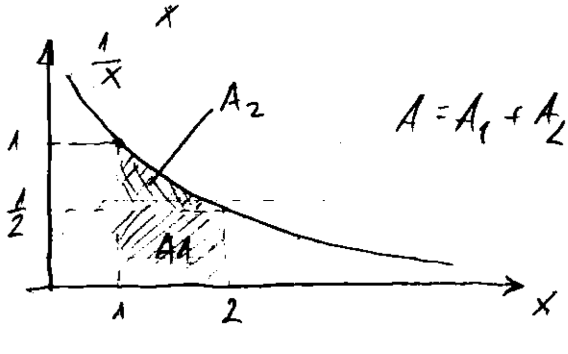


$$\ln(x)\doteq\int_1^x\frac{dt}{t} \qquad x>0$$



$$\ln(x)=\int_1^x\frac{1}{t}\,dt=-\int_x^1\frac{dt}{t}<0$$

EXAMPLE 1 $\qquad \frac{1}{2}<\ln 2<\frac{3}{4}$

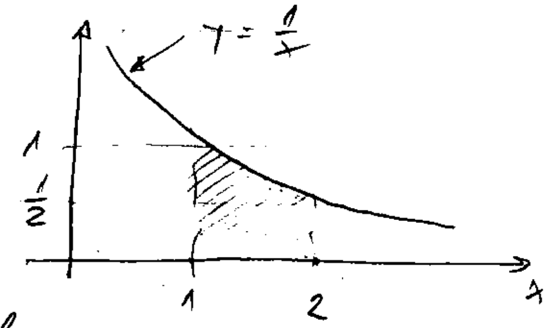


$A=A_1+A_2$

$A_1=1\cdot\frac{1}{2}=\frac{1}{2}$
$A_2<\left(1-\frac{1}{2}\right)\cdot\frac{1}{2}=\frac{1}{4}$
$A_1+A_2<\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$

$\ln 2 = \int_1^2 \frac{dt}{t} = A$; $\quad A = A_1 + A_2$

$\frac{1}{2} \le A \le A_1 + A_2 = \frac{3}{4}$ ($\frac{1}{2}$ — A_1)



• MIDPOINT RULE $\quad n = 10$

$$A = \sum_{i=1}^{n} f(x_i)\Delta x_i \qquad \Delta x_i = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n} = 0.1$$

$$x_i = 1 + \frac{\Delta x_i}{2}\cdot(2n-1) \qquad x_1 = 1 + \frac{0.1}{2}\cdot 1 = 1.05$$

$$x_2 = 1 + \frac{0.1}{2}\cdot 3 = 1 + \frac{0.3}{2} = 1.15$$

$$A = \sum_{i=1}^{10} f(x_i)\Delta x_i = \sum_{i=1}^{10} \frac{1}{x_i}\cdot\Delta x_i = 0.1\sum_{i=1}^{10}\frac{1}{x_i}$$

$$A = 0.1\left(\frac{1}{1.05} + \frac{1}{1.15} + \frac{1}{1.25} + \dots + \frac{1}{1.95}\right) \doteq 0.693$$

FTC1 $\quad g(x) = \int_a^x f(t)\,dt \quad a \le x \le b \quad$ $g'(x) = f(x)$

$$\frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x} \Rightarrow \frac{d}{dx}(\ln x) = \frac{1}{x}$$

③ LOWS OF LOGARITHM $\quad x, y > 0 \quad r$ - rational constant

1) $\ln(xy) = \ln x + \ln y$ $\quad$ 2) $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

3.) $\ln x^r = r \ln x$

$f = \ln xy \qquad e^f = x\cdot y$

Proof: $\quad f(x) = \ln(ax)$; $\quad f'(x) = \frac{1}{ax}\cdot a = \frac{1}{x}$

IZVODOT OD ln(x) i ln(ax) E IS ⇒ TIE SE RAZLIKUVAA ZA KONSTANTA!!

1.) $\ln(ax) = \ln x + C \qquad \ln(2x) = \ln 2 + \ln x = \ln x + 0.69$

$x = 1 \quad \ln(a) = \ln(1) + C = 0 + C \qquad C = \ln(a)$

$\ln(ax) = \ln(x) + \ln(a) \Rightarrow \ln(xy) = \ln(x) + \ln(y)$

2.) $x = \frac{1}{y} \Rightarrow \ln\frac{1}{y} + \ln y = \ln\left(\frac{1}{y}\cdot y\right) = \ln 1 = 0$

$\ln\left(\frac{1}{y}\right) = -\ln y \qquad \ln\left(x\cdot\frac{1}{y}\right) = \ln(x) + \ln\left(\frac{1}{y}\right) = \ln(x) - \ln y$

3) $\ln(x^r) = \ln(\underbrace{x\cdot x\cdot\ldots\cdot x}_{r}) = \underbrace{\ln(x) + \ln(x) + \dots \ln(x)}_{r} = r\cdot\ln(x)$

2.089
2.002

(a) $y = \ln(x)$ $\quad \lim_{x\to\infty} \ln(x) = \infty \quad \lim_{x\to 0^+} \ln(x) = -\infty$

$\ln x^r = r \cdot \ln x \qquad x = 2;\ r = n;\ \ln 2^n = n \cdot \ln 2$

$\ln 2 > 0 \Rightarrow n \to \infty \Rightarrow n \ln 2 \to \infty \quad \ln(x) \nearrow \ \ln'(x) = \frac{1}{x} > 0$

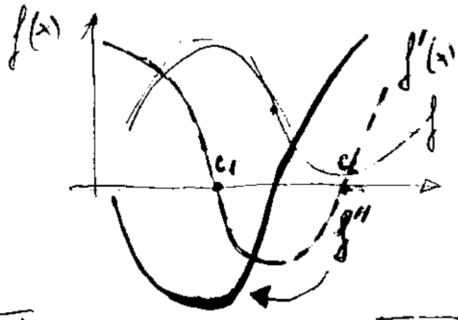
(b) $x = \frac{1}{t} \Rightarrow x \to 0 \quad t \to \infty$

$$\lim_{x\to 0} \ln(x) = \lim_{t\to\infty} \ln\left(\frac{1}{t}\right) = \lim_{t\to\infty}(-\ln(t)) = -\lim_{t\to\infty}(\ln(t)) = -\infty$$

- $y = \ln(x) \quad x > 0 \quad \frac{dy}{dx} = \frac{1}{x} > 0 \quad \frac{d^2y}{dx^2} = -\frac{1}{x^2} < 0$

Concavity Test

If $f''(x) > 0$ for all x in interval $I \Rightarrow f(x)$ is concave up
If $f''(x) < 0$ for all x in interval $I \Rightarrow f(x)$ is concave downward



$f''(c_1) < 0 \quad f(c_1) = 0 \Rightarrow f(x)$ MAXIMUM
$f''(c_2) > 0 \quad f'(c_2) = 0 \Rightarrow f(x)$ MINIMUM

ex6 $f(x) = x^4 - 4x^3$ ex7 $f(x) = x^{2/3}(6-x)^{1/3}$ ex8 $f(x) = e^{1/x}$

$y = e^x \qquad x = \ln(y) \qquad \frac{dx}{dx} = \frac{1}{y}\frac{dy}{dx} \qquad \frac{dy}{dx} = y$

$\boxed{\frac{dy}{dx} = e^x} \qquad \frac{d}{dx}(a^x) = \frac{d}{dx}\left(e^{x\ln a}\right) = e^{x\ln a} \ln a = a^x \ln a$

Inverse function exist if the function is one-to-one

$\boxed{y = f(x) \qquad f^{-1}(y) = x}$

- GENERAL LOGARITHMIC FUNCTIONS

$a > 0 \quad a \neq 1 \qquad f(x) = a^x$ - one-to-one function

$y = \log_a x \qquad \boxed{x = a^y} \qquad \frac{dx}{dx} = (\ln a) a^y \frac{dy}{dx}$

$\boxed{\frac{dy}{dx} = \frac{1}{\ln a (a^y)} = \frac{1}{x \ln a}} \quad x = e^{y \ln a} \quad \frac{dx}{dx} = \left(e^{y\ln a}\right) \ln a \frac{dy}{dx} = \ln a\, a^y \frac{dy}{dx}$

$\boxed{\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}}$ $\qquad y = a^x \quad \frac{dy}{dx} = \left(e^{x\ln a}\right)' = e^{x \ln a} \ln a$

$= a^x \cdot \ln a$

$\frac{d}{dx}(a^x) = a^x \ln a$!

$e = \lim_{x\to 0}(1+x)^{1/x}$

$f(x) = \ln(x)$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{x\to 0}\frac{f(1+x)-f(1)}{x} =$$

$$= \lim_{x\to 0}\frac{\ln(1+x)-\ln(1)}{x} = \lim_{x\to 0}\frac{1}{x}\ln(1+x) = \lim_{x\to 0}\ln(1+x)^{1/x}$$

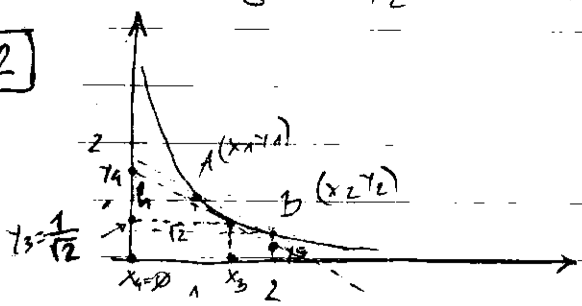
$$f'(x) = \frac{1}{x}\Big|_{x=1} = 1 = \lim_{x\to 0}\ln(1+x)^{1/x}$$

$$e^1 = e^{\lim_{x\to 0}\ln(1+x)^{1/x}} = \lim_{x\to 0} e^{\ln(1+x)^{1/x}} = \lim_{x\to 0}\ln(1+x)^{1/x}$$

ex1 $A_1 = \frac{1}{2}\cdot\frac{1}{\frac{3}{2}} = \frac{1}{2}\cdot\frac{2}{3}$ $\qquad A_2 = \frac{1}{2}\cdot\frac{1}{2}\cdot\left(1-\frac{2}{3}\right) = \frac{1}{12}$

$A = A_1 + A_2 = \frac{2}{6} + \frac{1}{12} = \frac{4+1}{12} = \frac{5}{12}$

Ex2



(a) $f(x) = \frac{1}{x}$

secant line AB:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$y - 1 = \frac{\frac{1}{2}-1}{2-1}(x-1)$; $\quad y = -\frac{1}{2}(x-1) + 1$

$f'(x) = -\frac{1}{x^2} = -\frac{1}{2} \qquad x^2 = 2 \qquad \boxed{x_3 = \sqrt2}$

$y_3 = \frac{1}{x_3} = \frac{1}{\sqrt2}$ $\quad y_4 = h + y_3 = h + \frac{1}{\sqrt2} = \frac{\sqrt2}{2} + \frac{1}{\sqrt2}$

$\frac{h}{\sqrt2} = \frac{1}{2}$ $\quad \boxed{h = \frac{\sqrt2}{2}}$ $\quad y_4 = \frac{\sqrt2}{2} + \frac{\sqrt2}{2} = 2\sqrt2/2 = \sqrt2$

tangent line:

$$y - y_3 = \frac{y_4 - y_3}{x_4 - x_3}(x - x_3); \quad y - \frac{1}{\sqrt2} = \frac{\sqrt2 - \frac{1}{\sqrt2}}{0 - \sqrt2}(x - \sqrt2)$$

$$y = \frac{\sqrt2}{2} + \frac{\frac{2-1}{\sqrt2}}{-\sqrt2}(x - \sqrt2) = \frac{\sqrt2}{2} - \frac{1}{2}(x - \sqrt2) = -\frac{1}{2}(x - \sqrt2) + \frac{\sqrt2}{2}$$

(b) $\ln(2) > 0.66$ $\qquad \ln(2) = \int_1^2 \frac{1}{x}dx$

$y = -\frac{1}{2}(x - \sqrt{2}) + \frac{\sqrt{2}}{2}$ $\quad x_5 = 2 \Rightarrow y_5 = -\frac{1}{2}(2-\sqrt{2}) + \frac{\sqrt{2}}{2} = -1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}$

$y_5 = \sqrt{2} - 1$

Area under secant in interval $1 \div 2$

$A = A_1 + A_2$; $\quad A_1 = 1 \cdot y_5 = 1 \cdot (\sqrt{2} - 1)$; ~~$A_2 = (1 - y_5) \cdot 1/2$~~

~~$A_2 = (1 - \sqrt{2} + 1) \cdot \frac{1}{2} = \frac{\sqrt{2}}{2}$~~

~~$A = A_1 + A_2 = (\sqrt{2} - 1 + \frac{\sqrt{2}}{2}) = \frac{2\sqrt{2} - 2 + \sqrt{2}}{2} = \frac{3\sqrt{2} - 2}{2} = 1.121$~~

$A_2 = (y_6 - y_5) \cdot \frac{1}{2}$ $\quad y_6 = ?$ $\quad y_6 = -\frac{1}{2}(1 - \sqrt{2}) + \frac{\sqrt{2}}{2} = -\frac{1}{2} + \sqrt{2}$

$A_2 = \left(-\frac{1}{2} + \sqrt{2} - \sqrt{2} + 1\right) \cdot \frac{1}{2} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$A = A_1 + A_2 = \sqrt{2} - 1 + \frac{1}{4} = \sqrt{2} - \frac{3}{4} = 0.66$

$\ln 2 > 0.66$

---

ex 1 (b) Midpoint rule $n = 10$ $[\ln 1.5] = ?$

$a = 1$, $b = 1.5$ $\quad \Delta x = \frac{1.5 - 1}{n} = \frac{0.5}{10} = 0.05$

$S = \sum_{i=1}^{n} f(x_i) \Delta x_i$ $\quad f(x_i) = \frac{1}{x_i}$ $\quad x_i = 1 + \frac{\Delta x}{2}(2i - 1)$

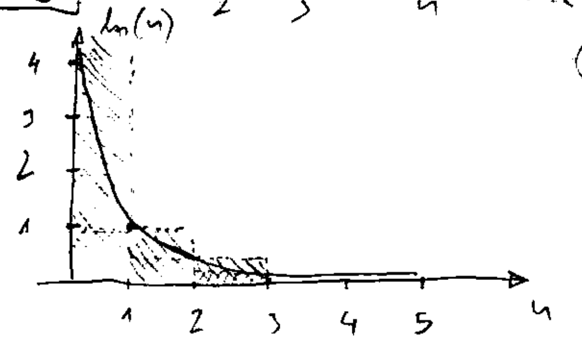
$x_1 = 1 + 0.025 = 1.025$

$x_2 = 1 + 3 \cdot 0.025 = 1.075$

$S = \sum_{i=1}^{10} \frac{0.05}{1 + 0.025 \cdot (2i - 1)} = 0.05 \left(\frac{1}{1.025} + \frac{1}{1.075} + \dots \frac{1}{1.475} \right) = 0.405\,41$

---

ex. 3. $\frac{1}{2} + \frac{1}{3} + \dots \frac{1}{n} < \ln(n) < 1 + \frac{1}{2} + \frac{1}{3} + \dots \frac{1}{n-1}$



(1) Rightpoint rule

$a = 0$, $b = n$

$\Delta x = \frac{b - a}{n} = 1$

$S = \sum_{i=1}^{n} f(x_i) \cdot \Delta x =$

$f(x_i) = \frac{1}{x_i}$; $\quad x_i = 1 + \Delta x (i - 1)$; $\quad S = \sum_{i=1}^{n} \frac{\Delta x}{1 + \Delta x (i - 1)}$

$S_R = \sum_{i=1}^{n} \frac{1}{1 + (i - 1)} = \sum_{i=1}^{n} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \dots \frac{1}{n}$

(2) Leftpoint rule ; $\quad \Delta x = \frac{b - a}{n} = 1$

$S_L = \sum_{i=1}^{n} \frac{1}{i - 1} \cdot \Delta x = \sum_{i=1}^{n} \frac{1}{i - 1} = \frac{1}{0} (\to \infty) + \frac{1}{1} + \dots + \frac{1}{n - 1}$

EXCLUDING FIRST TERM FROM S_R AND S_L SUM:

(1.) $S_{R_1} = \sum_{i=2}^{n} \frac{1}{i} = \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$

(2.) $S_{L_1} = \sum_{i=2}^{n} \frac{1}{i-1} = 1 + \frac{1}{2} + \dots + \frac{1}{n-1}$

RIGHT POINT: $a = 1 \quad b = n \qquad \Delta x = \frac{n-1}{n} = \left(1 - \frac{1}{n}\right) \approx 1$

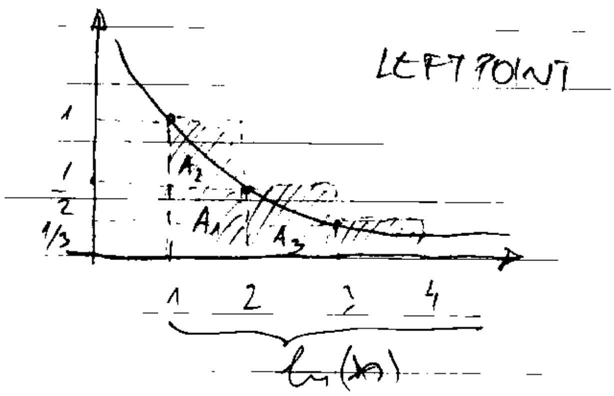
$S_R = \sum_{i=2}^{n} \frac{1}{i} \cdot \Delta x$ (with $\Delta x \to 1$) $\qquad x_i = 1 + i\Delta x = 1 + i$

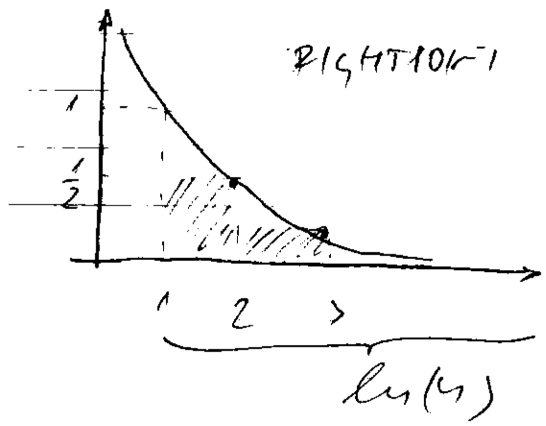
$S_R = \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$

LEFTPOINT:

$S_L = \sum_{i=2}^{n} \frac{1}{i-1} \cdot \Delta x$ (with $\Delta x \to 1$) $= 1 + \frac{1}{2} + \dots + \frac{1}{n-1}$

$S_R < \ln(n) < S_L$





ex 4 $\ln 2 = \int_1^2 \frac{dx}{x} = A_1 + A_2$

$A_1 = 1 \cdot \frac{1}{2} = \frac{1}{2} \qquad A_2 < \left(1 - \frac{1}{2}\right)\frac{1}{2} = \frac{1}{4} \qquad A_1 = A_1 + A_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} < 1$

$3 = A_1 + A_2 + A_3 \qquad A_3 = A_{31} + A_{32} \qquad A_{31} = \frac{1}{3} \cdot 1 = \frac{1}{3}, \quad A_{32} < \left(\frac{1}{2} - \frac{1}{3}\right) \cdot \frac{1}{2}$

$A_{32} = \frac{3-2}{6} \cdot \frac{1}{2} = \frac{1}{12} \qquad A_3 < \frac{1}{3} + \frac{1}{12} = \frac{4+1}{12} = \frac{5}{12}$

$3 = \frac{3}{4} + \frac{5}{12} = \frac{9+5}{12} = \frac{14}{12} > 1$

$\ln 2 \approx \frac{3}{4} \qquad \ln(3) \approx \frac{7}{6} \qquad \ln(e) = 1$

$\ln 2 < \ln(e) < \ln(3) \Rightarrow \boxed{2 < e < 3}$

ex 5: $a^{x-y} = e^{(x-y)\ln a} = e^{x\ln a} e^{-y\ln a} = a^x \cdot a^{-y} = a^x / a^y$

e^x/e^y : $\ln e^x/e^y = \ln e^x - \ln e^y = \ln e^{x-y} \Rightarrow e^x/e^y = e^{x-y}$

ex 7) $(a \cdot b)^x = e^{x \ln a \cdot b} = e^{x\ln a + x\ln b} = e^{x\ln a} \cdot e^{x \ln b} = a^x \cdot b^x$

ex10 $\log_a(xy) = z \quad a^z = xy \qquad a^z = e^{z\ln a}; \; e^{z\ln a} = xy$

$\ln(xy) = w \quad e^w = xy$

$\log_a(xy) = \log_a e^{z\ln a}; \quad \log_a xy = \log e$

$\log_a x = \frac{\ln x}{\ln 2}; \quad \log_a e^{x\ln a} \qquad a^x = e^{x\ln a}$

(1°) $a^{x+y} = a^x \cdot a^y$ (2°) $a^{x-y} = a^x / a^y$ (3°) $(a^x)^y = a^{xy}$ (4°) $(ab)^x = a^x b^x$

(a) $\log_a a^{x+y} = x+y \quad ; \quad \log_a xy$

$\log_a x = r \; ; \; \log_a y = s \; ; \; a^r = x \; ; \; a^s = y$

$x \cdot y = a^r \cdot a^s = a^{r+s} \; ; \; \log_a a^{r+s} = r+s = \log_a x + \log_a y$

(b) $\log_a(x/y) = \log a^r/a^s = \log_a a^{(r-s)} = r-s = \log_a x - \log_a y$

(c) $\log_a(x^y) = \log_a(a^{r \cdot y}) = r \cdot y = y \cdot r = \underline{y \cdot \log_a x}$

$\log_a x = r \; ; \; x = a^r = e^{r \cdot \ln a}$

$\log_a x = \frac{\ln x}{\ln a} \; ; \; \log_a x = \log_a e^{r \cdot \ln a} = r \cdot \ln a \cdot \log_a e$

$x = a^r / \ln \quad \ln(x) = r \cdot \ln a \quad \boxed{r = \frac{\ln(x)}{\ln a}}$

$\boxed{\log_a x = \frac{\ln(x)}{\ln(a)}}$

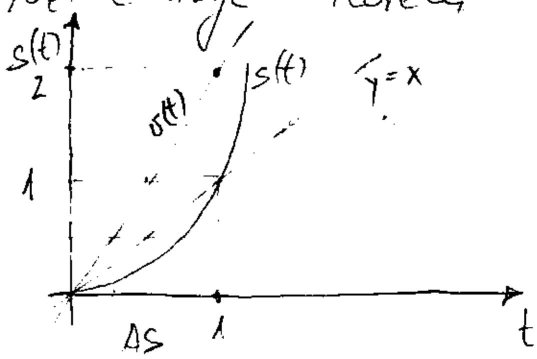
REVIEW

$S = \sum_{i=1}^{n} f(x_i)\Delta x \; ; \quad \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i$

FTC (1°) $g(x) = \int_a^x f(t)\,dt \qquad g'(x) = f(x) \qquad \boxed{a \le x \le b}$

(2°) $\int_a^b f(t)\,dt = F(b) - F(a) \qquad F' = f(x)$

Net Change Theorem $\qquad \int_a^b \frac{ds}{dt}\,dt = s(b) - s(a)$

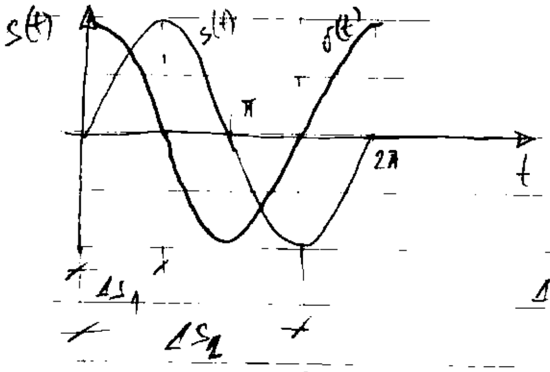


$s(t) = t^2 \quad t > 0 :$

$v(t) = \frac{ds(t)}{dt} = 2 \cdot t$

$\int_0^1 v(t)\,dt = \int_0^1 2t\,dt = 2 \cdot \frac{t^2}{2}\Big|_0^1 = 2 \cdot \frac{1}{2} = 1$

$\Delta s = s(1) - s(0) = 1^2 - 0 = 1$



$s(t) = \sin(t)$

$\Delta S_1 = s\left(\frac{\pi}{2}\right) - s(0) = 1 - 0 = 1$

$\Delta S_2 = s\left(\frac{3\pi}{2}\right) - s(0) = -1 - 0 = -1$

$\Delta S_2 = \int_0^{3\pi/2} v(t)\,dt = \int_0^{3\pi/2} \cos(t)\,dt = +\sin(t)\Big|_0^{3\pi/2}$

~~$\Delta S_2 = \cos(t)\Big|_0^{3\pi/2} = \cos(0) - \cos\frac{3\pi}{2} = 1 - 0 = 1$~~ $\quad \Delta S_2 = \sin\left(\frac{3\pi}{2}\right) - \sin(0) = -1$

$\int_0^{3\pi/2} |v(t)|\,dt = \int_0^{\pi/2} \cos(t)\,dt + \int_{\pi/2}^{3\pi/2} -\cos(t)\,dt = \sin\frac{\pi}{2} - \left(\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}\right) = 1 - (-2) = 3$

$$\boxed{\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(g(x))\,dg(x)}$$ substitution rule

$\int_1^2 \frac{x\,dx}{1+x^2} = \left| \begin{array}{l} u = 1+x^2 \\ du = 2x\,dx \\ x = 1 \quad u = 2 \\ x = 2 \quad u = 5 \end{array} \right| = \frac{1}{2}\int_{u(1)}^{u(2)} \frac{du}{u} = \frac{1}{2}\int_2^5 \frac{du}{u} = \frac{1}{2}\left[\ln(5) - \ln(2)\right]$

$f(x) = \frac{1/2}{1+x^2} \qquad f(g(x)) = \frac{1/2}{1+g(x)} \qquad 2x\,dx = g'(x)\,dx$

$\int_a^b \frac{x\,dx}{1+x^2} = \int_a^b \frac{1/2}{1+x^2}\, 2x\,dx = \int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(g(x))\,dg(x)$

$F' = f$

$\int_a^b f(g(x))\,g'(x)\,dx = \int_a^b F'(g(x))\,g'(x)\,dx = F(g(x))\Big|_a^b + c$

PROBLEMS

$\lim_{x\to 3}\left(\frac{x}{x-3}\int_3^x \frac{\sin t}{t}\,dt\right) = \lim_{x\to 3}\frac{x}{x-3}\left[\int_3^0 \frac{\sin t}{t}\,dt + \int_0^x \frac{\sin t}{t}\,dt\right]$

$= \lim_{x\to 3}\frac{x}{x-3}\left[\int_0^x \frac{\sin t}{t}\,dt - \int_0^3 \frac{\sin t}{t}\,dt\right] = \lim_{x\to 3}\frac{x\left[Si[x] - Si[3]\right]}{x-3} = \varnothing$

$u = \frac{1}{x-3}; \quad x \to 3; \quad u \to \infty; \quad x - 3 = \frac{1}{u} \quad x = \frac{1}{u} + 3$

$\lim_{u\to\infty} \frac{\frac{1}{u}+3}{\frac{1}{u}+3-3}\int_3^{1/u+3} \frac{\sin(t)}{t}\,dt = \lim_{u\to\infty} u\left(\frac{1}{u}+3\right)\int_3^{1/u+3} \frac{\sin(t)}{t}\,dt$

$= \lim_{u\to\infty}(1+3u)\int_3^{\frac{1}{u}+3} \frac{\sin(t)}{t}\,dt$

$$\lim_{x\to 3} \frac{x}{x-3}\int_3^x \frac{\sin(t)}{t}\,dt = \lim_{x\to 3} x \lim_{x\to 3}\frac{F(x)-F(3)}{x-3}$$

$$\left| F(x)=\int_0^x \frac{\sin(t)}{t}\,dt \right| = \lim_{x\to 3} x\; F'(x)\Big|_{x\to 3} = 3\cdot\frac{\sin x}{3}\Big|_{x\to 3} = \sin(3)$$

$$\lim_{x\to 1}\frac{\ln x}{x-1}$$

L'Hospitals Rule: $f(t)$ $g(t)$ ARE DIFFERENTIABLE $g'(t)\neq 0$ $x\neq a$

$\lim_{x\to a} f(t)=0$ $\lim_{x\to a} g(t)=0$ OR $\lim_{x\to a} f(t)=\pm\infty$ $\lim_{x\to a} g(t)=\pm\infty$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

ex.: 1° $\lim_{x\to 1}\frac{\ln x}{x-1} \overset{(0/0)}{=} \lim_{x\to 1}\frac{\frac{1}{x}}{1} = \lim_{x\to 1}\frac{1}{x} = 1$

2° $\lim_{x\to\infty}\frac{\ln x}{x-1} \overset{(\infty/\infty)}{=} \lim_{x\to\infty}\frac{1}{x} = 0$

3° $L=\lim_{x\to 3}\frac{x}{x-3}\int_3^x \frac{\sin(t)}{t}\,dt = \lim_{x\to 3}(x)\cdot\lim_{x\to 3}\frac{f(x)}{g(x)} \overset{(0/0)}{=} \lim_{x\to 3}\frac{f'(x)}{g'(x)}\cdot\lim_{x\to 3}(x)$

$f'(x) = \frac{d}{dx}\left[\int_3^x \frac{\sin(t)}{t}\,dt\right] = \frac{\sin(x)}{x}$ $\qquad g'(x)=1$

$L = 3\cdot\lim_{x\to 3}\frac{f'(x)}{g'(x)} = 3\lim_{x\to 3}\frac{\frac{\sin(x)}{x}}{1} = 3\cdot\lim_{x\to 3}\frac{\sin(x)}{x} = 3\cdot\frac{\sin(3)}{3}$

$L=\sin(3)$

PROOF OF L'HOSPITAL'S RULE

If $f(a), g(a)=0$ $\quad g'(a)\neq 0$ THEN

$$\lim_{x\to a}\frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} = \frac{\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}{\lim_{x\to a}\frac{g(x)-g(a)}{x-a}} = \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x\to a}\frac{f(x)}{g(x)}$$

ex.2 $\lim_{x\to\infty}\frac{e^x}{x^2} = \left| \begin{matrix} x\to\infty & e^x\to\infty \\ x\to\infty & x^2\to\infty \\ \infty/\infty & \end{matrix}\right| = \lim_{x\to\infty}\frac{e^x}{2x} = \lim_{x\to\infty}\frac{e^x}{2} = \infty$

ex.3 $\lim_{x\to\infty}\frac{\ln x}{\sqrt[3]{x}} = \lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{3}x^{\frac{1}{3}-1}} = \lim_{x\to\infty}\frac{3}{x\cdot x^{-\frac{2}{3}}} = \lim_{x\to\infty}\frac{3x^{2/3}}{x} =$

$= \lim_{x\to\infty} 3\cdot\frac{2}{3}\frac{x^{\frac{2}{3}-1}}{1} = \lim_{x\to\infty} 2\cdot\frac{1}{x^{1/3}} = \lim_{x\to\infty}\frac{2}{\sqrt[3]{x}} = 0$

PROBLEMS ① $x\sin(\pi x) = \int_0^{x^2} f(t)\,dt$ $\quad f(t)$ continuous

$f(4)=?$

$$x\sin(\pi x) = \int_0^{x^2} f(t)\,dt \quad \Big/ \frac{d}{dx} \qquad \sin(\pi x) + x\cdot\cos(\pi x)\cdot\pi =$$

$$= f(x^2)\cdot 2x$$

$f(4) = ?$ $\qquad 2\cdot x\, f(x^2) = \sin(\pi x) + \pi x\cos(\pi x) \Big|_{x=2}$ $\qquad 2\cdot 2\, f(4) = \sin(2\pi) + 2\pi\cdot\cos(2\pi)$ ($\sin(2\pi)\to 0$, $\cos(2\pi)\to 1$)

$$4\cdot f(4) = 2\pi\cdot 1 \qquad \boxed{f(4) = \frac{\pi}{2}}$$

② $y = f(x)$ PASSES THROUGH $(1,1)$ $\qquad \int_0^1 f'(x)\,dx = ?$

$$\int_0^1 f'(x)\,dx = f(1) - f(0) = 1 - f(0) = 1 \quad (f(0)\to 0)$$

③ $$\int \frac{dx}{1+x^4} = \left| \begin{array}{l} u = x^2 \\ du = 2x\,dx \\ x = \sqrt{u} \end{array} \right| = \int \frac{1}{2x}\,\frac{2x\cdot dx}{1+(x^2)^2} = \frac{1}{2}\underbrace{\int \frac{1}{\sqrt{u}}\,\frac{du}{1+u^2}}_{I_1}$$

$$I = \int \frac{dx}{1+x^2} = \left| \begin{array}{l} x = \operatorname{arctg} u \\ u = tg\, x \end{array} \quad \frac{du}{dx} = (1+tg^2(x)) = 1+u^2 \right|$$

$u = \operatorname{arctg}(x)$; $\quad x = tg(u)$ $\qquad \frac{dx}{dx} = (1+tg^2(u))\frac{du}{dx}$

$dx = (1+x^2)\,du$ $\qquad \frac{dx}{1+x^2} = du$

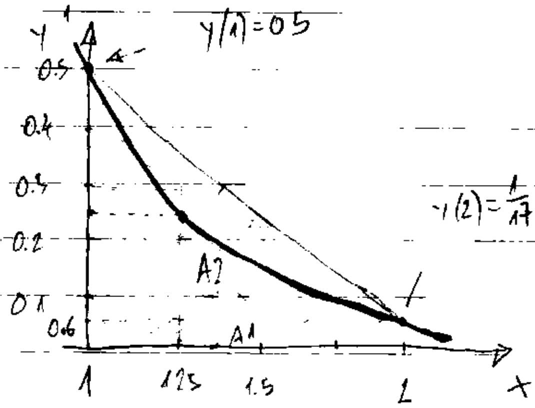
$$I = \int du = u = \operatorname{arctg}(x)$$

$$I_1 = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,\frac{du}{1+u^2} = \left| \int v\,du = u\cdot v - \int u\,dv \right|$$

$$I_1 = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,d[\operatorname{arctg}(u)] = \operatorname{arctg}(u)\,\frac{1}{\sqrt{u}} - \underbrace{\int \operatorname{arctg}(u)\,d\left(\frac{1}{\sqrt{u}}\right)}_{I_2}$$

$$I_2 = \int \operatorname{arctg}(u)\left(-\frac{1}{2}\right)u^{-\frac{1}{2}-1}\,du = \frac{1}{2}\int \frac{\operatorname{arctg}(u)}{\sqrt[3]{u^2}}\,du$$

$$\int_1^2 \frac{dx}{1+x^4} = ?$$



$A = A_1 + A_2$; $\quad A_1 = \frac{1}{17}\cdot 1 = \frac{1}{17}$

$$A_2 = \left(\frac{1}{2} - \frac{1}{17}\right)\cdot\frac{1}{2} = \frac{17-2}{34}\cdot\frac{1}{2} = \frac{15}{68}$$

$$A_1 = \frac{1}{17} < \int_1^2 \frac{dx}{1+x^4} < A = \frac{1}{17} + \frac{15}{68} =$$

$$= \frac{4+15}{68} = \frac{19}{68}$$

(4) $f(x) = (2cx - x^2)/c^3 \qquad c > 0$

$f(x) = -\frac{1}{c^3}x^2 + \frac{2cx}{c^3}$; $\quad f'(x) = -\frac{1}{c^3}\cdot 2x + 2\frac{1}{c^2} = 0$

$\frac{2}{c^3}x = \frac{2}{c^2}$ $\quad \boxed{x = c} \Longrightarrow \boxed{f(c) = 0}$

$f''(x) = -\frac{2}{c^3} < 0 \quad \Rightarrow$

(5) $f(x) = \int_0^{g(x)} \frac{dt}{\sqrt{1+t^3}} \qquad g(x) = \int_0^{\cos x} [1 + \sin(t^2)]\,dt \qquad f'(x) = ?$

$f'(x) = \frac{1}{\sqrt{1+g^3(x)}}\,\frac{d\,g(x)}{dx}$; $\quad \frac{d\,g(x)}{dx} = [1 + \sin(\cos^2(x))](-\sin(x))$

$g(x) = \int_0^{\cos(x)} dt + \int_0^{\cos(x)} \sin(t^2)\,dt = \cos(x) + I_1$

$I_1 = \int_0^{\cos(x)} \sin(t^2)\,dt = \left| \begin{array}{l} u = t^2 \\ du = 2t\,dt \\ dt = \frac{du}{2t} \end{array} \quad dt = \frac{du}{2\sqrt{u}} \right|$

$u = \cos(t^2)$; $\quad du = -\sin(t^2)\cdot 2t\,dt$; $\quad -\frac{du}{2t} = \sin(t^2)\,dt$

$t^2 = \arccos(u)$; $\quad t = \sqrt{\arccos u}$; $\quad -\frac{du}{2\sqrt{\arccos u}} = \sin(t^2)\,dt$

$I_1 = -\int \frac{du}{2\sqrt{\arccos(u)}} = \left| \begin{array}{l} v = \arccos u \\ dv = -\frac{1}{\sqrt{1-u^2}}\,du \end{array} \right. \qquad u = \cos(v) \qquad \frac{du}{dv} = -\sin(v) = -\sqrt{1-\cos^2 v}$

$dv = -\frac{du}{\sqrt{1-u^2}} \qquad du = -\sqrt{1-\cos^2 v}\,dv$

$I_1 = \int \frac{\sqrt{1-\cos^2 v}}{\sqrt{v}}\,dv = \int \sqrt{\frac{1}{v} - \frac{\cos^2 v}{v}}\,dv = \int \frac{\sin v}{\sqrt{v}}\,dv$

$f'(x) = \frac{-\sin(x)[1 + \sin(\cos^2(x))]}{\sqrt{1 + \left(\cos(x) + \sqrt{\frac{\pi}{2}}\,\mathrm{FresnelS}\left(\sqrt{\frac{2}{\pi}}\cos(x)\right)\right)^3}}$

(6) $f(x) = \int_0^x x^2 \sin(t^2)\,dt \qquad f'(x) = ?$

$f(x) = x^2 \int_0^x \sin(t^2)\,dt \qquad f'(x) = 2x \underbrace{\int_0^x \sin(t^2)\,dt}_{I} + x^2 \cdot \sin(x^2)$

$\mathrm{Fresnel}S = \int_0 \sin\left(\frac{\pi}{2}t^2\right)dt$

$I = \left| \begin{array}{l} t^2 = \frac{\pi}{2}u^2 \\ dt\,(2t) = \frac{\pi}{2}2u\,du \end{array} \right. \quad dt = \frac{\pi}{2t}u\,du \quad dt = \frac{\pi}{2t}\sqrt{\frac{2}{\pi}}\,t\,du \quad \boxed{u = \sqrt{\frac{2}{\pi}}\,t} \quad dt = \sqrt{\frac{\pi}{2}}\,du$

$t = 0 \quad u = 0$; $\quad t = x \quad u = \sqrt{\frac{2}{\pi}}x$; $\quad I = \int_0^{\sqrt{\frac{2}{\pi}}x} \sin\left(\frac{\pi}{2}u^2\right)\sqrt{\frac{\pi}{2}}\,du$

$$I = \sqrt{\frac{\pi}{2}} \int_0^{\sqrt{\frac{2}{\pi}}x} \sin\left(\frac{\pi}{2}u^2\right) du = \sqrt{\frac{\pi}{2}}\ \text{FresnelS}\left(\sqrt{\frac{2}{\pi}}x\right)$$

$$f'(x) = 2x \sqrt{\frac{\pi}{2}}\ \text{FresnelS}\left(\sqrt{\frac{2}{\pi}}x\right) + x^2 \sin(x^2)$$

⑦ $L = \lim_{x \to 0} \frac{1}{x} \int_0^x (1-\tan(2t))^{1/t} dt = \lim_{x \to 0} (1-tg(2x))^{1/x} =$

$= \lim_{x \to 0} e^{\frac{1}{x}\ln(1-tg(2x))} = e^{\lim_{x \to 0} \frac{\ln(1-tg(2x))}{x}}$ → ⊛

$⊛ = |0/0| = \lim_{x \to 0} \frac{-\frac{1}{1-tg(2x)} \cdot \frac{1}{\cos^2(2x)} \cdot 2}{1} = -2 \lim_{x \to 0} \frac{1}{(1-tg(2x))\cos^2(2x)}$ (with $tg(2x) \to 0$, $\cos^2(2x) \to 1$)

$⊛ = -2$ $\quad$ $L = e^{-2}$

⑧ $L = \lim_{t \to 0} \frac{\int_0^t \sin(t^2)\,dt}{\frac{1}{2} t \cdot \sin(t^2)} = |0/0| = \lim_{t \to 0} 2 \cdot \frac{\sin(t^2)}{\sin(t^2) + t\,2t \cdot \cos(t^2)}$

~~$= \lim_{t \to 0} 2 \frac{\sin(t^2)\,2}{\sin(t^2) + 2t^2 \sin(t^2)} \neq \lim_{t \to 0} \frac{2}{1+2t^2} = 2$~~

$L = 2 \lim_{t \to 0} \frac{\sin(t^2)}{\sin(t^2) + 2t^2\cos(t^2)} = 2 \lim_{t \to 0} \frac{2t\cos(t^2)}{2t\cos(t^2) + ⊛}$

$⊛ = \left[2t^2\cos(t^2)\right]' = 4t \cdot \cos(t^2) - 2t^2 \cdot 2t \cdot \sin(t^2)$

$L = 2 \lim_{t \to 0} \frac{2t\cos(t^2)}{2t\cos(t^2) + 4t \cdot \cos(t^2) - 4t^3 \cdot \sin(t^2)} =$

$= 2 \lim_{t \to 0} \frac{\cos(t^2)}{\cos(t^2) + 2 \cdot \cos(t^2) - 2t^2 \sin(t^2)} = 2 \cdot \frac{1}{1+2} = \frac{2}{3}$

⑨ $I = \int_a^b (-x^2+x+2)\,dx \quad (a,b) = ?$ tless $I = MAX$

$\frac{dI}{db} = 0 \quad \frac{d}{db} \int_a^b (-x^2+x+2)\,dx = 0 \quad -b^2+b+2 = 0$

$b^2 - b - 2 = 0$

$b_{1,2} = \frac{+1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = \begin{cases} 2 \\ -1 \end{cases}$

$\frac{dI}{da} = 0 \quad \frac{d}{da}\left[-\int_b^a (-x^2+x+2)\,dx = 0\right] \quad a^2 - a - 2 = 0$

$a_{1,2} = \begin{cases} 2 \\ -1 \end{cases}$

$$I = \int_{-\pi/2}^{\pi} \sin(x)\,dx = \left| \begin{array}{l} y = -x \\ dy = -dx \\ \sin(-x) = -\sin(x) \\ x = -\pi/2 \quad y = \pi/2 \qquad x = \pi \quad y = -\pi \end{array} \right| = \int_{\pi/2}^{-\pi} \sin(x)\,dx$$

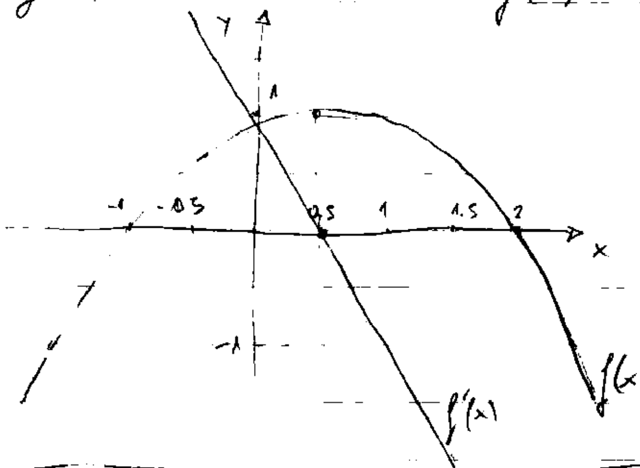
$$I = \left| \begin{array}{lll} y = \frac{\pi}{2} - x & dy = -dx & x = -\pi/2 \quad y = \pi \\ \sin\left(\frac{\pi}{2} - y\right) = \cos(y) & & x = \pi \quad y = -\frac{\pi}{2} \end{array} \right| = -\int_{\pi} \cos(y)\,dy$$

$$f(x) = -x^2 + x + 2 \qquad f'(x) = -2x + 1 = -2\left(x - \frac{1}{2}\right)$$

$$f'(x_0) = 0 \qquad -2x + 1 = 0 \qquad \boxed{x = \frac{1}{2}}$$

$$f(x_0) = -\frac{1}{4} + \frac{1}{2} + 2 = \frac{2-1}{4} + 2 = 2 + \frac{1}{4} = \frac{9}{4}$$

$$\boxed{f''(x_0) = -2 < 0 \Rightarrow MAX}$$



(i,k) k-th user connected to ith base station

P_{ik} - transmitting Power	s_{ik} - message signal
r_{ik} - transmitting Rate	w^j_{ik} - weight of j^{th} diversity branch

OUTPUT OF THE COMBINER FOR USER (i,k)

$$y_{ik} = \sum_{j=1}^{M} (w^j_{ik})^* x^j_i \qquad x^j_i = \sum_{n=1}^{N} \sum_{l=1}^{K_n} \sqrt{P_{nl}}\, a^j_{(n,l)i}\, s_{nl} + n^j_i$$

$a^j_{(n,l)i}$ - array gain between user (n,l) and base station i on j^{th} diversity route

n^j_i - noise at the j^{th} diversity branch of i^{th} base station

d^j_{ik} - received signal of user (i,k) at j^{th} diversity branch

$$d^j_{ik} = \sqrt{P_{ik}}\, a^j_{(ik)i}\, s_{ik}$$

$\boxed{SINR}$ (per bit) $\qquad \epsilon_{ik} = \frac{E_b}{I_0} = \frac{w^H_{ik} \Omega_{ik} w_{ik}}{w^H_{ik} \Phi_i w_{ik} - w^H_{ik} \Omega_{ik} w_{ik}} \frac{f_w}{r_{ik}}$

$\Phi_i = E(x_i x_i^H)$ → TOTAL RECEIVED SIGNAL (CORRELATION MATRIX)

$\Omega_{ik} = E(d_{ik} d^H_{ik})$ → received signal of interest

- Message signals are uncorrelated with zero mean

$$E(|s_{ik}|^2) = 1 \Rightarrow \Omega_{ik} = P_{ik}\, a_{(ik)i}\, a^H_{(ik)i}\,; \quad \Phi_i = \sum_{n,l} P_{nl}\, a_{(n,l)i}\, a^H_{(n,l)i} + N_i I$$

$$\epsilon_{ik} = \frac{P_{ik}\, w^H_{ik}\, a_{(i,k)i}\, a^H_{(i,k)i}\, w_{ik}}{\sum_{(n,l) \neq (i,k)} P_{nl}\, w^H_{ik}\, a_{(n,l)i}\, a^H_{(n,l)i}\, w_{ik} + N_i\, w^H_{ik} w_{ik}} \quad \frac{f_w}{r_{ik}}$$

WITH MVDR combining for user (i,k)

$$\epsilon_{ik} = P_{ik}\left(a^H_{(i,k)i} \left(\Phi_i - \Omega_{ik} \right)^{-1} a_{(i,k)i} \right) \; f w / \nu_{ik}$$

Q_{ik} - SINR REQUIREMENT OF USER (i,k)

$$\epsilon_{ik} \geq Q_{ik}$$

P^b_{ik} - power of user (i,k) for base station assignment b

T^b - TOTAL POWER OF base station assignment b

$b = 1..B$ - B TOTAL DIFFERENT BS assignment (net configs)

T_{min} - MINIMAL TOTAL POWER

$P^b = \{ P^b_{ik} \}$ - set of all users' power for BS assignment b

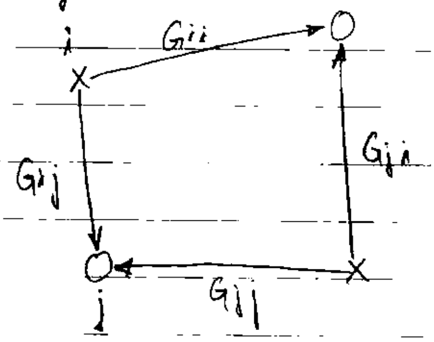
P_{min} = set that results in minimal power

JOINT OPTIMAL POWER CONTROL AND BEAMFORMING

G_{ij} - LINK GAIN between transmitter i and receiver j

P_i - transmitter power

$G_{ji} P_j$ - power received at receiver i from transmitter j



interfering signal at i

$$I_i = \sum_{j \neq i} G_{ji} P_j$$

CIR - Carrier to Interference

$$\Gamma_i = \frac{G_{ii} P_i}{\sum_{j \neq i} G_{ji} P_j} \geq \gamma_0$$

2. $P \geq \gamma_0 F P$, $P = [P_1, P_2, \ldots, P_M]^T$ - POWER VECTOR

$$[F]_{ij} = \begin{cases} 0 & j = i \\ \frac{G_{ji}}{G_{ii}} > 0 & j \neq i \end{cases}$$

$$\Gamma_i = \frac{G_{ii} P_i}{\sum_{j \neq i} G_{ji} P_j + N_i}$$ (thermal NOISE)

$$\Gamma_i \geq \gamma_0 \quad 1 \leq i \leq M$$

$$[I - \gamma_0 F] P \geq u \; ; \quad u_i = \frac{\gamma_0 N_i}{G_{ii}} \; ; \quad 1 \leq i \leq M$$

- Power Control Problem:

$$\text{minimize } \sum_i P_i$$
$$\text{s.t. } [I - \gamma_0 F] P \geq u$$

98, 141, 306, 372

$$\hat{P} = [I - \gamma_0 F]^{-1} u$$

- BEAMFORMING:

$$\Gamma_i = \frac{P_i G_{ii}}{\sum_{j \neq i} G_{ji} G_{ai}(w_i, a_{ji}) P_j + N_i w_i^H w_i}$$

SNR

(10) Use integral to estimate

$$S = \sum_{i=1}^{100\,000} \sqrt{i} \qquad I = \int_1^{100\,000} \sqrt{x}\,dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\Bigg|_1^{10^5} = \frac{2}{3}\sqrt{x^3}\Bigg|_1^{10^5}$$

$$I = \frac{2}{3}\left(\sqrt{10^{15}} - 1\right) =$$

(11)
$$\int_0^u |x|\,dx = \int_0^u x\,dx = \frac{x^2}{2}\Bigg|_0^u = \frac{u^2}{2} \qquad u \geq 0$$

$$\int_a^b |x|\,dx = |\,0 < a < b\,| = \int_a^b x\,dx = \frac{x^2}{2}\Bigg|_a^b = \frac{1}{2}(b^2 - a^2)$$

(12)
$$I = \frac{d^2}{dx^2}\int_0^x \left(\int_1^{\sin t} \sqrt{1+u^4}\,du\right) dt = \frac{d}{dx}\left[\frac{d}{dx}\int_0^x \left(\int_1^{\sin t} \sqrt{1+u^4}\,du\right) dt\right] =$$

$$= \frac{d}{dx}\left[\int_1^{\sin x} \sqrt{1+u^4}\,du\right] = \sqrt{1+\sin^4 x} \cdot \frac{d}{dx}(\sin x) = \underline{\underline{\cos(x)\sqrt{1+\sin^4(x)}}}$$

$$I\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}\sqrt{1+\left(\frac{1}{2}\right)^4} = \frac{\sqrt{3}}{2}\sqrt{\frac{17}{16}} = \frac{\sqrt{51}}{8}$$

$$I\left(\frac{\pi}{3}\right) = \frac{1}{2}\sqrt{1+\left(\frac{\sqrt{3}}{2}\right)^4} = \frac{1}{2}\sqrt{1+\frac{9}{16}} = \frac{1}{2}\sqrt{\frac{16+9}{16}} = \frac{5}{8} \,//$$

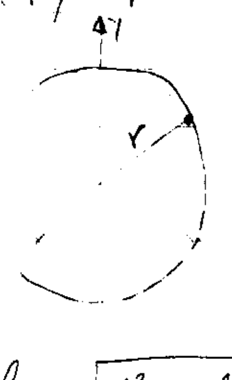
(13)
$$\int_0^x f(t)\,dt = [f(x)]^2 \quad \Big/ \frac{d}{dx}$$

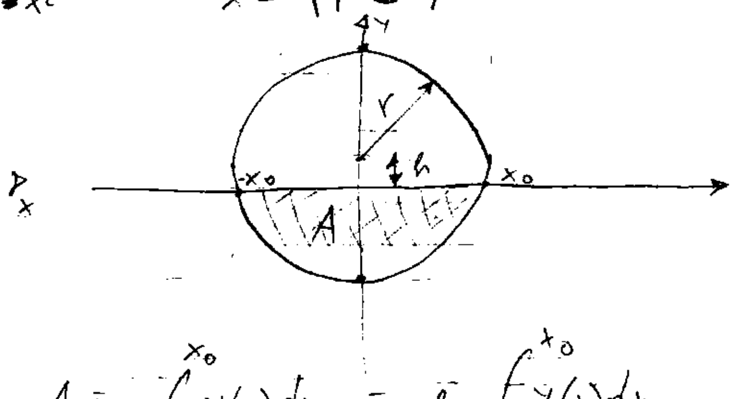
$$f(x) = 2f(x)\frac{df(x)}{dx} \qquad \frac{df(t)}{dt} = \frac{1}{2} \qquad f(x) = \frac{1}{2}\int dx$$

$$f(x) = \frac{x}{2} \qquad \int_0^x \frac{t}{2}\,dt = \frac{x^2}{2}\cdot\frac{1}{2} = \frac{x^4}{4} = (f(x))^2$$

$$\left(\cos^2(x)\right)' = 2\cos(x)\cdot -\sin(x) = -2\sin(x)\cos(x)$$

(14)
$$x^2 + y^2 = r^2 \qquad y = \sqrt{r^2 - x^2} \qquad x = \sqrt{r^2 - y^2}$$





$$y = h - \sqrt{r^2 - x^2} \qquad A = \int_{-x_0}^{x_0} y(x)\,dx = 2\int_0^{x_0} y(x)\,dx$$

$$0 = h - \sqrt{r^2 - x_0^2}\,; \quad r^2 - x_0^2 = h^2\,; \quad x_0 = \sqrt{r^2 - h^2}$$

$$A = 2\int_0^{\sqrt{r^2-h^2}} \left[h - \sqrt{r^2 - x^2}\right] dx$$

$$A = 2\int_0^{\sqrt{r^2-l_1^2}} \left(l_1 - \sqrt{r^2-x^2}\right) dx \quad ; \quad \frac{dA}{dl_1} = \emptyset \, , \quad \frac{d}{dl_1} 2\int_0^{\sqrt{r^2-l_1^2}} \left(l_1-\sqrt{r^2-x^2}\right) dx =$$

$$= 2\left[l_1 - \sqrt{r^2-(r^2-l_1^2)}\right] \cdot \frac{d}{dl_1}\left(\sqrt{r^2-l_1^2}\right) = \frac{1}{2} 4 l_1 \left(r^2-l_1^2\right)^{\frac{1}{2}-1} \cdot (-2l_1)$$

$$= \frac{-4l_1^2}{\sqrt{r^2-l_1^2}}$$

$$\frac{dA}{dl_1} = 2\cdot\frac{d}{dl_1}\Big[\underbrace{l_1\int_0^{\sqrt{r^2-l_1^2}} dx}_{(*)} - \underbrace{\int_0^{\sqrt{r^2-l_1^2}} \sqrt{r^2-x^2}\, dx}_{(**)}\Big]$$

$$(*) = \frac{d}{dl_1}\left[l_1\sqrt{r^2-l_1^2}\right] = \sqrt{r^2-l_1^2} + \frac{l_1}{2}\left(r^2-l_1^2\right)^{-\frac{1}{2}}(-2l_1) = \sqrt{r^2-l_1^2} - \frac{l_1^2}{\sqrt{r^2-l_1^2}}$$

$$(*) = \frac{r^2-l_1^2-l_1^2}{\sqrt{r^2-l_1^2}} = \frac{r^2-2l_1^2}{\sqrt{r^2-l_1^2}}$$

$$(**) = \frac{d}{dl_1}\left[\int_0^{\sqrt{r^2-l_1^2}} \sqrt{r^2-x^2}\, dx\right] = \sqrt{r^2-r^2+l_1^2}\ \frac{1}{2}\frac{1}{\sqrt{r^2-l_1^2}}(-2l_1) = -\frac{l_1^2}{\sqrt{r^2-l_1^2}}$$

$$2\left[(*)-(**)\right] = 2\left[\frac{r^2-2l_1^2}{\sqrt{r^2-l_1^2}} + \frac{l_1^2}{\sqrt{r^2-l_1^2}}\right] = \frac{r^2-l_1^2}{\sqrt{r-l_1^2}}$$

$$= 2\sqrt{r^2-l_1^2}$$

$$\left(r^2\pi = 2\int_0^{\sqrt{r^2-l_1^2}} (l_1 - \sqrt{r^2-x^2}\, dx \right) \Big/ \frac{d}{dr}$$

$$A = 2\int_0^{\sqrt{r^2-l_1^2}} l_1\, dx - 2\int_0^{\sqrt{r^2-l_1^2}} \sqrt{r^2-x^2}\, dx = 2l_1\sqrt{r^2-l_1^2} - 2\left[\frac{x}{2}\sqrt{r^2-x^2} + \frac{r^2}{2}\arcsin\frac{x}{r}\right]_0^{\sqrt{r^2-l_1^2}}$$

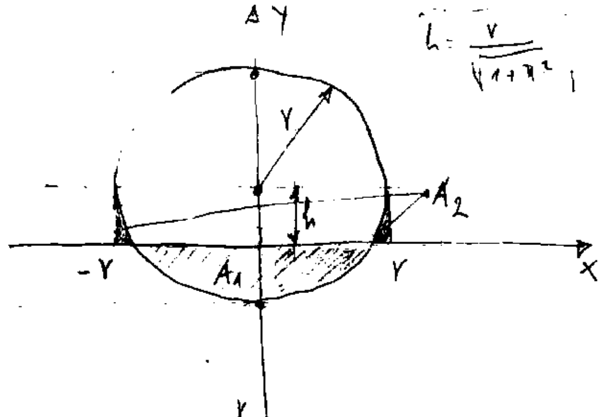
$k = 0, \ldots \infty$

$$A = 2l_1\sqrt{r^2-l_1^2} - \left[\sqrt{r^2-l_1^2}\cdot\sqrt{r^2-r^2+l_1^2} + r^2\cdot\arcsin\frac{\sqrt{r^2-l_1^2}}{r}\right] + 2k\pi$$

$$\boxed{A = 2l_1\sqrt{r^2-l_1^2} - l_1\sqrt{r^2-l_1^2} - r^2\arcsin\frac{\sqrt{r^2-l_1^2}}{r}}$$

$$\boxed{A = l_1\sqrt{r^2-l_1^2} - r^2\arcsin\frac{\sqrt{r^2-l_1^2}}{r}}$$

$$A = \emptyset \Rightarrow \quad l_1\sqrt{r^2-l_1^2} = r^2\arcsin\left(\frac{\sqrt{r^2-l_1^2}}{r}\right)$$



$h = \frac{r}{\sqrt{1+\pi^2}}$,

$y = h - \sqrt{r^2 - x^2}$

$r = 2 \Rightarrow y = h - \sqrt{4 - x^2}$

$y = \varnothing$

$h = \sqrt{r^2 - x^2}$, $x = \sqrt{r^2 - h^2}$

$\int \sqrt{r^2 - x^2}\,dx = \frac{x}{2}\sqrt{r^2 - x^2} + \frac{r^2}{2}\arcsin\left(\frac{x}{r}\right)$

$$A_T = A_1 + A_2 = \int_{-r}^{r}\left(h - \sqrt{r^2 - x^2}\right)dx = 2\int_{0}^{r}\left(h - \sqrt{r^2 - x^2}\right)dx$$

$$A_T = 2\left[hx\Big|_0^r - \left(\frac{x}{2}\sqrt{r^2 - x} + \frac{r^2}{2}\arcsin\left(\frac{x}{r}\right)\right)\Big|_0^r\right] = 2\left[h\cdot r - \frac{r^2}{2}\arcsin(1)\right] \quad \frac{\pi}{2}$$

$$A_T = 2\left[h\cdot r - \frac{r^2\pi}{4}\right] = 2h\cdot r - r^2\frac{\pi}{2}$$

$A_T = \varnothing$. $2h\cdot r = r^2\frac{\pi}{2}$; $h = \frac{r\pi}{4}$

$$A_1 = h\sqrt{r^2 - h^2} - r^2\arcsin\frac{\sqrt{r^2 - h^2}}{r}$$

$h = \varnothing$; $A_1 = -r^2\arcsin\frac{r}{r} = -r^2\cdot\frac{\pi}{2}$

(15) $$\underbrace{\int_0^x f(u)(x-u)\,du}_{I_1} = \underbrace{\int_0^x\left(\int_0^u f(t)\,dt\right)du}_{I_2}$$

$\int_0^x x f(u)\,du - \int_0^x u f(u)\,du = x\,[F(x) - F(0)]$

$x - u = t$; $-du = dt$; $u = \varnothing$ $t = x$; $u = x$ $t = \varnothing$ | $-\int_x^{0} t f(x-t)\,dt = \int_0^x t f(x-t)\,dt$

$$\frac{d}{dx}(I_1) = \frac{d}{dx}\left[\int_0^x x f(u)\,du - \int_0^x u f(u)\,du\right] = x f(x) - x f(x) = \varnothing$$

$$\frac{d}{dx}(I_2) = \int_0^x f(t)\,dt$$

• $f(u) = x^2$

I_1 : $\int_0^x u^2(x-u)\,du = x\int_0^x u^2\,du - \int_0^x u^3\,du = x\frac{x^3}{3} - \frac{x^4}{4} = \frac{4x^4 - 3x^4}{12} = x^4/12$

$I_2 = \int_0^x\left(\int_0^u f(t)\,dt\right)du = \int_0^x F(u)\,du$; $F'(u) = f(u)$

$$I_1 = x\int_0^x f(u)\,du - \int_0^x u f(u)\,du = x\int_0^x F'(u)\,du - \int_0^x u F'(u)\,du$$

$$= x\left(F(x) - F(0)\right) - \underbrace{\int_0^x u F'(u)\,du}_{(*)} \Rightarrow (*) = u\cdot F(u)\Big|_0^x - \int_0^x F(u)\,du$$

$\int u\,dv = uv - \int v\,du$

$$\int_0^x f(u)(x-u)\,du = \int_0^x \left(\int_0^u f(t)\,dt \right) du$$

$$x\int_0^x f(u)\,du - \int_0^x u f(u)\,du = \int_0^x \left(\int_0^u f(t)\,dt \right) du$$

$$xF(x) - \int_0^x u f(u)\,du = \int_0^x F(u)\,du \;, \quad xF(x) - \underbrace{\int_0^x u f(u)\,du}_{\circledast} = F(x) - F(0)$$

$$\circledast = \int_0^x u\,dF(u) = uF(u)\Big|_0^x - \int_0^x F(u)\,du = xF(x) - \int_0^x F(u)\,du$$

$$\int \underbrace{x}_{f(x)}\,dx = \underbrace{\frac{x^2}{2}}_{=F(x)} \qquad \frac{dF(x)}{dx} = \frac{2x}{2} = x$$

$$\int xe^x\,dx = \int x\,de^x = xe^x - \int e^x\,dx = xe^x - e^x$$

$$\int xe^{-x}\,dx = \int x\,d(-e^{-x}) = -xe^{-x} + \int e^{-x}\,dx = -e^{-x} - xe^{-x} = -(1+x)e^{-x}$$

$$\int e^{-x}\,dx = \left| \begin{array}{l} -x = u \\ -dx = du \end{array} \right| = -\int e^u\,du = -e^u = -e^{-x}$$

$$xF(x) - xF(x) + \int_0^x F(u)\,du = \int_0^x F(u)\,du \qquad \boxed{F(x) = F(x)}$$

$$x\int_0^x f(u)\,du - \int_0^x \underbrace{u f(u)\,du}_{= u\,dF(u)} = \int_0^x \left(\int_0^u \underbrace{f(t)\,dt}_{= dF(t)} \right) du$$

$$x\cdot F(x) - \left[uF(u)\Big|_0^x - \int_0^x F(u)\,du \right] = \int_0^x F(u)\,du$$

$$\Rightarrow xF(x) - xF(x) + \int_0^x F(u)\,du = \int_0^x F(u)\,du$$

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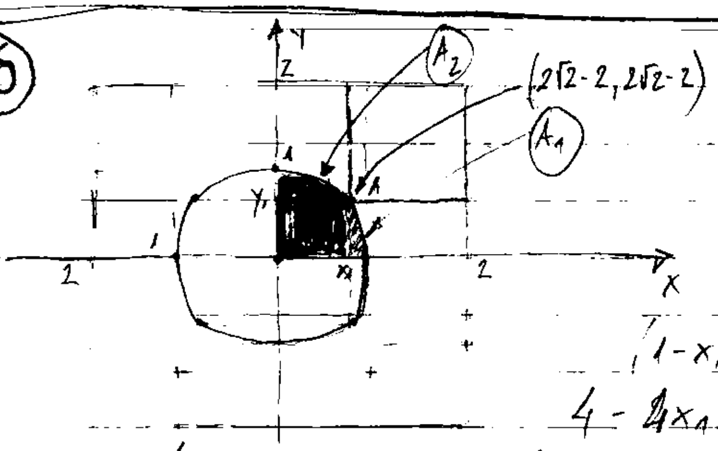
(16)



$$x_1^2 + y_1^2 = z^2 \qquad \boxed{z = 1}$$

$$\boxed{A = 8\cdot A_1 + 2x_1 \cdot 2y_1}$$

$$(1-x_1) + 1 = z \Rightarrow 2 - x_1 = \sqrt{x_1^2 + y_1^2}$$

$$4 - 4x_1 + x_1^2 = x_1^2 + y_1^2 \qquad \boxed{y = \sqrt{4-4x}}$$

$$A_1 = \int_0^1 \sqrt{4-4x}\,dx = \sqrt{4}\int_0^1 \sqrt{1-x}\,dx = \left| \begin{array}{l} u = 1-x \\ du = -dx \\ x=0 \; u=1 \\ x=1 \; u=0 \end{array} \right| = -\sqrt{2}\int_1^0 \sqrt{u}\,du$$

$A_1 = \sqrt{2} \int_1^0 \sqrt{u}\, du = 2 \frac{u^{3/2}}{3/2}\Big|_0^1 = \frac{2\sqrt{4}}{3} u\sqrt{u}\Big|_0^1 = \frac{2\sqrt{4}}{3}(1-0)$

$A = 4 \cdot A_1 = \ldots$ $\quad A_1 = \frac{4}{3}$

$1 - y_1 + 1 = \sqrt{x_1^2 + y_1^2}$; $(2-y_1)^2 = x_1^2 + y_1^2$; $4 - 4y_1 + y_1^2 = x_1^2 + y_1^2$

$x = \sqrt{4-4y} = 2\sqrt{1-y}$

$x^2 = 4 - 4y$, $y^2 = 4 - 4x$ $\Rightarrow$ $y = 1 - \frac{x^2}{4}$, $y = 2\sqrt{1-x}$

$x^2 = 4 - 4\sqrt{4-4x} = 4 - 8\sqrt{1-x}$ $\qquad x^2 - 4 = -8\sqrt{1-x}$ $\quad ()^2$

$x^4 - 8x^2 + 16 = 64(1-x)$ $\qquad x^4 - 8x^2 + 64x - 64 + 16 = 0$

$x^4 - 8x^2 + 64x - 48 = 0$ $\Rightarrow$ $x_0 = 2\sqrt{2} - 2$

$\rightarrow$ $4 - 4y = x^2$ $\quad 4y = 4 - x^2$ $\quad y = 1 - \frac{x^2}{4}$

$y_0 = \sqrt{4 - 4(2\sqrt{2} - 2)} = \sqrt{4 - 8\sqrt{2} + 8} = \sqrt{12 - 8\sqrt{2}}$

$A = A_1 + A_2 = \underbrace{\int_{2\sqrt{2}-2}^{1} 2\sqrt{1-x}\, dx}_{A_1} + \underbrace{\int_0^{2\sqrt{2}-2} \left(1 - \frac{x^2}{4}\right) dx}_{A_2}$

$A_1 = \left| \begin{matrix} u = 1-x & du = -dx \\ x = 2\sqrt{2}-2 & u = -1-2\sqrt{2}+2 \\ & u = 3-2\sqrt{2} \\ x = 1 & u = 1-x = 0 \end{matrix} \right| = -2\int_{3-2\sqrt{2}}^{0} \sqrt{u}\, du = 2\int_0^{3-2\sqrt{2}} \sqrt{u}\, du = \frac{4}{3} u\sqrt{u}\Big|_0^{3-2\sqrt{2}}$

$A_1 = \frac{4}{3}(3-2\sqrt{2})\sqrt{3-2\sqrt{2}} = 0.095$, $\quad A_2 = \int_0^{2\sqrt{2}-2} \left(1 - \frac{x^2}{4}\right) dx = 0.78$

$A_1 + A_2 = 0.8758$ $\qquad A_{TOT} = 4 \cdot (A_1 + A_2)$ $\qquad A_{TOT} = 3.5032$

(17) $\lim_{n\to\infty} \left(\frac{1}{\sqrt{n}\sqrt{n+1}} + \frac{1}{\sqrt{n}\sqrt{n+2}} + \ldots + \frac{1}{\sqrt{n}\sqrt{n+n}} \right)$

$\frac{1}{\sqrt{n^2 + n \cdot i}} = \frac{1}{n\sqrt{1 + \frac{i}{n}}} = \frac{\frac{1}{n}}{\sqrt{1 + \frac{i}{n}}}$

$\lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{\frac{1}{n}}{\sqrt{1 + \frac{i}{n}}} \right) = \quad S = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{\sqrt{1 + \frac{i}{n}}}$

$\lim_{n\to\infty} \frac{1}{\sqrt{n}\sqrt{n+n}} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}}}{\sqrt{2n}} = \lim_{n\to\infty} \frac{1}{\sqrt{2n^2}} = 0$?

$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \left| \begin{matrix} \Delta x = \frac{1}{n} \\ b = 1 \quad a = 0 \end{matrix} \right| = \int_0^1 \frac{dx}{\sqrt{1+x}}$

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)\,dx \qquad \boxed{\Delta x = \frac{b-a}{n}}$$

$$I=\int_0^1 \frac{1}{\sqrt{1+x}}\,dx = \left| \begin{matrix} u=1+x \\ du=dx \\ x=0 \;\; u=1 \\ x=1 \;\; u=2 \end{matrix} \right| = \int_1^2 \frac{du}{\sqrt{u}} = \left.\frac{u^{-\frac12+1}}{\frac12}\right|_1^2 = 2\sqrt{u}\Big|_1^2$$

$$\boxed{I = 2\sqrt{2}-2}$$

(18) ANY NUMBER „c" $\quad f_c(x) < (x-c)^2 \quad f_c(x) < (x-c-2)^2$

$g(c)=\int_0^1 f_c(x)\,dx \qquad -2\le c\le 2 \qquad$ find MAX & MIN of $g(c)$

$c=-2\ldots1 \quad \boxed{c=-2}\; g(c) = \int_0 x^2\,dx = \frac{x^3}{3} = \frac{?}{3}$

$\boxed{c=-1}\; g(c) = \int_0^1 (x-1)^2\,dx = \int_{-1} u^2\,du = \frac{u^3}{3} = -\frac{(-1)^3}{3} = \frac{1}{3}$ MIN

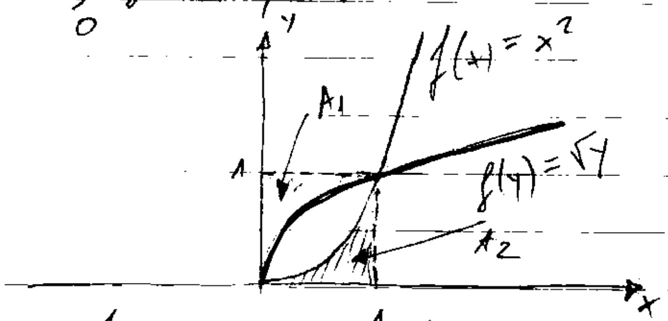
$c>1 \quad \boxed{c=2}\; g(c) = \int_0^1 (x-2)^2\,dx = \int_{-2}^{-1} u^2\,du = \left.\frac{u^3}{3}\right|_{-2}^{-1}$

$-\frac{1}{3} + \frac{2^3}{3} = \frac{8-1}{3} = \frac{7}{3}$ MAX

$$\boxed{\text{Max}\{g(c)\} = \frac{7}{3} \qquad \text{Min}\{g(c)\} = \frac{1}{3}}$$

(19) $f(0)=0 \quad f(1)=1 \quad f'(x)>0 \qquad A_2=\int_0^1 f(x)\,dx = \frac{1}{3}$

$\int_0^1 f^{-1}(y)\,dy = ?$



$f(x) = x^2 \qquad y = x^2$

$\int_0^1 x^2\,dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}$

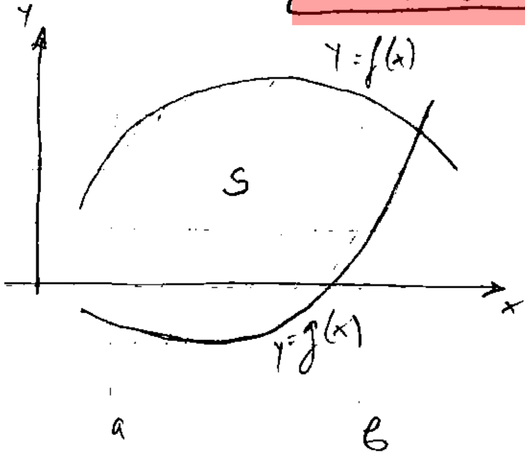
$x = \sqrt{y}$

$$A_1 = \int_0^1 f^{-1}(y)\,dy = \int_0^1 (1-f(x))\,dx = \int_0^1 dx - \int_0^1 f(x)\,dx = 1 - \int_0^1 f^{-1}(x)\,dx$$

$$A_1 = \int_0^1 f^{-1}(y)\,dy = \int_0^1 f(x)\,dx = \frac{1}{3}$$

$\hookrightarrow 1 - \int_0^1 \sqrt{x}\,dx = 1 - \left.\frac{2}{3}x\sqrt{x}\right|_0^1 =$
$= 1 - \frac{2}{3}\cdot 1 = \frac{1}{3}$

APLICATIONS OF INTEGRATION



$$\sum_{i=1}^{n} [f(x_i) - g(x_i)] \Delta x$$

$f(x) \geq g(x) \ \forall x \in [a, b]$

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)] \Delta x$$

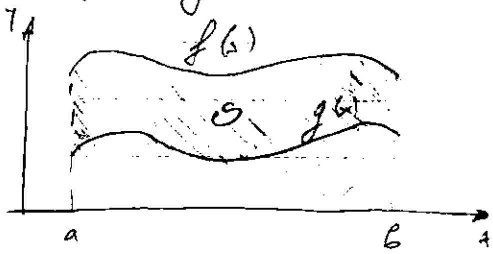
$$A = \int_a^b [f(x) - g(x)] \, dx$$

$x^2 - 2x + 1 = 0$

$x_{1,2} = \frac{+2 \pm \sqrt{4-4}}{2}$

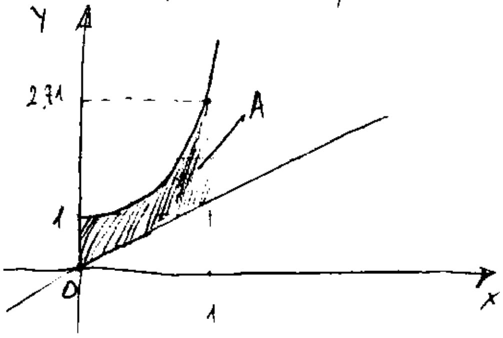
$x_{1,2} = 1$

If: $f(x), g(x) \geq 0$ it is obvious:

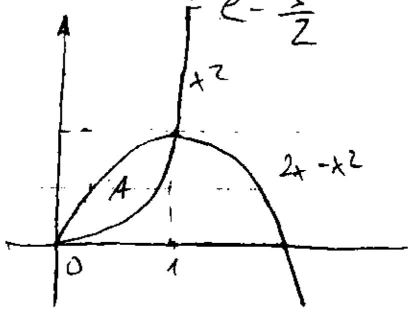


$$S = A_f - A_g = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b [f(x) - g(x)]\,dx$$

Ex. 1 $y = e^x$, $y = x$ $\quad x = 0,\ x = 1$



$$\int_0^1 (e^x - x)\,dx = e^x\Big|_0^1 - \frac{x^2}{2}\Big|_0^1 = e - 1 - \frac{1}{2} = e - \frac{3}{2}$$



Ex. 2 $y = x^2$; $y = 2x - x^2$ $\quad y = (2 - x)x$

$x^2 = 2x - x^2$; $2x^2 = 2x$ $\quad x = 1 \quad x = 0$

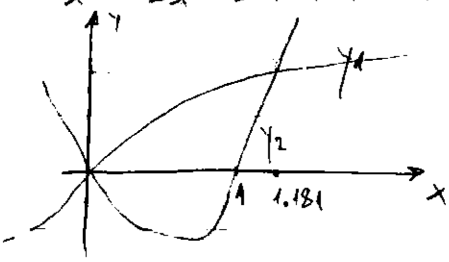
$$A = \int_0^1 (2x - x^2 - x^2)\,dx = \int_0^1 (2x - 2x^2)\,dx = 2\int_0^1 (x - x^2)\,dx = 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1$$

$$A = 2\left(\frac{1}{2} - \frac{1}{3}\right) = 2 \cdot \frac{1}{6} = \frac{1}{3}$$

Ex. 3 $y_1 = \frac{x}{\sqrt{x^2+1}}$ $\quad y_2 = x^4 - x$ $\quad \frac{x}{\sqrt{x^2+1}} = x^4 - x \quad (\ )^2$

$\frac{x^2}{x^2+1} = x^8 - 2x^5 - x^2$; $x^2 = (x^2+1)(x^8 - 2x^5 + x^2)$

$x^{10} - 2x^7 - 2x^5 + x^4 = 0$ $\quad x^4(x^6 - 2x^3 - 2x + 1) = 0$ $\quad x = 0$



$$I_1 = \int_0^{1.181} \frac{x}{\sqrt{x^2+1}}\,dx = \frac{1}{2}\int_0^{1.181} \frac{d(x^2)}{\sqrt{x^2+1}} = \frac{1}{2}\int_1^{(1.181)^2+1} \frac{du}{\sqrt{u}}$$

$$= \frac{1}{2} \cdot \frac{\sqrt{u}}{\frac{1}{2}}\Big|_1^{(1.181)^2+1} = \sqrt{(1.181)^2+1} = 1.5475 - 1 = 0.5475$$

$$I_2 = \int_0^{1.181} (x^4 - x)\,dx = \left(\frac{x^5}{5} - \frac{x^2}{2}\right)\Big|_0^{1.181} = \frac{(1.181)^5}{5} - \frac{(1.181)^2}{2} = -0.2378$$

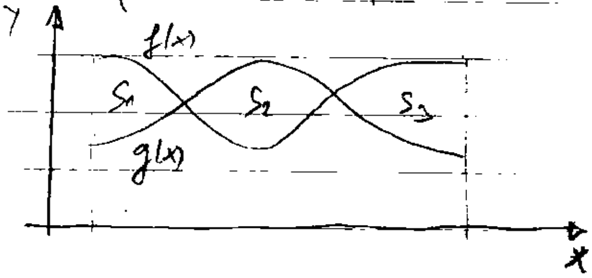
$$A = I_1 - I_2 = 0.5475 + 0.2378 = 0.7853$$

$$I_1 = \int_0^{1.181} \frac{x}{\sqrt{x^2+1}}\,dx = \left|\begin{array}{l} u = x^2+1 \\ du = 2x\,dx \\ x\,dx = \frac{1}{2}du \\ x = 1.181 \quad u = (1.181)^2+1 \end{array}\right| = \int_1^{(1.181)^2+1} \frac{\frac{1}{2}du}{\sqrt{u}} = \sqrt{u}\Big|_1^{1.181^2+1}$$

$$I_1 = \sqrt{1.181^2+1} - 1 = 0.5475$$

Ex. 4 $A_1 = \sum_{i=1}^{n} f_1(x_i)\,\Delta x$; $\Delta x = 2$ $\quad n = 8 \quad \Delta x = \frac{16-0}{8} = 2$

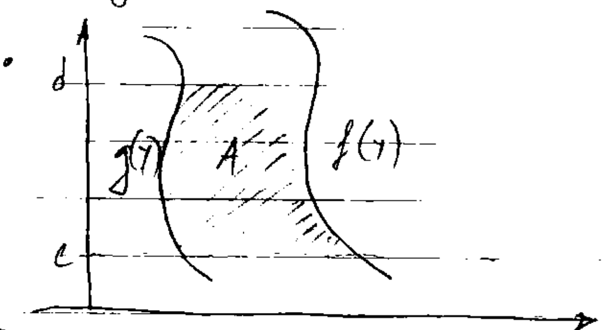
$$A_1 = (20 + 30 + 42.5 + 48 + 55 + 58 + 62.5 + 65)\cdot 2$$



$$A = \int_a^b |f(x) - g(x)|\,dx$$

Ex. 5 $y_1 = \sin(x)$ $\quad y_2 = \cos(x)$ $\quad x = 0$; $x = \frac{\pi}{2}$ $\qquad$ AREA = ?

$$\int_0^{\pi/2} |\cos(x) - \sin(x)|\,dx = \sqrt{2} - 2$$



$$A = \int_c^d [f(y) - g(y)]\,dy$$

$$A = \int_c^d (x_R - x_L)\,dy$$

Ex 6 $y = x - 1 \qquad y^2 = 2x + 6$

$x = y + 1 \qquad x = \frac{y^2 - 6}{2}$

Exercises **1** $y = 5x - x^2$ $\qquad y' = 5 - 2x = 0$ $\qquad$ $x = 2.5$

$y = x$

9 $y = \frac{1}{x}$; $y = \frac{1}{x^2}$ $\quad x = 2$ $\qquad$ **19** $x = 1 - y^2$; $x = y^2 - 1$

$y = \pm\sqrt{1-x} \qquad y = \pm\sqrt{x+1}$

$1 - x = 1 + x \qquad 2x = 0 \qquad$ $x = 0$

20 $\sin(\pi x/2)$; $y = x$ $\qquad$ **21** $y = \cos x$, $y = \sin 2x$, $x = 0$ $\; x = \pi/2$

$\sin(2x) = \cos x \qquad x = ?$ $\qquad$ $\sin(2x) = $ ~~$\sin^2(x) - \cos^2(x)$~~

~~$\sin^2(x) - 1 + \sin^2(x)$~~ $= \sin(x)$ $\qquad$ $2\sin^2(x)$ ~~$- \sin(x) - 1 = 0$~~

$2y^2 - y - 1 = 0 \qquad$ ~~$y_{1,2} = \{1, -1/2$~~

$\sin(x_1) = 1 \quad x_1 = \frac{\pi}{2} \qquad \sin(x_2) = -\frac{1}{2} \qquad x_2 = -\frac{\pi}{6}$?

$\sin(2x) = \sin x \cdot \cos x + \sin x \cos x = 2 \sin x \cos x$ $\quad$ $\boxed{\pi,\ \frac{5\pi}{6}}$

$2 \sin x \cdot \cos x = \cos x \qquad \sin x = \frac{1}{2} \qquad \boxed{x = \frac{\pi}{6}}$

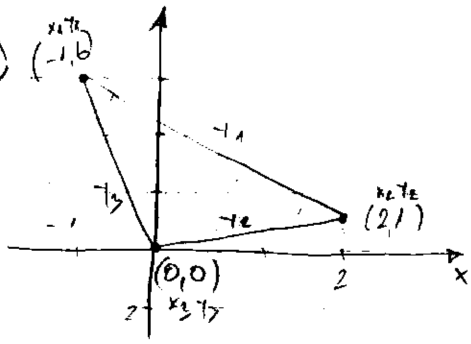
$2 \sin x \cos x - \cos x = 0 \qquad \cos(x)(2\sin x - 1) = 0 \qquad x = \frac{\pi}{6} \quad x = 0$

22 $y_1 = \sin(2x) \qquad y_2 = \sin(x)$

$2 \sin x \cos x = \sin x \qquad (2\cos x - 1)\sin x = 0 \qquad x = 0 \quad x = \pi$

$\cos x = \frac{1}{2} \qquad \boxed{x = \frac{\pi}{3}}$

27



$y - 6 = \frac{1-6}{2+1}(x+1)$

$y = -\frac{5}{3}(x+1) + 6 \quad (1)$

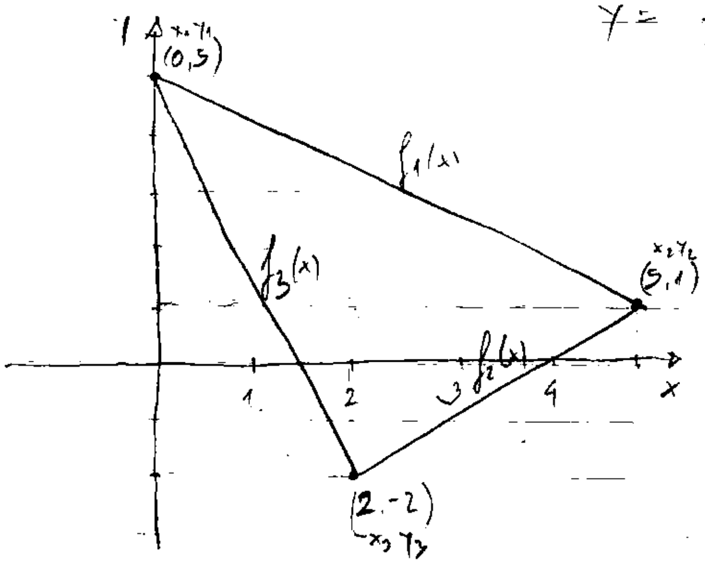
$y - 6 = \frac{0-6}{0+1}(x+1) \to y = -6(x+1)+6$

$y = -6x \quad (2)$

$y - 1 = \frac{0-1}{0-2}(x-2); \; y - 1 = \frac{1}{2}(x-2)$

$y = \frac{1}{2}x - 1 + 1 = \frac{1}{2}x \quad (3)$

28



$f_1(x) - 5 = \frac{1-5}{5-0}(x-0)$

$f_1(x) = -\frac{4}{5}x + 5 \quad (1)$

$f_2(x) - 1 = \frac{-2-1}{2-5}(x-5)$

$f_2(x) = x - 5 + 1 = x - 4 \quad (2)$

$f_3(x) - 5 = \frac{-2-5}{2-0}(x-0)$

$f_3(x) = -\frac{7}{2}x + 5 \quad (3)$

31 $y_1 = \sin^2\left(\frac{x\pi}{4}\right); \quad y_2 = \cos^2\left(\frac{x\pi}{4}\right)$

$a = 0 \qquad b = 1 \qquad \Delta x = \frac{b-a}{n} = \frac{1}{4} = 0.25 \qquad \frac{\Delta x}{2} = 0.125$

$S = \sum_{i=1}^{n} \left(y_1(x_i) - y_2(x_i)\right)\Delta x \qquad x_i = (2i-1)\frac{\Delta x}{2}$

$x_1 = 0.125 \quad x_2 = 0.375$

$x_3 = 0.625 \quad x_4 = 0.875$

38 ① $x - 2y^2 \geq 0$

② $1 - x - |y| \geq 0$

① $x \geq +2y^2 \qquad y \leq \sqrt{\frac{x}{2}}$

② $1 - x \geq |y| \qquad |y| \leq -x + 1$

① $x - 2y^2 \geq 0 \qquad x \geq 2y^2$

② $1 - x - |y| \geq 0 \qquad x \leq 1 - |y|$

[39]

t	0	1	2	3	4	5	6	7	8	9	10
v_c	0	[illegible]	68.8	[illegible]	116.2	[illegible]	148.4	[illegible]	174.2	[illegible]	193.6
v_k	0	[illegible]	54.3	[illegible]	89.5	[illegible]	117.3	126.1	136.4	143.7	149.[illegible]

$$20\left[\frac{\text{miles}}{\text{hours}}\right] = 20\,\frac{5280}{3600}\,\frac{\text{feet}}{\text{sec}} \qquad n=5 \quad \Delta t = \frac{10-0}{n} = 2$$

$$\Delta s = \sum_{i=1}^{n}\left[v_k(t_i) - v_c(t_i)\right]\Delta t \qquad t_i = (2i-1)\frac{\Delta t}{2} = (2i-1)$$

$t_1 = 1;\ t_2 = 3;\ t_3 = 5; \dots t_5 = 9$

$$\Delta s = 2\sum_{i=1}^{5}\left[v_k(t_i) - v_c(t_i)\right] =$$

$$1\,\text{sec} = \frac{1}{3600}\,h \qquad \Delta t = \frac{\frac{10}{3600}}{5} = \frac{1}{360}\cdot\frac{1}{5} = \frac{1}{1800}$$

$$\Delta s = \frac{1}{1800}\sum_{i=1}^{5}\left[v_k(t_i) - v_c(t_i)\right] =$$

(46) $$\int_1^a \frac{1}{x^2}\,dx = \frac{x^{-2+1}}{-2+1}\Big|_1^a = -\frac{1}{x}\Big|_1^a = \frac{1}{x}\Big|_a^1 = \left(1 - \frac{1}{a}\right)$$

$$1 - \frac{1}{a} = \frac{3}{8} \qquad \frac{1}{a} = 1 - \frac{3}{8} = \frac{5}{8} \qquad \boxed{a = \frac{8}{5} = 1.6}$$

(47) $$2\int_{-c}^{c}(c^2 - x^2)\,dx = 2\left[c^2(2c) - \frac{x^3}{3}\Big|_{-c}^{c}\right] = 2\left[2c^3 - \left(\frac{c^3}{3} + \frac{c^3}{3}\right)\right]$$

$$= 2\left[2c^3 - 2\frac{c^3}{3}\right] = 2c^3\left(2 - \frac{2}{3}\right) = 2\cdot c^3\,\frac{6-2}{3} = \frac{8c^3}{3}$$

$$\frac{8c^3}{3} = 576 \qquad c^3 = \frac{576\cdot 3}{8} = 216 \qquad c = \sqrt[3]{216} = 6$$

(46.b.) $$y = \frac{1}{x^2} \qquad x^2 = \frac{1}{b} \qquad x = \frac{1}{\sqrt{b}} \qquad y = b$$

$$\int_1^4 \frac{1}{x^2}\,dx = \frac{3}{4} \qquad \int_1^{1/\sqrt{b}}\left(\frac{1}{x^2} - b\right)dx = \frac{1}{2}\,\frac{3}{4}$$

$$b = \frac{1}{x^2} \qquad x = \frac{1}{\sqrt{b}}$$

(48) $0 < c < \pi/2 \qquad y_1(x) = \cos(x); \qquad y_2(x) = \cos(x-c)$

$\cos(x) = \cos(x-c) = \cos x \cdot \cos c + \sin x \cdot \sin c \qquad ; \quad A = \cos c$

$\cos(x) = A \cdot \cos(x) + A\sqrt{1-\cos^2(x)} \qquad (1-A)\cos(x) = A\sqrt{1-\cos^2(x)} \quad ()^2$

$(1-A)^2 \cdot \cos^2(x) = A^2(1-\cos^2(x)) \quad ; \quad (1-2A+A^2)\cos^2(x) = A^2 - A^2\cos^2(x)$

$\cos^2(x) - 2\cos^2(x) + A^2\cos^2(x) = A^2 - A^2\cos^2(x)$

$(1-2A+2A^2)\cos^2(x) = A^2 \qquad \cos^2(x) = \dfrac{A^2}{2A^2-2A+1}$

$$\cos(x) = \frac{A}{\sqrt{2A^2-2A+1}} = \frac{\cos(c)}{\sqrt{2\cos^2(c) - 2\cos(c) + 1}} \quad (*)$$

$(*) \quad \cos^2(c) + \cos^2(c) - 2\cos(c) + 1 = \cos^2(c) + (\cos(c) - 1)^2$

$\cos(x) = \cos x \cos c + \sin x \sin c$

$(1-\cos c)\cos(x) = \sin x \sin c \qquad \operatorname{tg} x = \dfrac{1-\cos(c)}{\sin(c)}$

$$\boxed{x_0 = \operatorname{arctg}\left[\frac{1-\cos(c)}{\sin(c)}\right]}$$

$$A_1 = \int_0^{x_0}[\cos(x) - \cos(x-c)]\,dx = +\sin(x)\Big|_0^{x_0} - \int_0^{x_0}\cos(x-c)\,dx = \left| \begin{matrix} u = x-c \\ du = dx \\ x=0 \quad u=-c \\ x = x_0 \quad u = x_0 - c \end{matrix} \right| =$$

$$= \sin(x_0) - \sin(0) - \int_{-c}^{x_0-c}\cos(u)\,du = \sin(x_0) - \sin(x_0-c) + \sin(-c)$$

$A_1 = \sin(x_0) - \sin(x_0 - c) - \sin(c)$

$x_0 = \frac{\pi}{4} \qquad A_1 = \frac{\sqrt{2}}{2} - \sin\left(\frac{\pi}{4} - \frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right) = \frac{\sqrt{2}}{2} + \sin\frac{\pi}{4} - 1 = \sqrt{2} - 1$

$c = \frac{\pi}{2}$

$y = \cos(x-c) \qquad x - c = \arccos y \qquad \boxed{x_1 = c + \arccos(y)}$

$x_2 = \pi$

$$A_2 = \int_c^{\frac{\pi}{2}+c}[1 - \cos(x-c)]\,dx + \underbrace{\left[\pi - \left(\frac{\pi}{2} + c\right)\right] \cdot 1}_{(*)}$$

$\left| \begin{matrix} u = x-c & du = dx \\ x = c & u = 0 \\ x = \frac{\pi}{2}+c & u = \frac{\pi}{2} \end{matrix} \right|$

$$A_2 = (*) + x\Big|_c^{\frac{\pi}{2}+c} - \int_c^{\frac{\pi}{2}+c}\cos(x-c)\,dx = \frac{\pi}{2} + c - c - \int_0^{\pi/2}\cos(u)\,du = \frac{\pi}{2} - \sin(u)\Big|_0^{\pi/2}$$

$A_2 = (*) + \frac{\pi}{2} - \sin\left(\frac{\pi}{2}\right) = \pi - \frac{\pi}{2} - c + \frac{\pi}{2} - 1 \qquad \pi - c - 1$

$A_1 = A_2 \Rightarrow \quad \boxed{\sin(x_0) - \sin(x_0 - c) - \sin(c) = \pi - c - 1} \qquad c = ?$

$$\boxed{x_0 = \operatorname{arctg}\left[\frac{1-\cos(c)}{\sin(c)}\right]}$$

$$\sin(x_0) - \sin(x_0 - c) - \sin(c) = \pi - c - 1$$

$$\frac{\sin(x_0)}{\cos(x_0)} = \tan x_0 = \frac{1-\cos(c)}{\sin(c)} \qquad \sin(c) = (1-\cos(c))\frac{\cos(x_0)}{\sin(x_0)}$$

$$\sin(x_0) - \sin(x_0)\cos(c) + \sin(c)\cos(x_0) - \sin(c) = \pi - c - 1$$

$$\sin(x_0)(1-\cos(c)) + \cos(x_0)\sin(c) = \pi - c - 1 + \sin(c)$$

$$\cos(x_0)\left[tg(x_0)(1-\cos(c)) + \sin(c)\right] = \pi - c - 1 + \sin(c)$$

$$\cos(x_0)\left[\frac{[1-\cos(c)]^2}{\sin(c)} + \frac{\sin^2(c)}{1}\right] = \pi - c - 1 + \sin(c)$$

$$\cos(x_0)\,\frac{1 - 2\cos(c) + \cos^2(c) + \sin^2(c)}{\sin(c)} = \pi - c - 1 + \sin(c)$$

$$\cos(x_0)\,\frac{2(1-\cos(c))}{\sin(c)} = 2\cos(x_0)\, tg\, x_0 = 2\cos(x_0)\frac{\sin(x_0)}{\cos(x_0)}$$

$$2\sin(x_0) = \pi - c - 1 + \sin(c) \qquad \boxed{x_0 = \arcsin\frac{\pi - c - 1 + \sin(c)}{2}}$$

$$2\sin(x_0) - \sin(c) = \pi - c - 1$$

$$2\sin(x_0) - (1-\cos(c))\frac{\cos(x_0)}{\sin(x_0)} = \pi - c - 1$$

$$\frac{2\sin^2(x_0) - \cos(x_0) + \cos(c)\cdot\cos(x_0)}{\sin(x_0)} = (\pi - c - 1$$

$$\boxed{\mathrm{arctg}\frac{1-\cos(c)}{\sin(c)} = \arcsin\frac{\pi - c - 1 + \sin(c)}{2}}$$

$$\cos(x_0) = \frac{\cos(c)}{\sqrt{2\cos^2(c) - 2\cos(c) + 1}}$$

$$\sqrt{1-\cos^2(x_0)} = \frac{\pi - c - 1 + \sin(c)}{2}$$

$$1 - \cos^2(x_0) = \frac{(\pi - c - 1 + \sin(c))^2}{4}; \quad 1 - \frac{\cos^2(c)}{2\cos^2(c) - 2\cos(c) + 1} =$$

$$= \frac{2\cos^2(c) - 2\cos(c) + 1 - \cos^2(c)}{2\cos^2(c) - 2\cos(c) + 1} = \frac{(\pi - c - 1 + \sin(c))^2}{4}$$

$$\int_0^{c/2}[\cos x - \cos(x-c)]\,dx = -\int^{\pi}\cos(x-c)\,dx \qquad \begin{matrix} x = \frac{\pi}{2}+c & u = \frac{\pi}{2} \\ x = \pi & u = \pi - c\end{matrix}$$

$$\sin(x)\Big|_0^{c/2} - \int_0^{c/2}\cos(x-c)\,dx = -\int_{\frac{\pi}{2}+c}^{\pi}\cos(x-c)\,dx \qquad \begin{matrix} x - c = u \\ du = dx \\ x = 0 \quad u = -c \\ x = \frac{c}{2} \quad u = -\frac{c}{2}\end{matrix}$$

$$\sin\left(\frac{c}{2}\right) - \int_{-c}^{-c/2}\cos(u)\,du = -\sin(u)\Big|_{\pi/2}^{\pi - c}$$

$$\sin\left(\frac{c}{2}\right) - \sin(u)\Big|_{-c}^{-c/2} = -\sin(\pi - c) + 1 \; ; \quad \sin\left(\frac{c}{2}\right) - \left(-\sin\left(\frac{c}{2}\right) + \sin(c)\right) = -\sin(\pi - c) + 1$$

$$2\sin\left(\frac{c}{2}\right) - \sin(c) = -\sin(c) + 1$$

$$\sin\left(\frac{c}{2}\right) = \frac{1}{2} \quad \Rightarrow \frac{c}{2} = \frac{\pi}{6} \quad \boxed{c = \frac{\pi}{3}}$$

$$\sin(x_0) - \sin(x_0 - c) - \sin(c) = 1 - \sin(c)$$

$$\sin(x_0) - \sin(x_0 - c) = 1$$

$$\sin(x_0) - \sin(x_0)\cdot\cos(c) + \sin(c)\cdot\cos(x_0) = 1$$

$$\sin(x_0)\,(1 - \cos(c)) + \sin(c)\cdot\cos(x_0) = 1$$

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(49) $\frac{x}{x^2+1} = mx \qquad 1 = mx^2 + m \; ; \; mx^2 + m - 1 = 0$

$$x^2 = \frac{1-m}{m} \qquad x_0 = \pm\sqrt{\frac{1-m}{m}} \qquad \boxed{0 < m < 1}$$

$$y_1 = mx \; ; \qquad y_2 = \frac{x}{x^2+1}$$

$$A = 2\int_0^{\sqrt{\frac{1-m}{m}}} \left(\frac{x}{x^2+1} - mx\right) dx \; ; \quad A_1 = \left| \begin{array}{l} x^2+1 = u \\ 2x\,dx = du \\ x = 0 \quad u = 1 \\ x = x_0 \quad u = x_0^2+1 \end{array} \right| = \int_1^{x_0^2+1} \frac{du}{u} = \ln(u)\Big|_1^{x_0^2+1}$$

$$\boxed{A_1 = \ln\left(\left(\sqrt{\frac{1-m}{m}}\right)^2 + 1\right) - 0} \qquad A_1 = \ln\left(\frac{1-m}{m} + 1\right) = \ln\left(\frac{1-m+m}{m}\right) = \ln\left(\frac{1}{m}\right)$$

$$A_2 = 2\int_0^{\sqrt{\frac{1-m}{m}}} mx\,dx = 2m\,\frac{x^2}{2}\Big|_0^{\sqrt{\frac{1-m}{m}}} = m\left(\frac{1-m}{m} - 0\right) = \underline{1-m}$$

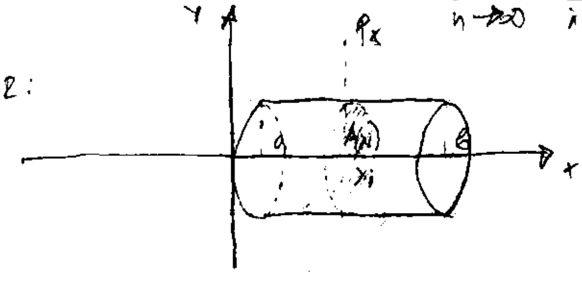
$$A = A_1 - A_2 = \ln\left(\frac{1}{m}\right) - (1-m) = \ln\left(\frac{1}{m}\right) - 1 + m = \underline{m - \ln(m) - 1}$$

VOLUMES

$$V(S_i) \approx A(x_i^*)\Delta x \; ; \qquad V = \sum_{i=1}^{n} A(x_i^*)\Delta x$$

Def. $x = a \ldots b \qquad V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\,dx$

CYLINDER:



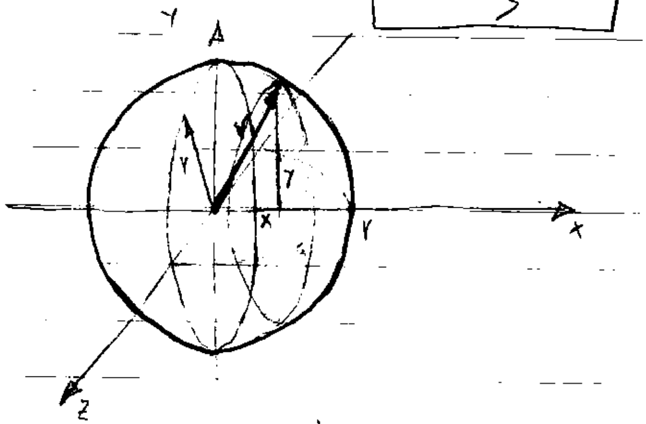
$A(x) = A = r^2\pi \Rightarrow$ CIRCULAR CYLINDER

$$V = \int_a^b A\,dx = A(b-a) = A \cdot h$$

$$\boxed{V_C = A \cdot h = r^2\pi h}$$

Ex. 1 Show that the volume of a sphere is:

$$V = \frac{4}{3}\pi r^3$$



$$V = 2\int_0^r A(x)\,dx$$

$$A(x) = y^2 \cdot \pi$$

$$y = \sqrt{r^2 - x^2}$$

$$A(x) = \pi (r^2 - x^2)$$

$$V = 2\int_0^r \pi (r^2 - x^2)\,dx = 2\pi \left(r^2 x \Big|_0^r - \frac{x^3}{3}\Big|_0^r \right)$$

$$V = 2\pi \left(r^3 - \frac{r^3}{3} \right) = 2\pi \, \frac{3r^3 - r^3}{3} = 2\pi \cdot \frac{2r^3}{3} = \frac{4\pi r^3}{3}$$

$$z^2 = R^2 - x^2 - y^2 \qquad z = \sqrt{R^2 - x^2 - y^2}$$

$$\int_0^R \underbrace{\int_0^R \sqrt{R^2 - x^2 - y^2}\,dx}_{I_1}\,dy$$

$$I_1 = \int_0^R \sqrt{(R^2 - y^2) - x^2}\,dx = \frac{R^2}{2} \arcsin\left(\frac{x}{R}\right) + \frac{x}{2}\sqrt{R^2 - x^2}$$

ex. 2 $y = \sqrt{x}$ $\qquad A(x) = y^2 \cdot \pi = x \cdot \pi$

$$\int_0^1 A(x)\,dx = \int_0^1 x\pi\,dx = \frac{x^2 \pi}{2}\Big|_0^1 = \frac{\pi}{2}$$

ex. 3 $y = x^3 \quad y = 8 \quad x = 0$

$$x = y^{1/3} = \sqrt[3]{y}$$

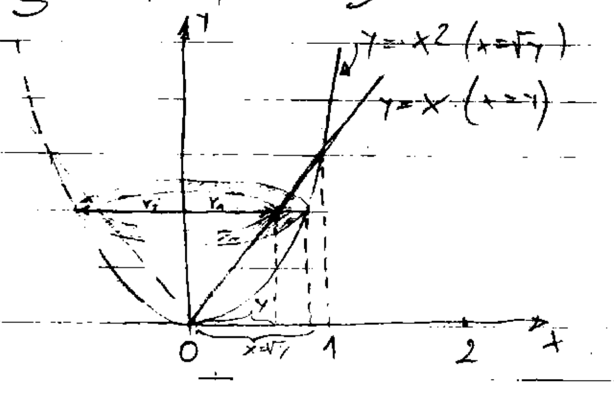
$$V = \int_0^8 A(y)\,dy \qquad A(y) = x^2 \pi = \sqrt[3]{y^2}\,\pi$$

$$V = \int_0^8 \sqrt[3]{y^2}\,\pi\,dy = \pi \frac{y^{\frac{2}{3}+1}}{\frac{2}{3}+1}\Bigg|_0^8 = 3\pi \frac{\sqrt[3]{y^5}}{5}\Bigg|_0^8 = \frac{3\pi}{5}\, y\sqrt[3]{y^2}\Bigg|_0^8$$

$$V = \frac{3\pi}{5}\, 8\sqrt[3]{64} = \frac{3\pi}{5} \cdot 8 \cdot 4 = \frac{3 \cdot 32\pi}{5} = \frac{96}{5}\pi$$

ex. 4

ROTATED BY (Y)



$$y_1 = x, \qquad y_2 = x^2$$

$$A = (r_2^2 - r_1^2)\,\pi$$

$$x_2 = \sqrt{y_2} = r_2 \qquad x_1 = y_1 = r_1$$

$$A(y) = \left[(\sqrt{y})^2 - y^2 \right] \pi$$

$$V = \pi \int_0^1 (y - y^2)\, dy = \pi \left(\frac{y^2}{2} - \frac{y^3}{3} \right)\Bigg|_0^1 = \pi \left(\frac{1}{2} - \frac{1}{3} \right) = \frac{3-2}{6}\pi = \frac{\pi}{6}$$

$A(x) = (x^2 - x^4)\pi$ ROTATED BY x

$$V = \int_0^1 (x^2 - x^4)\pi\, dx = \pi \left(\frac{x^3}{3} - \frac{x^5}{5} \right)\Bigg|_0^1 = \pi \frac{5-3}{15} = \frac{2\pi}{15}$$

ex. 5 $A(x) = (r_2^2 - r_1^2)\pi$

$y_1 = x$; $y_2 = x^2$

$r_2 = 2 - y_2 = 2 - x^2$

$r_1 = 2 - y_1 = 2 - x$

$$V = \int_0^1 A(x)\, dx = \int_0^1 \left[(2 - x^2)^2 - (2 - x)^2 \right] dx = \frac{8\pi}{15}$$

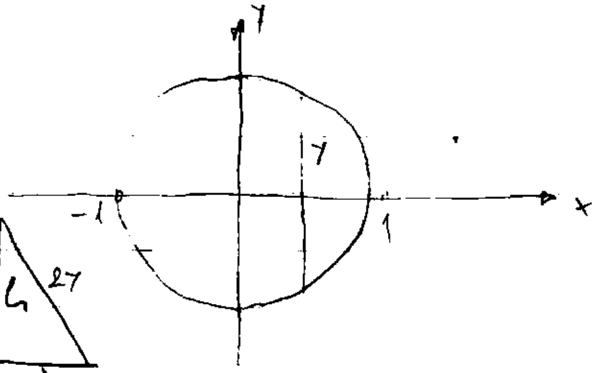
ex. 6 $A(y) = (r_2^2 - r_1^2)\pi$

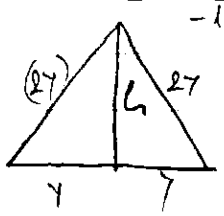
$r_1 = 1 + y$

$r_2 = 1 + \sqrt{y}$

$$V = \int_0^1 \left[(1 + \sqrt{y})^2 - (1 + y)^2 \right] dy = \frac{\pi}{2}$$

ex. 7





$x^2 + y^2 = 1 \qquad y = \sqrt{1 - x^2}$

$A(x) = 2y \cdot h/2 = y \cdot h$

~~$h^2 = y^2 - \left(\frac{y}{2}\right)^2 = y^2 - \frac{y^2}{4}$~~

~~$h^2 = \frac{3y^2}{4} \Rightarrow h = \frac{\sqrt{3}}{2} \cdot y$~~

$h^2 = (2y)^2 - y^2 = 4y^2 - y^2 = 3y^2$

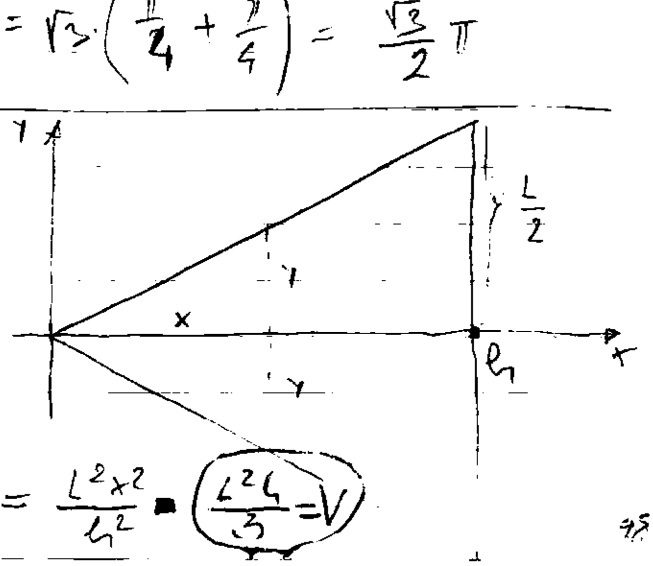
$A(x) = y \cdot h = \sqrt{3} y \cdot y = \sqrt{3}\, y^2 = \sqrt{3} \left(\sqrt{1 - x^2} \right)^2 = \sqrt{3} \cdot (1 - x^2)$

$$V = \int_{-1}^{1} A(x)\, dx = \sqrt{3} \int_{-1}^{1} \sqrt{1 - x^2}\, dx = \sqrt{3} \left(\frac{x}{2} \sqrt{1 - x^2} + \frac{1}{2} \arcsin(x) \right)\Bigg|_{-1}^{1}$$

$$V = \sqrt{3} \left[\left(\frac{1}{2} \sqrt{1-1} + \frac{1}{2} \cdot \frac{\pi}{2} \right) - \left(0 + \frac{1}{2} \left(-\frac{\pi}{2} \right) \right) \right] = \sqrt{3} \cdot \left(\frac{\pi}{4} + \frac{\pi}{4} \right) = \frac{\sqrt{3}}{2} \pi$$

$$V = \sqrt{3} \int_{-1}^{1} (1 - x^2)\, dx = \underline{\frac{4}{3}\sqrt{3}}$$

ex. 8

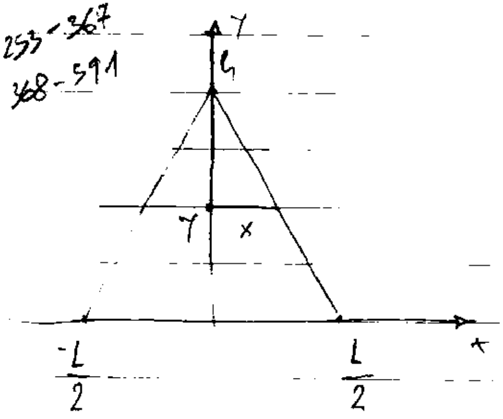


$$V = \int_0^h A(x)\, dx$$

$\frac{h}{x} = \frac{L/2}{y} \qquad y = \frac{L}{2h} \cdot x$

$$A(x) = 2y \cdot 2y = 4y^2 = 4 \cdot \frac{L^2}{4h^2} x^2 = \frac{L^2 x^2}{h^2} \rightarrow \boxed{\frac{L^2 h}{3} = V}$$

253 - 367
368 - 591



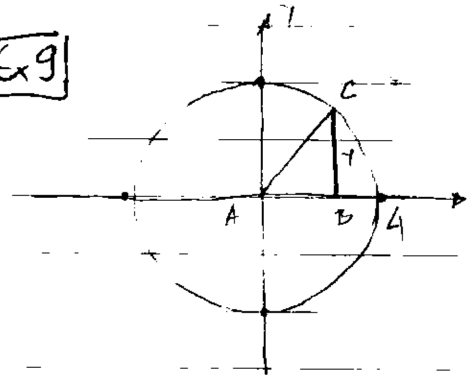
$$\frac{\frac{L}{2}}{x} = \frac{h}{(h-y)} \qquad \frac{L}{2x} = \frac{h}{(h-y)}$$

$$x = \frac{L(h-y)}{2h} \qquad A(x) = 2x \cdot 2x = 4x^2.$$

$$A(x) = 4 \cdot \frac{L^2(h-y)^2}{4h^2} = \frac{L^2(h-y)^2}{h^2}.$$

$$V = \int_0^h \frac{L^2(h-y)^2}{h^2}\,dy = \frac{L^2h}{3}$$

Ex 9



$$A(x) = y \cdot h/2 = h \cdot \frac{1}{2}\sqrt{4^2 - x^2}$$

$$\frac{h}{y} = tg\,\alpha \;; \quad \alpha = 30°$$

$$\frac{h}{y} = tg(30) = \frac{\sin 30}{\cos 30} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{3}$$

$$h = \frac{\sqrt{3}}{3} \cdot y$$

$$A(x) = \frac{1}{2}y \cdot h = \frac{1}{2}\frac{\sqrt{3}}{3}y^2 = \frac{1}{2}\frac{\sqrt{3}}{3}\left(\sqrt{4^2 - x^2}\right)^2 = \frac{1}{2}\frac{\sqrt{3}}{3}(16 - x^2)$$

$$V = \frac{\sqrt{3}}{3}\int_{-4}^{4}(16 - x^2)\,dx = \frac{128}{9}\sqrt{3}$$

Exercises

exc 7

$$y_1 = x^2 \qquad y_2^2 = x \qquad y_2 = \sqrt{x} \;; \quad A(x) = (r_2^2 - r_1^2)\pi$$

$$\pi\int_0^1 (x - x^4)\,dx = \frac{3\pi}{10}$$

exc. 8

$$y_2 = 1 \;; \quad y_1 = \sec(x) \;; \qquad A(x) = (y_1^2 - y_2^2)\pi = (\sec^2(x) - 1)\pi$$

$$V = -\pi\int_{-1}^{1}(1 - \sec^2(x))\,dx = -\pi\int_{-1}^{1}\left(1 - \frac{1}{\cos^2(x)}\right)dx = \pi\int_{-1}^{1}\frac{1 - \cos^2(x)}{\cos^2(x)}\,dx$$

$$V = \pi\int tg^2(x)\,dx$$

$$(\sec(x))' = \left(\frac{1}{\cos(x)}\right)' = (-1\cos^{-1-1}(x))(-\sin(x))$$

$$(\sec(x))' = \frac{\sin(x)}{\cos^2(x)} = \sec(x)\,tg(x)$$

$$(tg(x))' = \left(\frac{\sin(x)}{\cos(x)}\right)' = \frac{\cos(x)\cdot\cos(x) + \sin(x)\sin(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$$

$$V = +2\pi\int_0^1(\sec^2(x) - 1)\,dx = 2\pi\,(tg(x) - x)\Big|_0^1 = 2\pi\,(tg(1) - 1)$$

exc. 9 $y_1 = \sqrt{x}$; $y_2 = \frac{1}{2}x$; $x_1 = y^2$; $x_2 = 2y$

$\sqrt{x} = \frac{1}{2}x \quad ()^2 \qquad x = \frac{1}{4}x^2 \qquad \frac{1}{4}x^2 - x = 0 \qquad x^2 - 4x = 0$

$x(x-4) = 0 \qquad x = 0 \qquad x = \underline{4} \Rightarrow y(4) = \sqrt{4} = \underline{2}$

$A(y) = (r_2^2 - r_1^2)\pi \quad , \quad A(y) = [(2y)^2 - (y^2)^2]\pi$

$V = \pi\int_0^2 (4y^2 - y^4)\,dy \qquad V = \pi\int_0^2 (4y^2 - y^4)\,dy = \pi\left(4\frac{y^3}{3} - \frac{y^5}{5}\right)\Big|_0^2$

$V = \pi\left(4\cdot\frac{8}{3} - \frac{32}{5}\right) = \pi\frac{32\cdot5 - 32\cdot3}{15} = \frac{64\pi}{15}$

exc. 11 $A(x) = (r_1^2 - r_2^2)\pi \qquad y_2 = \sqrt{x}$; $y_1 = x$

$A(x) = [(1-x)^2 - (1-\sqrt{x})^2]\pi$

$V = \int_0^1 [(1-x)^2 - (1-\sqrt{x})^2]\pi\,dx = \frac{\pi}{6}$

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0713573353

exc. 17 $x_1 = \sqrt{y}$; $x_2 = y^2 \qquad y_1 = x^2$; $y_2 = \sqrt{x}$

$A(y) = [(1 + \sqrt{y})^2 - (1 + y^2)]\pi \qquad \int_0^1 A(y)\,dy =$

exc 18 $A = (r_2^2 - r_1^2)\pi \qquad r_2 = 4$; $r_1 = y\Big|_{y=2..4}$; $r_2 = 2\Big|_{y=0..2}$

$A(y) = (16 - y^2)\pi \qquad y = 2..4$

$V = \int_0^2 (16-4)\pi\,dy + \int_2^4 (16 - y^2)\pi\,dy$

ROTATED BY $x = 1$:

• $y = 0..2 \qquad r_2 = 4 - 1 = 3$; $r_1 = 2 - 1 = 1$

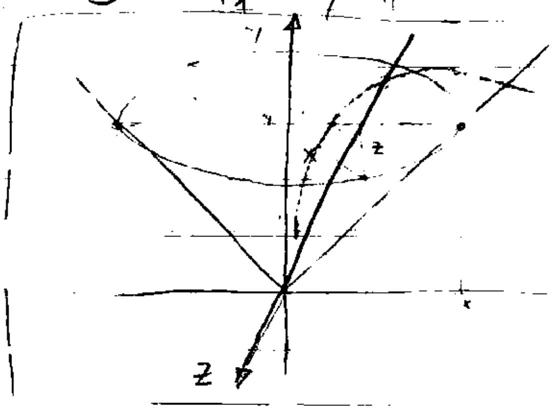
$V_1 = \int_0^2 (9 - 1)\pi\,dy = 8\pi y\Big|_0^2 = 16\pi$

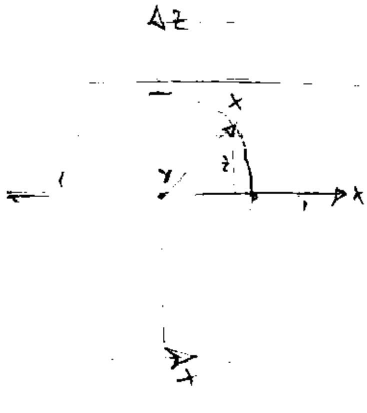
• $y = 2..4 \qquad r_2 = 4 - 1 = 3 \qquad r_1 = y - 1$

$V_2 = \pi\int_2^4 [9 - (y-1)^2]\,dy$

$V = V_1 + V_2$

$V = 76\frac{\pi}{3}$

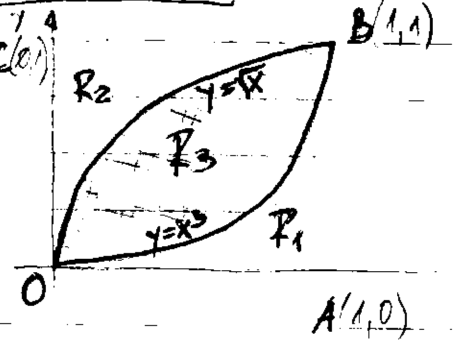




MAPLE VOLUME_OF_REVOLUTION

VolumeOfRevolution(x^2+1, $x=0..1$)

Exercise 19



$y_1(x) = \sqrt{x}$

$y_2(x) = x^3$

21 $y_1 = x^3$ $\quad y_2 = (-x+2)^3$

$x_1 = \sqrt[3]{y}$ $\quad y^{1/3} = -x+2$

$x_2 = 2 - \sqrt[3]{y}$

$A(y) = r^2\pi$; $\quad 2r = 2 - \sqrt[3]{y} - \sqrt[3]{y} = 2(1-\sqrt[3]{y})$; $r = 1-\sqrt[3]{y}$

circular cylinder : $r^2\pi \cdot h = 1^2 \cdot \pi \cdot 1 = \pi$

(19) $= \frac{\pi}{7}$ $\quad$ (23) $= \frac{\pi}{2}$ $\quad$ (27) $= \frac{5\pi}{14}$ $\quad$ $\frac{\pi}{7} + \frac{\pi}{2} + \frac{5\pi}{14} = \frac{2+7+5}{14}\pi = \pi$

~~(30) $= \frac{10\pi}{21}$ $\quad$ (22) $= \frac{5\pi}{14}$~~ $\quad$ (26) $= \frac{\pi}{6}$ $\quad$ (30) + (22) + (26) $= \pi$

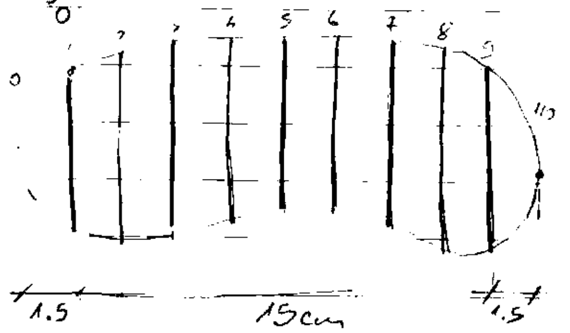
31 $\tan^3(x) = 1$ $\quad \tan(x) = 1$ $\quad x = \frac{\pi}{4}$

43 $\pi \int_0^1 (y^4 - y^8)\, dy$

$y_1 = \sqrt{x}$ $\quad x_1 = y_1^2$; $\quad y_2 = \sqrt[4]{x}$ $\quad x_2 = y^4$

44 ~~$\pi \int_0^{\pi/2} [(1+\cos x)^2 - 1^2]\, dx$~~

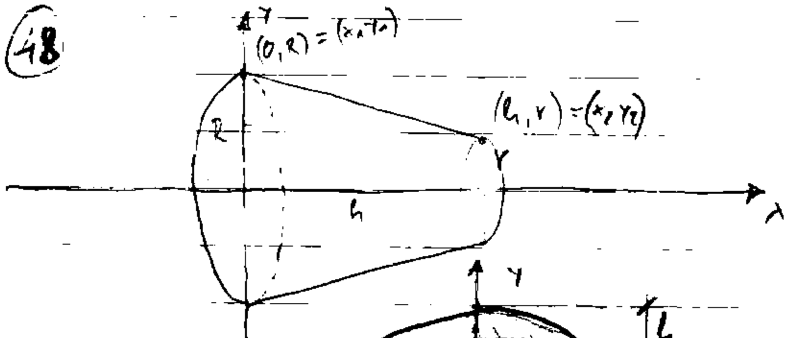
(45)



$n = 5$ $\quad \Delta x = \frac{b-a}{n} = \frac{15-0}{5} = 3cm$

$V = \sum_{i=1}^{5} A(x_i)\Delta x = 3 \cdot \sum_{i=1}^{5} A(x_i)$

(48)

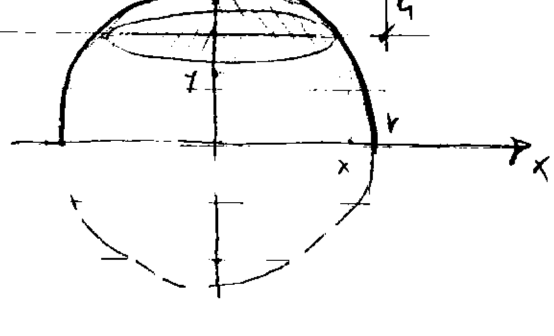


$y - R = \frac{r-R}{h-0}(x-0)$

$y = \frac{r-R}{h}x + R$

$V = \int_0^h \pi y^2(x)\, dx$

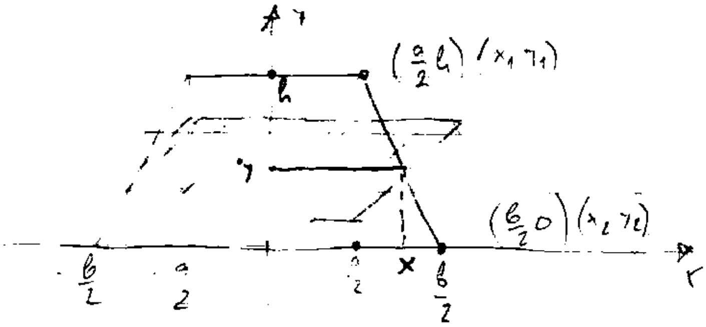
(49)



$y^2 + x^2 = r^2$

$x = \sqrt{r^2 - y^2}$

$y = \sqrt{r^2 - x^2}$



$$y - h = \frac{0-h}{\frac{b}{2}-\frac{a}{2}}\left(x - \frac{a}{2}\right)$$

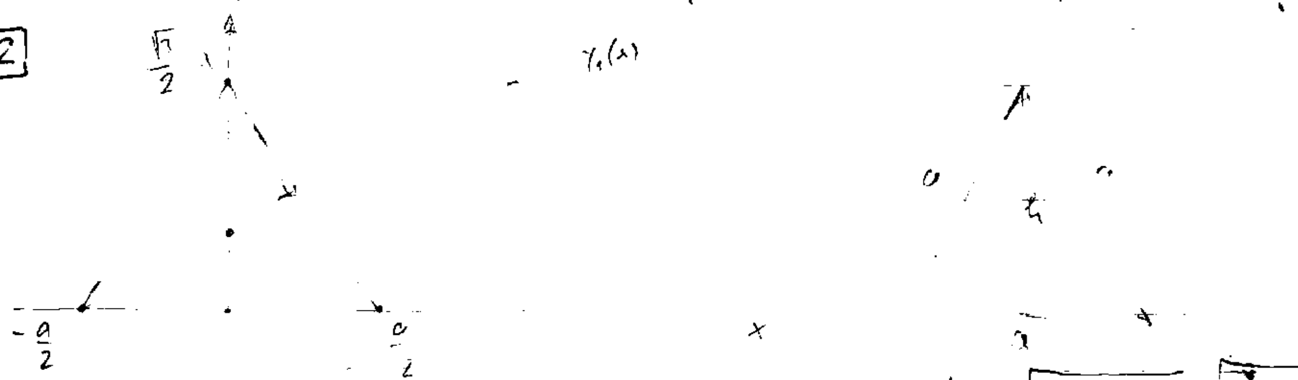
$$y - h = \frac{2h}{a-b}\left(x - \frac{a}{2}\right)$$

$$\boxed{x = \frac{a-b}{2h}(y-h) + \frac{a}{2}}$$

$$A(y) = (2x)(2x) = 4x^2 \qquad V = 4\int_0^h \left[\frac{a-b}{2h}(y-h) + \frac{a}{2}\right]^2 dy$$

$$\frac{a-b}{2h}y - \frac{a-b}{2} + \frac{a}{2} = \frac{a-b}{2h}y - \frac{a}{2} + \frac{b}{2} + \frac{a}{2} = \frac{a-b}{2h}y + \frac{b}{2}$$

52



$$h = \sqrt{a^2 - \frac{a^2}{4}} = \sqrt{\frac{3a^2}{4}}$$

$$h = \frac{\sqrt{3}}{2}a$$

$\boxed{a = 1}$ $\quad \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{3}{4} + \frac{1}{4} = 1$

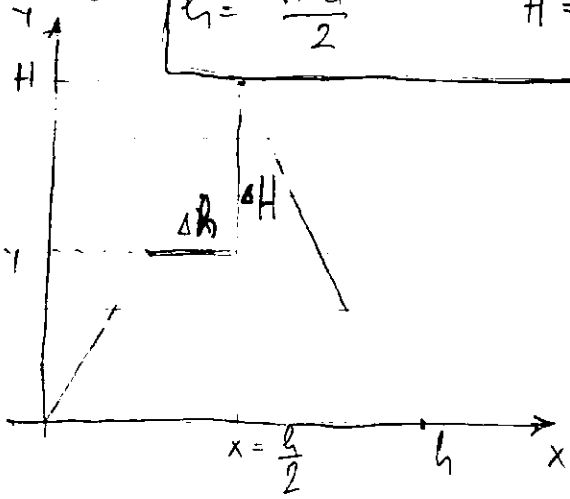
$$P = \frac{a \cdot h}{2} = \frac{a \cdot \frac{\sqrt{3}}{2}a}{2} = \frac{a^2\sqrt{3}}{4}$$

0,29 · 3
0,87

$$h = \frac{\sqrt{3}a}{2};\ H = \sqrt{h^2 - \left(\frac{h}{2}\right)^2} = \sqrt{\left(\frac{a\sqrt{3}}{2}\right)^2 - \left(\frac{a\sqrt{3}}{4}\right)^2} = a\sqrt{\frac{3}{4} - \frac{3}{16}} = a\sqrt{\frac{12-3}{16}} = \sqrt{\frac{9}{16}} = \frac{3a}{4}$$

$$\int_0^H A(y)\,dy \qquad y = \emptyset \qquad A(\emptyset) = \frac{a \cdot h}{2}$$

$$\boxed{h = \frac{\sqrt{3}a}{2} \qquad H = \frac{3a}{4}}$$



$$\frac{\Delta H}{\Delta h} = \frac{H}{h/2} \qquad \frac{\Delta H}{\Delta h} = \frac{2H}{h}$$

$$\Delta h = \frac{h}{2H}\Delta H = \frac{h}{2H}(H - y)$$

$$2\Delta h = \frac{\sqrt{3}\Delta a}{2} \qquad \Delta a = \frac{4\Delta h}{\sqrt{3}}$$

$$A(y) = \frac{\Delta a \cdot 2\Delta h}{2} = \Delta h \cdot \frac{4\Delta h}{\sqrt{3}} = \frac{4\sqrt{3}}{3}\Delta h^2$$

$$V = \frac{\sqrt{3}}{16}a^3 \qquad A(y) = \frac{4\sqrt{3}}{3}\left[\frac{h}{2H}(H-y)\right]^2$$

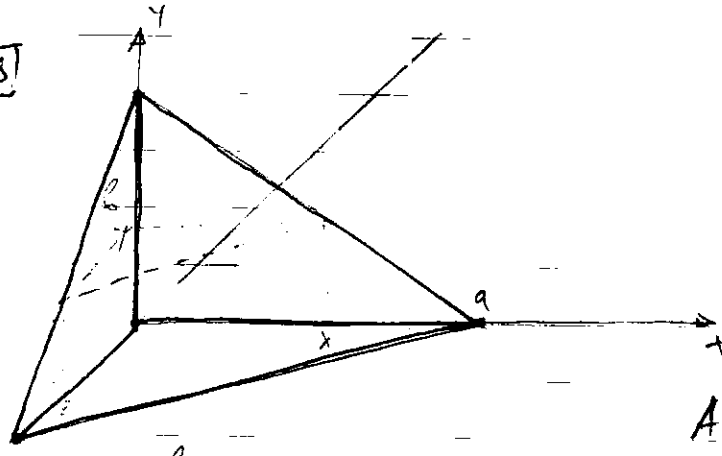
$\Delta h = \frac{h}{2H}(H-y)$; $\Delta h = \frac{h}{2}\left(1-\frac{y}{H}\right)$

$\Delta a = \frac{4\Delta h}{\sqrt{3}} = \frac{4}{\sqrt{3}}\frac{h}{2}\left(1-\frac{y}{H}\right) = \left| h = \frac{\sqrt{3}}{2}a \right| = \frac{4}{\sqrt{3}}\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2}a\left(1-\frac{y}{H}\right)$

$$\boxed{\Delta a = a\left(1-\frac{y}{H}\right)}$$

$$V = \int_0^H A(y)\,dy = \int_0^H \frac{4\sqrt{3}}{3}\left[\frac{h}{2H}(H-y)\right]^2 dy = \frac{\sqrt{3}}{16}a^3 = \frac{\sqrt{3}}{12}a^2\cdot H$$

53



$a=3$; $b=4$; $c=5$; [cm]

$V = ?$

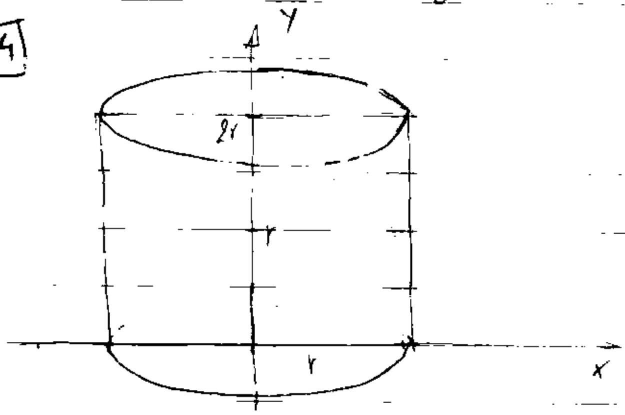
$\frac{a}{b} = \frac{x}{y} = \frac{3}{4}$; $x = \frac{3}{4}y$

$\frac{c}{b} = \frac{z}{y} = \frac{5}{4}$; $z = \frac{5}{4}y$

$A(y) = \frac{x\cdot z}{2} = \frac{1}{2}\frac{3}{4}\cdot\frac{5}{4}y^2 = \frac{15}{32}y^2$

$$V = \int_0^b A(y)\,dy = \int_0^4 \frac{15}{32}y^2\,dy = \frac{15}{32}\frac{y^3}{3}\Big|_0^4 = 5\,\frac{64}{32} = 2\cdot 5 = 10$$

54



$V = r^2\pi\cdot h = r^2\pi\, 2r = 2r^3\pi$

$x = r$

$$V = \int_0^{2r} x^2(y)\pi\,dy = \int_0^{2r} r^2\pi\,dy = r^2\pi\, y\Big|_0^{2r}$$

$V = 2r^3\pi$

PRESECITE NE SU KVADRATI !?

$x^2+y^2=r^2$ $\quad A(x) = (2y)(2y) = 4y^2 = 4(r^2-x^2)$

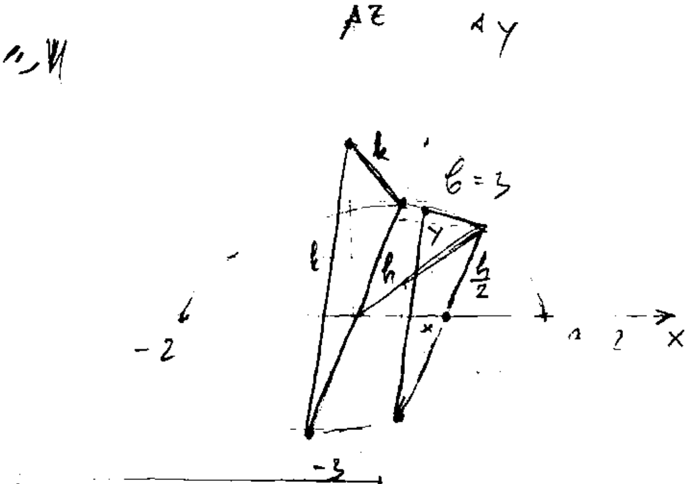
$$V = \int_{-r}^{r} A(x)\,dx = \int_{-r}^{r} 4(r^2-x^2)\,dx = 2\int_0^r 4(r^2-x^2)\,dx = \frac{16r^3}{3}$$

55 $\quad 9x^2+4y^2=36$ $\qquad 4y^2 = 36-9x^2$ $\qquad y = \sqrt{9-\frac{9x^2}{4}}$

$$\boxed{y = 3\sqrt{1-\left(\frac{x}{2}\right)^2}}$$

$\frac{9x^2}{36}+\frac{4y^2}{36} = 1$ $\qquad \frac{x^2}{4}+\frac{y^2}{9} = 1$

$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ $\qquad a=2$, $b=3$



$h^2 = k^2 + k^2$

$h = \sqrt{2}\,k$ $\quad P = k \cdot k / 2 = \frac{k^2}{2}$

$y = h/2 = \frac{\sqrt{2}\,k}{2}$

$y = 3\sqrt{1 - \left(\frac{x}{2}\right)^2}$

$\frac{\sqrt{2}}{2} k = 3\sqrt{1 - \left(\frac{x}{2}\right)^2}$

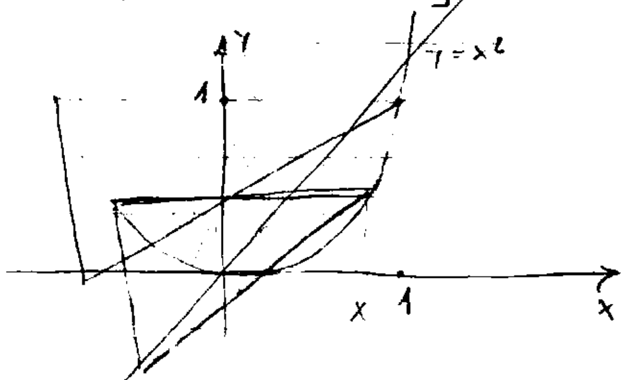
$k = \frac{6}{\sqrt{2}}\sqrt{1 - \left(\frac{x}{2}\right)^2} = 3\sqrt{2}\sqrt{1 - \left(\frac{x}{2}\right)^2}$

$k = 3\sqrt{2 - \frac{x^2}{2}}$

$A(x) = \frac{k^2}{2} = \frac{1}{2}\, 9\left(2 - \frac{x^2}{2}\right) = \frac{9}{4}(4 - x^2)$

$A(x) = 9\left(1 - \left(\frac{x}{2}\right)^2\right)$

(56) $R = \{(x,y) \mid x^2 \le y \le 1\}$



$y = x^2 \quad x = \sqrt{y}$

$A(y) = \frac{2x \cdot h}{2} = x \cdot h$

$h^2 + \left(\frac{a}{2}\right)^2 = a^2 \quad h^2 = a^2 - \frac{a^2}{4}$

$h = \frac{a\sqrt{3}}{2}$; $a = 2x$; $h = x\sqrt{3}$

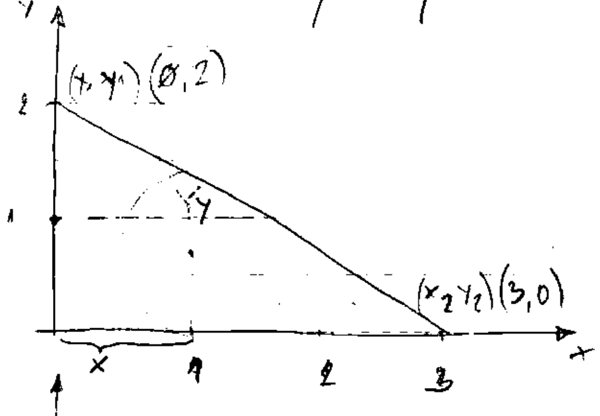
$A(y) = x(y) \cdot x(y)\, \sqrt{3}$

$A(y) = (\sqrt{y})^2\, \sqrt{3} = \sqrt{3}\, y$

$V = \int_0^1 \sqrt{3}\, y \, dy = \sqrt{3}\, \frac{y^2}{2}\Big|_0^1 = \frac{\sqrt{3}}{2}$

(57) $A(y) = a \cdot a = |a = 2x| = 4x^2 = 4y$; $V = \int_0^1 4y\,dy = 4\,\frac{y^2}{2}\Big|_0^1 = 2$

(58)

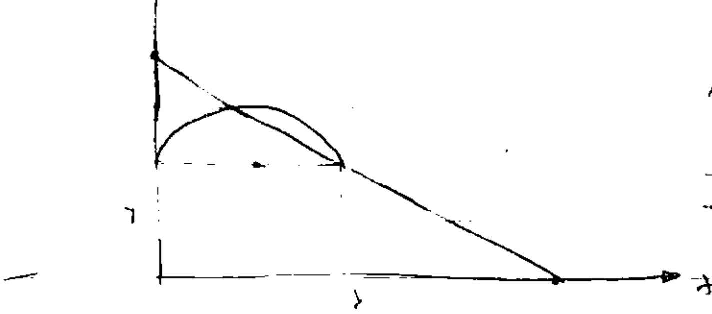


$y - 2 = \frac{0-2}{3-0}(x - 0)$

$y = -\frac{2}{3}x + 2 = -\frac{2}{3}(x - 3)$

$A(x) = \frac{y^2(x)}{4}\, \pi / 2$

$V = \frac{\pi}{8}\int_0^3 \left[-\frac{2}{3}(x-3)\right]^2 dx = 2\pi$

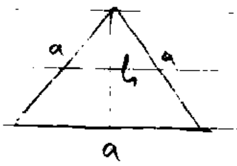


$A(y) = \left(\frac{x(y)}{2}\right)^2 \cdot \frac{\pi}{2}$

$\frac{2}{3}x = 2 - y \quad x = -\frac{3}{2}(y - 2)$

$V = \frac{\pi}{8}\int_0^2 \left[-\frac{3}{2}(y-2)\right]^2 dy = \frac{3\pi}{4}$

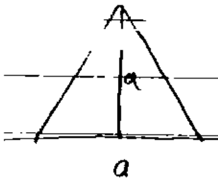
(59) $x = -\frac{3}{2}(y-2)$



$h = \sqrt{a^2 - \frac{a^2}{4}} = \frac{\sqrt{3}}{2} a \qquad P = a \cdot h/2$

$A(y) = x(y) \cdot h/2 = \frac{1}{2} x^2(y) \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4} x^2(y)$

$V = \int_0^2 \frac{\sqrt{3}}{4} x^2(y)\, dy = \frac{\sqrt{3}}{4} \int_0^2 \frac{9}{4}(y-2)^2\, dy =$

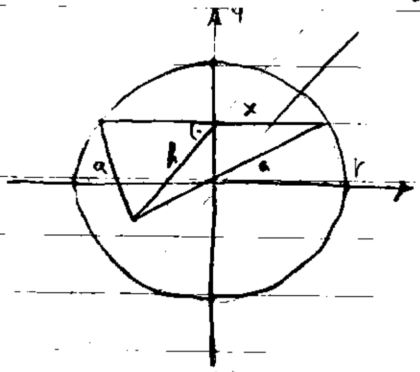


$A(y) = a \cdot h/2 = \frac{a^2}{2} = \frac{x^2(y)}{2}$

$\frac{1}{2} \int_0^2 x^2(y)\, dy = 3$

DRAGI JOVAN
265353
ŽIVE MASKOV

(60)



$x = \sqrt{r^2 - y^2}$

$A(y) = h \cdot x = h\sqrt{r^2 - y^2}$

$V = \int_{-r}^{r} h\sqrt{r^2 - y^2}\, dy = \frac{h \cdot r^2 \pi}{2}$

(61)

$x = R$

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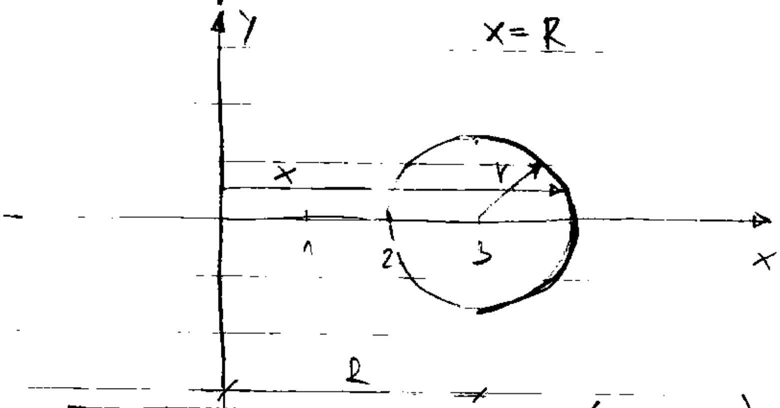


$y = \sqrt{r^2 - (x-R)^2}$

$r_2 = x = \sqrt{r^2 - (x-R)^2}$

$A_1(y) = (r_2^2 - r_1^2)\pi$

$r_1 = R$

$A_1 = (r^2 - (y-R)^2 - R^2)\pi$

~~$A_2(y) = (R^2 - r^2 + (y-R)^2)\pi$~~

~~$A(y) = A_1(y) + A_2(y) = [r^2 - (y-R)^2 - R^2 + R^2 - r^2 + (y-R)^2]\pi$~~

$A_2(y) = (r_2^2 - r_1^2)\pi; \quad r_2 = R; \quad r_1 = R - \sqrt{r^2 - (y-R)^2}$

$A_2(y) = \pi[R^2 - (R - \sqrt{r^2-(y-R)^2})^2] = \pi(R^2 - R^2 + 2R\sqrt{r^2-(y-R)^2} - (r^2 - (y-R)^2))$

$A(y) = A_1(y) + A_2(y) = \pi(r^2 - (y-R)^2 - R^2 + 2R\sqrt{r^2-(y-R)^2} - r^2 + (y-R)^2)\pi$

$A(y) = 2R\sqrt{r^2-(y-R)^2} - R^2$

$V = \int_{-r}^{r} A(y)\, dy = \int_{-r}^{r} (2R\sqrt{r^2-(y-R)^2} - R^2)\, dy$

$$y = \sqrt{r^2 - (x-R)^2} \qquad y^2 + (x-R)^2 = r^2$$

$$y^2 + x^2 - 2Rx + R^2 = r^2 \;\to\; (x-R)^2 = r^2 - y^2 \qquad x = R + \sqrt{r^2 - y^2}$$

$$\boxed{x(y) = R + \sqrt{r^2 - y^2}} \; ; \qquad A_1(y) = (x^2(y) - R^2)\pi$$

$$A_2(y) = [-(R - x(y))^2 + R^2]\pi \; ; \qquad A(y) = A_1(y) + A_2(y) = [x^2 - R^2 + R^2 - (R-x)^2]\pi$$

$$A(y) = \pi(x^2 - R^2 + 2Rx - x^2) = \pi(2Rx - R^2) = \pi\left(2R(R + \sqrt{r^2 - y^2}) - R^2\right)$$

$$\boxed{A(y) = \pi(R^2 + 2R\sqrt{r^2 - y^2})} \qquad V = \pi \int_{-r}^{r} (R^2 + 2R\sqrt{r^2 - y^2})\, dy$$

$$V = (2R^2 \cdot r + Rr^2\pi)\pi$$

OVA NE E DOBRO ZATOA:

$$x_1(y) = R + \sqrt{r^2 - y^2} \qquad x = f(x) = R + \sqrt{r^2 - y^2}$$

$$x_2(y) = R - \sqrt{r^2 - y^2} \qquad x = g(x) = R - \sqrt{r^2 - y^2}$$

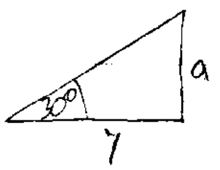
$$A(y) = (x_1^2 - x_2^2)\pi = \pi\left(R^2 + 2R\sqrt{r^2-y^2} + (r^2 - y^2) - R^2 + 2R\sqrt{r^2 - y^2} - (r^2 - y^2)\right)$$

$$A(y) = 4R\pi\sqrt{r^2 - y^2}$$

$$V = \int_{-r}^{r} 4R\pi\sqrt{r^2 - y^2}\, dy = 2\pi \int_{0}^{r} 4R\sqrt{r^2 - y^2}\, dy = 8\pi R \cdot \frac{r^2\pi}{4} = 2\pi^2 R r^2$$

62

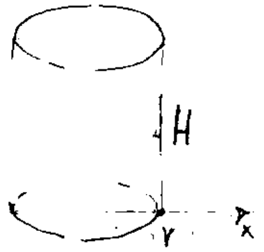
$$x = \sqrt{16 - y^2} \qquad tg(30^\circ) = \frac{a}{y} = \frac{\sin 30}{\cos 30} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{3}$$

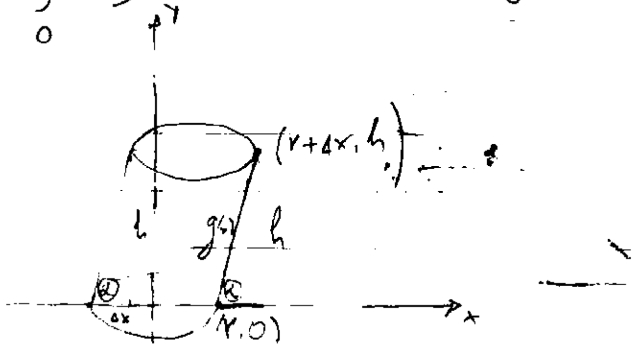


$$a = \frac{\sqrt{3}}{3} \cdot y \qquad A(y) = 2a \cdot x = 2\frac{\sqrt{3}}{3} y \cdot \sqrt{16 - y^2}$$

$$V = \int_0^4 \frac{2\sqrt{3}}{3} y \sqrt{16 - y^2}\, dy = \frac{128}{9}\sqrt{3}$$

63

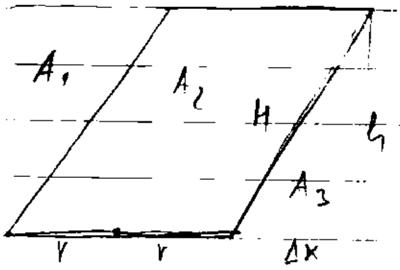




$$V = \int_0^h r^2\pi\, dy = r^2\pi y \Big|_0^h = r^2\pi h \qquad tg(\alpha) = \frac{h}{\Delta x} \qquad \Delta x = h / tg(\alpha)$$

$$g(x) = tg(\alpha)(x - r) = \frac{h}{\sqrt{H^2 - h^2}}(x - r) \qquad \frac{\sqrt{H^2 - h^2}}{h} y = x - r, \quad x = \frac{\sqrt{H^2 - h^2}}{h} \cdot y + r$$

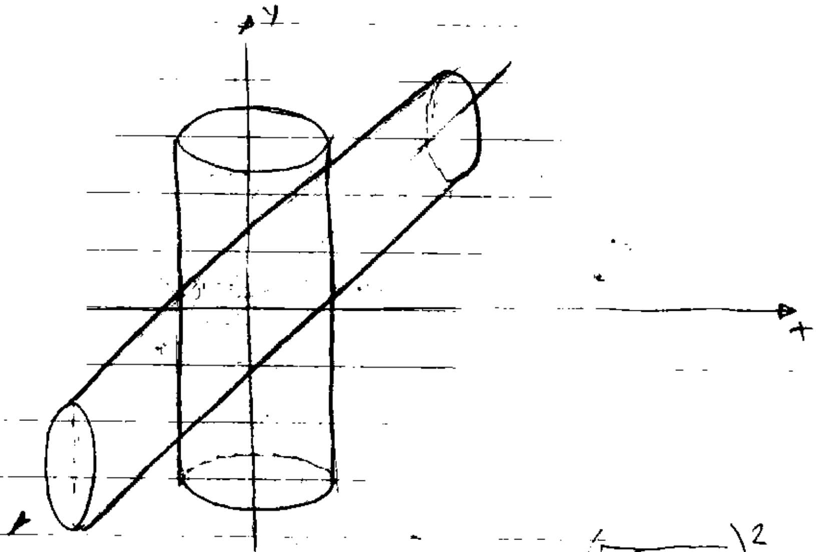
$$V = \int_0^h (x - r)^2 \pi\, dy = \pi \int_0^h \frac{H^2 - h^2}{h^2} \cdot y^2\, dy = \pi \frac{H^2 - h^2}{h^2} \frac{y^3}{3}\Big|_0^h = \frac{\pi}{3}(H^2 - h^2)h$$

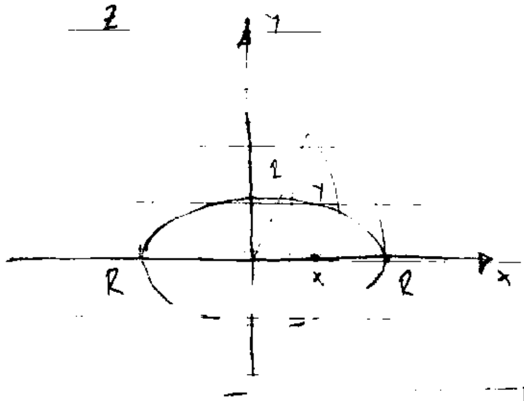


$$A = (\Delta x + 2r)\cdot h - 2\frac{\Delta x \cdot h}{2}$$

$$\boxed{A = \Delta x \cdot h + 2rh - \Delta x \cdot h = 2r\cdot h}$$

$$V = \int_0^h r^2\pi\, dy = r^2\pi h$$





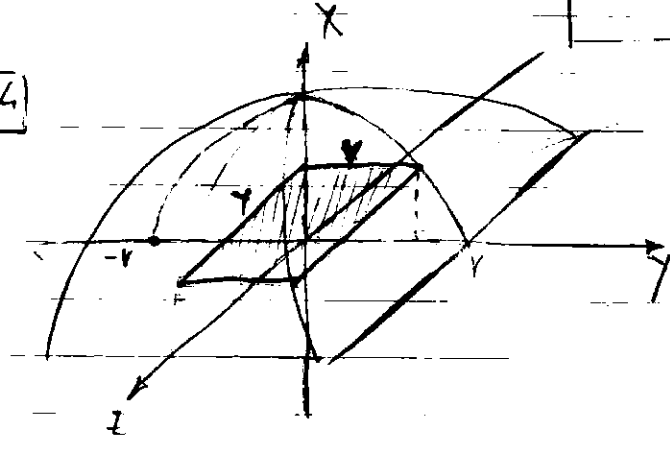
$$A(x) = \left(\sqrt{R^2 - x^2}\right)^2 \pi$$

$$\int_{-R}^{R} (R^2 - x^2)\pi\, dx = 2\int_0^R (R^2 - x^2)\pi\, dx$$

$$= 2\pi R^2 x \Big|_0^R - 2\pi \frac{x^3}{3}\Big|_0^R = 2\pi\left(R^3 - \frac{R^3}{3}\right)$$

$$= 2\pi \frac{2R^3}{3} = \frac{4\pi R^3}{3}$$

[c4]

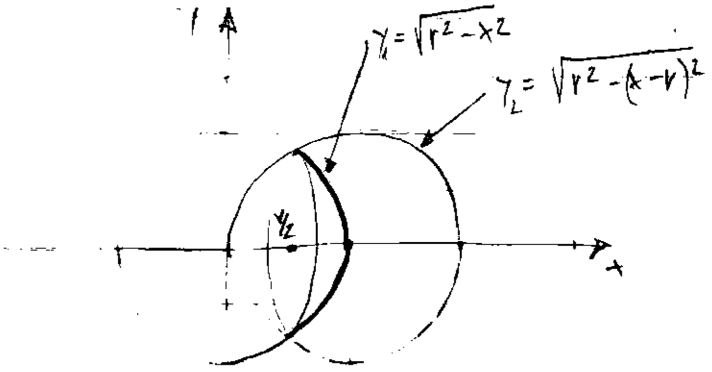


$$y = \sqrt{r^2 - x^2}$$

$$z = \sqrt{r^2 - y^2}$$

$$A(x) = 4\, y^2(x) = 4(r^2 - x^2)$$

$$V = \int_{-r}^{r} 4(r^2 - x^2)\, dx = \frac{16 r^3}{3}$$



$0 < x < \frac{r}{2}$ $\quad y_2 = \sqrt{r^2 - (x-r)^2}$

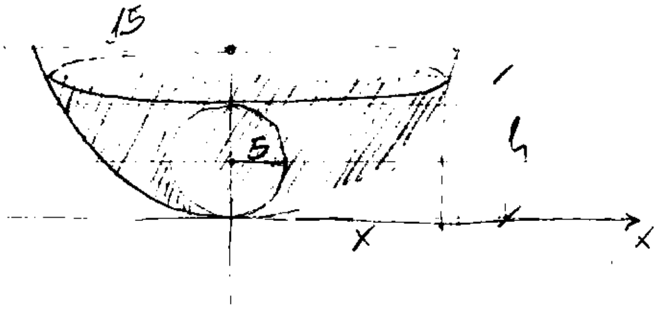
$\frac{r}{2} < x < r$ $\quad y_1 = \sqrt{r^2 - x^2}$

$$V = \int_0^{r/2} \pi \left(r^2 - (x-r)^2\right) dx + \int_{r/2}^{r} \pi \left(r^2 - x^2\right) dx$$

VOLUME OF THE CAP: $V_c = \frac{1}{3} \pi h^2 (3r - h) = \left/ h = \frac{r}{2} \right/ = \frac{1}{3} \pi \frac{r^2}{4} \left(3r - \frac{r}{2}\right)$

$V_c = \frac{r^2 \pi}{12} \cdot \frac{6r - r}{2} = \frac{r^2 \pi \cdot 5r}{24} = \frac{5\pi r^3}{24}$; $V = 2 V_c = \frac{5 \pi r^3}{12}$

66



$x = \sqrt{15^2 - (y-15)^2}$

if: $h \geq 10$ cm

$$V = \int_0^h \left(15^2 - (y-15)^2\right) \pi \, dy - \frac{4 \cdot 5^3 \pi}{3}$$

$\frac{4r^3\pi}{3}$ small sphere

$y = 30 + \sqrt{30^2 - x^2}$

$(y-30)^2 = 30^2 - y^2$ $\quad x = \sqrt{30^2 - (y-30)^2}$

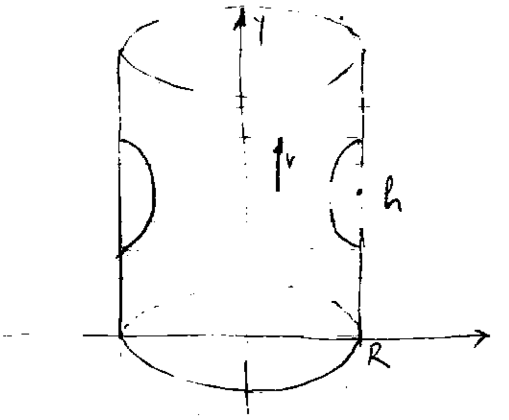
$V = -\frac{1}{3}\pi h^3 + 15\pi h^2 - \frac{500}{3}\pi$ $\qquad$ e.g.: $h = 30 \Rightarrow V(30) = \frac{13000}{3} \pi$ [cm^3]

if: $h \leq 20$ cm

$$V = \int_0^h \left(30^2 - (y-30)^2\right) \pi \, dy - \int_0^h \left(10^2 - (y-10)^2\right) \pi \, dy = \underline{\underline{40\pi h^2}}$$

eg. $V(8) = 640\pi$ [cm^3]

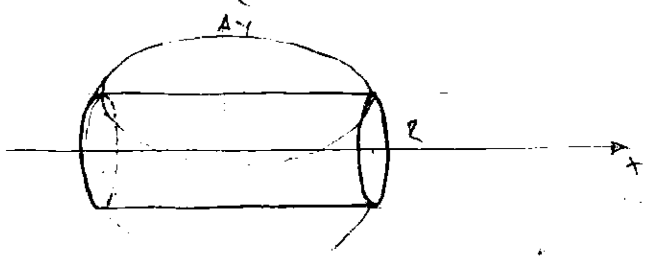
67



$V = V_1 + V_2$

$V_1 = R^2 \pi \cdot (h - 2r)$

$r^2 \pi \cdot 2R$
$\pi \cdot 6 =$

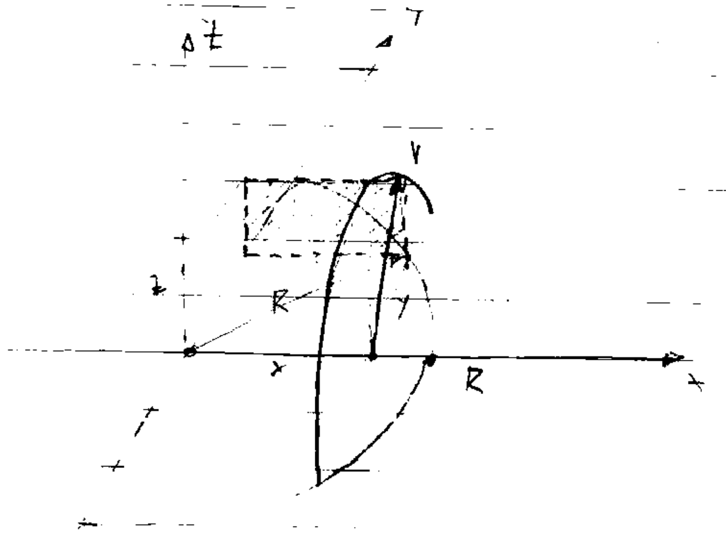


$V_2 = R^2 \pi \cdot 2r - r^2 \pi \cdot 2R$

$V = V_1 + V_2 = R^2 \pi h - R^2 \pi \cdot 2r + R^2 \pi 2r - r^2 \pi \cdot 2R = R^2 \pi h - r^2 \pi \cdot 2R$

$x = \sqrt{R^2 - y^2}$ $\qquad z = \sqrt{r^2 - y^2}$; $\quad A(y) = x \cdot z = \sqrt{R^2 - y^2} \sqrt{r^2 - y^2}$

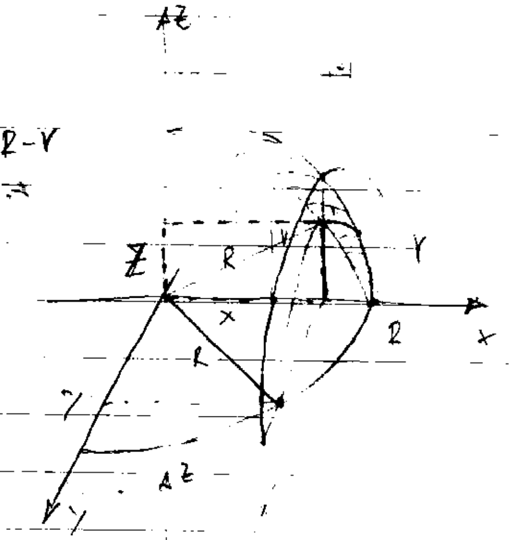
$$V = \int_{-r}^{r} A(y) \, dy = \int_{-r}^{r} \sqrt{R^2 - y^2} \sqrt{r^2 - y^2} \, dy =$$



$$A(z) = z \cdot x$$
$$z = \sqrt{r^2 - y^2}$$
$$x = \sqrt{R^2 - y^2}$$
$$V = \int_{-r}^{r} 4 \sqrt{r^2 - y^2}\,\sqrt{R^2 - y^2}\, dy$$
$$= 8 \int_{0}^{r} \sqrt{r^2 - y^2}\,\sqrt{R^2 - y^2}\, dy$$

[68]

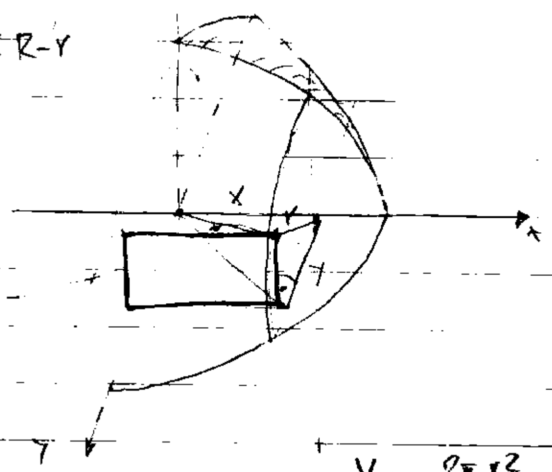
$x \geq R - r$



$$V = V_1 + V_2$$
$$V_1 = 2 \int_{R-r}^{R} \left(\sqrt{R^2 - x^2}\right)^2 \pi \, dx$$
$$V_1 = 2\pi r^2 \left(R - \frac{r}{3}\right) = 2\frac{\pi r^2}{3}(3R - r)$$

$x < R - r$

$R -$



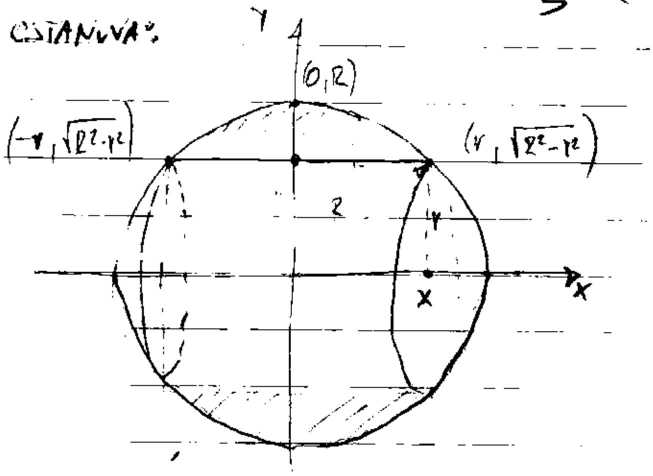
$$x = \sqrt{R^2 - r^2}$$
$$z = \sqrt{r^2 - y^2}$$
$$V_2 = \int_{-r}^{r} 4 \sqrt{R^2 - r^2} \cdot \sqrt{r^2 - y^2}\, dy =$$
$$= 8\sqrt{R^2 - r^2} \int_{0}^{r} \sqrt{r^2 - y^2}\, dy$$
$$V_2 = 2\sqrt{R^2 - r^2} \cdot r^2 \pi = 2x \cdot r^2 \pi$$

VOLUME OF CYLINDER

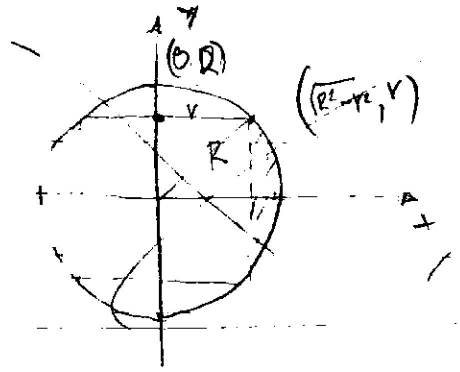
$$V = \frac{2\pi r^2}{3}(3R - r) + 2\sqrt{R^2 - r^2}\, r^2 \pi$$

REVIDENO!

• OSTANVVA:



$$y = \sqrt{R^2 - x^2} = r_2 \; ; \quad r_1 = r \; ;$$
$$A(x) = \pi(r_2^2 - r_1^2) = 2 \int^{\sqrt{R^2 - r^2}} \pi (R^2 - x^2 - r^2)\, dx$$
$$V = 2 \int_{0}^{\sqrt{R^2 - r^2}} \pi (R^2 - x^2 - r^2)\, dx$$
$$= \frac{4\pi}{3}(R^2 - r^2)^{3/2}$$

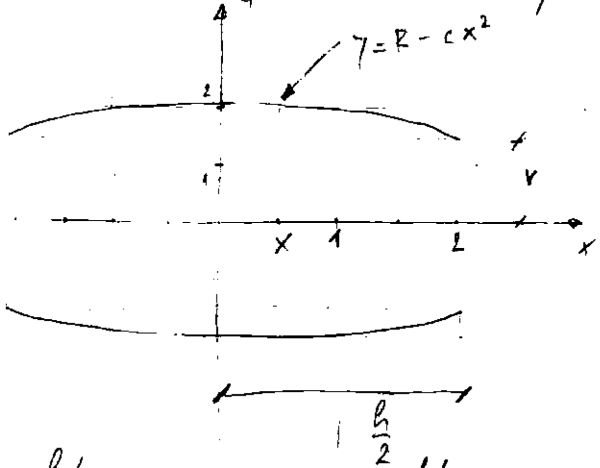


$x = \sqrt{R^2-x^2} = r_2$;

$$2\int_0^{\sqrt{R^2-r^2}} \pi\left(R^2-x^2\right)dx$$

69 $R = 2\,m$; $h = 2\,m$, $c = 0.5$;

$$\frac{y = R - cx^2}{r = R - d} \quad ; \quad d = \frac{ch^2}{4}$$



(a) $y\left(\frac{h}{2}\right) = R - c\,\frac{h^2}{4} = r$

$r = R - c\,\frac{h^2}{4} = R - d$

$d = \frac{ch^2}{4}$

(b) $A(x) = \pi y^2$; $c\cdot x^2 = R - y$

$x = \sqrt{\frac{R-y}{c}}$

$$V = 2\int_0^{h/2} \pi\left(R - cx^2\right)^2 dx = 2\int_0^{h/2} \pi\left(R - \frac{4d}{h^2}\cdot x^2\right)^2 dx$$

$V = \frac{1}{3}\pi h\left(2R^2 + r^2 - \frac{2}{5}d^2\right)$ $\qquad r = R - d \rightarrow d = R - r$

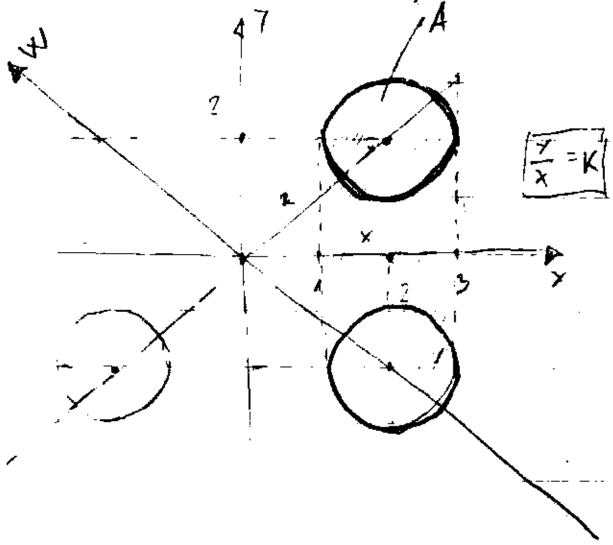
$$V = 2\left(\frac{1}{10}\pi d^2 h - \frac{1}{3}\pi R d h + \frac{1}{2}\pi R^2 h\right) = 2\pi h\left(\frac{d^2}{10} - \frac{Rd}{3} + \frac{R^2}{2}\right)$$

$$= 2\pi h\left(\frac{d^2}{10} - \frac{R(R-r)}{3} + \frac{R^2}{2}\right) = \pi h\left(\frac{d^2}{10} - \frac{R^2}{3} + \frac{Rr}{3} + \frac{R^2}{2}\right)$$

$$= 2\pi h\left(\frac{d^2}{10} + \frac{3R^2 - 2R^2}{6} + \frac{R\cdot r}{3}\right) = \pi h\left(\frac{d^2}{5} + \frac{R^2}{3} + \frac{2R\cdot r}{3}\right) =$$

$$= \frac{\pi h}{3}\left(\frac{3d^2}{5} + R^2 + 2R\cdot r\right)$$

70



$\frac{y}{x} = K$

$$V_1 = \int_{2-r}^{2+r} A(x)\,dx = \pi\int_1^3 \left(2 + \sqrt{1-(x-2)^2}\right)^2 - \left(2 - \sqrt{1-(x-2)^2}\right)^2 dx = 4\pi^2$$

$z = x^2 + y^2 = x^2 + K^2x^2 = x\sqrt{1+K^2}$

F. $\boxed{K = 2}$ $\qquad z = x\sqrt{2}$

$$V_2 = \pi\int_{2\sqrt{2}-r}^{2\sqrt{2}+r} \left(2\sqrt{2} + \sqrt{1-(x-2\sqrt{2})^2}\right)^2 - \left(2\sqrt{2} - \sqrt{1-(x-2\sqrt{2})^2}\right)^2 dx$$

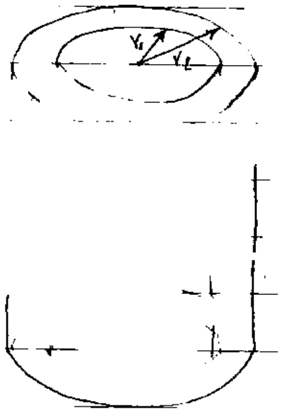
$V_2 = \sqrt{2}\,V_1$ $\qquad \boxed{V_2 = \sqrt{K^2+1}\,V_1}$

$f(x) > g(x) \quad [a,b]$

$$V = \pi \int_a^b [f(x)+k]^2 - [g(x)+k]^2 \, dx = \pi \int_a^b f^2(x) + 2k f(x) + k^2 - g^2(x) - 2k g(x) - k^2 \, dx$$

$$V = \underbrace{\pi \int_a^b [f^2(x) - g^2(x)] \, dx}_{V_1} + 2k\pi \underbrace{\int_a^b [f(x) - g(x)] \, dx}_{A} = V_1 + 2k\pi A$$

VOLUMES BY CYLINDRICAL SHELLS

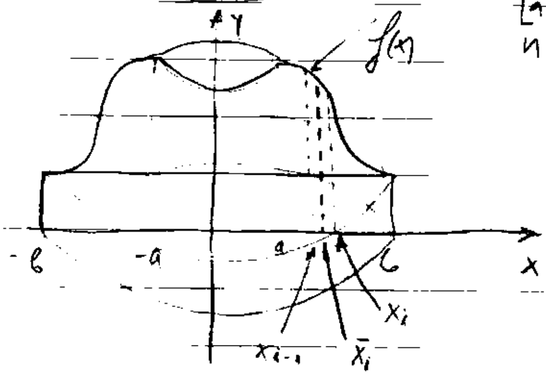


$V = V_2 - V_1 = r_2^2 \pi \cdot h - r_1^2 \pi h = \pi h \cdot (r_2^2 - r_1^2)$

$V = 2\pi h \frac{r_1 + r_2}{2} (r_2 - r_1) = 2\pi r h \cdot \Delta r$

$\Delta r = r_2 - r_1 \qquad r = \frac{r_1 + r_2}{2} \qquad \boxed{V = 2\pi r h \cdot \Delta r}$

V = [circumference] [height] [thickness]



$[a,b]$

n - subintervals $[x_{i-1}, x_i] \quad \Delta x \quad \bar{x}_i = \frac{x_i + x_{i-1}}{2}$

$V_i = 2\pi \bar{x}_i f(\bar{x}_i) \Delta x$

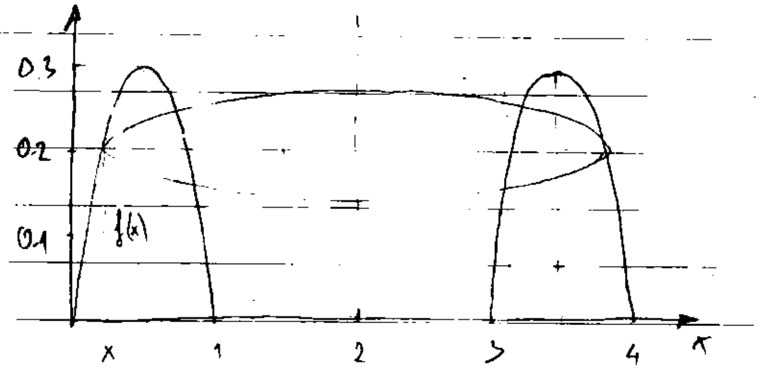
$V = \sum_{i=1}^{n} 2\pi \bar{x}_i f(\bar{x}_i) \Delta x = \sum_{i=1}^{n} V_i$

$\lim_{n \to \infty} \sum_{i=1}^{n} 2\pi \bar{x}_i f(\bar{x}_i) \Delta x = \int_a^b 2\pi x f(x) \, dx$

$V = \int_a^b 2\pi x f(x) \, dx \qquad 0 \le a < b$

EXAMPLE 1 $\quad y(x) = 2x^2 - x^3 \qquad V = 2 \int_0^2 2\pi x f(x) \, dx = \frac{32\pi}{5}$

Ex 4 $\quad y = x - x^2; \; y = 0; \; x = 2$



$$\int_0^1 2\pi (2-x) f(x) \, dx$$

EXERCISES [1] $y = x(x-1)^2$ [2] $y = \sin(x^2)$; $x^2 = \mathrm{sqrt}(\arcsin(y))$

$V = \int_0^{\sqrt{\pi}} 2\pi x \sin(x^2) \, dx \qquad \int_0^1 \pi \arcsin(y) \, dy$

$$V=\int_1^2 2\pi x \ln(x)\,dx = 2\pi\int_1^2 x\ln(x)\,dx = 2\pi\left(\frac{x^2}{2}\cdot\ln(x)\Big|_1^2 - \int_1^2 \frac{x^2}{2}\,d\ln(x)\right) =$$

$$= 2\pi\left(2\cdot\ln(2) - \int_1^2 \frac{x^2}{2}\cdot\frac{1}{x}\,dx\right) = 2\pi\left(2\ln(2) - \frac{1}{2}\int_1^2 x\,dx\right) =$$

$$2\pi\left(2\ln(2) - \frac{1}{2}\,\frac{x^2}{2}\Big|_1^2\right) = 2\pi\left(2\ln(2) - \frac{1}{4}(4-1)\right) = 2\pi\left(2\ln(2) - \frac{3}{4}\right)$$

$$\boxed{V = -\frac{3}{2}\pi + 4\pi\ln 2}$$

27 $n=4$; $y=tg(x)$ $\quad 0\le x\le \pi/4$

$$\left[0, \frac{\pi}{16}, \frac{\pi}{8}, \frac{3\pi}{16}, \frac{\pi}{4}\right] \qquad \Delta x = \frac{b-a}{n} = \frac{\frac{\pi}{4}-0}{4} = \frac{\pi}{16}$$

$$x_i = \left[\frac{\pi}{32}, \frac{3\pi}{32}, \frac{5\pi}{32}, \frac{7\pi}{32}\right]; \qquad V = \sum_{i=1}^{n} 2\pi x_i f(x_i)\Delta x$$

28 $n=5$ $\quad \Delta x = \frac{b-a}{n} = \frac{12-2}{5} = 2$

$$V = \sum_{i=1}^{n} 2\pi x_i \cdot f(x_i)\cdot\Delta x = 2\pi\Delta x\left[x_1 f(x_1) + x_2 f(x_2) + \dots + x_5 f(x_5)\right]$$

29 $\int_0^3 2\pi x^5\,dx = \int_0^3 2\pi x\, x^4\,dx$

31 $x = 1-y^2$ $\qquad y = \sqrt{1-x}$

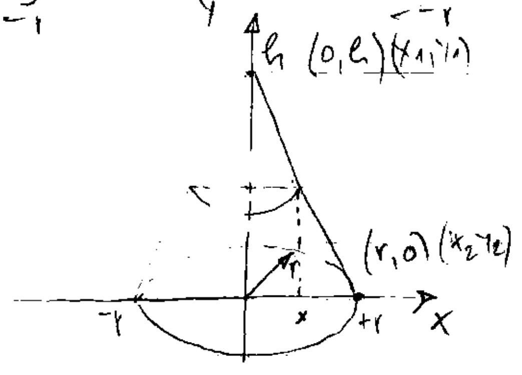
41 $x^2+(y-1)^2 = 1$; $\qquad (y-1)^2 = 1-x^2 \qquad y = 1\pm\sqrt{1-x^2}$

$$V = \frac{4\pi r^3}{3} \qquad \boxed{V = \frac{4\pi}{3}} \qquad x = \sqrt{1-(y-1)^2}$$

44 $V = 2\int_{R-r}^{R+r} 2\pi x\sqrt{r^2-(x-R)^2}\,dx = \left|\begin{matrix} u = x-R \\ du = dx \\ x = R+r \quad u = R+r-R = r \\ x = R-r \quad u = R-r-R = -r\end{matrix}\right| = 2\int_{-r}^{r} 2\pi(u+R)\sqrt{r^2-u^2}\,du$

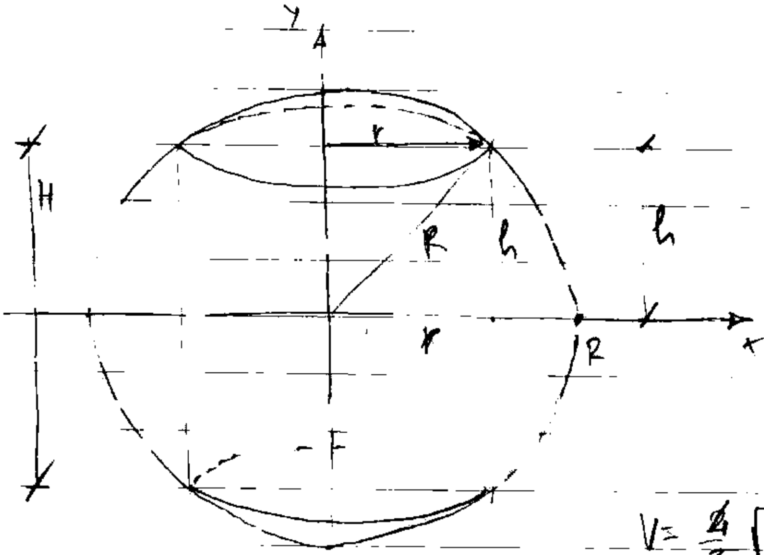
$$V = 2\int_{-r}^{r} 2\pi R\sqrt{r^2-u^2}\,du + 2\cdot 2\pi\int_{-r}^{r} u\sqrt{r^2-u^2}\,du = 2\cdot 2\pi R\cdot\frac{r^2\pi}{2} = 2\pi^2 R r^2$$

45



$$y - h = \frac{0-h}{r-0}(x-0) \qquad \boxed{y = -\frac{h}{r}x + h}$$

$$y = -\frac{h}{r}(x-r)$$



$r = \sqrt{R^2 - h^2}$

$y = \sqrt{R^2 - x^2}$; $x = \sqrt{R^2 - y^2}$

$$V = 2\int_r^R 2\pi x \sqrt{R^2 - x^2}\, dx$$

$$V = \frac{4}{3}\sqrt{R^2 - r^2}\,\pi R^2 - \frac{4}{3}\sqrt{R^2 - r^2}\,\pi r^2$$

$$V = \frac{4\pi}{3}\sqrt{R^2 - r^2}\,(R^2 - r^2) = \frac{4\pi}{3}(R^2 - r^2)^{3/2} = \frac{4\pi}{3}\left(R^2 - (R^2 - h^2)\right)^{3/2}$$

$$V = \frac{4\pi}{3}(h^2)^{3/2} = \frac{4\pi}{3}h^3$$

$$h = \frac{H}{2}\ ;\ V = \frac{4\pi}{3}\frac{H^3}{8} = \frac{\pi H^3}{6}$$

$R = 2$; $r = 1$; $h = \sqrt{4 - 1} = \sqrt{3}$ $\qquad V = \frac{4\pi}{3}\sqrt{3}^3 = \frac{4\pi}{3} 3\sqrt{3} = 4\pi\sqrt{3}$

$$V = \frac{4\pi R^3}{3} - r^2\pi \cdot H - V_{cap}$$

$$V_{cap} = 2\int_h^R \pi\left(\sqrt{R^2 - y^2}\right)^2 dy = \frac{4}{3}\pi R^3 + \frac{2}{3}\pi h^3 - 2\pi R^2 h$$

$$V_{cap} = \frac{1}{3}\pi (R - h)^2 (3R - R + h) = \frac{\pi}{3}(R^2 - 2Rh + h^2)(2R + h)$$

CAP FORMULA

$$V_{cap} = \frac{4}{3}\pi R^3 + \frac{2}{3}\pi\frac{H^3}{8} - 2\pi R^2 \cdot \frac{H}{2}$$

$$V = \frac{4\pi R^3}{3} - r^2 \cdot \pi \cdot H - \frac{4\pi R^3}{3} - \frac{2\pi H^3}{24} + \frac{2\pi R^2 H}{2}$$

$$= - \left(R^2 - \frac{H^2}{4}\right)\cdot \pi \cdot H - \frac{\pi H^3}{12} + \pi R^2 H$$

$$= - R^2\pi \cdot H + \frac{\pi H^3}{4} - \frac{\pi H^3}{12} + \pi R^2 H = \frac{3\pi H^3 - \pi H^3}{12}$$

$$= \frac{2\pi H^3}{12} = \frac{\pi H^3}{6}$$

64 WORK

Second Newton Law:

$$F = m \cdot \frac{d^2 s}{dt^2}\ ; \quad 1\,N \Leftrightarrow kg\,\frac{m}{s^2}$$

$W = F \cdot d$ work = Force · Distance

$1\,J \Leftrightarrow N \cdot m$

ex 1 (a) BOOK 1.2 KG, DESK 0.7m HIGH, $g = 9.8\ m/s^2$

$W = d \cdot F$ $\quad F = m \cdot \frac{d^2 s}{dt^2}$ $\quad F = m \cdot g = 1.2 \cdot 9.8$

$F = 11.76\ [N]$, $\quad W = 8.232\ [J]$

(b) 20 lb 6 [ft] $\quad 1\ [ft] = 0.305\ [m] \quad 1\ [lb] = 0.454\ [kg]$

$W_i \doteq f(x_i^*)\Delta x \qquad W \doteq \sum_{i=1}^{n} f(x_i)\Delta x$

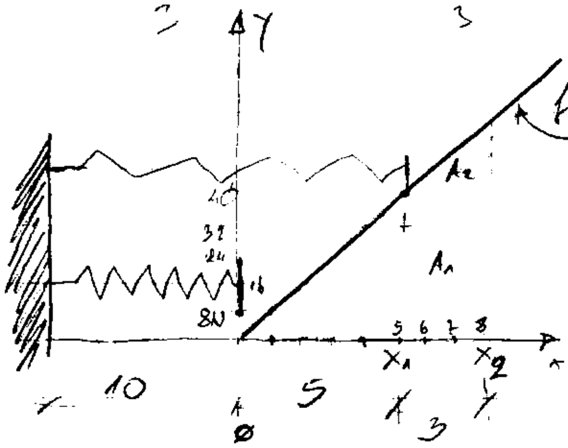
$W = \lim_{n \to \infty} \sum_{i=1}^{\infty} f(x_i)\Delta x = \int_a^b f(x)\,dx$

ex 2 $f(x) = x^2 + 2x \qquad W = \int_1^3 f(x)\,dx = \left(\frac{x^3}{3} + x^2\right)\Big|_1^3 = \frac{27}{3} + 9 - \frac{1}{3} - 1)$

$W = \frac{27 + 27 - 1 - 3}{3} = \frac{54 - 4}{3} = \frac{50}{3}$

$1\ m = 100\ cm$, $1\ cm = 10^{-2}\ m$

ex 3



$f(x) = k \cdot x$

$40N = k \cdot 5 \cdot 10^{-2}\ m$

$k = \frac{40}{5} \cdot 10^2 = 8 \cdot 10^2 = 800 \left[\frac{N}{m}\right]$

$f(x_2) = 800 \cdot 8 \cdot 10^{-2} = 64N$

$f(x_1) = 40N$

~~$W = [f(x_2) - f(x_1)] \cdot 3 \cdot 10^{-2} = 24 \cdot 3 \cdot 10^{-2}$~~

~~$W = 72 \cdot 10^{-2} = 0.72\ J$~~

$W = \int_{0.05}^{0.08} k \cdot x\,dx = k\frac{x^2}{2}\Big|_{0.05}^{0.08} = 1.56\ [J]$

$W = A_1 + A_2 = 40 \cdot 3 \cdot 10^{-2} + \frac{1}{2}(24 \cdot 3 \cdot 10^{-2}) = \frac{120}{100} + \frac{72}{100} \cdot \frac{1}{2} = 1.2 + \frac{0.72}{2}$

$W = 1.2 + 0.36 = 1.56\ [J]$

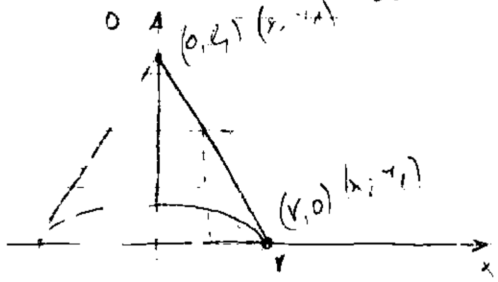
ex 4 200 lb CABLE 100 [ft] LONG

$F = (100 - x)$

~~$W = \int f(x)\,dx$~~

PODOLZNA SILA – $g = \frac{200}{100} = 2\ \frac{lb}{ft}$ $\qquad \Delta W = g \cdot x$

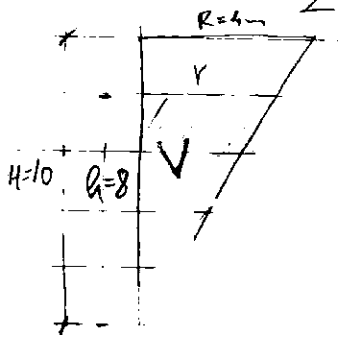
$W = \int_0^{100} g \cdot x\,dx = 2\left[\frac{lb}{ft}\right] \cdot \frac{x^2}{2}\Big|_0^{100} = (10^2)^2 = 10^4 = 10{,}000\ lb\text{-}ft$



$y - h = \frac{-h}{r} \cdot x \qquad y = h - \frac{h}{r}x$

$V = \int_0^r 2\pi x \cdot \left(h - \frac{h}{r}x\right) dx = \frac{\pi h r^2}{2}$

ex 5 $V = \frac{\pi r^2 \cdot h}{2}$ 14:32 01.01.27



$\frac{H}{R} = \frac{h}{r}$ $r = \frac{R}{H} \cdot h = \frac{4}{10} \cdot 8 = \frac{16}{5} m = 3.2m$

$V = \frac{\pi r^2 \cdot h}{2} = \frac{1024}{5}\pi$

- density of the water $\gamma = 1000 \frac{kg}{m^3}$

$m = \gamma \cdot V = 1000 \frac{1024}{25}\pi = 40960\pi$ kg

$F = m \cdot g = 40.960\pi \cdot 9.8$ $W = F \cdot H = 4.0141 \cdot 10^6 \pi [J]$

~~SEPAK~~ NE SITE DELOVI TREBA DA SE KRENAT ZA H

$\frac{r_i}{10-x_i} = \frac{4}{10}$ $r_i = \frac{4}{10}(10-x_i) = \frac{2}{5}(10-x_i)$

$V_i \doteq \pi r_i^2 \cdot \Delta x = \frac{4\pi}{25}(10-x_i)^2 \Delta x$

$m_i = \gamma \cdot V_i = 1000 \cdot V_i = 160\pi (10-x_i)^2 \Delta x$

$F_i = m_i g \approx 9.8 \cdot 160\pi (10-x_i)^2 \Delta x = 1570\pi (10-x_i)^2 \Delta x$

$W_i = F_i \cdot x_i = 1570\pi x_i (10-x_i)^2 \Delta x$

$W = \lim_{n\to\infty} \sum_{i=1}^{n} 1570\pi x_i (10-x_i)^2 \Delta x = \int_2^{10} 1570\pi x (10-x)^2 dx$

exc 1 $W = ?$ $d = 8m$ $F = 900N$ $W = F \cdot d = 900 \cdot 8 = 7200 J$

exc 2 $W = ?$ $m = 60 kg$ $F = m \cdot g = 60 \cdot 9.8 = 588$
$W = d \cdot F = 1176$ J $d = 2m$

exc 3 $F = \frac{10}{(1+x)^2}$ pounds $d = 9$ ft

$\int_0^9 \frac{10\, dx}{(1+x)^2} = 9$ ft-lb

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14:32

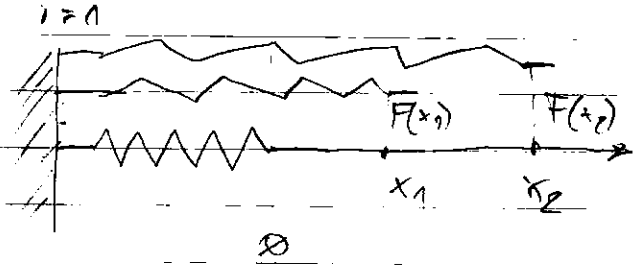
exc 4 $F = \cos(\pi x/3)$

exc 5 $W = \lim_{n\to\infty} \sum_{i=1}^{n} F(x_i) \cdot \Delta x$ $\Delta x = \frac{b-a}{n} = \frac{8-0}{4} = 2$

$x = 0..4$
$F(x) = \frac{30}{4} \cdot x$ $F(1) = \frac{30}{4} = 7.5$; $F(3) = \frac{90}{4} = 22.5$
$x = 4..8$ $F(x) = 30$

$W \approx \sum_{i=1}^{4} F(x_i) \cdot \Delta x = [7.5 + 22.5 + 30 + 30] \cdot 2$

exc 7

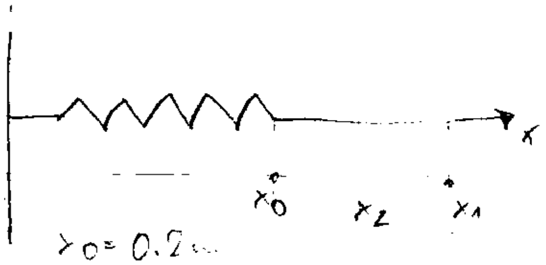


$F(x_1) = k \cdot x$
$10 \text{ lb} = k \cdot \frac{4}{12}$ ft
$k = \frac{120}{4} \frac{lb}{ft}$
$k = 30 \left[\frac{lb}{ft}\right]$

$$W = \int_0^{6/12} F(x)\,dx = \int_0^{0.5} 30 \cdot x\,dx = 30\frac{x^2}{2}\Big|_0^{0.5} = 15\left(\frac{1}{4}\right) = \frac{15}{4}\ [\text{ft}\cdot\text{lb}]$$

exc8 $F(x_1) = F(0.3) = 25\,N$ $\qquad F(0.3) = K(x_1 - x_2) = 0.1K$ [m]

$0.10K = 25\,N \qquad K = 250\left[\frac{N}{m}\right]$



$x_0 = 0.2$ m
$x_1 = 0.3$ m

$$W = \int_0^{0.05} 250\,x\,dx = 250\frac{x^2}{2}\Big|_0^{0.05}$$

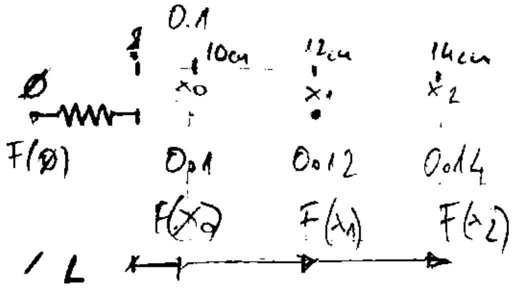
$$W = \frac{250}{2}(0.05^2 - 0) = 0.3125\ J$$

11 $$2J = \int_0^{0.12} kx\,dx = k\frac{x^2}{2}\Big|_0^{0.12} = k\cdot 0.0072 \qquad k = 277.8\left[\frac{N}{m}\right]$$

$F(x_0) = 30N = k\cdot x_0$; $\quad x_0 = \frac{F(x)}{k} = \frac{30}{277.8} = 0.108$ m

$x_0 = 10.8$ cm

12 ~~$6 = F(x)\int_{0.1}^{0.12} kx\,dx = k\frac{x^2}{2}\Big|_{0.1}^{0.12} = k\cdot 0.0022 \cdot 10 \quad k = \frac{6}{0.0022} = 2727.2727$~~



~~$10J = \int_{0.1}^{0.12} kx\,dx = 0.0026k$~~

~~$k = 3846.154$ [N/m]~~

~~$0.1\cdot k\cdot 0.0022 = 6N$~~

~~$F = 6 - k\cdot 0.0022$~~

$$6 = \int_{0.1-L}^{0.12-L} kx\,dx = 0.0002\,k(11 - 100L) \qquad (1°)$$

$$10 = \int_{0.12-L}^{0.14-L} kx\,dx = 0.0002\,k(13 - 100L) \qquad (2°)$$

$(1°) \rightarrow \quad k = \frac{30000}{11 - 100L}$

$(2°) \quad 50000 = \frac{30000}{11-100L}(13 - 100L) \qquad 5(11 - 100L) = 3(13 - 100L)$

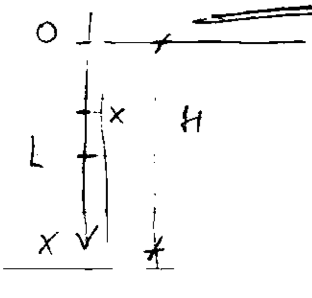
$55 - 500L = 39 - 300L \qquad 55 - 39 = 500L - 300L$

$16 = 200L \qquad L = \frac{16}{200} = \frac{8}{100} = \frac{2}{25} = 0.08\,m = 8\,cm$

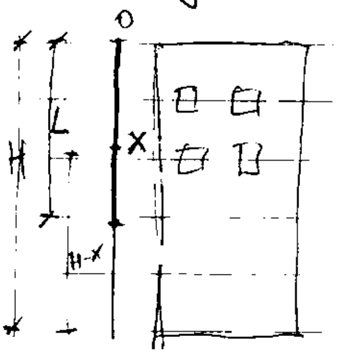
13 $L = 50$ ft ; $\delta = 0.5$ lb/ft ; $H = 120$ ft

$W = F\cdot d \qquad W = \int_0^L F(x)\,dx$

$F(x) = g\cdot(x\cdot\delta)$; $\quad W = \int_0^L g(L - x)\,dx =$



$L = 50\,ft \quad m_l = 0.5\ lb/ft \quad H = 120\,ft$



$F = g \cdot m \qquad W = F \cdot d$

$\Delta F = g \cdot \Delta m \qquad \Delta W = \Delta F \cdot x = g \cdot \Delta m \cdot x$

$\Delta F = m_l \cdot \Delta x \qquad \Delta W = g\, x \cdot m_l \cdot \Delta x$

$W = \sum_{i=1}^{n} g \cdot x_i \cdot m_l \cdot \Delta x$

$W = \int_0^L g \cdot x \cdot m_l \cdot dx = \cdot 0.5 \int_0^L x\, dx = 9.8 \cdot 0.5 \frac{x^2}{2}\Big|_0^L$

$W = \cdot 0.5 \frac{1}{2}(50^2) = \ 0.5 \cdot 0.5 \cdot 2500 = 625\ ft\text{-}lb$

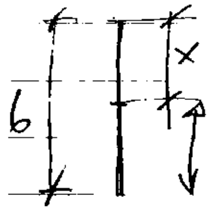
$W = \int_0^L 0.5 x\, dx = 625\ ft\text{-}lb \qquad W_1 = \int_0^{L/2} 0.5 x\, dx = \frac{625}{4}\ ft\text{-}lb$

$W_2 = \int_{L/2}^{L} 0.5 \cdot 25\, dx = \int_{25}^{50} 0.5 \cdot 25\, dx = \frac{625}{2} \qquad W = W_1 + W_2$

$\frac{625 + 1250}{4} = \frac{1875}{4}$

14 $L = 10\,m \quad m = 80\,kg \quad m_l = \frac{80}{10} = \frac{4}{5} = 8\ \frac{kg}{m}$

$F = g \cdot m_l \qquad \int_0^6 g \cdot m_l (6-x)\, dx = 9.8 \cdot 8 \int_0^6 (6-x)\, dx = 1411{,}2\ J$



$\Delta W = \Delta F \cdot (H-x) = \underbrace{\Delta m}_{8 \cdot \Delta x} \cdot g\, (H-x)$

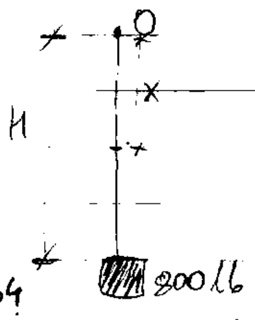
$\boxed{\delta = \frac{80}{10} = 8\ \frac{kg}{m}}$

$\Delta W = 8 \cdot g\, (H-x)\, \Delta x \Rightarrow W = \lim_{n\to\infty} \sum_{i=1}^{n} 8 \cdot g\, (H-x)\, \Delta x$

$W = \int_0^6 8 \cdot g\, (H-x)\, dx$

15 Cable weighs 2 lb/ft, 800 lb coal

$H = 500\,ft \to$ depth of the mineshaft

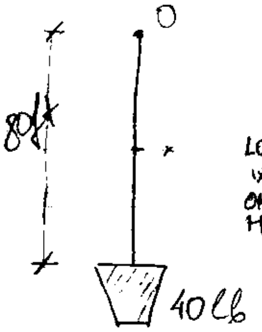


$W = 800 \cdot 500 + \int_0^H 2 \cdot x\, dx = 40 \cdot 10^4 +$

$+ 2 \cdot \frac{x^2}{2}\Big|_0^{500} = 4 \cdot 10^5 + (5 \cdot 10^2)^2 = 4 \cdot 10^5 + 25 \cdot 10^4 =$

$= 6.5 \cdot 10^5 = 650000\ lb\text{-}ft$

[16] bucket weights 4 lb
well depth $H = 80\ ft$



$v = 2\ ft/sec$; $v = \frac{ds}{dt}$ $t = \frac{s}{v} = \frac{80}{2} = 40\ sec$ — time to draw bucket to the top!

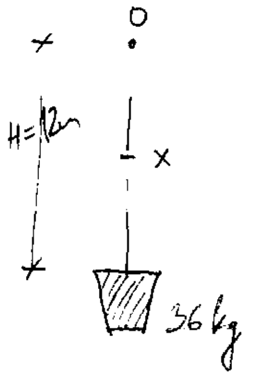
water leaks at rate 0.2 lb/s

LOST WEIGHT ON HEIGHT H $\rightarrow G_H = 0.2\ lb/s \cdot 40 s = \underline{8\ lb}$

$$G_x = 0.2\left[lb/s\right] \cdot \frac{(H-x)}{v}\left[sec\right] = \frac{0.2(H-x)}{2} = 0.1(H-x)\ [lb]$$

$$W = \int_0^{80}\left[44 - 0.1(80-x)\right]dx = 3200\ lb\text{-}ft$$

[17] bucket weight $\underline{10 kg}$, rope weights 0.8 kg/m ;

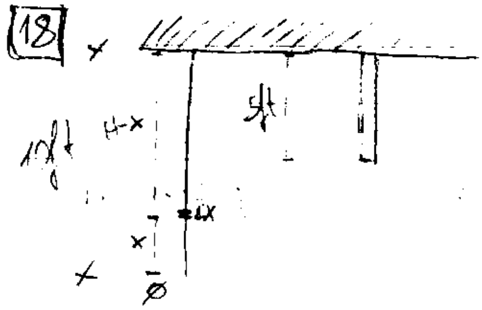


$M_x = \frac{36}{H} \cdot x$; $x=0$ $M_x = \emptyset$ $x = H$ $M_x = 36 kg$

$$W = g\int_0^H\left(\frac{36}{H}x + 0.8x\right)dx + \overbrace{10 \cdot 12 \cdot g}^{bucket} = 3857\ J$$

$$M = \frac{36}{12}\cdot x + 0.8x + 10 = (3.8x + 10)$$

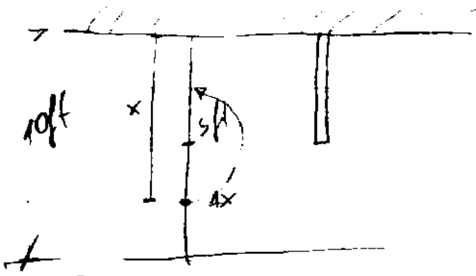
[18]



chain weights 25 lb $m_l = 2.5\ lb/ft$

$\Delta W = m_l \cdot \Delta x \cdot (H-x)$

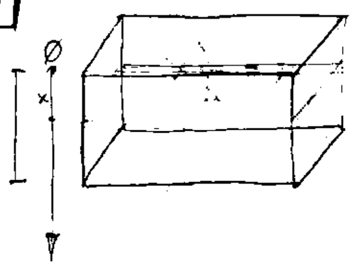
$$W = \int_0^{H/2} 2.5(H-x)\,dx = 2.5\int_0^5 (H-x)\,dx = 93.75\ ft\text{-}lb$$



$\Delta W = m_l \cdot \Delta x \cdot 2(x-5)$

$$W = \int_5^{10} 2.5 \cdot 2(x-5)\,dx = 5\int_5^{10}(x-5)\,dx = 62.5\ [ft\text{-}lb]$$

[19]

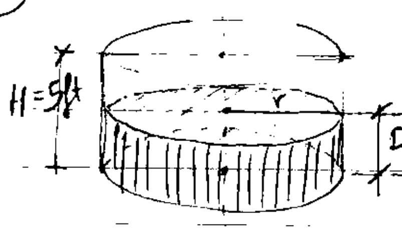


$m_i = 2 \cdot 1 \cdot \Delta x \cdot 1000 = 2000\,\Delta x$

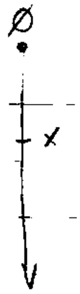
$F_i = g \cdot m_i = 9.8 \cdot 2000 \cdot \Delta x$

$$W = \int_0^{0.5} g \cdot m_i\, x\, dx = \int_0^{0.5} 9.8 \cdot 2000\, x\, dx = 2450\ J$$

(20)



$2r = 24\ ft \quad r = 12\ ft \; ; \; \gamma = 62.5\ lb/ft^3$



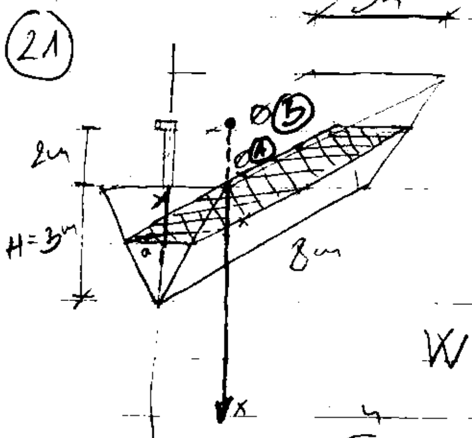
$\Delta W_i = 62.5 \cdot r^2 \pi \cdot \Delta x$

$$W = \int_1^5 x \cdot 62.5\, r^2 \pi\, dx = 62.5 r^2 \pi \int_1^5 x\,dx$$

$W = 1.08 \cdot 10^5 \pi$ ft-lb

↗ VIDI - Reserie (25)

(21)



(A) $\frac{3}{(3-x)} = \frac{1.5}{a} \qquad a = \frac{1.5}{3}(3-x) = \frac{1}{2}(3-x)$

$V = 2a \cdot 8 \cdot \Delta x = 8(3-x)\Delta x$

$\Delta m = \gamma \cdot V = 8000(3-x)\Delta x$

$\Delta F = g \Delta m$

$W_i = \Delta F \cdot x_i \; ; \quad W' = \sum_{i=1}^{n} \Delta F \cdot x_i =$

$= \sum_{i=1}^{n} g \cdot x 8000(3-x)\Delta x = 9.8 \cdot 8000 \int_0^3 x(3-x)\,dx$

$= 3.528 \cdot 10^5$ J ~~~~ $1.0584 \cdot 10^6$ J

(B)

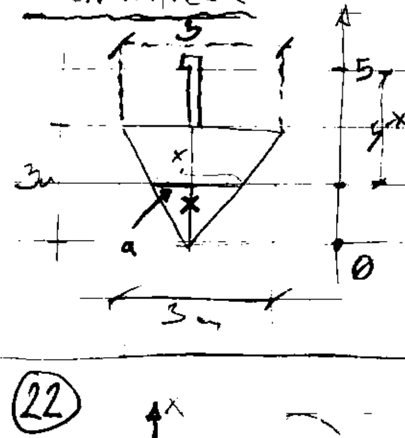
$\frac{3}{3-(x-2)} = \frac{1.5}{a} \qquad a = \frac{1}{2}(3-x+2) = \frac{1}{2}(5-x)$

$\Delta V = 2a \cdot 8 \cdot \Delta x = 8(5-x)\Delta x \qquad \Delta m = 8000(5-x)\Delta x$

$\Delta F = g\Delta m \quad W_i = g \Delta m \cdot x_i$

$W = 9.8 \int_2^5 8000(5-x)x\,dx =$ ~~$3.528 \cdot 10^5$ J~~ $= 1.0584 \cdot 10^6$ J

ALTERNATIVEZE



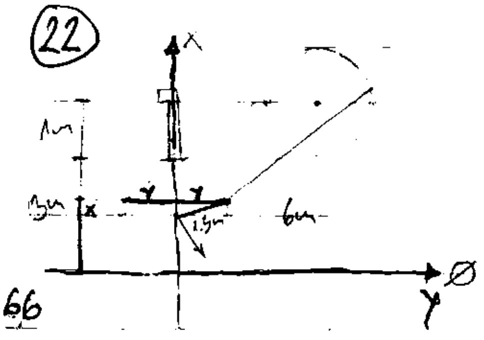
$\Delta V = 8 \cdot x \Delta x \qquad \Delta m = 8000 x \Delta x$

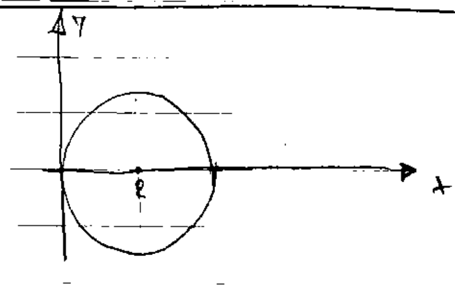
$\Delta F = 9.8 \cdot 8000 x \Delta x \qquad \Delta W = 9.8 \cdot 8000 x \Delta x (5-x)$

$$W = \int_0^3 9.8 \cdot 8000 \cdot x(5-x)\,dx =$$

$\left[\frac{3}{x} = \frac{1.5}{a} \qquad a = \frac{1}{2}x\right]$

(22)





$y = \sqrt{R^2 - A - R)^2}$

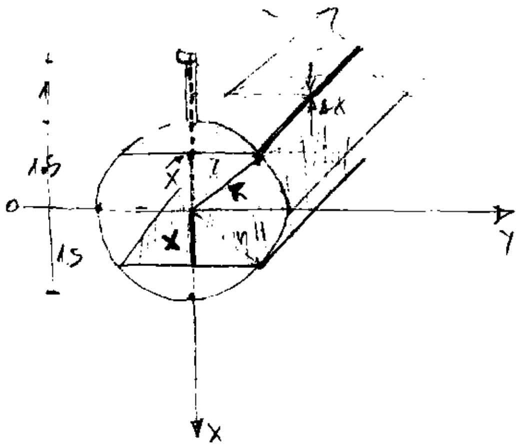
$\Delta V = (2 \cdot y)\ 6\ \Delta x$

$N = 12\sqrt{(1.5)^2 - (x-1.5)^2}\ \Delta x$

$\Delta m = \gamma \cdot \Delta V = 12000\sqrt{1.5^2-(x-1.5)^2}$; $\Delta F = \Delta m \cdot g$; $W_i = \Delta m \cdot g\,(4-x)$

$$W = \int_0^3 9.8 \cdot 12.000\sqrt{1.5^2-(x-1.5)^2}\cdot(4-x)\,dx = \underline{1.039 \cdot 10^6\ J}$$

ALTERNATIVELY:

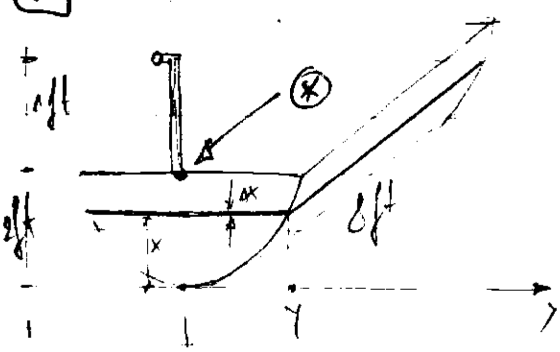


$$W_i = 9.8\cdot 1000\,\overbrace{\underbrace{\left(2\sqrt{1.5^2-x^2}\right)}_{2y}\cdot 6\cdot \Delta x}^{\Delta V}\cdot(2.5+x)$$

$$W = \int_{-1.5}^{1.5} 9.8\ 12\,000\sqrt{1.5^2-x^2}\,(2.5+x)\,dx$$

$$W = \underline{1.039\ 10^6\ J}$$

[23]



$\Delta m = 62.5\cdot\Delta V$;

$\Delta V = \Delta x\cdot 8\cdot 2y = 16\,\Delta x\sqrt{2^2-(x-2)^2}$

$W_i = 62.5\cdot 16\,\Delta x\sqrt{4-(x-2)^2}\cdot\underline{(3-x)}$

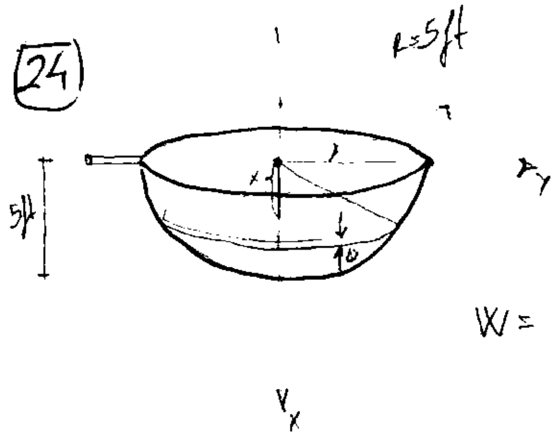
$$W = \int_0^2 1000\sqrt{4-(x-2)^2}\,(3-x)\,dx$$

$$W = 1000\left(\pi + \frac{8}{3}\right)\ [ft\text{-}lb]$$

AKO $x=0$ SE STAVI VO (*) TOGAŠ PROMENLIVATA Ḱ ... :

$W_i = 62.5\ 8\cdot 2\sqrt{4-x^2}\ \Delta x(x+1)$; $\boxed{W = \int_0^2 1000\sqrt{4-x^2}\,(x+1)\,dx}$

[24]



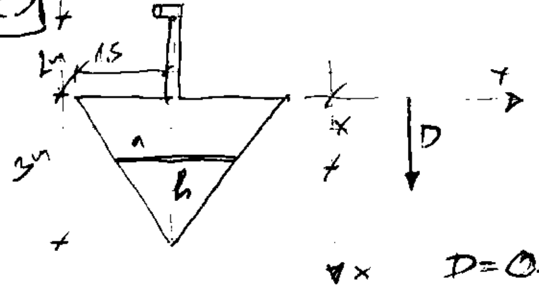
$\Delta m = 62.5\cdot\Delta V$

$\Delta V = \underbrace{\left(\sqrt{25-x^2}\right)^2}_{r^2}\pi\cdot\Delta x$

$W_i = 62.5\,\pi\,(25-x^2)\cdot\Delta x\cdot x$

$$W = \int_0^5 62.5\,\pi\,(25-x^2)\,x\,dx = 30675.616\ ft\text{-}lb$$

[25]



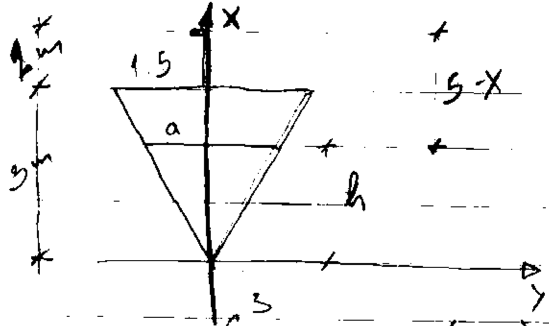
$\frac{3}{1.5} = \frac{3-x}{a}$ $\quad a = \frac{1}{2}(3-x)$

$\Delta V = 2a\cdot 8\ \Delta x = 8(3-x)\,\Delta x$; $\Delta m = 8000(3-x)\,\Delta x$

$$W = \int_0^3 g\cdot 8000(3-x)(x+2)\,dx = 1.0584\cdot 10^6\ J$$

$D = 0.971 \leftarrow W = \int_0^D g\cdot 8000(3-x)(x+2) = 4.7\ 10^5\ J$

$h = 3-D \doteq 2\ m$



$$\frac{3}{1.5} = \frac{h}{a} \qquad \boxed{a = \frac{1}{2}x}$$

$5 - x$

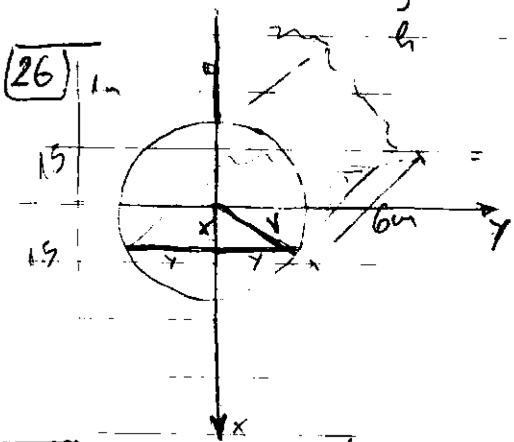
$\Delta m = 1000\,\Delta V$; $\Delta V = 2a \cdot 8 = 8x$

$\Delta m = 8000x$; $W_i = g \cdot \Delta m\,(5-x)$

$$W = \int_0^3 g\,8000x\,(5-x)\,dx = 1.06 \cdot 10^6 \text{ J}$$

$$W_0 = 4.7 \cdot 10^5 = \int_h^3 9.8 \cdot 8000x\,(5-x)\,dx \qquad \boxed{h \approx 2}$$

26



$\Delta m = 920 \cdot \Delta V$; $\Delta V = 2 \cdot \sqrt{r^2 - x^2} \cdot 6 \cdot \Delta x$

$\Delta m = 920 \cdot 12\,\Delta x\,\sqrt{(1.5)^2 - x^2}$

$W_i = 9.8 \cdot \Delta m \cdot (2.5 + x)$

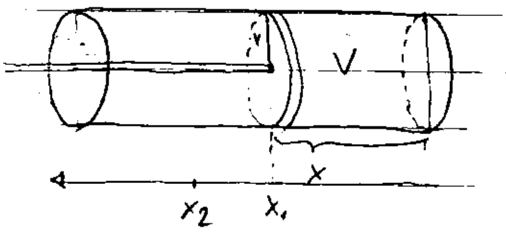
$$W = \int_0^{1.5} 920 \cdot 9.8 \cdot 12 \cdot \sqrt{(1.5)^2 - x^2} \cdot (2.5 + x)\,dx$$

$W \approx 6 \cdot 10^5$ J

27 - Pressure at any given time is function of volume $P = P(V)$

- Force exerted by the gas is : $\boxed{F = \pi r^2 P}$
- Show that work done by the gas for extention of volume FROM V_1 TO V_2 IS:

$$\boxed{W = \int_{V_1}^{V_2} P\,dV}$$



$\Delta W = \Delta F \cdot \Delta x = \underbrace{\pi r^2 \Delta x}_{\Delta V} \cdot P(\Delta V)$

$\Delta F = \pi r^2 P(\Delta V)$

$$W = \int_{V_1}^{V_2} P(V)\,dV$$

$$W = \int_{x_1}^{x_2} F(x)\,dx = \int_{x_1}^{x_2} \pi r^2 P(V)\,dx = \left/ \begin{array}{ll} V = \pi r^2 x & \\ dV = \pi r^2 dx & \\ x = x_1 & V = V_1 = \pi r^2 x_1 \\ x = x_2 & V = V_2 = \pi r^2 x_2 \end{array} \right/ = \int_{\pi r^2 x_1}^{\pi r^2 x_2} P(V)\,dV \Rightarrow$$

$$W = \int_{V_1}^{V_2} P(V)\,dV$$

28 Steam Engine $\quad P \cdot V^{1.4} = k$

- ENGINE STARTS AT $P_0 = 160$ lb/in² $\quad V_0 = 100$ in³
- $V_1 = 800$ in³

$$W = \int_{V_0}^{V_1} P(V)\,dV \qquad \boxed{1.4 = 1 + \frac{2}{5} = \frac{7}{5}}$$

$$k = 160 \frac{lb}{in^2} \cdot (100\, in^3)^{7/5} = 160 \cdot 100^{7/5} \left[lb \cdot in^{\frac{14}{5}-2}\right] = 1{,}01 \cdot 10^5 \left[lb \sqrt[5]{in^4}\right]$$

$$P(V) = \frac{k}{V^{7/5}}$$

$$W = \int_{100}^{800} \frac{1{,}01 \cdot 10^5}{V^{7/5}}\, dV = 22.588 \ [in\text{-}lb]$$

1 foot = 12 inch

$$W = \frac{22.588}{12} [ft \cdot lb] = 1.882 \ [ft \cdot lb]$$

$$P_0 = 160\ lb/in^2 = \left| 1\ in = \frac{1}{12} ft \right| = 160 \cdot 144\ lb/ft$$

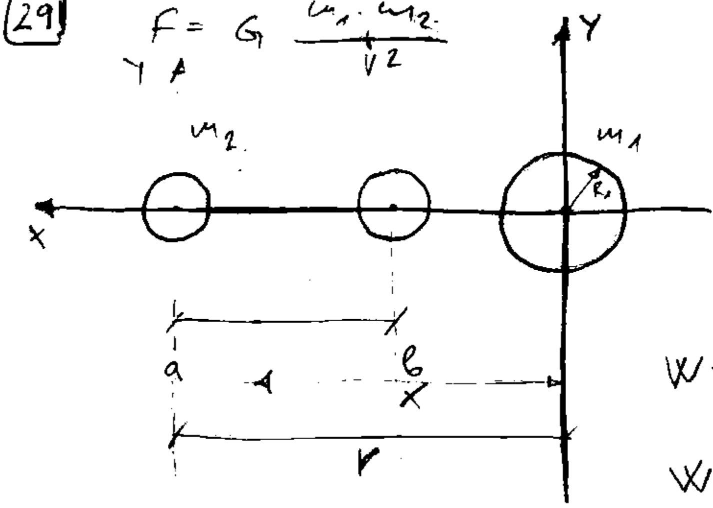
$$V_0 = 100\ in^3 = \left| 1\ in = \frac{1}{12} ft \right| = \frac{100}{1728} ft^3 \qquad V_1 = \frac{800}{1728} ft^3$$

$$k = P_0 \cdot V_0^{7/5} = 160 \cdot 144 \cdot \left(\frac{100}{1728}\right)^{7/5} = 426{,}502$$

$$W = \int_{100/1728}^{800/1728} \frac{426{,}502}{V^{7/5}}\, dV = 1.882{,}42 \ [ft \cdot lb]$$

29

$$F = G \frac{m_1 \cdot m_2}{r^2}$$



$$W = \int_b^a F(x)\, dx$$

PRETPOSTAVI KAKO DA TREBA ... DA SAVLADA GRAVITACIONATA SILA, OD POZICIJA b SE PRENESE VO (a)

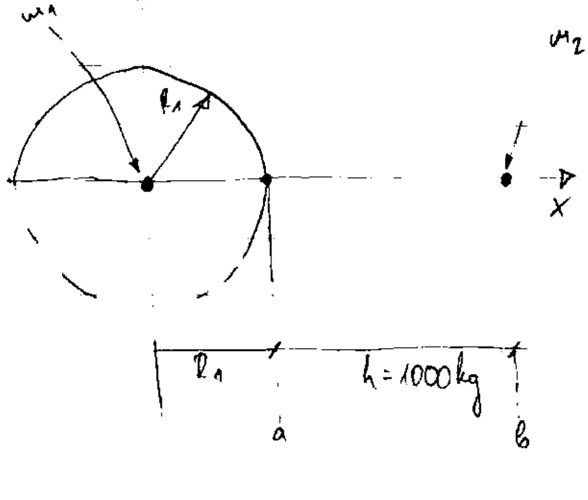
$$W = \int_b^a G \frac{m_1 m_2}{x^2}\, dx$$

$$W = G m_1 m_2 \frac{x^{-2+1}}{-2+1}\Big|_b^a = G m_1 m_2 \frac{(-1)}{x}\Big|_b^a$$

$$W = G m_1 m_2 \left(\frac{1}{b} - \frac{1}{a}\right) = \frac{G m_1 m_2 (a-b)}{ab}$$

30

- W = ? TO LAUNCH 1000kg satelite TO 1000 km HIGH ORBIT
- $m_1 = 5.98 \cdot 10^{24}$ kg ; $R_1 = 6.37 \cdot 10^6$; $G = 6.67 \cdot 10^{-11} N \frac{m^2}{kg^2}$



$$F = G \frac{m_1 m_2}{r^2}$$

$$W = \int_a^b G \frac{m_1 m_2}{x^2}\, dx = G m_1 m_2 \frac{1}{x}\Big|_b^a$$

$$W = G m_1 m_2 \left(\frac{1}{a} - \frac{1}{b}\right)$$

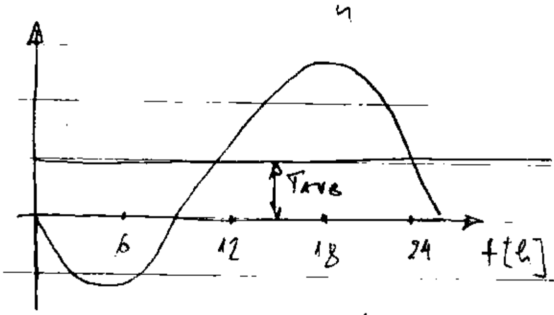
$a = R_1 = 6.37 \cdot 10^6 [m]$ $b = a + 10^6 = 7.37 \cdot 10^6 [m]$

$$W = \underbrace{6.67 \cdot 10^{-11}}_{G} \cdot \underbrace{5.98 \cdot 10^{24}}_{m_1} \cdot \underbrace{10^3}_{m_2} \cdot \left(\frac{1}{6.37 \cdot 10^6} - \frac{1}{7.37 \cdot 10^6}\right)$$

$$W = 8{,}496 \cdot 10^9\ J \doteq 8.5\ GJ$$

071279178 **6.5 AVERAGE VALUE OF A FUNCTION**

$$y_{ave} = \frac{y_1 + y_2 + \dots y_n}{n}$$



$y = f(x) \quad a \le x \le b$

$\Delta x = \frac{b-a}{n} \; ; \quad n = \frac{b-a}{\Delta x}$

$$y_{ave} = \frac{f(x_1^*) + f(x_2^*) \dots + f(x_n^*)}{n}$$

$$y_{ave} = \frac{1}{b-a}\left(f(x_1^*) + f(x_2^*) + \dots f(x_n^*)\right)\Delta x$$

$$y_{ave} = \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\Delta x \; ; \quad \lim_{n\to\infty}\frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\Delta x = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\boxed{f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

mean value theorem for derivatives

ex. 1 $f(x) = 1 + x^2 \qquad x = [-1, 2]$

MEAN VALUE THEOREM FOR INTEGRALS

If f is continuous on interval $[a,b]$, then there exist number „c" in $[a,b]$ such that:

$$\int_a^b f(x)\,dx = (b-a)f(c) \qquad \boxed{f(c) = f_{avg}(x)}$$

ex. 2 $f(x) = 1 + x^2 \qquad \int f(x)\,dx = f(c)[2+1] \qquad c = ?$

$f(c) : f_{ave} = 2 \Rightarrow 1 + c^2 = 2 \qquad \boxed{c = \pm 1}$

ex. 3 $\frac{\Delta s}{\Delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$ ~~AVERAGE~~ VELOCITY OF THE CAR

$$v_{ave} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} v(t)\,dt = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} s'(t)\,dt = \frac{1}{t_2 - t_1}\left[s(t_2) - s(t_1)\right]$$

Exercises

2 $f(x) = \frac{1}{x} \quad [1, 4] \qquad f_{ave} = \frac{1}{4-1}\int_1^4 f(x)\,dx$

$$f_{ave} = \frac{1}{3}\ln x \Big|_1^4 = \frac{1}{3}\left[\ln 2^2 - \ln 1\right] = \frac{2}{3}\ln 2$$

5 $f(t) = t e^{-t^2}$ $[0, 5]$ $\qquad f_{ave} = \frac{1}{5}\int_0^5 t e^{-t^2}\,dt$

$$\int t e^{-t^2}\,dt = \frac{1}{2}\int e^{-t^2}\,d(t^2) = t^2 e^{-t^2} - \int t^2\, d e^{-t^2} = t^2 e^{-t^2} -$$

$de^{-t^2} = e^{-t^2}(-2t)\,dt$

$$\int t e^{-t^2} dt = \left| \begin{array}{l} u = e^{-t^2} \\ du = -2t e^{-t^2} dt \\ t e^{-t^2} dt = -\frac{du}{2} \end{array} \right| = -\frac{1}{2} \int du = -\frac{1}{2} e^{-t^2}$$

$$f_{AVE} = \frac{1}{5} \left(-\frac{1}{2}\right) e^{-t^2} \Big|_0^5 = -\frac{1}{10}\left(e^{-25} - 1\right) = \frac{1}{10}\left(1 - e^{-25}\right)$$

7 $h(x) = \cos^4(x) \sin(x)$; $[0, \pi]$

$$\frac{1}{\pi} \int_0^{\pi} \cos^4(x) \sin(x) dx = -\frac{1}{\pi} \int_0^{\pi} \cos^4(x)\, d\cos(x) = -\frac{1}{\pi} \frac{\cos^5(x)}{5} \Big|_0^{\pi}$$

$$= -\frac{1}{5\pi}(-1 - 1) = \frac{2}{5\pi}$$

11 $2\sin(x) - \sin(2x) = \frac{4}{\pi}$ $\qquad \left(\frac{1}{\pi} \int_0^{\pi} 2\sin(x) - \sin(2x)\, dx = \frac{4}{\pi} \right)$

$2\sin(x) - 2\sin x \cdot \cos x = \frac{4}{\pi}$ $\qquad \sin x (1 - \cos x) = \frac{2}{\pi}$

$\sqrt{1 - \cos^2 x}\,(1 - \cos x) = \frac{2}{\pi}$ $\qquad (1 - u^2)(1 - u)^2 = \frac{4}{\pi^2}$

$(1 - u)(1 + u)(1 - u)^2 = \frac{4}{\pi^2}$ $\qquad (1 + u)(1 - u)^3 = \frac{4}{\pi^2}$

$(1 - u^2)(1 - 2u + u^2) = \frac{4}{\pi^2}$ $\qquad 1 - 2u + u^2 - u^2 + 2u^3 - u^4 = \frac{4}{\pi^2}$

$u^4 - 2u^3 + 2u + \frac{4}{\pi^2} - 1 = 0$ $\qquad u = [-0.95, 0.32]$

$\cos(x) = -0.95 \Rightarrow c = 2.81$

$\cos(x) = 0.32 \Rightarrow c = 1.24$

13 $\int_1^3 f(x) dx = 8$ $\qquad \frac{1}{b-a} \int_a^b f(x) dx = f(c)$

$\int_1^3 f(x) = (3 - 1) f(c)$ $\qquad f(c) = 4 \Rightarrow \int_1^3 f(x) = 2 \cdot 4 = 8$

$f(x) = 4$ for $x = c$ $\qquad f(c) = f_{AVE}$

14 $b = ?$ $\quad f_{AVG} = 3$ $\quad f(x) = 2 + 6x - 3x^2$ $\quad [0, b]$

$$\frac{1}{b-a} \int_0^b f(x) dx = \frac{1}{b} \int_0^b (2 + 6x - 3x^2) dx = \frac{1}{b}(2b + 3b^2 - b^3)$$

$f_{AVG} = 2 + 3b - b^2$ $\qquad b^2 - 3b - 2 + 3 = 0$ $\qquad b^2 - 3b + 1 = 0$

$$b_{1,2} = \frac{3 \pm \sqrt{9 - 4}}{2} = \frac{3}{2} \pm \frac{1}{2}\sqrt{5}$$

15

x	20	25	30	35	40	45	50
f(x)	42	38	31	29	35	48	60

$$f_{AVG} = \frac{1}{b-a} \sum_{i=1}^{n} f(x_i^*) \Delta x$$

$$\Delta x = \frac{b-a}{n} = \frac{50-20}{3} = \frac{30}{3} = 10$$

$$x_i = b + \frac{(2i-1)\Delta x}{2} \qquad b=20;\ x_1 = 20 + \frac{2-1}{2} 10 = \underline{\underline{25}}$$

$$x_2 = 20 + \frac{4-1}{2}\Delta x = 20 + 15 = \underline{\underline{35}};\quad x_3 = 20 + \frac{5}{2}\cdot 10 = \underline{\underline{45}}$$

$$f_{AVG} = \frac{1}{30} \sum_{i=1}^{3} f(x_i^*) \Delta x = \frac{1}{3}\left[38 + 29 + 48\right] = \frac{1}{3} 115$$

$$\boxed{f_{AVG} = 38.333}$$

20 $s = \frac{1}{2} g t^2$; velocity after time T is v_T

$$\int v(t)\, dt = s(t)$$

$$v_{AVE} = \frac{1}{T}\int_0^T v(t)\,dt = \frac{1}{T}\int_0^T s'(t)\,dt = \frac{1}{T}\int_0^T \frac{1}{2} g \cdot 2t\, dt$$

$$v_{AVE} = \frac{1}{T} g \int_0^T t\, dt = \frac{1}{T} g \frac{t^2}{2}\Big|_0^T = \frac{1}{T} g \frac{T^2}{2}$$

$$v = \frac{ds}{dt} = g\cdot t \qquad \boxed{v_T = g\cdot T} \qquad \boxed{v_{AVE} = \frac{gT}{2} = \frac{v_T}{2}}$$

$$v = \frac{\Delta s}{\Delta t} = \frac{\frac{1}{2} g T^2 - 0}{T - 0} = \frac{1}{2} g t = v_{AVG}$$

$$s_{AVG} = \frac{1}{T}\int_0^T \frac{1}{2} g t^2\, dt = \frac{1}{2T} g \frac{t^3}{3}\Big|_0^T = \frac{gT^2}{6}$$

$$s = \frac{gt^2}{2} = \frac{1}{2g}(gt)^2 = \frac{v^2}{2g} \qquad 2gs = v^2 \qquad \boxed{v = \sqrt{2gs}}$$

$$v_{AVG}^{s} = \frac{1}{s(T) - s(0)} \int_0^{\frac{gT^2}{2}} v(s)\, ds = \frac{1}{\frac{gT^2}{2}} \int_0^{\frac{gT^2}{2}} \sqrt{2gs}\, ds = \frac{2}{gT^2}\sqrt{2g}\, \frac{s^{\frac{1}{2}+1}}{\frac{1}{2}+1}\Big|_0^{g\frac{T^2}{2}}$$

$$s(0) = 0,\quad s(T) = \frac{gT^2}{2} \qquad = \frac{2}{gT^2}\sqrt{2g}\,\frac{\left(\frac{gT^2}{2}\right)^{\frac{3}{2}}}{\frac{3}{2}} = \frac{4}{gT^2}\sqrt{2g}\,\frac{\sqrt{g^3}\frac{T^3}{\sqrt{2^3}}}{3}$$

$$v^{s}_{avg} = \frac{2}{gT^2}\frac{\sqrt{2g/g}\sqrt{g}\ T^3}{3\sqrt{2}} = \frac{2}{g}\frac{\sqrt{2}(\sqrt{g})^2/g \cdot T}{3\sqrt{2}} = \frac{2}{3}gT = \frac{2}{3}v_T$$

21 T = 5 sec ⇒ DURATION OF FULL RESPIRATORY CYCLE

0.5 L/sec ⇒ MAXIMUM RATE OF AIR FLOW

$f(t) = 0.5\sin(2\pi t/5)$ RATE OF AIRFLOW

$$V(t) = \int_0^t f(t)\,dt = \int_0^t 0.5\sin(2\pi t/5)\,dt = 0.5\int_0^t \sin(2\pi t/5)\,dt$$

$$= \left| \begin{matrix} u = 2\pi t/5 \\ du = \frac{2\pi}{5}dt \end{matrix} \quad \begin{matrix} t=0 & u=0 \\ t=t & u=\frac{2\pi t}{5} \end{matrix} \right| = -0.5\left(\frac{5}{2\pi}\right)\cos(u)\Big|_0^{\frac{2\pi t}{5}}$$

$$V(t) = 0.5\cos(u)\Big|_{\frac{2\pi t}{5}}^{0} = \frac{2.5}{2\pi}\left(1-\cos\frac{2\pi t}{5}\right)$$

$$V(T) = \frac{2.5}{2\pi}\left(1-\cos\frac{2\pi 5}{5}\right) = \frac{2\cdot 2.5}{2\pi} = \frac{5}{2\pi}L$$

~~$V_{avg} = \frac{1}{T-0}\int_0^T \frac{2.5}{2\pi}\sin\left(\frac{2\pi t}{5}\right)dt = \frac{1}{T}\frac{2.5}{2\pi}\frac{5}{2\pi}\cos\left(\frac{2\pi t}{5}\right)\Big|_T^0 = \frac{1.5}{2\pi T}\left(1-\cos\frac{2\pi T}{5}\right)$~~

~~$V_{avg} = \frac{2.5}{2\pi T}(1+1) = \frac{5}{2\pi T} = \frac{1}{2\pi} = 0.16L$~~ (!)

$$V(t) = \frac{2.5}{2\pi}\left(1-\cos\frac{2\pi t}{5}\right)$$

$$V_{avg} = \frac{1}{T}\int_0^T V(t)\,dt = \frac{1}{2T}\frac{2.5}{\pi}\int_0^t\left(1-\cos\left(\frac{2\pi t}{5}\right)\right)dt = \frac{1}{2T}\frac{2.5}{\pi}\left(t - \frac{5}{2\pi}\sin\left(\frac{2\pi t}{5}\right)\right)\Big|_0^T$$

$$V_{avg} = \frac{2.5}{\pi}\frac{1}{2T}\left(T - \frac{5}{2\pi}\sin\left(\frac{2\pi T}{5}\right)\right) = \frac{2.5}{\pi}0.5\left(1-\frac{5}{2\pi T}\sin\left(\frac{2\pi T}{5}\right)\right)$$

if: T = 5 sec ⇒ $V_{avg} = \frac{2.5}{\pi}0.5\left(1-\frac{5}{2\pi 5}\cdot\sin(2\pi)\right) = 0.5L\cdot\frac{2.5}{\pi}$

→ $V_{avg} = \frac{1.25}{\pi}\left(1-\frac{5}{2\pi T}\sin\left(\frac{2\pi T}{5}\right)\right)$

$$\boxed{V_{avg} = \frac{1.25}{\pi}L = 0.398L}$$

22 $v(r) = \frac{P}{4\eta l}(R^2 - r^2)$ velocity of blood that flows in blood vessel with length l and radius R!

$$v_{avg} = \frac{1}{R}\int_0^R \frac{P}{4\eta l}(R^2-r^2)\,dr = \frac{PR^2}{6\eta l}\ ; \quad v_{max} = \frac{PR^2}{4\eta l}$$

$$v_{avg} = \frac{4}{4}\frac{PR^2}{6\eta l} = \frac{2}{3}\frac{PR^2}{4\eta l} = \frac{2}{3}v_{max}$$

[23] $\int_a^b f(t)\,dt = f'(c)(b-a)$ — mean value theorem for integrals

$f'(c) = \dfrac{f(b)-f(a)}{b-a}$ } mean value theorem

$\forall\ F'(x) = f(x)$

$F(x) = \int_a^x f(t)\,dt$ $\qquad F'(c) = \dfrac{F(b)-F(a)}{b-a}$

$f(c) = \dfrac{\int_a^b f(t)\,dt - \int_a^a f(t)\,dt}{b-a}$ → 0

$\int_a^b f(t)\,dt = (b-a)f(c)$

[24] $f_{avg}[a,b]$ – average of f in interval $[a,b]$

$a<c<b$

show: $f_{avg}[a,b] = \dfrac{c-a}{b-a} f_{avg}[a,c] + \dfrac{b-c}{b-a} f_{avg}[c,b]$.

$f_{avg}[a,b] = \dfrac{1}{b-a}\int_a^b f(t)\,dt = \dfrac{1}{b-a}\left[\int_a^c f(t)\,dt + \int_c^b f(t)\,dt\right]$

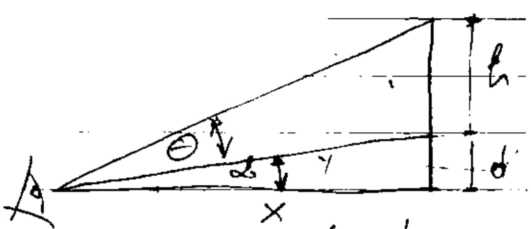
$= \dfrac{1}{b-a}\left[\dfrac{c-a}{c-a}\int_a^c f(t)\,dt + \dfrac{b-c}{b-c}\int_c^b f(t)\,dt\right]$

$f_{avg}[a,c]$ $\qquad f_{avg}[c,b]$

$= \dfrac{c-a}{b-a} f_{avg}[a,c] + \dfrac{b-c}{b-a} f_{avg}[c,b]$

APPLIED PROJECT: Where to sit at the Movies

[58] CH 4.7



$x = ?$ $\quad \theta = \theta_{max}$

$y = \sqrt{d^2 + x^2}$

$tg\,\alpha = \dfrac{d}{x}$ $\qquad \alpha = arctg\,\dfrac{d}{x}$

$tg(\theta+\alpha) = \dfrac{h+d}{x}$ $\qquad \dfrac{tg\theta + tg\alpha}{1 - tg\theta \cdot tg\alpha} = \dfrac{h+d}{x}$

$tg\theta + tg\alpha = \dfrac{h+d}{x}(1 - tg\theta \cdot tg\alpha)$; $tg\theta + \dfrac{d}{x} = \dfrac{h+d}{x}\left(1 - \dfrac{d}{x} tg\theta\right)$

$$\mathrm{tg}\,\theta + \frac{d}{x} = \frac{h+d}{x} - \frac{d(h+d)}{x^2}\,\mathrm{tg}(\theta)$$

$$\mathrm{tg}\,\theta\left[1 + \frac{d(h+d)}{x^2}\right] = \frac{h+d}{x} - \frac{d}{x}$$

$$\mathrm{tg}\,\theta\left[\frac{x^2 + dh + d^2}{x^2}\right] = \frac{h}{x} \quad \mathrm{tg}\,\theta = \frac{h\cdot x}{x^2 + dh + d^2}$$

$$(\mathrm{tg}\,x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x \cdot \cos x + \sin x \cdot \sin x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\theta(x) = \mathrm{arctg}\,\frac{hx}{x^2 + dh + d^2} \qquad \theta(u) = \mathrm{arctg}\,(u)$$

$$\theta'(x) = \frac{1}{1+u^2}\frac{du}{dx}$$

$$y(x) = \mathrm{arctg}\,x \qquad x = \mathrm{tg}\,y \;\Big/\, \frac{d}{dx} \Rightarrow 1 = \frac{1}{\cos^2 y}\frac{dy}{dx}$$

$$1 = \left(\frac{\sin^2(y) + \cos^2(y)}{\cos^2(y)}\right)\frac{dy}{dx} \Rightarrow 1 = \left(\mathrm{tg}^2(y) + 1\right)\cdot\frac{dy}{dx}$$

$$y'(x) = \frac{1}{1+x^2}$$

$$\theta'(x) = 0 \qquad \frac{1}{1 + \frac{hx}{x^2+dh+d^2}}\cdot\left(\frac{hx}{x^2+dh+d^2}\right)' = 0$$

$$\frac{x^2+dh+d^2}{x^2+dh+d^2+hx}\;\frac{h(x^2+dh+d^2) - hx(2x)}{(x^2+dh+d^2)^2} = 0$$

$$(x^2+dh+d^2)(hx^2 + dh^2 + d^2h - 2hx^2) = 0$$

$$(x^2+dh+d^2)(hx^2 - dh^2 - d^2h) = 0$$

$$x^2 + dh + d^2 = 0 \quad ??$$

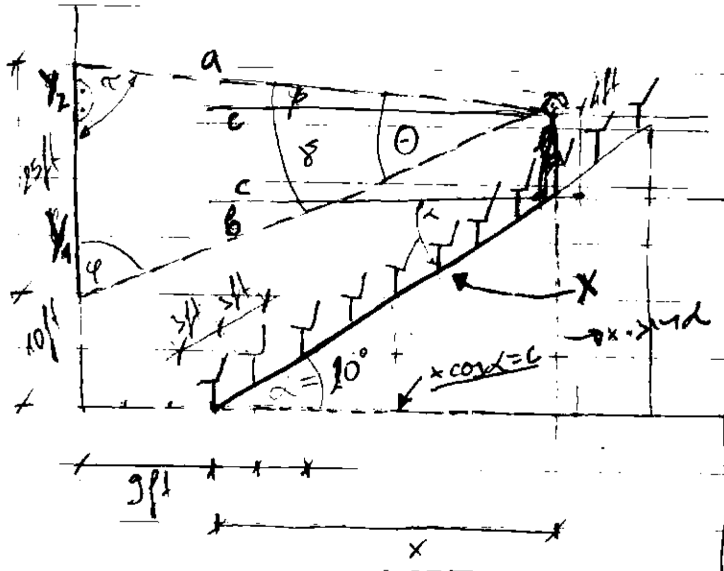
$$hx^2 - dh^2 - d^2h = 0 \qquad x^2 = d^2 + dh \qquad x = \sqrt{d^2+dh}$$

$$\theta(u) = \mathrm{arctg}\,(u), \qquad \frac{d\theta(u)}{dx} = \frac{d\theta(u)}{du}\frac{du}{dx} \quad \text{chain rule}$$

MOŽE I VAKA:

$$\theta = \mathrm{arctg}\,\frac{h+d}{x} - \mathrm{arctg}\,\frac{d}{x}; \quad \frac{d\theta}{dx} = \frac{1}{1+\left(\frac{h+d}{x}\right)^2}\cdot\left(-\frac{h+d}{x^2}\right) - \frac{1}{1+\left(\frac{d}{x}\right)^2}\cdot\left(-\frac{d}{x^2}\right)$$

$$\frac{d\theta}{dx} = 0 \Rightarrow x = \sqrt{d^2+dh} = \sqrt{d(d+h)}$$



• $\alpha = 20°$; $0 \le x \le 60$ ft

• NUMBER OF ROWS = 21

$$\theta = \arccos\left(\frac{a^2+b^2-625}{2ab}\right)$$

$$\cos\theta = \frac{a^2+b^2-625}{2ab}$$

~~$\cos\alpha = \frac{x}{c}$~~

~~$\theta = \beta + \gamma$~~

~~$tg\,\beta = \frac{a}{x+9ft}$~~ ~~$tg\,\gamma = \frac{b}{x+9}$~~ ~~$\cos\theta = \cos(\beta+\gamma)$~~

~~$\cos\theta = \cos\beta\cdot\cos\gamma - \sin\beta\cdot\sin\gamma$~~

$c = x\cdot\cos(\alpha)$; $\theta = \beta + \gamma$

$$a^2 = (9 + x\cos\alpha)^2 + (31 - x\sin\alpha)^2$$
$$b^2 = (9 + x\cos\alpha)^2 + (x\sin\alpha - 6)^2$$

$$y_2 = 10 + 25 - 4 - x\sin\alpha = 31 - x\sin\alpha$$
$$y_1 = 25 - y_2 = 25 - 31 + x\sin\alpha = x\sin\alpha - 6$$

$$a^2 = (9+c)^2 + y_2^2 = (9 + x\cos\alpha)^2 + (31 - x\sin\alpha)^2$$
$$b^2 = (9+c)^2 + y_1^2 = (9 + x\cos\alpha)^2 + (x\sin\alpha - 6)^2$$

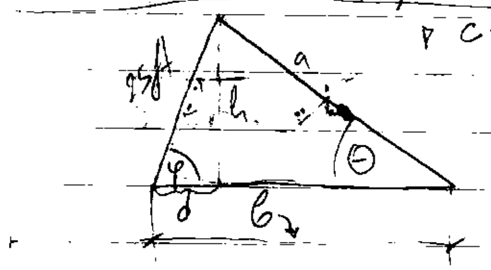
$\theta = \beta + \gamma$; $\cos\beta = \left(\frac{a}{c+9}\right)^{-1} = \left(\frac{a}{x\cos\alpha + 9}\right)^{-1}$; $\sin\beta = \left(\frac{a}{31 - x\sin\alpha}\right)^{-1}$

$\sin\gamma = \left(\frac{b}{x\sin\alpha - 6}\right)^{-1}$

$\cos\gamma = \frac{b}{x\cos\alpha + 9}$ $\cos\theta = \cos\beta\cdot\cos\gamma - \sin\beta\cdot\sin\gamma$

$$\cos\theta = \left(\frac{a\cdot b}{(x\cos\alpha+9)^2}\right)^{-1} - \left(\frac{a\cdot b}{(31 - x\sin\alpha)(x\sin\alpha - 6)}\right)^{-1} = \text{~~}ab\,\frac{(31-x\sin\alpha)(x\sin\alpha-6)-(x\cos\alpha+9)^2}{(x\cos\alpha+9)^2(31-x\sin\alpha)(x\sin\alpha-6)}\text{~~}$$

$c + d = x\cos\alpha + 9$



$$\frac{b\cdot h}{2} = \frac{(x\cos\alpha + 9)\,25}{2} \qquad h = \frac{25(x\cos\alpha + 9)}{b}$$

$$\frac{d}{25} = \cos\varphi \qquad d = 25\cos\varphi$$

$$\cos\theta = \frac{(x\cos\alpha + 9)^2}{ab} - \frac{(31 - x\sin\alpha)(x\sin\alpha - 6)}{ab}$$

$$\sin\gamma = \frac{y_1}{b} = \frac{25 - y_2}{b} = \frac{25 - 31 + x\sin\alpha}{b} = \frac{x\sin\alpha - 6}{b}$$

$$\cos\Theta = \frac{(x\cos\alpha + 9)^2 - (31x\sin\alpha + 186 - x^2\sin^2\alpha + 6x\sin\alpha)}{ab}$$

$$\cos\Theta = \frac{(x\cos\alpha + 9)^2 - 37x\sin\alpha + x^2\sin^2\alpha + 186}{ab}$$

$$a^2 + b^2 = 2(9 + x\cos\alpha)^2 + 961 - 2\cdot 31x\sin\alpha + x^2\sin^2\alpha$$
$$+ x^2\sin^2\alpha - 12x\sin\alpha + 36 = 2(9 + x\cos\alpha) + 997 - 74x\sin\alpha + 2x^2\sin^2\alpha$$
$$= 2(9 + x\cos\alpha)^2 + 372 - 2\cdot 37x\sin\alpha + 2x^2\sin^2\alpha + 625 \quad (997)$$
$$= 2\left[(9 + x\cos\alpha)^2 + 186 - 37x\sin\alpha + x^2\sin^2\alpha\right] + 625$$

$$\circledast = \frac{a^2 + b^2 - 625}{2}$$

$$\cos\Theta = \frac{a^2 + b^2 - 625}{2ab} \qquad \Theta = \arccos\frac{a^2 + b^2 - 625}{2ab}$$

7. Techniques of Integration

- $\int \sec x \,\mathrm{tg}\, x \, dx = \int \frac{1}{\cos x}\frac{\sin x}{\cos x}\,dx = -\int \frac{d\cos(x)}{\cos^2(x)} = -\frac{\cos^{-2+1}(x)}{-2+1}$

$$= \frac{1}{\cos(x)} = \sec(x) + C$$

$$I = \int \frac{1}{x^2 + a^2}\,dx = \frac{1}{a}\,\mathrm{arctg}\left(\frac{x}{a}\right) + C$$

$y = \mathrm{arctg}\, x \qquad x = \mathrm{tg}\, y \qquad \frac{dx}{dx} = \frac{\sin^2 y + \cos^2 y}{\cos^2 y}\frac{dy}{dx} = (1 + \mathrm{tg}^2 y)\frac{dy}{dx}$

$$1 = (1 + \mathrm{tg}^2 y)\frac{dy}{dx} \qquad \frac{dy}{dx} = \frac{1}{1 + x^2}$$

$$I = \left| \frac{x}{a} = u \quad \frac{dx}{a} = du \right| = \int \frac{1}{a}\frac{1}{\left(\frac{x}{a}\right)^2 + 1}\, a\, d\left(\frac{x}{a}\right) = \frac{1}{a}\int \frac{1}{\left(\frac{x}{a}\right)^2 + 1}\, d\left(\frac{x}{a}\right)$$

$$I = \frac{1}{a}\,\mathrm{arctg}\left(\frac{x}{a}\right) + C$$

- $y = \arcsin(x) \qquad x = \sin y \qquad 1 = \cos y \frac{dy}{dx} \qquad \frac{dy}{dx} = \frac{1}{\cos y}$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}$$

$$\int \frac{1}{\sqrt{1 - x^2}}\,dx = \arcsin(x) + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \left| \frac{x}{a} = u \quad \frac{dx}{a} = du \right| = \int \frac{a}{a}\frac{1}{\sqrt{1 - \frac{x^2}{a^2}}}\, d\left(\frac{x}{a}\right) = \frac{1}{1}\arcsin\left(\frac{x}{a}\right) + C$$

7.1 Integration by Parts

$$\frac{d\, f(x)g(x)}{dx} = f'(x)g(x) + f(x)g'(x) \quad \Big/\int$$

$$\int [f'(x)g(x) + f(x)g'(x)]\,dx = f(x)g(x)$$

$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx}$$

$u = f(x)$ $\quad v = g(x)$

$du = f'(x)dx$ $\quad dv = g'(x)dx$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Ex 1 $I = \int x \sin x\,dx = ?$ (x: u, $\sin x\,dx$: dv)

$u = x \to du = dx$

$v = -\cos x$ $\quad dv = \sin x\,dx$

$I = -x\cdot\cos(x) + \int \cos x\,dx = -x\cos x + \sin x = \sin x - x\cos x$

$f(x) = x$; $f'(x) = 1$

$g'(x) = \sin(x)$; $g(x) = -\cos(x)$

$I = f(x)\cdot g(x) - \int f'(x)g(x)dx = -x\cos(x) + \int 1\cdot\cos(x)\,dx =$
$= \sin(x) - x\cos(x) + C$

$\frac{dI(x)}{dx} = \cos(x) - [\cos(x) - x\sin(x)] = \cos(x) - \cos(x) + x\sin(x)$

Ex 2 $\int \ln(x)\,dx = \left| \begin{matrix} u = \ln x \\ du = \frac{dx}{x} \\ v = x \quad dv = dx \end{matrix} \right| = x\cdot\ln(x) - \int x \frac{dx}{x}$
$= x\ln(x) - x + C$

Ex 3 $I = \int t^2 e^t dt = \left| \begin{matrix} u = t^2 ; du = 2t\,dt \\ dv = e^t dt ;\ v = \int e^t dt = e^t \end{matrix} \right| = t^2 e^t - \int e^t \cdot 2t\,dt$ (the last integral $= 2\cdot I'$)

$I' = \int e^t t\,dt = \left| \begin{matrix} u = t \quad du = dt \\ dv = e^t dt \quad v = e^t \end{matrix} \right| = te^t - \int e^t dt = te^t - e^t$

$I = t^2 e^t - 2[te^t - e^t] = t^2 e^t - 2te^t + 2e^t + C$

Ex 4 $I = \int e^x \sin(x)\,dx = \left| \begin{matrix} u = \sin(x) \quad du = \cos(x)dx \\ dv = e^x dx \Rightarrow v = e^x \end{matrix} \right| \left| \begin{matrix} u = e^x \quad du = e^x dx \\ dv = \sin(x)dx \quad v = \cos(x) \end{matrix} \right|$

$I = \sin(x)\cdot e^x - \int e^x \cos(x)\,dx = \left| \begin{matrix} u = \cos(x) \quad du = -\sin(x)\,dx \\ dv = e^x dx \quad v = e^x \end{matrix} \right|$ (the integral $\int e^x\cos(x)\,dx$ is I_1)

$I_1 = \cos(x)\,e^x + \int e^x \sin(x)\,dx$

$I = \sin(x)e^x - \cos(x)e^x - \int e^x \sin(x)\,dx$ (the last integral is I) $\qquad 2I = \sin(x)e^x - \cos(x)e^x$

$$I = \frac{e^x}{2}\left(\sin(x) - \cos(x)\right) + C$$

$$I=\int e^x\left(\frac{1}{2j}\left(e^{jx}-e^{-jx}\right)\right)dx=\frac{1}{2j}\left(\int e^{x(1+j)}dx-\int e^{+x(1-j)}dx\right)$$

$$I=\frac{1}{2j}\left(\frac{1}{1+j}e^{x(1+j)}-\frac{1}{1-j}e^{x(1-j)}\right)=\frac{e^x}{2j}\left(\frac{e^{jx}}{1+j}-\frac{e^{-jx}}{1-j}\right)$$

$$=\frac{e^x}{2j}\;\frac{(1-j)e^{jx}-(1+j)e^{-jx}}{1-j^2}=\frac{e^x}{4j}\left(e^{jx}-je^{jx}-e^{-jx}-je^{-jx}\right)=$$

$$=\frac{e^x}{2\cdot 2j}\left[\left(e^{jx}-e^{-jx}\right)-j\left(e^{jx}+e^{-jx}\right)\right]=\frac{e^x}{2}\Bigg[\underbrace{\frac{e^{jx}-e^{-jx}}{2j}}_{\sin x}-\underbrace{\frac{e^{jx}+e^{-jx}}{2}}_{\cos x}\Bigg]$$

$$I=\frac{e^x}{2}\left(\sin(x)-\cos(x)\right)$$

$f(x)=F'(x)$ $\qquad f'(x)=\left(e^x\sin(x)\right)'=e^x\cdot\sin(x)+\cos(x)e^x$

$x=0$ $\qquad f'(x)=1\cdot 0+1\cdot 1=1>0 \Rightarrow F(0)$ – MINIMUM

integration by parts of definite integrals

$$\int_a^b f(x)g'(x)dx=f(x)g(x)\Big|_a^b-\int_a^b g(x)f'(x)dx$$

Ex. 5 $I=\int_0^1 \tan^{-1}(x)dx=\int_0^1 \operatorname{arctg}(x)dx=\left|\begin{matrix}u=\operatorname{arctg}(x) & du=\frac{dx}{1+x^2}\\ dv=dx & v=x\end{matrix}\right|$

$$I=x\cdot\operatorname{arctg}(x)\Big|_0^1-\int_0^1\frac{x\,dx}{1+x^2}=\left(1\cdot\frac{\pi}{4}-0\right)-\frac{1}{2}\int_0^1\frac{d(1+x^2)}{1+x^2}=\frac{\pi}{4}-\frac{1}{2}\ln|1+x^2|\Big|_0^1$$

$$I=\frac{\pi}{4}-\frac{1}{2}\left(\ln 2-\ln 1\right)=\frac{\pi}{4}-\frac{1}{2}\ln(2)=\frac{1}{2}\left(\frac{\pi}{2}-\ln(2)\right)$$

Ex. 6 $I=\int\sin^n(x)dx=-\frac{1}{n}\cos(x)\sin^{n-1}(x)+\frac{n-1}{n}\int\sin^{n-2}(x)dx$

$$I=\int\sin^{n-1}(x)\sin(x)dx=\left|\begin{matrix}dv=\sin^{n-1}(x)dx\\ v=\int\sin^{n-1}(x)dx=\left|\begin{matrix}q=\sin(x)\\ dq=\cos(x)dx\end{matrix}\right|?\end{matrix}\right|$$

$v=\int\sin(x)dx=-\cos(x)$ $\qquad u=\sin^{n-1}(x)$ $\qquad du=(n-1)\sin^{n-2}(x)\cdot\cos(x)dx$

$$\rightarrow\quad v=\int q^{n-1}\frac{dq}{\cos x}=\int\frac{q^{n-1}}{\sqrt{1-q^2}}dq$$

$u=\sin^n(x)$; $du=-n\sin^{n-1}(x)\cdot\cos(x)dx$

$dv=dx$; $v=x$

$$I=\int\sin^n(x)dx=\int u\,dv=\sin^n(x)\cdot x+\int x\cdot n\sin^{n-1}(x)\cos(x)dx\,?$$

$$I = \int \sin^n(x)\,dx \qquad \int \sin^n(x)\,dx = -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$$

$$(1+z)^n = \sum_{k=0}^{\infty} \binom{n}{k} z^k$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

$$(a+b)^n = a^n\left(1+\frac{b}{a}\right)^n = a^n \sum_{k=0}^{\infty}\binom{n}{k}\left(\frac{b}{a}\right)^k = \sum_{k=0}^{\infty}\binom{n}{k} a^{n-k} b^k$$

$$I = \int \sin^{n-1}(x)\sin(x)\,dx$$

1° $u = \sin(x) \quad du = \cos(x)\,dx$

$dv = \sin^{n-1}(x)\,dx \quad v = \int \sin^{n-1}(x)\,dx$

$$I = \sin(x)\int \sin^{n-1}(x)\,dx - \int \cos(x)\left(\int \sin^{n-1}(x)\,dx\right)dx \qquad ?$$

2° $u = \sin^{n-1}(x) \quad du = (n-1)\sin^{n-2}(x)\cdot\cos(x)\,dx$

$dv = \sin(x)\,dx \quad v = -\cos(x)$

$$I = -\sin^{n-1}(x)\cdot\cos(x) + \underbrace{\int \cos(x)\cdot(n-1)\sin^{n-2}(x)\cos(x)\,dx}_{I_1} \quad (*)$$

3° $u = \sin^n(x) \quad du = n\sin^{n-1}(x)\cdot\cos(x)\,dx$

$dv = dx \quad v = x$

$$I = x\sin^n(x) - \int x\cdot n\sin^{n-1}(x)\cos(x)\,dx \qquad ?$$

$$I_1 = \int (n-1)\cos^2(x)\sin^{n-2}(x)\,dx = (n-1)\int \frac{\cos^2(x)}{\sin^2(x)}\sin^n(x)\,dx$$

$$y = ctg^2(x) \quad \frac{dy}{dx} = -\frac{1}{\sin^2(x)} \qquad \left(\frac{\cos(x)}{\sin(x)}\right)' = \frac{-\sin(x)\sin(x) - \cos(x)\cos(x)}{\sin^2(x)}$$

$$v = \int ctg^2(x)\,dx = \int \frac{\cos^2(x)}{\sin^2(x)}dx = \int \frac{1-\sin^2(x)}{\sin^2(x)}dx = \int\frac{dx}{\sin^2(x)} - x = -ctg(x) - x$$

$u = \sin^n(x) \quad du = n\sin^{n-1}(x)\cos(x)\,dx$

$$\frac{I_1}{n-1} = \sin^n(x)\cdot\left(\frac{\cos^2(x)}{\sin^2(x)} - x\right) + \int \left(\frac{\cos(x)}{\sin(x)} + x\right) n\sin^{n-1}(x)\cos(x)\,dx$$

$$\frac{I_1}{n-1} = \sin^n(x)\left(ctg^2(x) - x\right) + \int x\sin^{n-1}(x)\cos(x)\,dx + n\underbrace{\int \cos^2(x)\sin^{n-2}(x)\,dx}_{\frac{I_1}{n-1}}$$

$$\frac{I_1}{n-1} - \frac{nI_1}{n-1} = \frac{(1-n)I_1}{n-1} = -I_1 = \sin^n(x)\left(ctg^2(x) - x\right) + \underbrace{\int x\sin^{n-1}(x)\cos(x)\,dx}_{I_2}$$

$$I_2 = \int x\sin^{n-1}(x)\cos(x)\,dx = \left|\begin{array}{ll} u = \sin^{n-1}(x) & du = (n-1)\sin^{n-2}(x)\,dx \\ dv = x\cos(x)\,dx & v = \int x\cos(x)\,dx = \cos(x) + x\sin(x) \end{array}\right|$$

$I_2 = \sin^{n-1}(x)\left(\cos(x) + x\sin(x)\right) - (n-1)\int [\cos(x) + x\sin(x)]\sin^{n-2}(x)\,dx$ (?)

$I_2 = \sin^{n-1}(x)\left(\cos(x) + x\sin(x)\right) - (n-1)\int \cos(x)\sin^{n-2}(x)\,dx - (n-1)\int x\sin^{n-1}(x)\,dx$

(*) $I = -\sin^{n-1}(x)\cos(x) + \int (n-1)\sin^{n-2}(x)\cos^2(x)\,dx = \left| \cos^2(x) = 1 - \sin^2(x) \right| =$

$= -\sin^{n-1}(x)\cos(x) + \int (n-1)\sin^{n-2}(x)\,dx - (n-1)\int \sin^n(x)\,dx$ (I)

$I + (n-1)I = I + nI - I = \underline{nI} = -\sin^{(n-1)}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\,dx$

$$\boxed{I = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx}$$

$I = \int \sin^n(x)\,dx$

REDUCTION FORMULA

Ex. 1 $I = \int x\ln x\,dx = \left| \begin{matrix} u = \ln x & du = \frac{1}{x}dx \\ dv = x\,dx & v = \frac{x^2}{2} \end{matrix} \right| = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx$

$I = \frac{x^2}{2}\ln(x) - \frac{1}{2}\int x\,dx = \frac{x^2}{2}\ln(x) - \frac{x^2}{4} = \frac{x^2}{2}\left(\ln(x) - \frac{1}{2}\right)$

Ex. 2 $I = \int \theta\sec^2\theta\,d\theta = \left| \begin{matrix} (?)\ u = \sec^2\theta & du = \left(\frac{1}{\cos\theta}\right)' d\theta = -2\cos^{-3}\theta(-\sin\theta) \\ dv = \sec^2\theta\,d\theta & v = \int \sec^2\theta\,d\theta = tg\theta + C \\ u = \theta & du = d\theta \end{matrix} \right.$

$I = \theta\cdot tg\theta - \int tg\theta\,d\theta = \theta\,tg\theta - \int \frac{\sin\theta}{\cos\theta}\,d\theta = \theta\,tg\theta + \int \frac{d(\cos\theta)}{\cos\theta}$

$I = \theta\,tg\theta + \ln|\cos\theta|$

Ex 5 $\int r e^{r/2}\,dr = \left| \begin{matrix} u = r & du = dr \\ dv = e^{r/2}dr & v = \int e^{r/2}dr = 2e^{r/2} \end{matrix} \right|$

$2\cdot r e^{r/2} - \int 2e^{r/2}\,dr = 2re^{r/2} - 4e^{r/2} = 2e^{r/2}(r - 2) + C$

Ex. 7 $I = \int x^2\sin(\pi x)\,dx = \left| \begin{matrix} u = x^2 & du = 2x\,dx \\ v = \int \sin(\pi x)\,dx = \frac{1}{\pi}(-\cos(\pi x)) \end{matrix} \right|$

$I = -\frac{x^2}{\pi}\cos(\pi x) + \frac{2}{\pi}\underbrace{\int x\cdot\cos(\pi x)\,dx}_{I_1}$

$I_1 = \left| \begin{matrix} u = x & du = dx \\ dv = \cos(\pi x)\,dx & v = \int \cos(\pi x)\,dx = \frac{1}{\pi}\sin(\pi x) \end{matrix} \right|$

$I_1 = \frac{1}{\pi}x\sin(\pi x) - \frac{1}{\pi}\int \sin(\pi x)\,dx = \frac{1}{\pi}x\sin(\pi x) + \frac{1}{\pi^2}\cos(\pi x)$

$I = -\frac{x^2}{\pi}\cos(\pi x) + \frac{2}{\pi}\left(\frac{1}{\pi}x\sin(\pi x) + \frac{1}{\pi^2}\cos(\pi x)\right)$

$I = \frac{1}{\pi}\left(-x^2\cos(\pi x) + \frac{2}{\pi}x\sin\pi x + \frac{2}{\pi^2}\cos(\pi x)\right) + C$

Ex. 9 $I=\int \ln(2x+1)\,dx = \left| \begin{array}{ll} u=\ln(2x+1), & dv=dx \\ du=\frac{2}{2x+1}dx & v=x \end{array}\right| =$

$= x\ln(2x+1) - \underbrace{\int \frac{2x}{2x+1}dx}_{I_1}$

$I_1 = \int \frac{2x+1}{2x+1}dx - \int \frac{dx}{2x+1} = x - \frac{1}{2}\int \frac{d(2x+1)}{2x+1} = x - \frac{1}{2}\ln(2x+1)$

$I = x\ln(2x+1) - x + \frac{1}{2}\ln(2x+1) = \frac{1}{2}\ln(2x+1)(2x+1) - x + C$

Ex. 11 $I=\int \operatorname{arctg}(4t)\,dt = \left| \begin{array}{ll} u=\operatorname{arctg}(4t); & du=\frac{4}{1+16t^2}dt \\ dv=dt & v=t \end{array}\right|$

$I = t\operatorname{arctg}(4t) - 4\underbrace{\int \frac{t\,dt}{1+16t^2}}_{I_1} = t\operatorname{arctg}(4t) - \frac{4}{32}\ln(16t^2+1)$

$I_1 = \frac{1}{32}\int \frac{d(16t^2+1)}{1+16t^2} = \frac{1}{32}\ln(1+16t^2)$

$= t\operatorname{arctg}(4t - \frac{1}{8}\ln(16t^2+1) + C$

Ex. 13 $I=\int \ln^2(x)\,dx$ $\left| \begin{array}{ll} u=\ln^2(x) & dv=dx \\ du=2\frac{\ln(x)}{x}dx & v=x \end{array}\right|$

$I = x\ln^2(x) - \int x\cdot 2\frac{\ln(x)}{x}dx = x\ln^2(x) - 2\underbrace{\int \ln(x)\,dx}_{I_1}$

~~$3I = x\ln^2(x) \quad I = \frac{x}{3}\ln^2(x)+C$~~

$I_1 = \int \ln(x)\,dx = \left| \begin{array}{ll} u=\ln(x) & dU=\frac{1}{x}dx \\ dV=dx & V=x \end{array}\right| = x\ln(x) - \int x\cdot\frac{dx}{x} = x\ln(x) - x$

$I = x\ln^2(x) - 2(x\ln(x) - x) = x\ln^2(x) - 2x\ln(x) + 2x$

Ex. 15 $I=\int e^{2\theta}\sin(3\theta)\,d\theta$ $\quad \begin{array}{ll} u=\sin(3\theta) & du=3\cdot\cos(3\theta)\,d\theta \\ dv=e^{2\theta}d\theta & v=\int e^{2\theta}d\theta=\frac{1}{2}e^{2\theta} \end{array}$

$I = \frac{1}{2}e^{2\theta}\sin(3\theta) - \frac{1}{2}\int e^{2\theta}\cdot 3\cos(3\theta)\,d\theta = \frac{1}{2}e^{2\theta}\sin(3\theta) - \frac{3}{2}\underbrace{\int e^{2\theta}\cos(3\theta)\,d\theta}_{I_1}$

$I_1 = \left| \begin{array}{ll} u=\cos(3\theta) & v=\int e^{2\theta}d\theta \\ du=-3\sin(3\theta)\,d\theta & v=\frac{1}{2}e^{2\theta} \end{array}\right| = \frac{1}{2}e^{2\theta}\cos(3\theta) + \frac{3}{2}\underbrace{\int e^{2\theta}\sin(3\theta)\,d\theta}_{I}$

$I = \frac{1}{2}e^{2\theta}\sin(3\theta) - \frac{3}{2}\cdot\frac{1}{2}e^{2\theta}\cos(3\theta) - \frac{9}{4}I;\quad \frac{13I}{4_2} = \frac{1}{2}e^{2\theta}\sin(3\theta) - \frac{3}{4_2}e^{2\theta}\cos(3\theta)$

$I = \frac{2}{13}e^{2\theta}\sin(3\theta) - \frac{3}{13}e^{2\theta}\cos(3\theta)$

exc. 17 $\int \sinh(x)\,dx = \frac{1}{2}\int (e^{x} - e^{-x})\,dx = \frac{1}{2}(e^{x} + e^{-x}) = \cosh(x)$

$I = \int y \sinh y\, dy = \left| \begin{matrix} u = y & du = dy \\ v = \int \sinh y\, dy = \cosh(y) \end{matrix} \right| = y\cosh(y) - \int \cosh(y)\, dy$

$\boxed{I = y\cosh(y) - \sinh(y)}$

exc. 22 $I = \int_1^4 \sqrt{t}\ln(t)\,dt = \left| \begin{matrix} u = \ln(t) \\ du = \frac{1}{t}dt \end{matrix} \right.$ $\quad v = \int \sqrt{t}\,dt = \frac{1}{\frac{3}{2}}\cdot\sqrt{t^3} = \frac{2t\sqrt{t}}{3}$, $\quad v = \frac{2t\sqrt{t}}{3}$

$I = \frac{2t\sqrt{t}\ln(t)}{3}\Big|_1^4 - \int_1^4 \frac{2t\sqrt{t}}{3}\frac{1}{t}dt = \left(\frac{2\cdot 4\cdot 2\ln 4}{3} - \frac{2\cdot 0}{3}\right) - \frac{2}{3}\int_1^4 \sqrt{t}\,dt =$

$= \frac{2}{3}\left(7 - \frac{2}{3}(4\cdot 2 - 1)\right) = \frac{2}{3}\left(7 - \frac{14}{3}\right)$

$I = \left(\frac{32\cdot\ln(2)}{3} - 0\right) - \frac{2}{3}\,\frac{2}{3}t\sqrt{t}\Big|_1^4 = \frac{32}{3}\ln(2) - 0 - \frac{4}{9}(8-1)$

$I = \frac{32}{3}\ln(2) - 0 - \frac{28}{9} = \frac{32}{3}\ln(2) - \frac{28}{9}$

exc. 25 $I = \int_0^{1/2} \arccos(x)\,dx$ $\quad u = \arccos(x)$, $du = -\frac{1}{\sqrt{1-x^2}}dx$ $\quad dv = dx$, $v = x$

$y = \arccos x$; $x = \cos y$; $\quad 1 = -\sin(y)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-x^2}}$

$I = x\cdot\arccos(x)\Big|_0^{1/2} + \int_0^{1/2} \frac{x\,dx}{\sqrt{1-x^2}} = \frac{1}{2}\frac{\pi}{3} - 0 + \frac{(-1)}{2}\int_0^{1/2} \frac{d(1-x^2)}{\sqrt{1-x^2}}$

$I = \frac{\pi}{6} - \frac{1}{2}\,2\sqrt{(1-x^2)}\Big|_0^{1/2} = \frac{\pi}{6} - \left(\sqrt{\left(1-\frac{1}{4}\right)} - \sqrt{1}\right) = \frac{\pi}{6} - \left(\sqrt{\left(\frac{3}{4}\right)} - 1\right)$

$\boxed{I = \frac{\pi}{6} - \frac{\sqrt{3}}{2} + 1}$

exc. 29 $I = \int \cos(\ln x)\,dx$ $\quad u = \cos(\ln x)$, $du = -\sin(\ln x)\cdot\frac{1}{x}dx$ $\quad dv = dx$, $v = x$

$I = x\cdot\cos(\ln x) + \underbrace{\int \sin(\ln x)\,dx}_{I_1}$ $\quad \left| \begin{matrix} U = \sin(\ln x) & dV = dx \\ dU = \cos(\ln x)\frac{1}{x} & V = x \end{matrix} \right|$

$I_1 = x\sin(\ln x) - \underbrace{\int \cos(\ln(x))\,dx}_{I} = x\sin(\ln x) - I$

$I = x\cos(\ln x) + x\sin(\ln x) - I \Rightarrow I = \frac{x}{2}(\cos(\ln x) + \sin(\ln x))$

ALTERNATIVE:

$I = \int \cos(\ln x)\,dx \qquad w = \ln x \quad dw = \frac{dx}{x} \qquad x = e^w \quad dw = e^{-w} dx$

$dx = e^w\,dw$

$I = \int e^w \cos(w)\,dw = \left| \begin{matrix} u = \cos w & v = \int e^w dw \\ du = -\sin(w)dw & v = e^w \end{matrix} \right| =$

$= e^w \cos w + \underbrace{\int e^w \sin(w)\,dw}_{I_1} \qquad \begin{matrix} U = \sin w & dU = \cos w\,dw \\ dV = e^w dw & V = e^w \end{matrix}$

$I_1 = e^w \sin w - \underbrace{\int e^w \cos w\,dw}_{I} \; ; \; I = e^w \cos w + e^w \sin w - I$

$I = \frac{e^w}{2}\cos w + \frac{e^w}{2}\sin w = \frac{x}{2}\left(\cos(\ln x) + \sin(\ln x)\right)$ #

EXC. 30 $I = \int_0^1 \frac{r^3}{\sqrt{4+r^2}}\,dr \qquad \left| \begin{matrix} u = r^3 \quad du = 3r^2 dr \\ v = \int \frac{dr}{\sqrt{4+r^2}} = \frac{1}{2}\cdot ar \end{matrix} \right.$

$u = \frac{r^3}{\sqrt{4+r^2}} \qquad \frac{du}{dr} = \frac{3r^2\sqrt{4+r^2} - r^3\,\frac{1}{2}(4+r^2)^{\frac{1}{2}-1}}{4+r^2} =$

$= \frac{6r^2(4+r^2) - r^3}{(4+r^2)\sqrt{4+r^2}}$?

#

$u = r^2 \quad du = 2r\,dr$

$v = \int \frac{r\,dr}{\sqrt{4+r^2}} = \frac{1}{2}\int \frac{d(4+r^2)}{\sqrt{4+r^2}} = \frac{1}{2}\,\frac{1}{-\frac{1}{2}+1}\cdot\sqrt{4+r^2} = \sqrt{4+r^2}$

$I = r^2\sqrt{4+r^2}\Big|_0^1 - 2\int_0^1 \sqrt{4+r^2}\,r\,dr = 1\sqrt{5} - \int_0^1 \sqrt{4+r^2}\,d(4+r^2)$

$I = \sqrt{5} - \frac{2}{3}(4+r^2)\sqrt{4+r^2}\Big|_0^1 = \sqrt{5} - \frac{2}{3}(5\cdot\sqrt{5} - 4\cdot 2) = \sqrt{5} - \frac{10}{3}\sqrt{5} + \frac{16}{3}$

$I = \frac{3-10}{3}\sqrt{5} + \frac{16}{3} = \frac{16}{3} - \frac{7}{3}\sqrt{5}$

EXC. 31 $I = \int_1^2 x^4 (\ln x)^2\,dx \qquad \begin{matrix} u = x^4 \quad du = 3x^3 dx \\ v = \int \ln^2(x)\,dx = I_1 \end{matrix}$

$I_1 = \left| \begin{matrix} u = \ln^2(x) & dv = dx \\ du = 2\ln(x)\cdot\frac{1}{x} & v = x \end{matrix} \right| = x\ln^2(x) - 2\int \ln(x)\,dx$

$\int \ln(x)\,dx = \left| \begin{matrix} u = \ln x & v = x \\ du = \frac{1}{x}dx \end{matrix} \right| = x\ln(x) - \int x\frac{dx}{x} = x\ln(x) - x$

$3I$

$I_1 = x\ln^2(x) - 2x\ln(x) + 2x = v$

$I = x^4(x\ln^2(x) - 2x\ln(x) + 2x) - \int 3x^3(x\ln^2(x) - 2x\ln x + 2x)\,dx$

$4I = \left(x^5\ln^2(x) - 2x^5\ln(x) + 2x^5\right)\Big|_1^2 + 6\underbrace{\int x^4 \ln x\,dx}_{I_2} - 6\underbrace{\int x^4\,dx}_{6\frac{x^5}{5}\big|_1^2}$

$$I_2=\int x^4 \ln x\,dx = \left| \begin{matrix} v=\ln x & v=\int x^4 dx \\ dv=\frac{1}{x}dx & v=\frac{x^5}{5}\end{matrix}\right| = \frac{x^5}{5}\ln x - \int \frac{x^5}{5}\cdot\frac{1}{x}dx$$

$$I_2=\frac{x^5}{5}\ln(x)-\frac{1}{5}\int x^4dx=\frac{x^5}{5}\ln(x)-\frac{x^5}{25}=\frac{x^5}{5}\left(\ln(x)-\frac{1}{5}\right)\Big|_1^2$$

$$4I=x^5\left(\ln^2(x)-2\ln(x)+2\right)\Big|_1^2+\frac{6x^5}{5}\left(\ln(x)-\frac{1}{5}\right)\Big|_1^2-\frac{6x^5}{5}\Big|_1^2$$

$$I=\frac{x^5}{4}\left(\ln^2(x)-2\ln(x)+2\right)\Big|_1^2+\frac{3\cdot 6x^5}{2\cdot 4\cdot 5}\left(\ln(x)-\frac{6}{5}\right)\Big|_1^2$$

$$I=\frac{x^5}{2}\left(\frac{1}{2}\ln^2(x)-\ln(x)+1\quad\frac{3}{5}\ln(x)-\frac{18}{25}\right)\Big|_1^2$$

$$I=\frac{x^5}{2}\left(\frac{1}{2}\ln^2(x)-\frac{2}{5}\ln(x)+\frac{7}{25}\right)\Big|_1^2$$

$$I=\frac{32}{2}\left(\frac{1}{2}\ln^2(2)-\frac{2}{5}\ln(2)+\frac{7}{25}\right)-\frac{1}{2}\left(0-0+\frac{7}{25}\right)$$

(16)

$$\begin{array}{r}224\\175\\\hline 49\end{array}$$

$$I=8\ln^2(2)-\frac{32}{5}\ln(2)+\frac{112}{25}-\frac{7}{50}=8\ln^2(2)-\frac{32}{5}\ln(2)+\frac{49}{50}$$

ALTERNATIVE

$$I=\int_1^2 x^4\ln^2(x)dx \qquad u=\ln^2(x)\quad du=2\frac{\ln(x)}{x}dx \qquad v=\int x^4dx=\frac{x^5}{5}$$

$$I=\frac{x^5}{5}\ln^2(x)\Big|_1^2-\frac{2}{5}\int x^5\ln(x)dx = \frac{32}{5}\ln^2(2)-\frac{2}{5}\underbrace{\int x^4\ln(x)dx}_{I_1}$$

$$I_1=\left|\begin{matrix} v=\ln(x) & V=\int x^4 dx\\ dv=\frac{1}{x}dx & V=\frac{x^5}{5}\end{matrix}\right| = \frac{x^5}{5}\cdot\ln(x)-\int\frac{x^4}{5}\frac{1}{x}dx=\frac{x^5}{5}\ln(x)-\frac{x^5}{25}$$

$$I=\frac{32}{5}\ln^2(2)-\frac{2}{5}\left(\frac{x^5}{5}\ln(x)-\frac{x^5}{25}\right)\Big|_1^2=\frac{32}{5}\ln^2(2)-\frac{2x^5}{25}\left(\ln(x)-\frac{1}{5}\right)\Big|_1^2$$

$$=\frac{32}{5}\ln^2(2)-\frac{2\cdot 32}{25}\left(\ln(2)-\frac{1}{5}\right)+\frac{2}{25}\left(0-\frac{1}{5}\right)$$

$$=\frac{32}{5}\ln^2(2)-\frac{64}{25}\ln(2)+\frac{64}{125}-\frac{2}{125}=\frac{32}{5}\ln^2(2)-\frac{64}{25}\ln(2)+\frac{62}{125}$$

32 $I=\int_0^t e^s\sin(t-s)ds \qquad u=\sin(t-s)\quad du=-\cos(t-s)ds \qquad v=\int e^s ds=e^s$

$$I=e^s\sin(t-s)\Big|_0^t+\int_0^t\cos(t-s)e^s ds=\left(e^t\cdot 0-e^0\sin t\right)+\underbrace{\int_0^t e^s\cos(t-s)ds}_{I_1}$$

$$I_1=\left|\begin{matrix} v=\cos(t-s) & v=e^s\\ dv=+\sin(t-s)ds & \end{matrix}\right|=e^s\cdot\cos(t-s)\Big|_0^t-\underbrace{\int_0^t e^s\sin(t-s)ds}_{I}$$

$$2I=-\sin t+e^s\cdot\cos(t-s)\Big|_0^t=-\sin t+\left(e^t\cdot 1-1\cos t\right)=e^t-\sin t-\cos t$$

$$I=\frac{1}{2}\left(e^t-\sin t-\cos t\right)$$

Exc. 33 $I = \int \sin\sqrt{x}\, dx$ $\quad w = \sqrt{x}$, $dw = \frac{1}{2}\frac{1}{\sqrt{x}} dx = \frac{1}{2w} dx$

$dx = 2w\, dw$; $I = \int 2w \sin w\, dw = 2\int w \sin(w)\, dw$

$u = w \quad du = dw \quad v = \int \sin w\, dw = -\cos(w)$

$I = 2\left[-w\cos w + \int \cos w\, dw\right] = -2w\cos w + 2\sin w \Rightarrow$

$I = 2\sin\sqrt{x} - 2\sqrt{x}\cos\sqrt{x}$

Exc 35 $I = \int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3 \cos(\theta^2)\, d\theta$ $\quad w = \theta^2 \quad \theta = \sqrt{w}$, $dw = 2\theta\, d\theta = 2\sqrt{w}\, d\theta$, $d\theta = dw/2\sqrt{w}$

$I = \int \sqrt{w^3} \cos(w) \frac{dw}{2\sqrt{w}} = \frac{1}{2}\int w\cos(w)\, dw = \left| \begin{matrix} u = w & du = dw \\ v = \sin w & \end{matrix} \right|$

$I = \frac{1}{2}\left(w\sin w - \int \sin w\, dw\right) = \frac{1}{2}\left(w\sin w + \cos(w)\right)$

$I = \frac{1}{2}\left(\theta^2 \sin\theta^2 - \cos(\theta^2)\right)\Big|_{\sqrt{\pi/2}}^{\sqrt{\pi}} = \frac{1}{2}\left(\pi \sin\pi + \cos\pi\right) - \frac{1}{2}\left(\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2}\right)$ (with $\cos\pi = -1$)

$I = \frac{1}{2}(-1) - \frac{1}{2}\left(\frac{\pi}{2}\cdot 1 - 0\right)$ $\quad \boxed{I = -\frac{1}{2} - \frac{\pi}{4}}$?

$\theta = \sqrt{\frac{\pi}{2}} \quad w = \theta^2 = \frac{\pi}{2}$; $\theta = \sqrt{\pi} \quad w = \theta^2 = \sqrt{\pi}$)

$I = \frac{1}{2}\left(w\sin w - \cos w\right)\Big|_{\pi/2}^{\pi} = \frac{1}{2}\left(\pi\sin\pi + \cos\pi\right) - \frac{1}{2}\left(\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2}\right)$ (with $\sin\pi = 0$, $\cos\pi = -1$, $\sin\frac{\pi}{2} = 1$, $\cos\frac{\pi}{2} = 0$)

$\boxed{I = -\frac{1}{2} - \frac{\pi}{4}}$

Exc. 38 $I = \int \underbrace{x^{3/2}\ln(x)}_{f(x)}\, dx = \left| \begin{matrix} u = \ln(x) & v = \int x^{3/2} dx \\ du = \frac{1}{x}dx & v = \frac{2}{5}x^{5/2} \end{matrix} \right|$

$I = \frac{2}{5}x^{5/2}\ln(x) - \frac{2}{5}\int x^{3/2}\cdot\frac{1}{x}\, dx = \frac{2}{5}x^{5/2}\ln(x) - \frac{2}{5}\cdot\frac{2}{5}\cdot x^{5/2}$

$I = \frac{2}{5}x^{5/2}\left(\ln(x) - \frac{2}{5}\right)$

$f'(x) = \frac{3}{2}x^{\frac{1}{2}}\ln(x) + x^{3/2}\frac{1}{x} = \frac{3}{2}\sqrt{x}\ln(x) + \sqrt{x}$ $\quad f'(1) = 1 > 0$ μηκηση

$\int f(x)\, dx = F(x) \quad f(x) = F'(x) \quad f'(x) = F''(x)$

Exc. 40 $I = \int x^3 e^{x^2} dx = \left| \begin{matrix} x^2 = w & x = \sqrt{w} & dx = \frac{dw}{2\sqrt{w}} \\ 2x\,dx = dw & 2\sqrt{w}\,dx = dw & \end{matrix} \right|$

$I = \int w\sqrt{w}\, e^{w} \frac{dw}{2\sqrt{w}} = \frac{1}{2}\int w e^{w} dw = \left| \begin{matrix} u = w & du = dw \\ v = \int e^{w} dw = e^{w} & \end{matrix} \right|$

$I = \frac{1}{2}\left[w e^{w} - e^{w}\right] = \frac{e^{w}}{2}(w - 1) = \frac{e^{x^2}}{2}(x^2 - 1)$

Exc 41 $\boxed{\int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx}$

ⓐ
$I = \int \sin^2(x)\,dx = -\frac{1}{2}\sin(x)\cos(x) + \frac{1}{2}\int \sin^0(x)\,dx = \frac{x}{2} - \frac{1}{2}\sin(x)\cos(x)$

$\sin(2x) = 2\sin x\cos x \qquad \sin x\cos x = \frac{\sin(2x)}{2}$

$\boxed{I = \frac{x}{2} - \frac{1}{4}\sin(2x)}$ $\qquad \frac{x}{2} - \frac{1}{4}\sin(2x)$

ⓑ $\int \sin^4(x)\,dx = -\frac{1}{4}\sin^3(x)\cos(x) + \frac{3}{4}\int \sin^2(x)\,dx$

$I = -\frac{1}{4}\sin^3(x)\cos(x) + \frac{3x}{8} - \frac{3}{16}\sin(2x) + C$

Exc 42 ⓐ Proof $\boxed{I = \int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx}$

$I = \int \cos^{n-1}(x)\cos(x)\,dx\,; \quad u = \cos^{n-1}(x),\ du = -(n-1)\cos^{n-2}(x)\sin(x)\,dx$
$v = \int \cos(x)\,dx = \sin(x)$

$I = \sin(x)\cos^{n-1}(x) + (n-1)\int \cos^{n-2}(x)\sin^2(x)\,dx = \sin(x)\cos^{n-1}(x) + (n-1)\int \cos^{n-2}(x)\,dx - (n-1)I$

$I + (n-1)I = I + nI - I = \sin(x)\cos^{n-1}(x) + (n-1)\int \cos^{n-2}(x)\,dx$

$I = \frac{1}{n}\sin(x)\cos^{n-1}(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$

ⓑ $\int \cos^2(x)\,dx = \frac{1}{2}\cos(x)\sin(x) + \frac{1}{2}\int dx = \frac{1}{2}x + \frac{1}{2}\cos(x)\sin(x)$

$= \frac{1}{2}x + \frac{1}{4}\sin(2x)$

ⓒ $I = \int \cos^4(x)\,dx = \frac{1}{4}\cos^3(x)\sin(x) + \frac{3}{4}\int \cos^2(x)\,dx$

$I = \frac{1}{4}\cos^3(x)\sin(x) + \frac{3}{8}x + \frac{3}{16}\sin(2x) + C$

Exc. 43

$\int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$

ⓐ SHOW THAT: $\int_0^{\pi/2} \sin^n(x)\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}(x)\,dx$

$I = \int_0^{\pi/2} \sin^n(x)\,dx = \underbrace{-\frac{1}{n}\sin^{n-1}(x)\cos(x)\Big|_0^{\pi/2}} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}(x)\,dx$

$I_1 = -\frac{1}{n}\left(1^{(n-1)}\cdot 0 - 0^{n-1}\cdot 1\right) = 0 \qquad I = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}(x)\,dx$

$I = \int_0^{\pi/2} \sin^n(x)\,dx = \Big|\ u = \sin^{n-1}(x),\ du = (n-1)\sin^{n-2}(x)\cos(x)\,dx \qquad dv = \sin(x)\,dx,\ v = \int \sin(x)\,dx = -\cos(x)$

$I = -\sin^{n-1}(x)\cos(x)\Big|_0^{\pi/2} + (n-1)\int_0^{\pi/2} \cos^2(x)\sin^{n-2}(x)\,dx = (n-1)\int_0^{\pi/2} \sin^{n-2}(x)\,dx - (n-1)\int_0^{\pi/2} \sin^n(x)\,dx$

$I + (n-1)I = I + nI - I = nI \Rightarrow \boxed{I = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}(x)\,dx}$

ⓑ $I_1 = \int_0^{\pi/2} \sin^3(x)\,dx = \frac{3-1}{3}\int_0^{\pi/2} \sin(x)\,dx = -\frac{2}{3}\cos(x)\Big|_0^{\pi/2} = -\frac{2}{3}(0-1) = \frac{2}{3}$

$$I=\int_0^{\pi/2}\sin^n(x)\,dx = \frac{n-1}{n}\int_0^{\pi/2}\sin^{n-2}(x)\,dx \qquad \int_0^{\pi/2}\sin^3(x)\,dx = \frac{2}{3}$$

$$I_2 = \int_0^{\pi/2}\sin^5(x)\,dx = \frac{4}{5}\int_0^{\pi/2}\sin^3(x)\,dx = \frac{4}{5}\cdot\frac{2}{3} = \frac{8}{15}$$

$$I_3 = \int_0^{\pi/2}\sin^7(x)\,dx = \frac{7-1}{7}\underbrace{\int_0^{\pi/2}\sin^5(x)\,dx}_{I_2} = \frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}$$

$$I_n = \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{n-1}{n} \qquad n = 3,5,7,\dots,2k+1$$

$$\int_0^{\pi/2}\sin^{2n+1}(x)\,dx = \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{2n}{2n+1}$$

PI:

$$\int_0^{\pi/2}\sin^{2k+1}(x)\,dx = \frac{2k+1-1}{2k+1}\int_0^{\pi/2}\sin^{2k-1}(x)\,dx = \frac{2k}{2k+1}\cdot\frac{2k-2}{2k-1}\int_0^{\pi/2}\sin^{2k-3}(x)\,dx$$

$$= \frac{2k}{2k+1}\cdot\frac{2k-2}{2k-1}\cdots\frac{4}{5}\cdot\frac{2}{3}$$

44 $$\int_0^{\pi/2}\sin^{2n}(x)\,dx = \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots2n}\cdot\frac{\pi}{2}$$

$$\int_0^{\pi/2}\sin^2(x)\,dx = \frac{1}{2}\int_0^{\pi/2}\sin^0(x)\,dx = \frac{1}{2}\,\frac{\pi}{2}$$

$$\int_0^{\pi/2}\sin^4(x)\,dx = \frac{3}{4}\int_0^{\pi/2}\sin^2(x)\,dx = \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$\int_0^{\pi/2}\sin^{2k}(x)\,dx = \frac{2k-1}{2k}\int\sin^{2k-2}(x)\,dx = \frac{2k-1}{2k}\cdot\frac{2k-3}{2k-2}\cdots\frac{3}{4}\cdot\frac{1}{2}\,\frac{\pi}{2}$$

$n = 1,2,\dots k$

let $n = (k+1)$:

$$\int_0^{\pi/2}\sin^{2n}(x)\,dx = \int_0^{\pi/2}\sin^{2(k+1)}(x)\,dx = \frac{2k+2-1}{2k+2}\int_0^{\pi/2}\sin^{2k}(x)\,dx$$

$$= \frac{1}{2}\,\frac{3}{4}\,\frac{5}{6}\cdots\frac{2k-1}{2k}\cdot\frac{2k+1}{2k+2} \Rightarrow$$ HOLDS FOR $n=k+1$ BY INDUCTION IT HOLDS FOR ANY $n\geq1$

45 $I = \int \ln^n(x)\,dx = x\ln^n(x) - n\int\ln^{n-1}x\,dx$

$u = \ln^n(x) \qquad du = n\cdot\ln^{n-1}(x)\cdot\frac{1}{x}\,dx \qquad dv = dx \quad v = x$

$I = x\cdot\ln^n(x) - n\int\ln^{n-1}(x)\,\frac{1}{x}\cdot x\,dx = x\ln^n(x) - n\int\ln^{n-1}(x)\,dx$

46 $I = \int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$

$u = x^n \qquad du = n\,x^{n-1}\,dx \qquad v = \int e^x\,dx = e^x$

$I = x^n\cdot e^x - n\int x^{n-1}e^x\,dx$

47 $I = \int (x^2+a^2)^n\,dx = \frac{x(x^2+a^2)^n}{2n+1} + \frac{2na^2}{2n+1}\int (x^2+a^2)^{n-1}\,dx \qquad n\neq-\frac{1}{2}$

$u = (x^2+a^2)^n \qquad du = n\,(x^2+a^2)^{n-1}\cdot2x\,dx \qquad v = \int dx = x$

$I = x\,(x^2+a^2)^n - 2\int n\,(x^2+a^2)^{n-1}\cdot x^2\,dx$

$I = x(x^2+a^2)^n - 2n\int (x^2+a^2)^{n-1}(x^2+a^2)\,dx + 2n\cdot a^2\int (x^2+a^2)^{n-1}dx$

$I = x(x^2+a^2)^n - 2n\cdot I + 2na^2\int (x^2+a^2)^{n-1}dx$

$(2n+1)I = x(x^2+a^2)^n + 2na^2\int (x^2+a^2)^{n-1}dx$

$I = \frac{x(x^2+a^2)^n}{2n+1} + \frac{2na^2}{(2n+1)}\int (x^2+a^2)^{n-1}dx \qquad 2n+1\neq 0 \quad n\neq -\frac{1}{2}$

48 $I = \int \sec^n(x)\,dx = \frac{\tan(x)\sec^{n-2}(x)}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx$

$u = \sec^n(x) \quad du = n\sec^{n-1}(x)\,(-1)\cos^{-1-1}(x)(-\sin(x))\,dx$

$du = n\cdot\frac{1}{\cos^{n-1}(x)}\cdot\frac{1}{\cos^2(x)}\cdot\sin(x)\,dx = \frac{n}{\cos^n(x)}\cdot tg(x)\,dx$

$dv = dx \Rightarrow v = x$??

$u = \sec^{n-2}(x) \quad du = (n-2)\sec^{n-3}(x)\cdot(-1)\cos^{-2}(x)(-\sin(x))\,dx$

$dv = \sec^2(x)\,dx \quad v = \int \sec^2(x)\,dx = tg(x)$

$du = (n-2)\sec^{n-3}(x)\,\sec(x)\cdot tg(x)\,dx = (n-2)\sec^{n-2}(x)\,tg(x)\,dx$

$I = tg(x)\sec^{n-2}(x) - (n-2)\underbrace{\int \sec^{n-2}(x)\,tg^2(x)\,dx}_{I_2}$

$I_2 = \int \sec^{n-2}(x)\frac{\sin^2(x)}{\cos^2(x)}dx = \int \sec^{n-2}(x)\frac{1-\cos^2(x)}{\cos^2(x)}dx$

$I_2 = \int \sec^{n-2}(x)\cdot\frac{1}{\cos^2(x)}dx - \int \sec^{n-2}(x)\,dx = \underbrace{\int \sec^n(x)\,dx}_{I} - \int \sec^{n-2}(x)\,dx$

$I = tg(x)\sec^{n-2}(x) - (n-2)I + (n-2)\int \sec^{n-2}(x)\,dx \qquad I + nI - 2I = (n-1)I$

$$I = \frac{tg(x)\sec^{n-2}(x)}{(n-1)} + \frac{(n-2)}{(n-1)}\int \sec^{n-2}(x)\,dx \qquad n\neq 1$$

49 $I = \int \ln^3(x)\,dx \qquad \int \ln^n(x)\,dx = x\ln^n(x) - n\int \ln^{n-1}(x)\,dx$

$I = x\cdot\ln^3(x) - 3\underbrace{\int \ln^2(x)\,dx}_{I_1} \qquad I_1 = \left| \begin{matrix} u = \ln^2(x) & du = 2\ln(x)\frac{1}{x}dx \\ dv = dx & v = \int dx = x \end{matrix} \right|$

$I_1 = x\ln^2(x) - 2\int \ln(x)\cdot\frac{1}{x}\cdot x\,dx = x\ln^2(x) - 2\int \ln(x)\,dx = x\ln^2(x) - 2(x\ln(x) - x)$

$I_1 = x\ln^2(x) - 2x\ln(x) + 2x$

$I = x\ln^3(x) - 3(x\ln^2(x) - 2x\ln(x) + 2x) = x\ln^3(x) - 3x\ln^2(x) + 6x\ln(x) - 6x$

50 $I = \int x^4e^x\,dx \qquad \int x^ne^x\,dx = x^ne^x - n\int x^{n-1}e^x\,dx$

$I = x^4e^x - 4\int x^3e^x\,dx = x^4e^x - 4\left[x^3e^x - 3\underbrace{\int x^2e^x\,dx}_{I_1}\right]; \quad I_1 = x^2e^x - 2\int xe^x\,dx$

$I_1 = x^2e^x - 2(xe^x - e^x); \quad I = x^4e^x - 4(x^3e^x - 3(x^2e^x - 2xe^x + 2e^x)) =$

$= x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24e^x$

51 $y = x e^{-0.4x}$ $\quad y=0,\ x=5$

$$A = \int_0^5 x e^{-0.4x} dx \Rightarrow \quad I = \int x e^{cx} dx = \frac{1}{c^2}\int cx\, e^{cx} d cx$$

$$= \frac{1}{c^2}\int u e^{u} du = \frac{1}{c^2}\left(u e^{u} - e^{u}\right) = \left| \begin{matrix} u = cx \\ c = -0.4 \end{matrix} \right| =$$

$$= \frac{1}{(0.4)^2}\left(-0.4x e^{-0.4x} - e^{-0.4x}\right)\Big|_0^5 = -\frac{1}{0.16}\left(0.4x+1\right)e^{-0.4x}\Big|_0^5$$

$$= -\frac{1}{0.16}\left[(2+1)e^{-2} - (0+1)e^{0}\right] = -\frac{1}{0.16}\left(3\cdot e^{-2} - 1\right) = \frac{1}{0.16}\left(1 - 3e^{-2}\right)$$

$$\boxed{A = 6.25 - 18.75\, e^{-2}} \qquad \boxed{A = \frac{25}{4} - \frac{75}{4}e^{-2}}$$

52 $y_1 = 5\ln x \quad y_2 = x\ln x \qquad A = \int_1^5 (5\ln x - x\ln x)dx$

$$A = 5\int_1^5 \ln x\, dx - \int_1^5 x\ln x\, dx$$

$$I_1 = \int \ln x\, dx = \left| \begin{matrix} u = \ln x & du = \frac{1}{x}dx \\ v = x & \end{matrix} \right| = x\ln x - x$$

~~$\int \ln^n(x)dx = x\cdot \ln^n(x) - n\int \ln^{n-1}(x)dx$~~

~~$I_2 = \int \ln^2(x)dx = x\cdot\ln^2(x) - 2\int \ln(x)dx = x\ln^2(x) - 2(x\ln x - x)$~~

~~$A = 5(x\ln x - x)\Big|_1^5 + \left(-x\ln^2(x) + 2x\ln x - 2x\right)\Big|_1^5$~~

~~$A = \left(-x\ln^2(x) + 7x\ln(x) - 7x\right)\Big|_1^5 = \left(-5\ln^2(5) + 7\cdot 5\ln(5) - 35\right)$~~

~~$-(0 + 0 - 7) = -5\ln^2(5) + 35\ln(5) - 28$~~

$$I_2 = \int x\ln x\, dx = \left| \begin{matrix} u = \ln x & du = \frac{1}{x}dx \\ v = \int x\, dx = \frac{x^2}{2} & \end{matrix} \right| = \frac{x^2}{2}\ln(x) - \int \frac{x}{2}dx$$

$$I_2 = \frac{x^2}{2}\ln(x) - \frac{x^2}{4}$$

$$A = 5(x\ln x - x)\Big|_1^5 + \left(-\frac{x^2}{2}\ln(x) + \frac{x^2}{4}\right)\Big|_1^5 =$$

$$= 5\left[5\ln 5 - 5 + 1\right] + \left[\frac{25}{4} - \frac{25}{2}\ln(5) - \frac{1}{4} + 0\right] =$$

$$= 25\ln 5 - 20 + \frac{24}{4} - \frac{25}{2}\ln(5) = \frac{50-25}{2}\ln(5) + \frac{-80+24}{4}$$

$$A = \frac{25}{2}\ln(5) - \frac{56}{4} = \underline{\underline{\frac{25}{2}\ln(5) - 14}}$$

[53] & [54] $A = \int_0^{x_1} (\mathrm{arctg}(x) - x\sin(x))\,dx + \int_{x_1}^{x_2} (x\sin(x) - \mathrm{arctg}(x))\,dx$

check MAKE WORKSHEET 6

$I_1 = \int x\sin(x)\,dx = \left| \begin{matrix} u = x & du = dx \\ v = \int \sin(x)dx = -\cos(x) \end{matrix} \right| = -x\cos(x) + \int \cos(x)dx = \sin(x) - x\cos(x)$

$I_2 = \int \mathrm{arctg}(x)\,dx = \left| \begin{matrix} u = \mathrm{arctg}(x) \\ du = \frac{1}{1+x^2}dx \\ v = x \end{matrix} \right| = x\,\mathrm{arctg}(x) - \int \frac{x\,dx}{1+x^2}$

$I_2 = x\,\mathrm{arctg}(x) - \frac{1}{2}\int \frac{d(x^2+1)}{1+x^2} = x\,\mathrm{arctg}(x) - \frac{1}{2}\ln|1+x^2|$

$A = \left(x\,\mathrm{arctg}(x) - \frac{1}{2}\ln(1+x^2)\right)\Big|_0^{x_1} - \left(\sin(x) - x\cos(x)\right)\Big|_0^{x_1} +$

$+ \left(\sin(x) - x\cos(x)\right)\Big|_{x_1}^{x_2} - \left(x\,\mathrm{arctg}(x) - \frac{1}{2}\ln(1+x^2)\right)\Big|_{x_1}^{x_2}$

$I_1 = \int \mathrm{arctg}(3x)\,dx = \frac{1}{3}\int \mathrm{arctg}(u)\,du = \frac{1}{3}\left(3x\,\mathrm{arctg}(3x) - \frac{1}{2}\ln(1+9x^2)\right)$

$I_3 = x\,\mathrm{arctg}(3x) - \frac{1}{6}\ln(9x^2+1)$

[55] Use METHOD OF CYLINDRICAL SHELLS TO FIND VOLUME

$y = \cos(\pi x/2)$; $y = 0$ $0 \le x \le 1$

$V = \int_0^1 2\pi x \cdot \cos(\pi x/2)\,dx = \left| \begin{matrix} u = 2\pi x & du = 2\pi\,dx \\ v = \int \cos\left(\frac{\pi x}{2}\right)dx = \frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right) \end{matrix} \right|$

$V = 2\pi x \cdot \frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)\Big|_0^1 - \int_0^1 \frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right) 2\pi\,dx$

$V = 4x\sin\left(\frac{\pi x}{2}\right)\Big|_0^1 + 4\cos\left(\frac{\pi x}{2}\right) \cdot \frac{2}{\pi}\Big|_0^1 = \left[4x\sin\frac{\pi x}{2} + \frac{8}{\pi}\cos\left(\frac{\pi x}{2}\right)\right]_0^1$

$V = \left(4\sin\frac{\pi}{2} + \frac{8}{\pi}\cos\frac{\pi}{2}\right) - \left(4\sin(0) + \frac{8}{\pi}\cos(0)\right)$

$\boxed{V = 4 - \frac{8}{\pi}}$

[57] $y = e^{-x}$; $y = 0$; $x = -1$, $x = 0$ ABOUT $x = 1$

$V = \int_{-1}^{0} 2\pi(1-x)e^{-x}\,dx = 2\pi\int_{-1}^{0} e^{-x}dx - 2\pi\underbrace{\int_{-1}^{0} x e^{-x}dx}_{I_1}$

$I_1 = \int x e^{-x}dx = \left| \begin{matrix} u = x & du = dx \\ v = \int e^{-x}dx = -e^{-x} \end{matrix} \right| = -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x}$

$V = \left[-2\pi e^{-x} - 2\pi(-xe^{-x} - e^{-x})\right]\Big|_{-1}^{0} = 2\pi\left(xe^{-x} + e^{-x} - e^{-x}\right)\Big|_{-1}^{0} = 2\pi\left(xe^{-x}\right)\Big|_{-1}^{0} = 2\pi e^{-x}(x)\Big|_{-1}^{0}$

$V = 2\pi e^{-x} \cdot x\Big|_{-1}^{0} = (0 - 2\pi e^{1}(-1)) = \underline{\underline{2\pi e}}$

[58] $y=e^x$; $x=0$; $y=\pi$; ABOUT x-AXIS

$x=\ln y$

$$V=\int_1^{\pi} 2\pi y \ln y \, dy = \left| \begin{matrix} u=\ln y & du=\frac{1}{y}dy \\ v=\int y\,dy = \frac{y^2}{2} \end{matrix} \right| = 2\pi\left(\frac{y^2}{2}\ln y - \frac{1}{2}\int y\,dy\right)$$

$$V=2\pi\left(\frac{y^2}{2}\ln y - \frac{y^2}{4}\right) = \left(\pi y^2 \ln y - \frac{\pi}{2}y^2\right)\Big|_1^{\pi}$$

$$V=\pi\cdot\pi^2\ln\pi - \frac{\pi}{2}\pi^2 - \pi\cdot 0 + \frac{\pi}{2} = \pi^3\ln\pi - \frac{\pi^3}{2}+\frac{\pi}{2}$$

$e^{x_1}=\pi \quad x_1=\ln\pi$

[59] $f(x)=x^2\ln(x)$ $\quad f_{AVE}=?$ $\quad [1,3]$

$$f_{AVE}=\frac{1}{3-1}\int_1^3 x^2\ln(x)\,dx = \left| \begin{matrix} u=\ln(x) & du=\frac{1}{x} \\ v=\frac{x^3}{3} \end{matrix} \right| =$$

$$=\frac{1}{2}\left(\frac{x^3}{3}\ln(x)\Big|_1^3 - \int_1^3 \frac{x^2}{3}dx\right) = \frac{1}{2}\left(\frac{x^3}{3}\ln(x)-\frac{x^3}{9}\right)\Big|_1^3$$

$$f_{AVG}=\frac{27}{6}\ln(3)-\frac{27}{18}-\frac{1}{6}\ln 1+\frac{1}{18} = \frac{27}{6}\ln(3)-\frac{26}{18}$$

$$f_{AVG}=\frac{9}{2}\ln(3)-\frac{13}{9}$$

[60] m - initial mass of the rocket

r - RATE of consumtion of fuel

v_e - exaust gases are ejected with this velocity (relative to rocket)

VELOCITY OF THE ROCKET AT TIME „t" is:

$$v(t)=-gt-v_e\ln\left(\frac{m-rt}{m}\right)$$

$g=9.8\,\frac{m}{s^2}$; $m=30.000\,kg$; $r=160\,kg/s$; $v_e=3000\,m/s$

$h=?$ ONE MINUTE AFTER LIFT OFF

$$v(t)=\frac{dh}{dt}\ ; \quad h=\int_0^T v(t)\,dt=\int_0^T -gt-v_e\cdot\ln\left(\frac{m-rt}{m}\right)dt$$

$$h=-g\frac{t^2}{2}\Big|_0^T - \underbrace{v_e\int_0^T \ln\left(\frac{m-rt}{m}\right)dt}_{I_1}$$

$$I_1=\left| \begin{matrix} u=\frac{m-rt}{m} & du=-\frac{r}{m}dt \\ t=0 \quad u=1 & t=T \quad u=\frac{m-rT}{m} \end{matrix} \right| = -\frac{m}{r}\int_1^{\frac{m-rT}{m}} \ln u \, du = \left| \begin{matrix} v=\ln u \\ dv=\frac{1}{u}du \\ V=u \end{matrix} \right|$$

$$=-\frac{m}{r}\left(u\cdot\ln u - u\right)\Big|_1^{\frac{m-rT}{m}} = -\frac{m}{r}\frac{m-rT}{m}\left(\ln\frac{m-rT}{m}-1\right)+\frac{m}{r}\left(\ln 1-1\right)$$

$$I_1=-\frac{m-rT}{r}\left(\ln\frac{m-rT}{m}-1\right)-\frac{m}{r}$$

$$h_1 = -g\frac{T^2}{2} + v_e\left[\frac{m-rT}{r}\ln\left(\frac{m-rT}{m}\right) - \frac{m-rT}{r} + \frac{m}{r}\right]$$

$$h_1 = -\frac{gT^2}{2} + v_e\left[\frac{rT}{r} + \frac{m-rT}{r}\ln\left(\frac{m-rT}{m}\right)\right]$$

$$h_1 = -\frac{gT^2}{2} + v_e\left[T + \frac{m-rT}{r}\ln\left(\frac{m-rT}{m}\right)\right]$$

61 $v(t) = t^2 e^{-t}$ $\quad s(t) = \int_0^t x^2 e^{-x}\,dx = \left| \begin{matrix} u = x^2 \\ du = 2x\,dx \\ v = -e^{-x} \end{matrix} \right|$

$$s(t) = -x^2 \cdot e^{-x}\Big|_0^t + 2\int_0^t x e^{-x}\,dx = \left| \begin{matrix} U = x & dU = dx \\ V = \int e^{-x}dx = -e^{-x} \end{matrix} \right|$$

$$= -x^2 \cdot e^{-x}\Big|_0^t + 2\left(-xe^{-x} - e^{-x}\right)\Big|_0^t = \left(-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right)\Big|_0^t$$

$$s(t) = -e^{-x}\left(x^2+2x+2\right)\Big|_0^t = -e^{-t}\left(t^2+2t+2\right) + (2) = \underline{2-(t^2+2t+2)e^{-t}}$$

62 If $f(0) = g(0) = 0$ $\quad f'', g''$ are continuous show that:

$$\int_0^a f(x)g''(x)\,dx = f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x)\,dx$$

$$\frac{d\, f(x)g(x)}{dx} = f'(x)g(x) + f(x)g'(x) \quad \Big/ \int$$

$$f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

$$\frac{d^2 f(x)g(x)}{dx^2} = \left(f'(x)g(x) + f(x)g'(x)\right)' = f''(x)\cdot g(x) + \underline{g'(x)\cdot f'(x)}$$

$$+ \underline{f'(x)\cdot g'(x)} + f(x)g''(x) \quad \Big/ \int$$

$$f'(x)g(x) + f(x)g'(x) = \int f''(x)g(x)\,dx + 2\int g'(x)f'(x)\,dx + \int f(x)g''(x)\,dx$$

$$\int f(x)g''(x)\,dx = f'(x)g(x) + f(x)g'(x) - 2\int g'(x)f'(x)\,dx - \int f''(x)g(x)\,dx$$

$$\int_0^a f(x)g''(x)\,dx = f'(a)g(a) + f(a)g'(a) - 2\int_0^T g'(x)f'(x)\,dx - \int_0^a f''(x)g(x)\,dx$$

$f(0)=g(0)=0$

Proof: $I = \int_0^a f(x)\, g''(x)\,dx = f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x)\,dx$

$u = f(x) \quad du = f'(x)dx \quad dv = g''(x)dx \quad v = \int g''(x)dx = g'(x)$

$I = f(x)g'(x)\Big|_0^a - \underbrace{\int_0^a g'(x) f'(x)\,dx}_{I_1}$

$I_1 = \left| \begin{matrix} u = f'(x) & du = f''(x)dx \\ v = \int g'(x)dx = g(x) & \end{matrix} \right| = g(x) f'(x) - \int g(x) f''(x)\,dx$

$I = f(x)g'(x)\Big|_0^a - g(x)\cdot f'(x)\Big|_0^a + \int_0^a g(x) f''(x)\,dx$

$|f(0) = g(0) = 0| \Rightarrow \boxed{I = f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)\, g(x)\,dx}$

$$\boxed{\int f(x)g''(x)\,dx = f(x)g'(x) - f'(x)g(x) + \int g(x) f''(x)\,dx} \quad (*)$$

63 $f(1)=2$; $f(4)=7$; $f'(1)=5$; $f'(4)=3$; f'' is continuous

$I = \int_1^4 x f''(x)\,dx = \int_0^4 x f''(x)\,dx - \int_0^1 x f''(x)\,dx = \left| \begin{matrix} f(x) = x \\ g(x) = f(x) \end{matrix} \right|$

$= 4\cdot f'(x)\Big|_{x=4} - (x)'\cdot f(x)\Big|_{x=4} + \int_0^4 (x)'' f(x)\,dx - \left(1\cdot f'(1) - (x)'\Big|_{x=1}\cdot f(1) + \int_0^1 (x)'' f(x)\,dx\right) = 4\cdot f'(4) - 1\cdot f(4) - f'(1) + f(1) =$

$= 4\cdot 3 - 1.7 - 5 + 2 = 12 - 7 - 5 + 2 = 2$

$(*) \Rightarrow \int_1^4 x f''(x)\,dx = (x\cdot f'(x) - x'\cdot f(x))\Big|_1^4 + \int (x)''\cdot f(x)\,dx$

$= 4\cdot f'(4) - 1\cdot f(4) - (1\cdot f'(1) - 1\cdot f(1)) = 4\cdot f'(4) - 1\cdot f(4) - f'(1) + f(1)$

$= 4\cdot 3 - 7 - 5 + 2 = 2$ //

AlT:

$I = \int_1^4 x f''(x)\,dx$; $u = x \quad du = dx$; $v = \int f''(x)dx = f'(x)$

$I = x\cdot f'(x)\Big|_1^4 - \int_1^4 f'(x)\,dx = 4\cdot f'(4) - 1\cdot f'(1) - f(4) + f(1)$

$I = 4\cdot 3 - 1\cdot 5 - 7 + 2 = 2$

64 a) $\int f(x)dx = \left| \begin{matrix} u = f(x) & du = f'(x)dx \\ dv = dx & v = x \end{matrix} \right| = x \cdot f(x) - \int x \cdot f'(x) dx$

b) $\int_a^b f(x)dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)dy$

$y = f(x)$; $x = g(y)$;

(II) $\frac{dy}{dy} = \frac{df(x)}{dx} \frac{dx}{dy} \Rightarrow dy = f'(x)dx$.

$\int_a^b f(x)dx = x \cdot f(x)\Big|_a^b - \int_a^b x f'(x)dx = b \cdot f(b) - af(a) - \int_{a\ f(b)}^{b} x f'(x) dx = (*)$

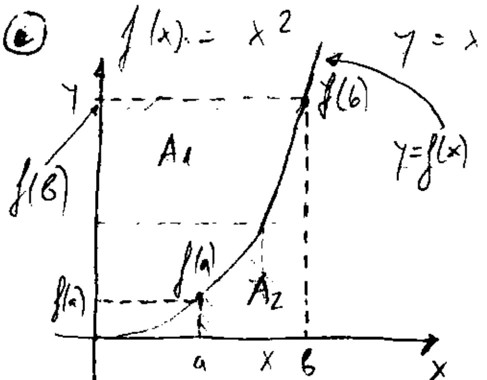
$= \left| \begin{matrix} x = g(y) \\ dx = g'(y)dy \\ y = f(b) \end{matrix} \right| \begin{matrix} f'(x) = (y)' \\ x = a \quad g(y) = a \\ y = f(a) \\ \Leftarrow x = b \quad g(y) = b \end{matrix} \Big| = bf(b) - af(a) - \int_{f(a)}^{} g(y) \cdot 1 \cdot g'(y)dy$

$g'(y) = \frac{dg(y)}{dy} = \frac{d(x)}{x} = 1$ $\qquad f'(x) = \frac{df(x)}{dx} = \frac{d(y)}{y}$ #

(*) $\Rightarrow$ $x = g(y)$; $dy = f'(x)dx$ (II), $y = f(x)$ $\qquad x = a \Rightarrow y = f(a)$; $x = b \Rightarrow y = f(b)$

$$\int_a^b f(x)dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y) \cdot dy$$

c) $f(x) = x^2$ $\qquad y = x^2 \Rightarrow x = \sqrt{y}$



$A_1 + A_2 = b \cdot f(b) - a f(a)$

$A_2 = b \cdot f(b) - a f(a) - A_1$

$A_1 = \int_{f(a)}^{f(b)} g(y)dy$

d) $I = \int_1^e \ln(x)dx$ $\qquad f(x) = \ln(x)$, $y = \ln(x)$; $x = e^y$; $g(y) = e^y$;

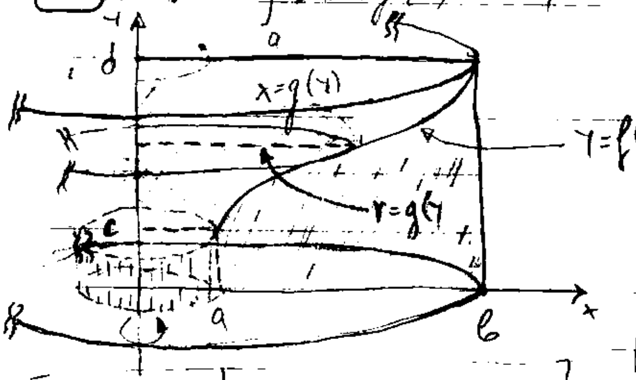
$I = e \cdot \ln(e) - 1 \cdot \ln(1) - \int_{\ln(1)=0}^{\ln(e)=1} e^y dy = e\ln(e) - e^y\Big|_0^1 = e\ln e - e + 1$

$I = e - e + 1 = 1$

ALTER: $I = \int_1^e \ln(x)dx = \left| \begin{matrix} u = \ln(x) \\ du = \frac{1}{x}dx \\ v = x \end{matrix} \right| = x\ln(x)\Big|_1^e - \int_1^e x \cdot \frac{dx}{x} =$

$= e\ln e - 1 \cdot \ln 1 - x\Big|_1^e = e - e + 1 = 1$

65 $V = \int_a^b 2\pi x f(x)\,dx$; $V = \pi b^2 d - \pi a^2 c - \int_c^d \pi g^2(y)\,dy$



$V = V_{cil1} - V_{cil2} - V_g =$

$= b^2\pi \cdot d - a^2\pi \cdot c - \int_c^d \pi \underbrace{g^2(y)}_{x^2}\,dy$ (*) $= I_1$

$\boxed{y = f(x)}$ $\boxed{g(y) = x}$

$I_1 = \int_c^d \pi g^2(y)\,dy$ $y = f(x)$ $dy = f'(x)\,dx$

$x = g(y)$ $\frac{dx}{dy} = g'(y)$ $dy = \frac{1}{g'(y)}\,dx$

$I_1 = \int_c^d \pi \cdot x^2 \cdot \frac{1}{x'}\,dx = \int_c^0 \pi x^2\,dx$ #

$I_1 = \int_c^d \pi g^2(y)\,dy = \left| \begin{matrix} u = g^2(y) \\ du = 2g(y)g'(y)\,dy \\ dv = \pi\,dy \quad v = \pi y \end{matrix} \right| = \pi y \cdot g^2(y)\Big|_c^d - \int_c^d 2\pi y\, g(y) g'(y)\,dy$

$I_1 = \pi d \cdot g^2(d) - \pi c \cdot g^2(c) - \int_c^d 2\pi y\, g(y) g'(y)\,dy = \pi b^2 \cdot d - \pi a^2 c - \int_c^d 2\pi y\, g(y) g'(y)\,dy$

(*) $V = b^2\pi d - a^2\pi c - \pi b^2 d + \pi a^2 c + \int_c^d 2\pi y\, g(y) g'(y)\,dy = \int_c^d 2\pi y\, g(y) g'(y)\,dy$

$V = \int_c^d 2\pi y\, g(y) g'(y)\,dy$ $x = g(y)$, $dx = g'(y)\,dy$ $y = c$ $x = g(c)$; $y = d$ $x = g(d)$

$V = \int_{g(c)}^{g(d)} 2\pi f(x) \cdot x \cdot \underbrace{g'(y)\,dy}_{dx} = \int_a^b 2\pi x f(x)\,dx$

ALT: $I_1 = \int_c^d \pi g^2(y)\,dy = \left| \begin{matrix} y = f(x) & dy = f'(x)\,dx \\ g(y) = x & y=c \; x=g(c) \\ & y=d \; x=g(d) \end{matrix} \right| = \int_{g(c)}^{g(d)} \pi x^2 f'(x)\,dx$

$I_1 = \int_a^b \pi x^2 f'(x)\,dx = \left| \begin{matrix} u = x^2 & du = 2x\,dx \\ v = \int f'(x)\,dx = f(x) \end{matrix} \right| = \pi \left[x^2 f(x)\Big|_a^b - \int_a^b 2x f(x)\,dx \right]$

$= \pi b^2 f(b) - \pi a^2 f(a) - \int_a^b 2\pi x f(x)\,dx = \pi b^2 d - \pi a^2 c - \int_a^b 2\pi x f(x)\,dx$

$V = b^2\pi d - a^2\pi c - \pi b^2 d + \pi a^2 c + \int_a^b 2\pi x f(x)\,dx = \int_a^b 2\pi x f(x)\,dx$

66 $I_n = \int_0^{\pi/2} \sin^n(x)\,dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2}(x)\,dx$.

$\int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\cdot\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$

$k = 2n+1$ $\quad \int_0^{\pi/2} \sin^{2n+1}(x)\,dx = \frac{2\cdot4\cdot6\cdots 2n}{3\cdot5\cdot7\cdots 2n+1}$ $\qquad \frac{2\cdot4}{3\cdot5} = \frac{8}{15} = 0.533$

$\int_0^{\pi/2} \sin^{2n}(x)\,dx = \frac{1\cdot3\cdot5\cdots 2n-1}{2\cdot4\cdot6\cdots 2n}\frac{\pi}{2}$ $\qquad \frac{1\cdot3}{2\cdot4} = \frac{3}{8} = 0.589$

(a) $I_{2n+2} \leq I_{2n+1} \leq I_{2n}$

$I_{2n} = \frac{1\cdot3\cdot5\cdots 2n-1}{2\cdot4\cdot6\cdots 2n}\frac{\pi}{2}$; $I_{2n+1} = \frac{2\cdot4\cdot6\cdots 2n}{3\cdot5\cdot7\cdots 2n+1}$

$I_{2n+2} = \frac{1\cdot3\cdot5\cdots 2n+1}{2\cdot4\cdot6\cdots 2n+2}\frac{\pi}{2}$

(1) $I_{2n} = \frac{2n+2}{2n+1}\cdot I_{2n+2}$

$\frac{I_{2n}}{I_{2n+2}} = \frac{\frac{1\cdot3\cdot5\cdots 2n-1}{2\cdot4\cdot6\cdots 2n}\frac{\pi}{2}}{\frac{1\cdot3\cdot5\cdots 2n-1\cdot 2n+1}{2\cdot4\cdot6\cdots 2n\cdot 2n+2}\frac{\pi}{2}} = \frac{2n+2}{2n+1} > 1$

$\boxed{I_{2n} > I_{2n+2}}$

$I_{2n+1} = \frac{1}{\frac{3\cdot5\cdot7\cdots 2n-1\; 2n+1}{2\cdot4\cdot6\cdots 2n\cdot2\cdot2n}} = \frac{1}{(2n+1)\,I_{2n}}\frac{\pi}{2}$

(2) $I_{2n+1} = \frac{1}{(2n+1)\,I_{2n}}\cdot\frac{\pi}{2}$

$I_{2n+2} = \frac{1}{\frac{2\cdot4\cdot6\cdots 2n\,(2n+2)}{1\cdot3\cdot5\cdots(2n-1)(2n+1)}}\frac{\pi}{2} = \frac{1}{(2n+2)\,I_{2n+1}}\cdot\frac{\pi}{2}$

($\frac{2}{\pi}\cdot I_{2n}$; I_{2n+1})

(3) $I_{2n+1} = \frac{\pi}{2}\frac{1}{(2n+2)\,I_{2n+2}}$

(1),(3) $\quad I_{2n+2} = \frac{2n+1}{2n+2} I_{2n}$

$I_{2n} - I_{2n+1} = I_{2n} - \frac{1}{(2n+1)I_{2n}}\frac{\pi}{2} = \frac{(2n+1)I_{2n}^2 - \frac{\pi}{2}}{(2n+1)I_{2n}}$

(2) $\quad 2n\cdot I_{2n+1} + I_{2n+1} = \frac{1}{I_{2n}}\frac{\pi}{2}$

$\frac{I_{2n}}{I_{2n+1}} = \frac{\frac{1\cdot3\cdot5\cdots 2n-1}{2\cdot4\cdot6\cdots 2n}\frac{\pi}{2}}{\frac{2\cdot4\cdot6\cdots 2n}{3\cdot5\cdot7\cdots 2n-1\cdot 2n+1}} = (2n+1)\,I_{2n}^2\,\frac{2}{\pi}$

(4), (7)

$$I_{2n+1} = \frac{1}{(2n+1) I_{2n}} \cdot \frac{\pi}{2}$$

$$I_{2n+2} = \frac{1}{(2n+2) I_{2n+1}} \cdot \frac{\pi}{2}$$

$$I_{2n} > I_{2n+2} = \frac{1}{(2n+2) I_{2n+1}} \cdot \frac{\pi}{2}$$

$$I_{2n} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \cdot \frac{\pi}{2}$$

$$I_{2n+1} = \frac{2 \cdot 4 \cdot 6 \cdots 2n}{3 \cdot 5 \cdot 7 \cdots 2n+1}$$

$$I_{2n+1} = \int_0^{\pi/2} \sin^{2n+1}(x)\,dx = \frac{2n+1-1}{2n+1} \int_0^{\pi/2} \sin^{2n-1}(x)\,dx = \frac{2n}{2n+1} \int_0^{\pi/2} \sin^{2n-1}(x)\,dx$$

$$I_{2n+2} = \int_0^{\pi/2} \sin^{2n+2}(x)\,dx = \frac{2n+2-1}{2n+2} \underbrace{\int_0^{\pi/2} \sin^{2n}(x)\,dx}_{I_{2n}}$$

$$I_{2n+2} = \frac{2n+1}{2n+2} I_{2n}$$

~~$I_{2n+1} = \int \sin^{2n+1}(x)\,dx = \quad u = \sin^{2n}$~~

$$0 \le x \le \frac{\pi}{2} \qquad 0 \le \sin(x) \le 1 \quad \Rightarrow$$

(a) $\sin^{2n+2}(x) \le \sin^{2n+1}(x) \le \sin^{2n}(x)$

(b) $$\frac{I_{2n+2}}{I_{2n}} = \frac{\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)}{2 \cdot 4 \cdot 6 \cdots 2n \cdot (2n+2)} \cdot \frac{\pi}{2}}{\frac{1 \cdot 3 \cdot 5 \cdots 2n-1}{2 \cdot 4 \cdot 6 \cdots 2n} \cdot \frac{\pi}{2}} = \frac{2n+1}{2n+2}$$

(c) $$\frac{2n+1}{2n+2} \le \frac{I_{2n+1}}{I_{2n}} \le 1$$

$$I_{2n+2} \le I_{2n+1} \le I_{2n} \qquad \frac{2n+1}{2n+2} I_{2n} \le I_{2n+1} \le I_{2n} \Big/ I_{2n}$$

$$\frac{2n+1}{2n+2} \le \frac{I_{2n+1}}{I_{2n}} \le 1$$

$$\lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = ? \qquad \lim_{n\to\infty} \frac{2n+1}{2n+2} = \frac{2}{2} = 1$$

$$\lim_{n\to\infty} 1 = 1$$

SQUEEZE THEOREM $\lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = 1$

(d) $\lim_{n\to\infty} \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots\frac{2n}{2n-1}\cdot\frac{2n}{2n+1} = \frac{\pi}{2}$

$I_{2n+1} = \frac{2\cdot4\cdot6\cdot8\cdots2n}{1\cdot3\cdot5\cdot7\cdots2n+1}$ $\quad I_{2n} = \frac{1\cdot3\cdot5\cdots2n-1}{2\cdot4\cdot6\cdots2n}\cdot\frac{\pi}{2}$

$\frac{I_{2n+1}}{I_{2n}} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\cdot\frac{2}{\pi}$

$\to 1$

$\lim_{n\to\infty}\frac{I_{2n+1}}{I_{2n}} = \frac{2}{\pi}$ $\quad \lim_{n\to\infty}\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdots\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}$

$(*) = \frac{\pi}{2}$ — $\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}$ Wallis' Product

(c) $\frac{\text{width}}{\text{height}} = \frac{1}{1};\ \frac{2}{1};\ \frac{2}{\frac{3}{2}} = \frac{2}{1}\cdot\frac{2}{3};\ \frac{\frac{5}{2}}{\frac{3}{2}} = \frac{5}{3} = \frac{2}{3}\cdot\frac{5}{2};\ \frac{\frac{5}{2}}{2} = \frac{5}{4}$

$k=1$ $\quad \frac{a}{h} = \frac{1}{1} = 1$

$k=2$ $\quad \frac{a}{h} = \frac{1+1}{1} = \frac{2}{1} = 2$ $\quad \frac{2k}{2k-1} = \frac{4}{3}$ $\quad n=1$

$k=3$ $\quad \frac{a}{h} = \frac{2}{1+\frac{1}{2}} = \frac{2}{\frac{3}{2}} = \frac{4}{3} = 1.33$ $\quad n=1$

$k=4$ $\quad \frac{a}{h} = \frac{2+\frac{2}{3}}{\frac{3}{2}} = \frac{\frac{8}{3}}{\frac{3}{2}} = \frac{16}{9} = 1.66$ $\quad n=2$ $\quad k=2n=4$

$k=5$ $\quad \frac{a}{h} = \frac{5/2}{\frac{3}{2}+\frac{1}{2.5}} = \frac{5/2}{\frac{3}{2}+\frac{2}{5}} = \frac{5/2}{\frac{15+4}{10}} = \frac{5/2}{19/10} = \frac{25}{19}$ $\quad n=2$

$k=6$ $\quad \frac{a}{h} = \frac{5/2+\frac{10}{19}}{19/10} = \frac{\frac{95+20}{38}}{19/10} = \frac{\frac{115}{38}}{19/10} = \frac{575}{361} = 1.593$ $\quad n=3$

a_{n-1}

~~$\frac{1}{1}\ \frac{2}{1}\ \frac{2}{3}\ \frac{5}{3}\ \frac{2}{4}\ \frac{11}{8}$~~

$2.75\cdot\Delta h = 1 \Rightarrow \Delta h = \frac{1}{2.75} = \frac{1}{\frac{11}{4}} = \frac{4}{11}$

$k=7$ $\quad \frac{a}{h} = \frac{115/38}{19/10+\Delta h} = \frac{\frac{115}{38}}{\frac{19}{10}+\frac{38}{115}} = \frac{\frac{115}{38}}{\frac{380+2185}{1150}} = \frac{\frac{115}{38}}{\frac{2565}{1150}} = 1.356$

$h_{(n-1)} + \frac{1}{a_{n-1}}$

GRS k(l)

~~$\frac{a_n}{h_n} = \frac{a_{n-1}}{h_{(n-1)}+\frac{1}{a_{n-1}}} = \frac{(a_{n-1})^2}{a_{n-1}\cdot h_{n-1}+1} = \frac{a_n^2}{a_n h_{n-1}+1}$~~

~~ex. $n=5$ $\quad \frac{(\frac{5}{2})^2}{\frac{5}{2}\cdot\frac{3}{2}+1} = \frac{\frac{25}{4}}{\frac{15}{4}+1} = \frac{\frac{25}{4}}{\frac{19}{4}} = \frac{25}{19}$~~

~~$\frac{5^2}{\frac{3}{2}\cdot 2+1}$ $\quad \frac{4}{4}$~~

(n=1)

$k = 2n$ step AREA = $2n$

$n=1 \Rightarrow k = 2n = 2$ AREA $= 2n = 2$

$$\frac{a}{h} = \frac{1 \cdot \frac{2n}{2n-1}}{1} = \frac{1 \cdot \frac{2}{1}}{\frac{3}{2}} = \frac{2}{1} = 2 \qquad \text{ratio}\left(\frac{a}{h}\right) = \frac{2n}{2n-1}$$

$$k = 2n+1 = 3 \quad \frac{a}{h} = \frac{2}{1 \cdot \frac{2n+1}{2n}} = \frac{2}{\frac{3}{2}} = \frac{4}{3} \qquad \text{ratio}\left(\frac{a}{h}\right) = \frac{2n}{2n+1}$$

$$\text{ratio}\left(\frac{a}{h}\right)_{2n} \cdot \text{ratio}\left(\frac{a}{h}\right)_{2n+1} = \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \Rightarrow \lim_{n\to\infty} \frac{2}{1}\,\frac{2}{3}\,\frac{4}{3} \cdots = \frac{\pi}{2}$$

n=2 $k = 2n$ $k = 4$

$$\frac{a}{h} = \frac{2 \cdot \frac{2n+1}{2n+1}}{\frac{3}{2}} = \frac{2 \cdot \frac{5}{4+1}}{\frac{3}{2}} = \frac{\frac{5}{2}}{\frac{3}{2}} = \frac{5}{3}$$ ~~$\frac{8}{3} \ldots \frac{16}{9} = 1.778$~~

$k = 2n+1$ $k = 5$

$$\frac{a}{h} = \frac{\frac{5}{2}}{\frac{3}{2} \cdot \frac{2n+1}{2n-1}} \quad \frac{5/2}{\frac{3}{2}\,\frac{5}{3}} = \frac{5}{4}$$

$$\frac{\frac{5}{2} \cdot \frac{7}{6}}{2} = \frac{\frac{35}{12}}{2} = \frac{35}{24}$$

n=3 $k = 2n$ $k = 6$

$$\frac{a}{h} = \frac{\frac{5}{2} \cdot \frac{2n}{2n-1}}{2} = \frac{\frac{5}{2} \cdot \frac{6}{5}}{2} = \frac{3}{2}$$

$$\frac{5}{2} + \frac{1}{4} = \frac{11}{8} \qquad \frac{5}{2} \cdot x = \frac{11}{8\,4} \qquad x = \frac{11}{5} = 2.2 \;(?)$$

~~$$\frac{5}{2} + \frac{5}{2} \cdot \frac{2n}{2n-1} = \frac{5}{2} + \frac{5}{2} \cdot \frac{6}{5} = \frac{5}{2} + \ldots$$~~

$$\frac{a}{h} = \frac{a_n}{h_n + \Delta h} \qquad \Delta h \cdot a_n = 1 \quad \Delta h = \frac{1}{a_n}$$

$$1.5 \cdot x = 1 \qquad x = \frac{1}{1.5} = \frac{2}{3} \qquad 2 + \frac{2}{3}$$

$$\frac{a}{h} = \frac{a_k}{h_{k-1} + \frac{1}{a_k}} = \frac{a_k^2}{a_k h_{k-1} + 1}$$

EXAMPLE: $k=3$ $a_k = 2$, $h_{k-1} = 1$ $\left(\frac{a}{h}\right)_3 = \frac{4}{2 \cdot 1 + 1} = \frac{4}{3}$

$$k=5 \quad \left(\frac{a}{h}\right)_5 = \frac{a_5^2}{a_5 \cdot h_4 + 1} = \frac{\left(\frac{5}{2}\right)^2}{\frac{5}{2} \cdot \frac{3}{2} + 1} = \frac{\frac{25}{4}}{\frac{15+4}{4}} = \frac{25}{19}$$

$$k=7 \quad \frac{a_7}{h_7} = \frac{a_7^2}{a_7 h_6 + 1} = \frac{\left(\frac{115}{38}\right)^2}{\frac{115}{38} \cdot \frac{19}{10} + 1} = \frac{\left(\frac{115}{38}\right)^2}{\frac{2185 + 380}{380}} = \frac{\left(\frac{115}{38}\right)^2}{\frac{2565}{380}}$$

$$\frac{a_7}{h_7} = 1.35683$$

$$k=2 \quad \frac{a_n}{h_n} = \frac{a_{n-1} + \frac{1}{h_n}}{h_n} = \frac{h_n a_{n-1} + 1}{(h_n)^2} = \frac{1 \cdot 1 + 1}{1^2} = 2$$

$$k=4 \quad \frac{a_4}{h_4} = \frac{a_3 \cdot h_4 + 1}{(h_4)^2} = \frac{[illegible] \cdot \frac{3}{2} + 1}{(3/2)^2} = \frac{3+1}{9/4} = \frac{16}{9}$$

$$\frac{2+\frac{2}{3}}{\frac{3}{2}} = \frac{\frac{6+2}{3}}{\frac{3}{2}} = \frac{16}{9}$$

(k=4) $$\frac{a}{b_n} = \frac{2\cdot\frac{2n}{2n-1}}{\frac{3}{2}} = \frac{2\cdot\frac{4}{3}}{\frac{3}{2}} = \frac{\frac{8}{3}}{\frac{3}{2}} = \frac{16}{9}$$ → $\frac{1}{9_k}$

MANUAL

$k=1$ $\frac{a}{b_n} = \frac{1}{1}$; $k=2$ $\frac{a}{b_n} = \frac{1+1}{1} = 2$; $k=3 \Rightarrow \frac{a}{b_n} = \frac{2}{1+\frac{1}{2}} = \frac{2}{\frac{3}{2}} = \frac{4}{3}$;

$k=4$ $\frac{a}{b_n} = \frac{2+\frac{1}{3/2}}{3/2} = \frac{2+\frac{2}{3}}{\frac{3}{2}} = \frac{\frac{8}{3}}{\frac{3}{2}} = \frac{16}{9}$; $k=5$ $\frac{a}{b_n} = \frac{\frac{8}{3}}{\frac{3}{2}+\frac{3}{8}} = \frac{\frac{8}{3}}{\frac{15}{8}} = \frac{64}{45}$

$k=6$ $\frac{a}{b_n} = \frac{8/3+\frac{8}{15}}{\frac{15}{8}} = \frac{\frac{48}{15}}{\frac{15}{8}} = \frac{384}{225}$; $k=7$ $\frac{a}{b_n} = \frac{\frac{48}{15}}{\frac{15}{8}+\frac{15}{48}} = \frac{\frac{48}{15}}{\frac{90+15}{48}} = \frac{\frac{48}{15}}{\frac{105}{48}}$

~~$a_n \cdot \frac{2n+1}{2n} = \frac{\frac{8}{3}\cdot\frac{5}{4}}{\frac{15}{8}} = \frac{40}{12}$~~

$k=5$ $k=2n+1$; $n=2$;

$$\frac{a_n}{b_{n-1}\,\frac{2n+1}{2n}} = \frac{8/3}{\frac{3}{2}\cdot\frac{5}{4}} = \frac{8/3}{\frac{15}{8}} = \frac{64}{45}$$

$k=6$ $k=2n$; $n=3$
$$\frac{a_{n-1}\cdot\frac{2n}{2n-1}}{b_n} = \frac{\frac{8}{3}\cdot\frac{6}{5}}{\frac{15}{8}} = \frac{\frac{48}{15}}{\frac{15}{8}} = \frac{384}{225}$$

$k=7$ $k=2n+1$; $n=3$
$$\frac{a_n}{b_{n-1}\cdot\frac{2n+1}{2n}} = \frac{\frac{48}{15}}{\frac{15}{8}\cdot\frac{7}{6}} = \frac{\frac{48}{15}}{\frac{105}{48}} = \frac{256}{175}$$

$k=2$, $n=1$: $\frac{2n}{2n-1}\Big|_{n=1} = \frac{2}{1}$; $k=3$, $n=1$: $\frac{2n+1}{2n} = \frac{3}{2}$; $k=4$, $n=2$: $\frac{2n}{2n-1} = \frac{4}{3}$

$$\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots \qquad \lim_{n\to\infty}(\ast) = \frac{\pi}{2}$$

$$\frac{\frac{a_5}{b_5}}{\frac{a_4}{b_4}} = \frac{4}{5} \qquad \frac{\frac{a_7}{b_7}}{\frac{a_6}{b_6}} = \frac{\frac{\frac{48}{15}}{\frac{15}{8}}\cdot\frac{6}{7}}{\frac{\frac{48}{15}}{\frac{15}{8}}} = \frac{6}{7}$$

CRC [1 1 0 0 1 1 0] $M = x^6 + x^5 + x^2 + x$
1 1 0 1 $P = x^3 + x^2 + 1$ $r = 3$

$M\cdot x^3 / P$ $7+4-1=10$
$T = M_x$ $R = \frac{T}{Q\cdot P}$ $Q = x^6 + x^? x$

LINEAR BLOCK CODES

$X_m = [x_{m1}, x_{m2}, \ldots x_{mk}]$ INPUT VECTOR INTO ENCODER

$C_m = [c_{m1}, c_{m2}, \ldots c_{mn}]$ OUTPUT OF THE ENCODER

$$\boxed{C_m = X_m G}$$

$$G = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ g_{k1} & g_{k2} & \cdots & g_{kn} \end{bmatrix}$$

⇒ G - GENERATOR MATRIX

$$c_{mj} = x_{m1} g_{1j} + x_{m2} g_{2j} + \ldots + x_{mk} g_{kj}$$

$$G = [I_k | P] = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & p_{11} & p_{12} & \cdots & p_{1n-k} \\ 0 & 1 & 0 & & 0 & p_{21} & p_{22} & \cdots & p_{2n-k} \\ 0 & 0 & 0 & \cdots & 1 & p_{k1} & p_{k2} & & p_{kn-k} \end{bmatrix} \} k$$

(k columns of identity, $n-k$ columns of P)

① HAMMING CODE

$$(n, k) = (2^m - 1, 2^m - 1 - m)$$

$m = 3$ $\quad 2^3 - 1 = 8 - 1 = 7$; $2^3 - 1 - 3 = 8 - 4 = 4$

$(n, k) = (7, 4)$ CODE ⊛ ⇒

$$\boxed{C_m H' = \emptyset}$$

H - PARITY CHECK MATRIX

↓ ALL ZERO ROW VECTOR WITH $n-k$ ELEMENTS

$G \cdot H' = \emptyset \Rightarrow k \times (n-k)$ $\emptyset$ MATRIX

ex:

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} = [I_4 | P]$$

$c_{m5} = x_{m1} + x_{m2} + x_{m3}$

$c_{m6} = x_{m2} + x_{m3} + x_{m4}$

$c_{m7} = x_{m1} + x_{m2} + x_{m4}$

$$\boxed{C_m = [x_{m1}, x_{m2}, x_{m3}, x_{m4}, c_{m5}, c_{m6}, c_{m7}]}$$

$$H = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \Leftarrow \boxed{H = [-P' \,|\, I_{n-k}]}$$

$C_m H'$

$x_{m1} + x_{m2} + x_{m3} + c_{m5} = \emptyset$

$x_{m2} + x_{m3} + x_{m4} + c_{m6} = \emptyset$

$x_{m1} + x_{m2} + x_{m4} + c_{m7} = \emptyset$

⊛ ⇒ $n = 2^m - 1$ $\quad m = n - k = 3$ element $\quad$ HAMMING CODE

DATA = 1011 $\quad n = 2^m - 1$, $m = 3$ $\quad n = 2^3 - 1 = 7$

$\boxed{k = 4}$ $\quad$ ↳ $k = n - m = 2^m - 1 - m = 2^3 - 1 - 3 = 4$

msg1 = 1011

$[h, g, n, k] = \text{hammgen}(n)$

$$\text{code} = [1\ 0\ 1\ 1] \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} = [P \mid I_k]$$

$1+1+1,\ 1+0+1+0,\ 0+0+1,\ 1,\ 0,\ 1,\ 1$

$[1, 0, 0, 1, 0, 1, 1]$

83 − 75 = 8+1   93 − 88 = 5

$$H = [I_{n-k} \mid P^T] = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

The binary primitive polynomial used to produce Hamming code is default primitive polynomial $GF(2^m)$

pol = gfprimdf(2,5)

pol = [2, 1, 1] $(2 + x + x^2)$

gfprimdf(3,2) = [1, 1, 0, 1] $2^3 + 2 + 1 = 11$ = primpoly(3)

$D^3 + D^1 + 1$

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Channels and PROPAGATION

(a) GAUSSIAN NA [illegible]
- [illegible]
- DECODE & FOR [illegible]

(b) PERIOD [illegible]

(c) [illegible] DEL ZO [illegible] FIZIK [illegible] ⇒ [illegible]

DIVERSITY – [illegible] ŠTA REPLIKA TO [illegible] PATERN TOGAŠ "[illegible]" DIVERSITY PROBLEM

. [illegible]

RAYLEIGH FEDING SE KORISTI KOGA NEMA LOS (line of sight), A RICIAN RASPREDELBA KOGA IMA LOS

• [illegible] GAIN G = SNAGATA NA [illegible] / $P_T / 4\pi\lambda^2$

• EFEKTIVNA POVRŠINA : $A = \frac{\lambda^2 G}{4\pi}$

ПРИЕМНА МОЌНОСТ: $P = W \cdot A$

$$P_R = \frac{P_T G_T}{4\pi d^2} A = \frac{P_T G_T}{4\pi d^2} \frac{\lambda^2 G_R}{4\pi}$$

$$[*] \quad \boxed{\frac{P_R}{P_T} = G_T G_R \left(\frac{\lambda}{4\pi d}\right)^2} \quad / \log$$

ФУНДАМЕНТАЛНА РАВЕНКА ЗА СЛАБЕЕЊЕ ВО СЛОБОДЕН ПРОСТОР

$$\frac{P_T}{P_R} = \frac{1}{G_T G_R} \left(\frac{4\pi d}{\lambda}\right)^2 \quad / \log$$

$$10\log \frac{P_T}{P_R} = -10\log(G_T) - 10\log(G_R) + 10\log\left(\frac{4\pi d \cdot c}{c \cdot \lambda}\right)^2$$

$$= -10\log G_T - 10\log(G_R) + 20\log\left(\frac{4\pi d}{c}\right) + 20\log\left(\frac{c}{\lambda}\right)$$

$$= -10\log(G_T) - 10\log(G_R) + 20\log(f) + 20\log d + 20\log\frac{4\pi}{c}$$

-147.56 dB

ИЗОТРОПНА АНТЕНА

$$L_B(0) = 20\log(f_{MHz}) + 20\log(d_{km}) + 20\log 10^6 + 20\log 10^3$$

$$-147.56 = 32.44\,dB + 20\log(f_{MHz}) + 20\log(d_{km})$$

$$SNR = \frac{P_R}{kTB} = \frac{P_T G_T G_R}{kTB} \left(\frac{c}{4\pi f d}\right)^2$$

- Густина на моќност:

$$W = \frac{|\vec{E}|^2}{Z_W}$$

Z_W = бранова импеданса

$$Z_W = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi\ \Omega$$

$$W = \frac{E^2}{120\pi} \qquad \frac{P_T G_T}{4\pi d^2} = \frac{E^2}{120\pi}$$

(30)

$$\boxed{E = \frac{\sqrt{30 P_T G_T}}{d}}$$

$$P_R = W \cdot A = \frac{E^2}{120\pi} \cdot \frac{\lambda^2 G_R}{4\pi} = \left(\frac{E\lambda}{2\pi}\right)^2 \frac{G_R}{120} \sim E^2$$

- Фазна разлика меѓу директниот и рефлектираниот зрак:

$$\Delta\phi = \frac{2\pi \Delta R}{\lambda} = \frac{4\pi h_T h_R}{\lambda d}$$

$$E = E_d\left[1 + \rho e^{-j\Delta\phi}\right] = E_d\left[1 + |\rho| e^{-j(\Delta\phi+\varphi)}\right]$$ FAZE VO TRIAGOLNIKOT

$$\psi \approx 0 \qquad \rho = \frac{\sin\psi - \sqrt{(\varepsilon_r - jx) - \cos^2\psi}}{\sin\psi + \sqrt{(\varepsilon_r - jx) - \cos^2\psi}} = \frac{-\sqrt{(\varepsilon_r - jx) - 1}}{\sqrt{(\varepsilon_r - jx) - 1}} = -1$$

$$E = E_d\left[1 - e^{-j\Delta\phi}\right] \qquad |E| = |E_d|\sqrt{(1-\cos\Delta\phi)^2 + \sin^2\Delta\phi} =$$
$$= |E_d|\sqrt{4\sin^2\left(\frac{\Delta\phi}{2}\right)} = |E_d|\cdot 2\sin\left(\frac{\Delta\phi}{2}\right)$$ PROVERKA

$$e^{jx} = \cos x + j\sin x \qquad \left|1 - e^{-j\Delta\phi}\right|^2 = \left|(1-\cos\Delta\phi) + j\sin\Delta\phi\right|^2 =$$
$$1 - 2\cos\Delta\phi + \cos^2\Delta\phi + \sin^2\Delta\phi = 1 - 2\cos\Delta\phi + 1 =$$
$$= 2(1 - \cos\Delta\phi) = (*)$$
$$\cos 2\frac{\Delta\phi}{2} = \cos\frac{\Delta\phi}{2}\cdot\cos\frac{\Delta\phi}{2} - \sin\left(\frac{\Delta\phi}{2}\right)\cdot\sin\left(\frac{\Delta\phi}{2}\right) =$$
$$= \cos^2\left(\frac{\Delta\phi}{2}\right) - \sin^2\left(\frac{\Delta\phi}{2}\right)$$

$$(*) = 2\left(1 - \cos^2\left(\frac{\Delta\phi}{2}\right) + \sin^2\left(\frac{\Delta\phi}{2}\right)\right) = 2\left(\cos^2\left(\frac{\Delta\phi}{2}\right) + \sin^2\left(\frac{\Delta\phi}{2}\right) - \cos^2\left(\frac{\Delta\phi}{2}\right) + \sin^2\left(\frac{\Delta\phi}{2}\right)\right) = 4\sin^2\left(\frac{\Delta\phi}{2}\right)$$

NA DA PROVERAM VOVO PISUVA !!

$$\Delta R = R_2 - R_1 \approx \frac{2h_T' \cdot h_R'}{d} \qquad \Delta\phi = \frac{2\pi\Delta R}{\lambda} = \frac{4\pi h_T' \cdot h_R'}{\lambda \cdot d}$$

$$P_R \sim 4|E_d|^2 \cdot \sin^2\left(\frac{2\pi h_T' h_R'}{\lambda \cdot d}\right) \overset{(*)}{=} 4P_T G_T G_R \left(\frac{\lambda}{4\pi d}\right)^2 \cdot \sin^2\left(\frac{2\pi h_T' h_R'}{\lambda \cdot d}\right)$$

$$d \gg h_T \quad d \gg h_R: \quad = 4G_T G_R P_T \left(\frac{\lambda}{4\pi d}\right)^2 \cdot \frac{(2\pi h_T' h_R')^2}{\lambda^2 d^2} = 4G_T G_R \frac{1}{16\pi^2 d^2}\cdot\frac{4\pi^2 h_T^2 h_R^2}{d^2}$$

$$P_R = P_T G_T G_R \frac{h_T^2 h_R^2}{d^4} = P_T G_T G_R \left(\frac{h_T h_R}{d^2}\right)^2 = P_T G_T G_R \left(\frac{h_T h_R}{d^2}\right)^2$$

$$\boxed{\frac{P_R}{P_T} = G_T G_R \left(\frac{h_T h_R}{d^2}\right)^2}$$ RAVENKA NA PROPAGACIJA NAD RAMNA POVRŠINA;

- VLIJANIE NA NERAMNINITE

$$\Delta l = 2d\sin\psi \qquad \Delta\theta = \frac{2\pi}{\lambda}\cdot\Delta l = \frac{4\pi d}{\lambda}\sin\psi$$

RAYLEIGH-OV KRITERIUM: $d_R \geq \frac{\lambda}{8\sin\psi}$

$$\Delta\theta \geq \frac{\pi}{2} \qquad \frac{4\pi d_R}{\lambda}\sin\psi \geq \frac{\pi}{2} \qquad d_R \geq \frac{\lambda}{8\sin\psi} = \frac{\lambda}{8\psi}$$

· STANDARDNA DEVIJACIJA NA TERENSKA NERAVNOST:

$$c = \frac{4\pi\sigma \sin\varphi}{\lambda} \approx \frac{4\pi\sigma\cdot\varphi}{\lambda}$$

$c < 0{,}1 \Rightarrow$ POVRŠINATA E RAMNA

$c > 10 \Rightarrow$ POVRŠINATA E NERAMNA

MULTIPLE INTEGRALS

$$\sum_{i=1}^{n} f(x_i^*)\Delta x \qquad \int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

Volumes and double integrals

$z = f(x,y)$ $\quad R = [a,b]\times[c,d] = \{(x,y)\in R^2,\ a\le x\le b,\ c\le y\le d\}$

$$S = \{(x,y,z)\in R^3 \mid 0\le z\le f(x,y),\ (x,y)\in R\}$$

$[a,b]$ $\quad [x_{i-1},x_i]$, $\quad \Delta x = \frac{b-a}{m}$

$[c,d]$ $\quad [y_{j-1},y_j]$; $\quad \Delta y = \frac{d-c}{n}$

Volume of the column (ij)

$f(x_{ij}^*, y_{ij}^*)\Delta A$; $\quad V \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$

$$V = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

726
1096
1325
818
1005

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_j)\,\Delta A$$

$$V = \iint_R f(x,y)\,dA$$

EX 1 $\quad R = [0,2]\times[0,2]$ $\quad V = ?$ $\quad n = 2 \quad \Delta x = 1$

$z = 16 - x^2 - 2y^2$ $\quad m = 2 \quad \Delta y = 1$

$$V \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_i, y_j)\Delta A = f(1,1)\cdot\Delta A + f(1,2)\Delta A +$$

$$+ f(2,1)\Delta A + f(2,2)\Delta A = 13\cdot 1 + 7\cdot 1 + 10\cdot 1 + 4\cdot 1 = 20 + 14 = 34$$

EX 2 $R = \{(x,y) \mid -1 \le x \le 1,\ -2 \le y \le 2\}$ $\qquad \iint_R \sqrt{1-x^2}\, dA$

$$V = \frac{R^2\pi}{2} \cdot (2\cdot 2) = \frac{\pi}{2}\cdot 4 = 2\pi$$

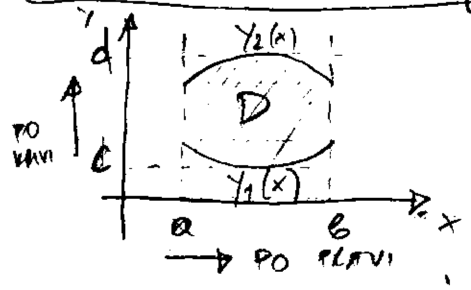
EX 1 $V = \int_0^2\int_0^2 (16 - x^2 - 2y^2)\, dx\, dy = \int_0^2 dx \left(16y - x^2 y - 2\frac{y^3}{3}\right)\Big|_0^2 =$

$$= \int_0^2 dx \left[\left(2\cdot 16 - 2x^2 - 2\cdot 8/3\right) - (0 - 0 - 0)\right] = \int_0^2 \left(32 - 2x^2 - 16/3\right) dx$$

$$= \int_0^2 \left(\frac{96-16}{3} - 2x^2\right) dx = \left(\frac{80}{3}x - 2\frac{x^3}{3}\right)\Big|_0^2 = \frac{160}{3} - \frac{16}{3} = \frac{144}{3}$$

$\boxed{V = 48}$

REŠAVANJE DVOJNI INTEGRALI OD ELEKTROMAGNETIKA



$$V = \iint_D f(x,y)\, dx\, dy \qquad D = \left\{(x,y) \;\middle|\; \begin{matrix} a < x < b \\ y_1(x) < y < y_2(x)\end{matrix}\right\}$$

$$V = \int_a^b dx \int_{y_1(x)}^{y_2(x)} f(x,y)\, dy = \int_c^d dy \int_{x_1(y)}^{x_2(y)} f(x,y)\, dx$$

SMENA NA PROMENLJIVI KAJ DVOEN INTEGRAL:

$x = x(u,v)$

$y = y(u,v)$

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = J$$

$$\iint_D f(x,y)\, dx\, dy = \iint_{D_1} f(u,v)\, |J|\, du\, dv$$

PRIMER: $x = \rho \cdot \cos\varphi$ $\quad u - \rho$

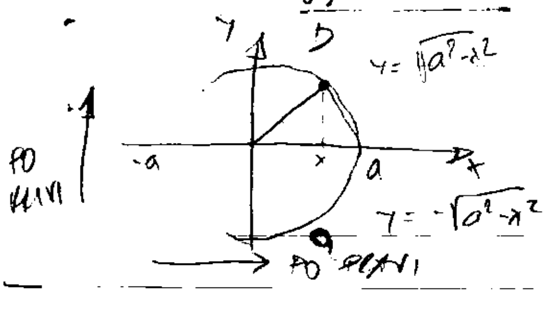
$y = \rho\cdot\sin\varphi$ $\quad v - \varphi$

$$\begin{vmatrix} \cos\varphi & -\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{vmatrix} = J$$

$$J = \rho\cdot\cos^2\varphi + \rho\sin^2\varphi = \rho\left(\cos^2\varphi + \sin^2\varphi\right) = \rho$$

$$\iint_D f(x,y)\, dx\, dy = \iint_{D_1} f(\rho\cos\varphi, \rho\sin\varphi)\, \rho\, d\rho\, d\varphi$$

PRIMER: $\iint_D \sqrt{x^2+y^2}\, dx\, dy \quad D: x^2 + y^2 \le a^2$



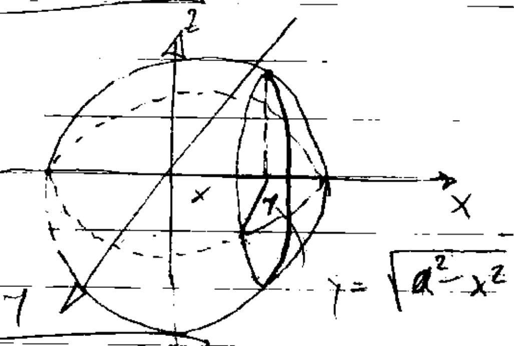
$$\iint_D \sqrt{x^2+y^2}\, dx\, dy = \int_{-a}^{a} dx \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy$$

$$\left.\begin{array}{l} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{array}\right\} \Rightarrow J = \rho \qquad V = \iint_D f(x,y)\,dx\,dy = \int\int f(\rho\cos\varphi, \rho\sin\varphi)\,\rho\,d\rho\,d\varphi$$

$$V = \int_0^{2\pi} d\varphi \int_0^a \rho \sqrt{\rho^2 \underbrace{(\cos^2\varphi + \sin^2\varphi)}_{1}}\, d\rho = \int_0^{2\pi} d\varphi \int_0^a \rho^2 d\rho = \int_0^{2\pi} \frac{a^3}{3} d\varphi$$

$$\boxed{V = \frac{2a^3\pi}{3}}$$

TOPKA:



$$V = \int y^2\pi\,dx = 2\int_0^a (a^2 - x^2)\pi\,dx$$

$$= 2\pi\left[a^2x - \frac{x^3}{3}\right]\Big|_0^a =$$

$$= 2\pi\left[a^3 - \frac{a^3}{3}\right] = 2\pi\,\frac{2a^3}{3}$$

$$\boxed{V = \frac{4\pi a^3}{3}}$$

TROJNI INTEGRALI

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iint_D dx\,dy \int_{z_1(x,y)}^{z_2(x,y)} dz$$

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{V_1} f\big(x(u,v,w), y(u,v,w), z(u,v,w)\big)\,|J|\,du\,dv\,dw$$

CILINDRIČNI KOORDINATI

$x = \rho\cos\varphi\,;\quad y = \rho\sin\varphi\,;\quad z = z$

$$\begin{array}{l} \rho - u \\ \varphi - v \\ z - w \end{array}$$

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{V_1} f(\rho\cos\varphi, \rho\sin\varphi, z)\,\rho\,d\rho\,d\varphi\,dz$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$\begin{bmatrix} 9 & 8 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 6 \end{bmatrix} = 9\cdot\begin{vmatrix} 1 & 3 \\ 0 & 6 \end{vmatrix} - 8\begin{vmatrix} 0 & 3 \\ 0 & 6 \end{vmatrix} + 4\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 9(6-0) = 54$$

DETERM

$$J = \begin{vmatrix} \cos\varphi & -\rho\sin\varphi & 0 \\ \sin\varphi & +\rho\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos\varphi\begin{vmatrix} \rho\cos\varphi & 0 \\ 0 & 1 \end{vmatrix} + \rho\sin\varphi\begin{vmatrix} \sin\varphi & 0 \\ 0 & 1 \end{vmatrix} =$$

$$= \cos\varphi(\rho\cos\varphi) + \rho\sin^2\varphi = \rho(\cos^2\varphi + \sin^2\varphi) = \rho$$

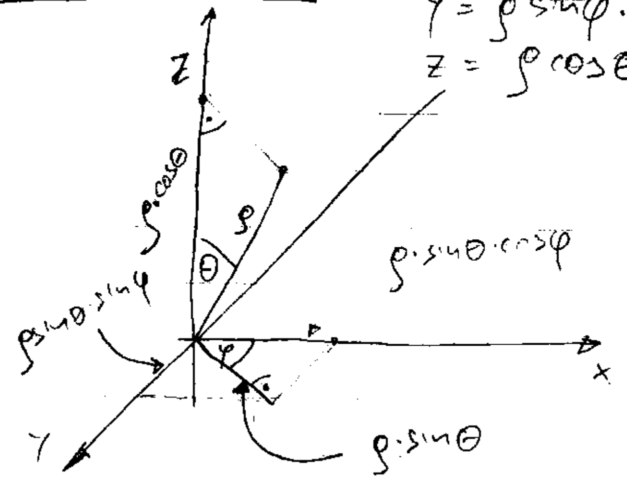
SFERIČNI KOORDINATI

$x = \rho\cos\varphi\cdot\sin\theta$ $\qquad 0 \le \varphi \le 2\pi$

$y = \rho\sin\varphi\cdot\sin\theta$ $\qquad 0 \le \theta \le \pi$

$z = \rho\cos\theta$

$u - \rho$, $v - \varphi$, $w - \theta$



$$j = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \theta} \end{vmatrix}$$

$$j = \begin{vmatrix} \cos\varphi\sin\theta ; & -\rho\sin\varphi\sin\theta ; & \rho\cos\varphi\cdot\cos\theta \\ \sin\varphi\cdot\sin\theta ; & \rho\cos\varphi\sin\theta ; & \rho\cdot\sin\varphi\cdot\cos\theta \\ \cos\theta ; & 0 ; & -\rho\sin\theta \end{vmatrix} = -\sin\theta\cdot\rho^2$$

$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$|j| = \sin\theta\,\rho^2$$

$$I = \iiint_V f(x,y,z)\,dx\,dy\,dz =$$

$$= \iiint_V f(\rho\cos\varphi\cdot\sin\theta,\ \rho\sin\varphi\cdot\sin\theta,\ \rho\cos\theta)\,\rho^2\sin\theta\cdot d\rho\,d\varphi\,d\theta$$

ARC LENGTH (PRESMETUVANJE NA DOLŽINA NA RAMNINSKA KRIVA SO KRIVOLINISKI INTEGRAL OD I RED)

$$L = \lim_{n\to\infty}\sum_{i=1}^{n}|P_{i-1}P_i|$$

$$|P_{i-1}P_i| = \sqrt{(x_i-x_{i-1})^2+(y_i-y_{i-1})^2} = \sqrt{\Delta x^2+\Delta y^2}$$

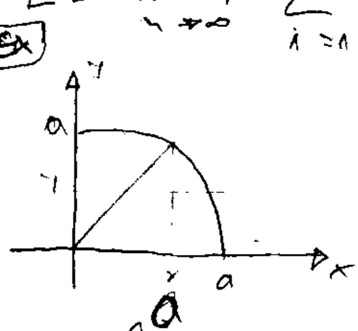
$$f(x_i)-f(x_{i-1}) = f'(x_i^*)(x_i-x_{i-1}) \quad ; \quad \Delta y = f'(x_i^*)$$

$$|P_{i-1}P_i| = \Delta x\sqrt{1+\frac{\Delta y^2}{\Delta x^2}} = \Delta x\sqrt{1+[f'(x_i^*)]^2}$$

$$L = \lim_{n\to\infty}\sum_{i=1}^{n}\Delta x\sqrt{1+[f'(x_i^*)]^2}$$

$$L = \int_a^b\sqrt{1+f'^2(x)}\,dx$$

Ex



$$f(x) = \sqrt{a^2-x^2} \qquad f'(x) = \frac{1}{2}\frac{1}{\sqrt{a^2-x^2}}(-2x)$$

$$f'(x) = -\frac{x}{\sqrt{a^2-x^2}} \qquad f'^2(x) = \frac{x^2}{a^2-x^2}\,dx$$

$$L = \int_0^a\sqrt{\frac{a^2-x^2+x^2}{a^2-x^2}}\,dx = a\int_0^a\frac{dx}{\sqrt{a^2-x^2}} = \int_0^a\frac{a\,d\left(\frac{x}{a}\right)}{\sqrt{1-\left(\frac{x}{a}\right)^2}} = a\arcsin\left(\frac{x}{a}\right)\Big|_0^a$$

$y = \arcsin(x)$

$x = \sin y$

$$1 = \cos y\cdot\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 x}} = \frac{1}{\sqrt{1-x^2}}$$

$$L = a \cdot \arcsin\left(\frac{x}{a}\right)\Big|_0^a = a\left(\arcsin(1) - \arcsin(0)\right) = \frac{a\pi}{2}$$

$$L_{arc} = 4 \cdot L = 4 \cdot \frac{a\pi}{8} = 2a\pi$$

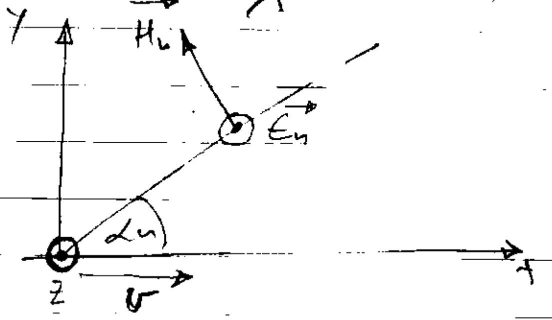
$[9 + 7^2 = 7 + 49 = 58]$

$$\Delta\phi = \frac{2\pi \Delta l}{\lambda} = \frac{2\pi d \cos\theta}{\lambda} \qquad \omega = \frac{\Delta\phi}{\Delta t} \qquad f = \frac{\omega}{2\pi} = \frac{\Delta\phi}{2\pi\Delta t}$$

$$f = \frac{2\pi d \cos\theta}{2\pi \Delta t \lambda} = \left(\frac{v}{\lambda}\right) \cdot \cos\theta = f_{d\,max} \cos\theta$$

$$f = \frac{v}{\lambda} \cdot \cos\theta$$

$$\omega_{dn} = \frac{2\pi v}{\lambda} \cos(\alpha_n) = \omega_{d\,max} \cos(\alpha_n)$$



$$\vec{e} = E_z \vec{i}_z$$

$$E_z = \sum_{n=1}^{N} \vec{E}_n = E_0 \sum_{n=1}^{N} C_n \cos(\omega t + \theta_n)$$

$$= E_0 \sum_{n=1}^{N} C_n \cos(\omega_c t + \omega_{dn} t + \phi_n)$$

θ_n → СЛУЧАЈНА ФАЗА НА n-ТА УПАДНА КОМПОНЕНТА

$E_0 C_n$ → АМПЛИТУДА НА n-ТА УПАДНА КОМПОНЕНТА

$$\sum_{n=1}^{N} \overline{C_n^2} = 1 \qquad \sigma^2 = \overline{\left(\xi - \bar{\xi}\right)^2}$$

RICIAN FADING

$$p(r, \theta) = \frac{r}{2\pi\sigma^2} e^{-\frac{r^2 + r_s^2 - 2 r r_s \cos\theta}{2\sigma^2}} \qquad p_r(r) = \int p(r,\theta)\, dr$$

$$p_r(r) = \frac{r}{\sigma^2} e^{-\frac{r^2 + r_s^2}{2\sigma^2}} I_0\left(\frac{r r_s}{\sigma^2}\right)$$

$$E_z(t) = T_c(t) \cdot \cos(\omega_c t) - T_s(t) \sin(\omega_c t)$$

$$T_c(t) = E_0 \sum_{n=1}^{N} C_n \cos(\omega_{dn} t + \phi_n)$$

$$T_s(t) = E_0 \sum_{n=1}^{N} C_n \sin(\omega_{dn} t + \phi_n)$$

} БАНДОЛОМЕНЛИВИ СЛУЧАЈНИ ПРОЦЕСИ

$$\overline{T_c(t)} = \overline{T_s(t)} = 0 \qquad \overline{T_c^2} = \overline{T_s^2} = \overline{|E_z|^2} = \frac{E_0^2}{2}$$

RAYLEIGH РАСПРЕДЕЛБА

$$E_z(t) = r(t) \cos\left[\omega_c t - \varphi(t)\right]$$

$$E_z(t) = \frac{T_c}{2}\left(e^{j\omega t} - e^{-j\omega t}\right) - \frac{T_s}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)$$

$$= \frac{e^{j\omega t}}{2}(T_c - T_s) - \frac{e^{-j\omega t}}{2}(T_c + T_s)$$

$$E_z(t) = T_c(t)\cdot\cos(\omega_c t) - T_s(t)\cdot\sin(\omega_c t)$$

$T_c(t) = R(t)\cdot\cos\varphi(t)$; $T_s(t) = R(t)\sin\varphi(t)$

$$E_z(t) = R(t)\cdot\cos\varphi(t)\cos(\omega_c t) - R(t)\sin\varphi(t)\cdot\sin(\omega_c t)$$

$$E_z(t) = R(t)\cdot\cos(\omega_c t + \varphi(t))$$

$$T_c^2(t) + T_s^2(t) = R^2(t)\left[\cos^2\varphi(t) + \sin^2\varphi(t)\right]$$

$$\boxed{R(t) = \sqrt{T_c^2(t) + T_s^2(t)}}$$

$$\frac{T_s}{T_c} = tg[\varphi(t)]$$

$$\boxed{\varphi(t) = \operatorname{arctg}\frac{T_s(t)}{T_c(t)}}$$

СУМА ОД ДВЕ ИЛИ ПОВЕЌЕ СЛУЧАЈНИ ПРОМЕНЛИВИ СО ПОЗНАТА ГУСТИНА НА ВЕРОЈАТНОСТ

$\xi_1, \xi_2, \dots \xi_n$ $\qquad \xi_1 + \xi_2 + \dots + \xi_n \Rightarrow$ ГАУСОВА ЗАКОНИТОСТ САМО АКО $n \to \infty$

$$p_\xi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}} \qquad m = \sum_i m_i \qquad \sigma^2 = \sum_i \sigma_i^2$$

НЕКА: $\boxed{n = 2}$

$\xi, \eta \qquad p_{\xi\eta}(x,y)$ } СЛУЧАЈНИ ПРОМЕНЛИВИ И НИВНА ЗДРУЖЕНА Ф-ЈА НА ВЕРОЈАТНОСТ

$\zeta = \xi + \eta \;;\quad z = x + y \qquad p_\zeta(z) = ?$

- ВОВЕДУВАЊЕ НА ПОМОШНА ПРОМЕНЛИВА ЗА ДА СЕ НАЈДЕ $|J|$

$\xi' = \xi \qquad z = x + y,\; x' = x \quad$ ИНВ. Ф-ЈА $\quad y = z - x,\; x = x'$

$$J = \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1 \qquad |J| = 1$$

$$p_{\xi'\zeta}(x', z) = \left.\frac{p_{\xi\eta}(x,y)}{|J|}\right|_{\substack{x = x' \\ y = z - x'}} = p_{\xi\eta}(x', z - x')$$

$$p_\zeta(z) = \int_{-\infty}^{\infty} p_{\xi'\zeta}(x', z)\,dx' = \int_{-\infty}^{\infty} p_{\xi\eta}(x', z - x')\,dx' = \left|\, x' = x \,\right|$$

$$= \int_{-\infty}^{\infty} p_{\xi\eta}(x, z - x)\,dx$$

ξ, η ARE STATISTICALY INDEPENDANT

$$p_{\xi\eta}(x, z - x) = p_\xi(x)\, p_\eta(z - x)$$

$$p_\zeta(z) = \int_{-\infty}^{\infty} p_\xi(x)\cdot p_\eta(z-x)\,dx = p_\xi(z) \circledast p_\eta(z-x) = \rho_{\xi\eta}(z)$$

• n - SLUČAJNIH PROMENLJIVIH:

$\xi_i,\ i = 1, 2, \ldots n$

$$\xi = \sum_{i=1}^{n} \xi_i \qquad \boxed{p_\xi(x) = p_{\xi_1}(x) * p_{\xi_2}(x) * \cdots * p_{\xi_n}(x)}$$

$$W_\xi(j\Omega) = \int_{-\infty}^{\infty} p_\xi(x)\, e^{+j\Omega x}\,dx$$ ⇒ KARAKTERISTIČNA FUNKCIJA = KONJUGOVANA FURIJEVA TRANSFORM.

$$W_\xi(j\Omega) = W_{\xi_1}(j\Omega)\cdot W_{\xi_2}(j\Omega)\cdots W_{\xi_n}(j\Omega)$$

$$p_\xi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} W_\xi(j\Omega)\, e^{-j\Omega x}\,d\Omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \prod W_{\xi_i}(j\Omega)\, e^{-j\Omega x}\,d\Omega$$ (**) ⇒

$n \to \infty \qquad p_\xi(x) \to$ GAUSOVA RASPODELA

• n - normalno raspodeljeni promenljivi

$\xi_i,\ i = 1, 2, \ldots n;\qquad p_{\xi_i}(x_i) = \frac{1}{\sqrt{2\pi\sigma_i^2}}\, e^{-\frac{(x-m_i)^2}{2\sigma_i^2}}$

$$W_{\xi_i}(j\Omega) = \int_{-\infty}^{\infty} p_{\xi_i}(x_i)\, e^{j\Omega x_i}\,dx_i = \frac{1}{\sqrt{2\pi\sigma_i^2}}\int_{-\infty}^{\infty} e^{-\frac{(x_i-m_i)^2}{2\sigma_i^2}}\cdot e^{j\Omega x_i}\,dx_i$$

$$= \frac{1}{\sqrt{2\pi\sigma_i^2}}\int_{-\infty}^{\infty} e^{-\frac{(x_i-m_i)^2}{2\sigma_i^2}}\, e^{j\Omega(x_i-m_i)}\cdot e^{j\Omega m_i}\,dx_i = \quad (*)$$

$$= \frac{e^{j\Omega m_i}}{\sqrt{2\pi\sigma_i^2}}\int_{-\infty}^{\infty} e^{-\frac{(x_i-m_i)^2 + j2\Omega\sigma_i^2(x_i-m_i)}{2\sigma_i^2}}\,dx_i$$

$$\left| x_i - m_i = u \quad dx_i = du \right| = \frac{e^{j\Omega m_i}}{\sqrt{2\pi\sigma_i^2}}\int_{-\infty}^{\infty} e^{\frac{-u^2 + 2j\Omega\sigma_i^2 u}{2\sigma_i^2}}\,du$$

$$I = \int e^{-\frac{u^2}{2\sigma^2}}\cdot e^{j\Omega u}\,du = \frac{1}{j\Omega}\int e^{-\frac{u^2}{2\sigma^2}}\,d e^{j\Omega u}$$

$$u = e^{-\frac{u^2}{2\sigma^2}} \qquad v = \int e^{j\Omega u}\,du = \frac{1}{j\Omega}\int e^{j\Omega u}\,d(j\Omega u) = \frac{e^{j\Omega u}}{j\Omega}$$

$$I = e^{-\frac{\mu^2}{2\sigma^2}} \frac{e^{j\Omega\mu}}{j\Omega} - \underbrace{\int \frac{e^{j\Omega\mu}}{j\Omega} \cdot d\left(e^{-\frac{\mu^2}{2\sigma^2}}\right)}_{I_1}$$

$$I_1 = \frac{1}{j\Omega}\int e^{j\Omega\mu} e^{-\frac{\mu^2}{2\sigma^2}} \frac{2\mu}{2\sigma^2}\, d\mu$$

$$I = \int_{-\infty}^{\infty} e^{-\frac{\mu^2}{2\sigma^2}+j\Omega\mu}\, d\mu = \frac{1}{2}\int_{-\infty}^{\infty} \underbrace{\cos(\Omega\mu)}_{\text{PARNA}}\, \underbrace{e^{-\frac{\mu^2}{2\sigma^2}}}_{\text{PARNA}}\, d\mu + \underbrace{\frac{1}{2}\int_{-\infty}^{\infty} \underbrace{\sin(\Omega\mu)}_{\text{NEPAR}}\, \underbrace{e^{-\frac{\mu^2}{2\sigma^2}}}_{\text{PAR}}\, d\mu}_{=0}$$

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \cos(\Omega\mu)\, e^{-\frac{\mu^2}{2\sigma^2}}\, d\mu$$

$$u = e^{-\frac{\mu^2}{2\sigma^2}} \quad du = -e^{-\frac{\mu^2}{2\sigma^2}} \frac{2\mu}{2\sigma^2}\, d\mu \quad v = \int \cos(\Omega\mu)\, d\mu = \frac{1}{\Omega}\sin(\Omega\mu)$$

$$I = \frac{1}{2}\left(e^{-\frac{\mu^2}{2\sigma^2}} \cdot \frac{1}{\Omega}\sin(\Omega\mu) + \int \frac{2\mu}{2\sigma^2} e^{-\frac{\mu^2}{2\sigma^2}} \cdot \frac{1}{\Omega} \sin(\Omega\mu)\, d\mu \right)$$

$$\int_{-\infty}^{\infty} e^{-\frac{\mu^2}{2\sigma^2}+j\Omega\mu}\, d\mu = \sqrt{2\pi\sigma^2}\, e^{-\frac{\sigma^2\Omega^2}{2}}$$

(*) $$W_{\xi_i}(j\Omega) = \frac{e^{j\Omega m_i}}{\sqrt{2\pi\sigma_i^2}} \int_{-\infty}^{\infty} e^{-\frac{\mu^2}{2\sigma_i^2}+j\Omega\mu}\, d\mu = \frac{e^{j\Omega m_i}}{\sqrt{2\pi\sigma_i^2}} \cdot \sqrt{2\pi\sigma_i^2}\, e^{-\frac{\sigma_i^2\Omega^2}{2}}$$

$$W_{\xi_i}(j\Omega) = e^{-\frac{\sigma_i^2\Omega^2}{2}} e^{j\Omega m_i} \qquad W_{\xi}(j\Omega) = \prod_{i=1}^{n} W_{\xi_i}(j\Omega) =$$

$$= e^{-\frac{\Omega^2}{2}\sum_{i=1}^{n}\sigma_i^2 + j\Omega\sum_{i=1}^{n} m_i} = e^{-\frac{\Omega^2\sigma^2}{2} + j\Omega m}$$

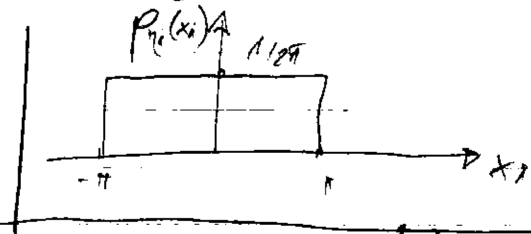
$$\boxed{\sigma^2 = \sum_{i=1}^{n}\sigma_i^2 \qquad m = \sum_{i=1}^{n} m_i}$$

• SUMA OD SINUSOIDI SO SLUČAJNA FAZA

$\xi_i = A_i \sin \eta_i$

$y_i = A_i \sin x_i$

$i = 1, 2, 3, \ldots, n$

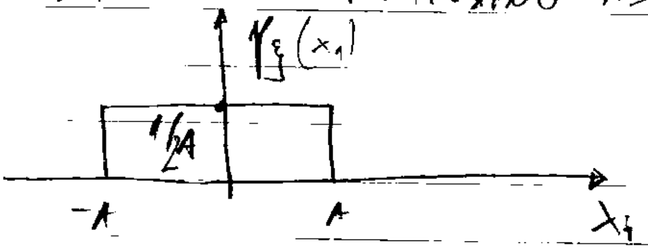


$$p_{\eta_i}(x_i) = \begin{cases} \frac{1}{2\pi} & -\pi \le x_i \le \pi \\ 0 & \text{other} \end{cases}$$

$$\xi = \sum_{i=1}^{n} \xi_i \qquad W_{\xi}(j\Omega) = e^{-\frac{\Omega^2\sigma^2}{2}} \qquad p_{\xi}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$

$$P_i = \frac{A_i^2}{2} = \left(\frac{A_i}{\sqrt{2}}\right)^2 = \frac{A_i^2}{2} \qquad P = \sigma^2 = \sum_{i=1}^{n} \sigma_i^2 = \sum_{i=1}^{n} \frac{A_i^2}{2}\,!$$

SUMA OD MNOŽHNO NEZAVISNI SLUČAJNI PROMENLIVI



$\xi_i \quad i = 1, 2, 3, \dots$

$$p_{\xi_i}(x_i) = \begin{cases} \frac{1}{2A} & -A \le x_i \le A \\ 0 & \text{otherwise} \end{cases}$$

$$W_{\xi_i}(j\Omega) = \int_{-\infty}^{\infty} p_{\xi_i}(x_i)\, e^{+j\Omega x_i}\, dx_i = \int_{-A}^{A} \frac{1}{2A} e^{j\Omega x_i}\, dx_i$$

$$W_{\xi_i}(j\Omega) = \frac{1}{2A}\,\frac{1}{j\Omega}\int_{-A}^{A} e^{j\Omega x_i}\, d(j\Omega x_i) = \frac{1}{2jA\Omega}\left(e^{+j\Omega A} - e^{-j\Omega A}\right)$$

$$W_{\xi_i}(j\Omega) = \frac{1}{A\Omega}\sin(\Omega A) = \frac{\sin \Omega A}{\Omega A}$$

RAYLEIGH - RASPREDELBA

$$E_z(t) = R(t)\cos[\omega_c t - \varphi(t)]$$

$$R(t) = \sqrt{I_c^2(t) + I_s^2(t)} \qquad \varphi(t) = \operatorname{arctg}\frac{I_s(t)}{I_c(t)}$$

$$f_{I}(x) = f_{I_c}(x) = f_{I_s}(x) = \frac{1}{\sqrt{2\pi b}}\, e^{-\frac{x^2}{2b}} \qquad b = \sigma^2 = \frac{E_0^2}{2}$$

$$f_R(r) = \frac{r}{b}\, e^{-\frac{r^2}{2b}} \qquad r > 0$$

$$p_{\xi\eta}(x,y) = p_\xi(x)\cdot p_\eta(y) \qquad p_\xi(x),\ p_\eta(y) \Rightarrow \text{GAUSSIAN DISTRIBUTION}$$

$$p_{\xi\eta}(x,y) = \frac{1}{2\pi\sigma^2}\, e^{-\frac{x^2+y^2}{2\sigma^2}} = \frac{1}{2\pi\sigma^2}\, e^{-\frac{r^2}{2\sigma^2}} \qquad r(t) = \sqrt{x^2(t) + y^2(t)}$$

$$x = r\cos\varphi \qquad y = r\sin\varphi \qquad |J| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r$$

$$dx\,dy = |J|\,dr\,d\varphi \qquad p_{\xi\eta}(r,\varphi) = |J|\, p_{\xi\eta}(x,y)$$

$$p_{\xi\eta}(r,\varphi) = \frac{r}{2\pi\sigma^2}\, e^{-\frac{r^2}{2\sigma^2}}$$

$$p(r) = \int_0^{2\pi} \frac{1}{2\pi\sigma^2}\, e^{-\frac{r^2}{2\sigma^2}}\, r\, d\varphi = \frac{r}{2\pi\sigma^2}\, e^{-\frac{r^2}{2\sigma^2}}\, 2\pi = \frac{r}{\sigma^2}\, e^{-\frac{r^2}{2\sigma^2}}$$

$$p(r) = \frac{r}{\sigma^2}\, e^{-\frac{r^2}{2\sigma^2}} \Rightarrow \text{RAYLEIGH DISTRIBUTION}$$

$$p(\varphi) = \int_0^\infty p(\varphi, r)\,dr = \int_0^\infty \frac{r}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}}\,dr = \frac{1}{2\pi\sigma^2} \underbrace{\int r e^{-\frac{r^2}{2\sigma^2}}\,dr}_{I}$$

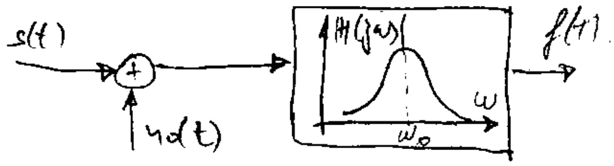
$$I = \left| \begin{matrix} u = e^{-\frac{r^2}{2\sigma^2}} & v = \int r\,dr \\ du = -e^{-\frac{r^2}{2\sigma^2}} \cdot \frac{2r}{2\sigma^2}\,dr & v = \frac{r^2}{2} \end{matrix} \right| = \frac{r^2}{2} e^{-\frac{r^2}{2\sigma^2}} \Big|_0^\infty + \int \frac{r^3}{2} e^{-\frac{r^2}{2\sigma^2}}\,dr$$

$$I = \frac{-2\sigma^2}{2} \int e^{-\frac{r^2}{2\sigma^2}}\, d\left(\frac{-r^2}{2\sigma^2}\right) = -\sigma^2 e^{-\frac{r^2}{2\sigma^2}} \Big|_0^\infty = -\sigma^2\left(\frac{1}{\infty} - 1\right) = \sigma^2$$

$$\boxed{p_\eta(\varphi) = \frac{1}{2\pi\sigma^2} \cdot \sigma^2 = \frac{1}{2\pi}}$$

$$p_{\xi\eta}(r, \varphi) = \underbrace{p_\xi(r)}_{\text{RAYLEIGH}} \cdot \underbrace{p_\eta(\varphi)}_{\text{UNIFORM}}$$

$$p_{\xi\eta}(r, \varphi) = \frac{r}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} = \underbrace{\frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}}_{p(r)} \cdot \underbrace{\frac{1}{2\pi}}_{p(\varphi)}$$



$f(t) = s(t) + n_d(t)$

$n_d(t)$ – nce – MODEL OF NOISE

$s(t) = A \cos(\omega_0 t)$

$$f(t) = A\cos\omega_0 t + x(t)\cos(\omega_0 t) - y(t)\sin(\omega_0 t) =$$
$$= [A + x(t)]\cos(\omega_0 t) - y(t)\sin(\omega_0 t) = z(t)\cos(\omega_0 t) - y(t)\sin(\omega_0 t)$$

$\overline{z(t)} = A \qquad z = A + x(t)$

$f(t) = r(t)\cdot\cos[\omega_0 t + \varphi(t)]$

$z(t) = r(t)\cdot\cos(\varphi(t))$

$y(t) = r(t)\cdot\sin(\varphi(t))$

$r(t) = \sqrt{z^2(t) + y^2(t)}$

$\varphi(t) = \operatorname{arctg}\frac{y(t)}{z(t)} \pm k\pi$

$$p_{r,\varphi}(r, \varphi) = r \cdot p_{\xi\eta}(z, y) = \frac{r}{2\pi\sigma^2} \cdot e^{-\frac{x^2 + y^2}{2\sigma^2}} = \frac{r}{2\pi\sigma^2} e^{-\frac{(z-A)^2 + y^2}{2\sigma^2}}$$

$$= \frac{r}{2\pi\sigma^2} e^{-\frac{(r\cos\varphi - A)^2 + r^2\sin^2\varphi}{2\sigma^2}} = \frac{r}{2\pi\sigma^2} e^{-\frac{r^2 + A^2 - 2rA\cos\varphi}{2\sigma^2}}$$

$$r^2\cos^2\varphi - 2rA\cos\varphi + A^2 + r^2\sin^2\varphi = r^2 + A^2 - 2rA\cos\varphi$$

$$p(r) = \int_0^{2\pi} p(r, \varphi)\,d\varphi = \frac{r}{2\pi\sigma^2} e^{-\frac{r^2 + A^2}{2\sigma^2}} \int_0^{2\pi} e^{\frac{2rA}{2\sigma^2}\cos\varphi}\,d\varphi$$

~~$p(r) = \frac{1}{2\pi\sigma^2} e^{-\frac{r^2+A^2}{\sigma^2}} \int^{2\pi} e^{\frac{2rA}{\sigma^2}}$~~ $= \frac{r}{\sigma^2} e^{-\frac{r^2+A^2}{\sigma^2}} \frac{1}{2\pi} \int_0^{2\pi} e^{\frac{rA}{\sigma^2}\cos\varphi}\,d\varphi$

$$\boxed{p(r) = \frac{r}{\sigma^2} \cdot e^{-\frac{r^2 + A^2}{\sigma^2}} \cdot I_0\left(\frac{r A}{\sigma^2}\right)}$$

RICIAN DISTRIBUTION

[redacted] SOVA [redacted]

PARAMETRI NA RAYLEIGH RASPODJELA:

$$\bar{R} = \int_0^\infty r f_R(r)\, dr \qquad f_R(r) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} = \frac{r}{6} e^{-\frac{r^2}{26}}$$

$$\bar{R} = \int_0^\infty \frac{r^2}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}\, dr = \int_0^\infty \frac{r^2}{6} e^{-\frac{r^2}{26}}\, dr$$

$$u = r \;;\; v = \int \frac{r}{6} e^{-\frac{r^2}{26}}\, dr = \int e^{-\frac{r^2}{26}}\, d\left(\frac{r^2}{26}\right)$$

$$v = - e^{-\frac{r^2}{26}} \;;\; du = dr$$

$$\bar{R} = - r e^{-\frac{r^2}{26}} \Big|_0^\infty + \underbrace{\int_0^\infty e^{-\frac{r^2}{26}}\, dr}_{I} = - \frac{r}{e^{\frac{r^2}{26}}} \Big|_0^\infty + I$$

$$\lim_{r\to\infty} \frac{(r)'}{\left(e^{\frac{r^2}{26}}\right)'} = \lim_{r\to\infty} \frac{1}{\frac{2r}{26} \cdot e^{\frac{r^2}{26}}} = 0 \qquad \lim_{r\to 0} \frac{r \to 0}{e^{\frac{r^2}{26}} \to 1} = 0$$

$$\bar{R} = \int_0^\infty e^{-\frac{r^2}{26}}\, dr = \left| \begin{matrix} r^2 = m \\ 2r\,dr = dm \\ dr = \frac{dm}{2r} \end{matrix} \quad dr = \frac{dm}{2\sqrt{m}} \right| = \frac{1}{2} \int_0^\infty e^{-\frac{m}{6}} \frac{dm}{2\sqrt{m}}$$

$$\bar{R} = \frac{1}{4} \int_0^\infty \frac{e^{-\frac{m}{6}}}{\sqrt{m}}\, dm$$

$$\int e^{-t^2}\, dt = \int e^{x^2 \ln} \quad \triangleright \tfrac{1}{2}$$

$$\bar{R} = \frac{1}{2} \left(\frac{1}{\sqrt{2\pi \cdot 6}} \int_0^\infty e^{-\frac{r^2}{6}}\, dr \right) \cdot \sqrt{2\pi \cdot 6} = \frac{1}{4} \sqrt{2\pi 6}$$

$$\bar{R} = \int_0^\infty e^{-\frac{r^2}{26}}\, dr = \left(\frac{1}{\sqrt{2\pi 6}} \int_0^\infty e^{-\frac{r^2}{26}}\, dr \right) \sqrt{2\pi 6} = \frac{\sqrt{2\pi 6}}{2}$$

$$\bar{R} = \frac{\sqrt{\pi 6}}{\sqrt{2}} = \frac{\sqrt{\pi \sigma^2}}{\sqrt{2}} = \ldots \qquad \operatorname{erf}(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-m^2}\, dm =$$

$$= \frac{\sqrt{2}\,1}{\sqrt{2}\sqrt{\pi}} \int_{-\infty}^{0} e^{-m^2}\, dm + \frac{1}{\sqrt{\pi}} \int_0^x e^{-m^2}\, dm =$$

$\frac{1}{2}$

$$\operatorname{erf}(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-u^2} du \qquad \boxed{u = \frac{x-m}{\sqrt{2}\sigma}}$$

$$\operatorname{erfc}(x) = \frac{1}{\sqrt{\pi}} \int_{x}^{\infty} e^{-u^2} du = 1 - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-u^2} du = 1 - \operatorname{erf}(x)$$

$$P(a \le \xi \le b) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_a^b e^{-\frac{(x-m)^2}{2\sigma^2}} dx = \frac{1}{\sqrt{\pi}} \int_a^b e^{-\frac{(x-m)^2}{2\sigma^2}} d\left(\frac{x-m}{\sqrt{2}\sigma}\right)$$

$$= \left| \begin{matrix} u = \frac{x-m}{\sqrt{2}\sigma} & x=a \;\; u = \frac{a-m}{\sqrt{2}\sigma} \\ du = \frac{1}{\sqrt{2}\sigma} dx & x=b \;\; u = \frac{b-m}{\sqrt{2}\sigma} \end{matrix} \right| = \frac{1}{\sqrt{\pi}} \int_{\frac{a-m}{\sqrt{2}\sigma}}^{\frac{b-m}{\sqrt{2}\sigma}} e^{-u^2} du$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\frac{b-m}{\sqrt{2}\sigma}} e^{-u^2} du - \frac{1}{\sqrt{\pi}} \int_0^{\frac{a-m}{\sqrt{2}\sigma}} e^{-u^2} du = \frac{1}{2}\left[\Phi\left(\frac{b-m}{\sqrt{2}\sigma}\right) - \Phi\left(\frac{a-m}{\sqrt{2}\sigma}\right)\right]$$

$$\boxed{\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du} \Rightarrow$$ GAUSOV INTEGRAL ZA GREŠKA

- $$\overline{R^2} = \int_0^\infty r^2 \frac{r}{b} e^{-\frac{r^2}{2b}} dr = 2b \int_0^\infty \frac{r^2}{2b} e^{-\frac{r^2}{2b}} d\left(\frac{r^2}{2b}\right) = 2b \int_0^\infty u\, e^{-u} du$$

$$= \left| \begin{matrix} u_1 = u & du_1 = du \\ v = \int e^{-u} du = -e^{-u} \end{matrix} \right| = 2b\left(-u e^{-u} + \int_0^\infty e^{-u} du\right)\Big|_0^\infty$$

$$\overline{R^2} = 2b(-1-u) e^{-u}\Big|_0^\infty = -2b\left[\lim_{u\to\infty} \frac{1+u}{e^u} - \lim_{u\to 0} \frac{1+u}{e^u}\right] = 2b$$

$$\sigma_1^2 = \overline{R^2} - \overline{R}^2 = 2b - \left(\frac{\sigma\sqrt{\pi}}{\sqrt{2}}\right)^2 = 2\sigma^2 - \frac{\pi\sigma^2}{2} = \left(2-\frac{\pi}{2}\right)\sigma^2$$

$$\boxed{\overline{R} = \sigma\sqrt{\frac{\pi}{2}} = 1.2533\sqrt{b} \;;\quad \overline{R^2} = 2b \;;\quad \sigma = \left(2-\frac{\pi}{2}\right)\sigma^2 = 0.4292 b}$$

MEDIANA: $$\frac{1}{2} = \int_0^x r e^{-\frac{r^2}{2}} dr \Rightarrow \boxed{r_m = 1.1774\sigma} = 1.1774\sqrt{b}$$

- $$RMS = \sqrt{\overline{R^2}} = \sqrt{2}\sqrt{b} = \sqrt{2}\sigma = 1.41\sigma$$

- SPEKTRALNA GUSTINA NA SREDNA MOĆNOST

$$r(t) = R(t)\cos[\omega_c t - \varphi(t)] = r_c(t)\cos(\omega_c t) - r_s(t)\sin(\omega_c t)$$

ТЕОРИЈА НА ИНФОРМАЦИИ – КОРЕЛАЦИОНИ ФУНКЦИИ И СПЕКТРИ

- $f(t) = f(t + nT) \qquad n = 0, 1, 2, \ldots \infty \qquad T = \frac{2\pi}{\omega}$

ДИРИХЛЕОВ УСЛОВ

$$\int_{-T/2}^{T/2} |f(t)|\,dt < \infty \qquad f(t) = \sum_{n=-\infty}^{\infty} F(jn\omega)\, e^{-jn\omega t}$$

$$F(jn\omega) = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-jn\omega t}\,dt$$

АВТОКОРЕЛАЦИОНА Ф-ЈА ЗА ПЕРИОДИЧНИ СИГНАЛИ

$$R_{11}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} f_1(t)\, f_1(t+\tau)\,dt = \overline{f_1(t)\, f_1(t+\tau)}$$

КОРЕЛАЦИЈА НА ПЕРИОДИЧНИ СИГНАЛИ: ОСНОВИ ТКА

$$f_1(t) = \sum_{n=-\infty}^{\infty} (F_n)_1\, e^{jn\omega_0 t} \qquad f_2(t) = \sum_{n=-\infty}^{\infty} (F_n)_2\, e^{jn\omega_0 t}$$

$$R_{12}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} f_1(t)\cdot f_2(\tau+t)\,dt = \sum_{n=-\infty}^{\infty} (F_n)_1^{*}(F_n)_2\, e^{jn\omega_0\tau}$$

$$(F_n)_1^{*}(F_n)_2 = \frac{1}{T}\int_{-T/2}^{T/2} R_{12}(\tau)\, e^{-jn\omega_0\tau}\,d\tau$$

ФУРИЕВ ТРАНСФОРМАЦИОНЕН ПАР

АВТОКОРЕЛАЦИЈА: $f_1(t) = f_2(t) = f(t)$

$$R_{11}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\cdot f(t+\tau)\,dt = \sum_{n=-\infty}^{\infty} |F_n|^2 e^{jn\omega_0\tau}$$

$$|F_n|^2 = \frac{1}{T}\int_{-T/2}^{T/2} R_{11}(\tau)\, e^{-jn\omega_0\tau}$$

АВТОКОРЕЛАЦИОНАТА ФУНКЦИЈА И ~~СПЕКТАРОТ~~ НА ~~МОЌТА~~ ЧИНАТ ФУРИЕВ ТРАНСФОРМАЦИОНЕН ПАР.

$$R_{11}(0) = \frac{1}{T}\int_{-T/2}^{T/2} f^2(t)\,dt = \sum_{n=-\infty}^{\infty} |F_n|^2$$

KONVOLUCIJA NA PERIODIČNI SIGNALI:

$$\rho_{12}(\tau) = \int_{-T/2}^{T/2} f_1(t) f_2(\tau - t)\, dt = \sum_{n=-\infty}^{\infty} (\bar{F}_n)_1 (\bar{F}_n)_2 e^{jn\omega_0 \tau}$$

$$(\bar{F}_{n1})(\bar{F}_{n2}) = \frac{1}{T} \int_{-T/2}^{T/2} \rho_{12}(\tau) e^{-jn\omega_0 t} dt$$

FUR. TRANS. PAR

KONVOLUCIJA SO DIRAKOVA PERIODIČNA FUNKCIJA

$$f_2(t) = \delta_T(\tau) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t} \qquad (\bar{F}_n)_2 = \frac{1}{T}$$

$$f(t) = \frac{E\tau}{T} \sum_{n=-\infty}^{\infty} \frac{\sin \frac{n\omega_0 \tau}{2}}{\frac{n\omega_0 \tau}{2}} e^{-jn\omega_0 t} = \left| f(t)\delta(t) \right| = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}$$

$$\rho_{12}(\tau) = \int_{-T/2}^{T/2} f_1(t) \cdot \delta_T(\tau - t)\, dt = \frac{1}{T} \sum_{n=-\infty}^{\infty} (\bar{F}_n)_1 e^{jn\omega_0 \tau} \quad !$$

$$R_{12}(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} f_1(t) \cdot f_2(t+\tau)\, dt = \frac{1}{T} \int_{-T/2}^{T/2} f_1(t) \sum_{n=-\infty}^{\infty} (\bar{F}_n)_2 e^{jn\omega_0 t} e^{jn\omega_0 \tau} dt$$

$$= \sum_{n=-\infty}^{\infty} (\bar{F}_n)_2 \cdot e^{jn\omega_0 \tau} \frac{1}{T} \int_{-T/2}^{T/2} f_1(t) e^{jn\omega_0 t} dt = \sum_{n=-\infty}^{\infty} (\bar{F}_n)_1^* (\bar{F}_n)_2 e^{jn\omega_0 \tau}$$

$$(\bar{F}_n)_1^*$$

$$\rho_{12} = \frac{1}{T} \int_{-T/2}^{T/2} f_1(t) \cdot f_2(\tau - t)\, dt = \int_{-T/2}^{T/2} f_1(t) \cdot \left(\sum_{n=-\infty}^{\infty} (\bar{F}_n)_2 e^{+jn\omega_0 \tau} e^{-jn\omega_0 t} \right) dt$$

$$= \sum_{n=-\infty}^{\infty} (\bar{F}_n)_2 e^{jn\omega_0 \tau} \frac{1}{T} \int_{-T/2}^{T/2} f_1(t) e^{-jn\omega_0 t} dt = \sum_{n=-\infty}^{\infty} (\bar{F}_n)_1 (\bar{F}_n)_2 e^{jn\omega_0 \tau}$$

$$(\bar{F}_n)_1$$

$$\bar{F}_n = \frac{1}{2}(a_n - jb_n) \qquad \bar{F}_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) \cdot e^{-jn\omega_0 t} dt = \frac{1}{T} \int f(t) (\cos(n\omega_0 t) + \sin n\omega_0 t) dt$$

$$\bar{F}_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) \cos(n\omega_0 t)\, dt - \frac{j}{T} \int_{-T/2}^{T/2} f(t) \sin n\omega_0 t\, dt$$

$$= \frac{1}{2} \left[\frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos(n\omega_0 t)\, dt - \frac{2j}{T} \int_{-T/2}^{T/2} f(t) \sin n\omega_0 t\, dt \right]$$

$$\bar{F}_n = \frac{1}{2}\left[a_n - j b_n\right] \qquad \bar{F}_n^* = \frac{1}{2}\left[a_n + j b_n\right] =$$

$$= \frac{1}{2}\left[\frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega_0 t)\,dt + \frac{2j}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega_0 t)\,dt\right.$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} f(t)\left[\cos(n\omega_0 t) + j\sin(n\omega_0 t)\right]dt = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{jn\omega_0 t}\,dt$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(n\omega_0 t) + b_n\sin n\omega_0 t$$

$$f(t) = \sum_{n=-\infty}^{\infty} \bar{F}_n e^{+jn\omega_0 t} = \sum_{n=-\infty}^{\infty} \frac{1}{2}(a_n - jb_n)\left(\cos n\omega_0 t + j\sin n\omega_0 t\right)$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2}\left(a_n\cos(n\omega_0 t) + ja_n\sin(n\omega_0 t)\right) - jb_n\cos n\omega_0 t + b_n\sin n\omega_0 t)$$

$$= \sum_{n=-\infty}^{-1} \frac{1}{2}\left(a_n\cos(n\omega_0 t) + ja_n\sin(n\omega_0 t)\right) - jb_n\cos n\omega_0 t + b_n\sin n\omega_0 t)$$

$$+ \frac{1}{2}a_0 - jb_0 + \sum_{n=1}^{\infty}\frac{1}{2}\left(a_n\cos n\omega_0 t + ja_n\sin(n\omega_0 t) - jb_n\cos n\omega_0 t + b_n\sin n\omega_0 t\right)$$

$n = -m$

$$= \sum_{m=1}^{\infty}\frac{1}{2}\left(a_{-m}\cos(-m\omega_0 t) + ja_{-m}\sin(-m\omega_0 t) - j b_{-m}\cos(-m\omega_0 t) + b_{-m}\sin(-m\omega_0 t)\right)$$

($a_{-m} \to a_m$, $\sin(-m\omega_0 t) \to -\sin(n\omega_0 t)$, $b_{-m} \to -b_n$)

$$+\frac{a_0}{2} + \sum_{n=1}^{\infty}\frac{1}{2}\left(a_n\cos n\omega_0 t + ja_n\sin(n\omega_0 t) - jb_n\cos n\omega_0 t + b_n\sin n\omega_0 t\right)$$

~~$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(n\omega_0 t) + jb_n\sin(n\omega_0 t)$$~~

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega_0 t)\,dt \qquad b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega_0 t)\,dt$$

$$\boxed{a_{-n} = a_n} \qquad \boxed{b_{-n} = -b_n}$$

$$= \sum_{n=1}^{\infty}\frac{1}{2}\left[a_n\cos(n\omega_0 t) - ja_n\sin(n\omega_0 t) + jb_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t)\right]$$

$$+\frac{a_0}{2} + \sum_{n=1}^{\infty}\frac{1}{2}\left[a_n\cos(n\omega_0 t) + ja_n\sin(n\omega_0 t) - jb_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t)\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t)$$

• СПЕКТРАЛНА АНАЛИЗА НА ПЕРИОДИЧНИ СИГНАЛИ

$$R_{11}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} f_1(t) f_1(t+\tau)\,dt = \sum_{n=-\infty}^{\infty} (\overline{F}_n)_1 (\overline{F}_n)_1^* e^{jn\omega_0\tau}$$

$$R_{11}(\tau) = \sum_{n=-\infty}^{\infty} |\overline{F}_n|^2 e^{jn\omega_0\tau} \qquad |\overline{F}_n|^2 = \frac{1}{T}\int_{-T/2}^{T/2} R_{11}(\tau) e^{-jn\omega_0\tau}\,d\tau$$

$$R_{11}(\tau) = \sum_{n=-\infty}^{\infty} |F_1(jn\omega)|^2 \cdot e^{jn\omega_0\tau}$$

$$|F_1(jn\omega)|^2 = \frac{1}{T}\int_{-T/2}^{T/2} R_{11}(\tau) e^{-jn\omega_0\tau}\,d\tau \qquad R_{11}(0) = \frac{1}{T}\int_{-T/2}^{T/2} f^2(t)\,dt$$

$$R_{11}(0) = \sum_{n=-\infty}^{\infty} |F_1(jn\omega)|^2$$ ⇒ СРЕДНА СНАГА НА ПЕРИОДИЧНИОТ СИГНАЛ

• МЕЃУКОРЕЛАЦИОНА ФУНКЦИЈА НА ДВА ПЕРИОДИЧНИ СИГНАЛИ СО ИСТА ~~...~~ ПЕРИОДА:

$$R_{12}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} f_1(t) f_2(t+\tau)\,dt = \sum_{n=-\infty}^{\infty} F_1(jn\omega) F_2^*(jn\omega) e^{jn\omega\tau}$$

• СПЕКТРАЛНА АНАЛИЗА НА АПЕРИОДИЧНИ СИГНАЛИ

$$\int_{-\infty}^{\infty} |f(t)|\,dt < \infty \qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega) e^{j\omega t}\,d\omega$$

$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\,dt \qquad R_{11}(\tau) = \int_{-\infty}^{\infty} f_1(t) f_1(t+\tau)\,dt$$

$$f_1(t+\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(j\omega) e^{+j\omega(t+\tau)}\,d\omega$$

$$R_{11}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_1(t)\left[\int_{-\infty}^{\infty} F_1(j\omega) e^{+j\omega t} e^{j\omega\tau}\,d\omega\right] dt =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(j\omega) e^{j\omega\tau}\left[\underbrace{\int_{-\infty}^{\infty} f_1(t) e^{j\omega t}\,dt}_{F_1^*(j\omega)}\right] d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F_1(j\omega)|^2 e^{j\omega\tau}\,d\omega$$

$$R_{11}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F_1(j\omega)|^2 e^{j\omega\tau}\,d\omega$$

ФУРИЕВ ТР. ПАР

$$R_{12}(\tau) = \int_{-\infty}^{\infty} f_1(t) f_2(t+\tau)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(j\omega) F_2^*(j\omega) e^{j\omega\tau}\,d\omega.$$

• USREDNUVANJE PO VREME I USREDNUVANJE PO ANSAMBL NA SLUČAJNI PROMENLIVI

VREMENSKA SREDNA VR. $i = 1, 2, 3, \ldots$

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f^{(i)}(t)\,dt = m$$

$$\overline{f(t)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)\,dt$$

$$f^{(1)}(-k\Delta t),\ f^{(2)}(-k\Delta t),\ \ldots,\ f^{(i)}(-k\Delta t),\ \ldots$$
$$x_k^{(1)},\ x_k^{(2)},\ \ldots,\ x_k^{(i)},\ \ldots$$

$$\overline{X_k} = \overline{\xi} = \int_{-\infty}^{\infty} x\, p_\xi(x)\,dx \qquad \boxed{\overline{f(t)} = \overline{\xi}}$$

SREDNA VR. PO VREME E EDNAKVA NA SR. VR. PO ANSAMBL

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)\,dt = \int_{-\infty}^{\infty} x\, p_\xi(x)\,dx \qquad \boxed{\overline{f(t)} = \overline{f(x)}}$$

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} F\{f(t)\}\,dt = \int_{-\infty}^{\infty} F(x)\, p_\xi(x)\,dx$$

• AVTOKORELACIONA FUNKCIJA NA SLUČAJNI PROMENLIVI:

$$\left[f^{(i)}(-k\Delta t),\ f^{(i)}(-k\Delta t+\tau)\right],\quad k = 0, 1, 2, \ldots$$

$$\overline{f^{(i)}(t)\, f^{(i)}(t+\tau)} = \lim_{T\to\infty} \frac{1}{T} \int_{-T}^{0} f^{(i)}(t)\cdot f^{(i)}(t+\tau)\,dt$$

$$R_f(\tau) = \overline{f(t)\, f(t+\tau)} = \lim_{T\to\infty} \frac{1}{T} \int_{-T}^{0} f(t)\, f(t+\tau)\,dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)\, f(t+\tau)\,dt$$

$R_f(\tau)$ – VREMENSKA KORELACIONA F-JA NA SL. PROCES $f(t)$

• $t = t_1 \qquad t = t_1 + \Delta t = t_1 + \tau = t_2$

$$\left[f^{(i)}(-k\Delta t),\ f^{(i)}(-k\Delta t+\tau)\right] \qquad \left[x_1^{(i)}, x_2^{(i)}\right] \qquad \xi_1, \xi_2$$

$$\overline{\xi_1 \xi_2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2\, p_{\xi_1\xi_2}(x_1, x_2, \tau)\,dx_1\,dx_2 = R_{\xi\xi}(\tau) = R_\xi(\tau)$$

• ZA ERGODIČEN PROCES $\overline{\xi_1\xi_2} = \overline{f(t)\, f(t+\tau)} \qquad R_{ff}(\tau) = R_{\xi\xi}(\tau)$

• OSOBINI NA AVTOKORELACIONITE FUNKCII:

$$R_{ff}(-\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)\cdot f(t-\tau)\,dt = \left| \begin{matrix} t-\tau = x & t=-T & x=-T-\tau \\ dt = dx & t=T & x=T-\tau \end{matrix} \right|$$

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T-\tau}^{T-\tau} f(x)\, f(x-\tau)\,dx = R_{ff}(\tau)$$

$$R_{ff}(0) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f^2(t)\,dt = \overline{f^2(t)} = \overline{\xi^2}$$

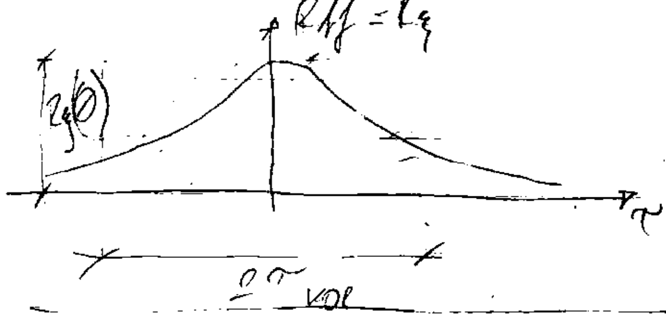
$$\tau \nearrow \;\to\; R_{ff}(\tau) \searrow \qquad \lim_{\tau\to\infty} R_{ff}(\tau) = \begin{cases} \overline{\xi}^2 & \text{АКО ИМА СР. ВР.} \\ 0 & \text{АКО НЕМА СР. ВР.} \end{cases}$$

$$\lim_{\tau\to\infty} R_{\xi\xi}(\tau) = \lim_{\tau\to\infty} \int_{-\infty}^{\infty}\!\!\int x_1 x_2\, p_{\xi_1\xi_2}(x_1,x_2)\,dx_1\,dx_2 = (*)$$

$\tau\to\infty$ АМПЛИТУДИТЕ НА СИГНАЛОТ СЕ СТАТИСТИЧКИ НЕЗАВ.

$$(*) = \int_{-\infty}^{\infty} x_1\, p_{\xi_1}(x_1)\,dx_1 \int_{-\infty}^{\infty} x_2\, p_{\xi_2}(x_2)\,dx_2 = \begin{cases} \overline{\xi}^2 & \text{ИМА СР. ВР.} \\ 0 & \text{НЕМА СР. ВР.} \end{cases}$$

$R_\xi(\tau \geq \tau_{kor}) = 0$ $\quad \tau_{kor}$ – ТРАЕЊЕ НА АВТОКОРЕЛАЦИЈАТА



$$2\tau_{kor}\cdot R_\xi(0) = \int_{-\infty}^{\infty} |R_{ff}(\tau)|\,d\tau$$

$$\tau_{kor} = \frac{1}{2R_\xi(0)}\int_{-\infty}^{\infty} |R_{ff}(\tau)|\,d\tau$$

МЕЃУКОРЕЛАЦИОНА Ф-ЈА НА СЛ. ПРОЦЕСИ:

$$R_{fg}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int f(t)\,g(t+\tau)\,d\tau$$

$$R_{\xi\eta}(\tau) = \int_{-\infty}^{\infty}\!\!\int_{-\infty}^{\infty} x\,y\, p_{\xi\eta}(x,y,\tau)\,dx\,dy$$

$$R_{\xi\eta}(\tau) = \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} x_i\,y_j\, p_{\xi\eta}(x_i,y_j,\tau)$$

$$R_{\xi\xi}(\tau) = \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} x_{1i}\,x_{2j}\, p_{\xi_1\xi_2}(x_{1i},x_{2j},\tau)$$

$$|R_{fg}(\tau)| < \frac{1}{2}\left[R_{ff}(0) + R_{gg}(0)\right]$$

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left[f(t)\pm g(t+\tau)\right]^2 > 0 \qquad \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f^2(t)\,dt \;+$$

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g^2(t+\tau)\,dt \pm \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(t)\,g(t+\tau)\,dt > 0$$

$$R_{ff}(0) + R_{gg}(0) \pm R_{fg}(\tau) > 0$$

$$|R_{fg}| < R_{ff}(0) + R_{gg}(0)$$

$$\lim_{\tau \to \infty} R_{fg}(\tau) = 0 \qquad \xi V \eta = 0$$

$$R_{\xi\eta}(\tau) = 0 \qquad \xi \text{ i } \eta \text{ SE NEKORELIRANI T.E. OKOMITI}$$

SGS

- SPEKTRALNA GUSTINA NA SNAGA NA SL. PROCESI

$$R_{ff}(\tau) = \int_{-\infty}^{\infty} f(t) \cdot f(t+\tau)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2 e^{j\omega\tau} d\omega \quad / 2T$$

$$\frac{1}{2T}\int_{-T}^{T} f(t) f(t+\tau)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|F(j\omega)|^2}{2T} e^{j\omega\tau} d\omega$$

$$R_T(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|F(j\omega)|^2}{2T} e^{j\omega\tau} d\omega = \hat{R}_f(\tau)$$

$$\hat{R}_f(\tau) = \overline{\hat{R}_T(\tau)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\overline{|F(j\omega)|^2}}{2T} e^{j\omega\tau} d\omega$$

$$R_f(\tau) = R_\xi(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\overline{|F(j\omega)|^2}}{2T} e^{j\omega\tau} d\omega$$

$$R_\xi(\tau) = \lim_{T\to\infty} \hat{R}_\xi(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\lim_{T\to\infty} \frac{\overline{|F(j\omega)|^2}}{2T}\right) e^{j\omega\tau} d\omega$$

$$\text{SGS: } \Phi_\xi(\omega) = \lim_{T\to\infty} \frac{\overline{|F(j\omega)|^2}}{2T}$$

VINER-HINČINOVA TEOREMA

$$R_\xi(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_\xi(\omega) e^{j\omega\tau} d\omega \qquad \Phi_\xi(\omega) = \int_{-\infty}^{\infty} R_\xi(\tau) e^{-j\omega\tau} d\tau$$

AVTOKORELACIONA FUNKCIJA ZA SLUČAJNI SIGNALI

$$R_\xi(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_\xi(\omega)\, d\omega = \overline{P} \qquad \text{VKUPNA SREDNA SNAGA}$$

$$\Phi_\xi(\omega) (=) \frac{W}{Hz} \Rightarrow R_\xi (=) W$$

$$R_\xi(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_\xi(\omega) \cos(\omega\tau)\, d\omega \qquad \Big\} \; \Phi_\xi(\omega) \text{ PARNA}$$

$$R_\xi(\tau) = \frac{1}{\pi}\int_{0}^{\infty} \Phi_\xi(\omega) \cos(\omega\tau)\, d\omega$$

$$\Phi_\xi(\omega) = 2\int_{0}^{\infty} R_\xi(\tau) \cos(\omega\tau)\, d\tau$$

МАТЕМАТИЧКА SGS: $\overline{P(\omega_1 \omega_2)} = 2\int_{\omega_1}^{\omega_2} \Phi(\omega) d\omega$

РЕАЛНА SGS: $P(f_1, f_2) = \int_{f_1}^{f_2} \phi(f) df \qquad \phi(f) = 2\Phi(\omega),\ \omega > 0$

МЕЃУКОРЕЛАЦИОНА ТЕОРЕМА ЗА СЛ. ПРОЦЕСИ

$\xi(t)$ и $\eta(t)$ СТАЦИОНАРНИ И ЕРГОДИЧНИ

$$\Phi_{\xi\eta}(\omega) = \int_{-\infty}^{\infty} R_{\xi\eta}(\tau) e^{-j\omega\tau} d\tau, \quad R_{\xi\eta}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_{\xi\eta}(\omega) e^{j\omega\tau} d\omega$$

$$R_{\xi\eta}(-\tau) = R_{\eta\xi}(\tau) \qquad \Phi_{\xi\eta}(\omega) = \Phi^{*}_{\xi\eta}(\omega) \qquad \Phi_{\xi\eta}(-\omega) = \Phi_{\xi\eta}(\omega)$$

SGS НА СУМА И ПРОИЗВОД ОД СЛ. ПРОЦЕСИ

$\xi(t)$ и $\eta(t)$ СТАЦИОНАРНИ СЛ. ПРОЦЕСИ

- $\zeta(t) = \xi(t) + \eta(t) \qquad z(t) = x(t) + y(t)$

$$R_{\zeta\zeta}(\tau) = \overline{\zeta(t)\zeta(t+\tau)} = \overline{(\xi(t)+\eta(t))(\xi(t+\tau)+\eta(t+\tau))} =$$

$$= \overline{\xi(t)\xi(t+\tau)} + \overline{\xi(t)\eta(t+\tau)} + \overline{\eta(t)\xi(t+\tau)} + \overline{\eta(t)\eta(t+\tau)}$$

$$R_{\zeta\zeta}(\tau) = R_{\xi\xi}(\tau) + R_{\xi\eta}(\tau) + R_{\eta\xi}(\tau) + R_{\eta\eta}(\tau)$$

АКО СТАТИСТ. ПРОЦЕСИ СЕ НЕКОРЕЛИРАНИ Т.Е. ОРТОГОНАЛНИ

$R_{\xi\eta}(\tau) = R_{\eta\xi}(\tau) = 0 \qquad \boxed{R_{\zeta\zeta}(\tau) = R_{\xi\xi}(\tau) + R_{\eta\eta}(\tau)}$

$\xi_i = 1, 2, \ldots n \qquad \xi = \sum_{i=1}^{n} \xi_i \qquad R_{\xi\xi}(\tau) = \sum_{i=1}^{n} R_{\xi_i}(\tau)$

$\Phi_\zeta(\omega) = \Phi_\xi(\omega) + \Phi_\eta(\omega) \qquad \Phi_\xi(\omega) = \sum_{i=1}^{n} \Phi_{\xi_i}(\omega)$

- $\zeta(t) = \xi(t)\cdot\eta(t) \qquad R_{\zeta\zeta}(\tau) = \overline{\xi(t)\cdot\eta(t)\,\xi(t+\tau)\,\eta(t+\tau)} = R_{\xi\xi}(\tau) R_{\eta\eta}(\tau)$

$$\mathcal{F}\{R_{\zeta\zeta}(\tau)\} = \mathcal{F}\{R_{\xi\xi}(\tau)\} * \mathcal{F}\{R_{\eta\eta}(\tau)\}$$

$$\Phi_{\zeta\zeta}(\omega) = \int_{-\infty}^{\infty} R_{\zeta\zeta}(\tau) e^{-j\omega\tau} d\tau = \int_{-\infty}^{\infty} R_{\xi\xi}(\tau) R_{\eta\eta}(\tau) \cdot e^{-j\omega\tau} d\tau$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_\xi(\Omega)\,\Phi_\eta(\omega-\Omega)\, d\Omega \qquad R_{\xi\xi} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_\xi(\omega) e^{+j\omega\tau} d\omega$$

$$\Phi_{\zeta\zeta}(\omega) = \Phi_\xi(\omega) * \Phi_\eta(\omega)$$

$\xi = \prod_{i=1}^{n} \xi_i \qquad R_{\xi\xi}(\tau) = \prod_{i=1}^{n} R_{\xi_i}(\tau)$

$$v(t) = R(t)\cos(\omega_0 t + \varphi(t)) = T_c(t)\cos(\omega_0 t) - T_s\sin(\omega_0 t)$$

$$\overline{v(t)\,v(t+\tau)} = \overline{\left(T_c(t)\cos(\omega_0 t) - T_s\sin(\omega_0 t)\right)\left(T_c(t+\tau)\cos(\omega_0 t) - T_s(t+\tau)\sin(\omega_0 t)\right)}$$

$$= \left| \begin{array}{l} \overline{T_c(t)T_s(t)} = 0 \\ \overline{T_s(t)T_c(t)} = 0 \end{array} \right| = \overline{T_c(t)\,T_c(t+\tau)}\cos(\omega_0 \tau) - \overline{T_s(t)\,T_s(t+\tau)}\sin(\omega_0 \tau)$$

$$= \Phi_{T_cT_c}(\tau)\cdot\cos(\omega_0 t) - \Phi_{T_sT_s}(\tau)\sin(\omega_0 t)$$

$$\boxed{\Phi_{T_cT_c} = \Phi_{T_sT_s} \qquad \Phi_{T_cT_s} = -\Phi_{T_sT_c}}$$ → AUTOKORELACIONI I MEĐUKORELACIONI FUNKCIJE NA SL FROM T_s I T_c

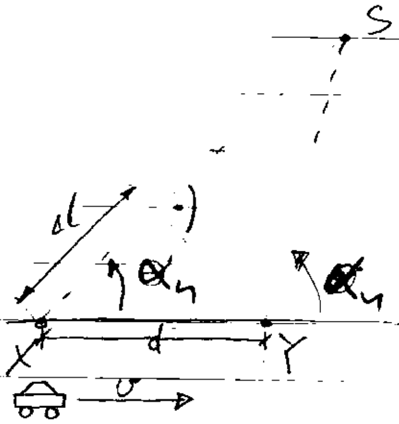
- $\varphi_n \neq \varphi_m \quad n \neq m$

$$\Phi_{T_cT_c}(\tau) = \overline{T_c(t)\,T_c(t+\tau)}\Big|_{\substack{\text{USREDNJAVATI} \\ \text{PO } \alpha_n \to \alpha}} = \frac{\epsilon_0^2}{2}\, E_{\alpha_n}\{\sin[\omega_{dmax}\tau\cos\alpha_n]\}$$

$$T_c = \epsilon_0\sum_{n=1}^{N} C_n\cos(\omega_{dn}t + \varphi_n) \qquad T_s = \epsilon_0\sum_{n=1}^{N} C_n\sin(\omega_{dn}t + \varphi_n)$$

$T_c(t)$, $T_s(t)$ GAUSOVI RASPODELI $\overline{T_c^2} = \frac{\epsilon_0^2}{2}$

The Mobile PROPAGATION CHANNEL $T_c \equiv I(t)$ $T_s \equiv Q(t)$



$$\Delta\phi = \frac{2\pi\Delta l}{\lambda} = \frac{2\pi\, d\cdot\cos\theta}{\lambda} = \frac{2\pi\cdot v\cdot\Delta t\cos\theta}{\lambda}$$

$$\left(v = \frac{d}{\Delta t}\right) \quad f_d = \frac{1}{2\pi}\frac{\Delta\phi}{\Delta t} = \frac{v}{\lambda}\cdot\cos\alpha_n$$

$$f_{dmax} = \frac{v}{\lambda}; \qquad \boxed{f_d = f_{dmax}\cdot\cos\alpha_n}$$

$\alpha_n = 0 \quad f_d = f_{dmax}$

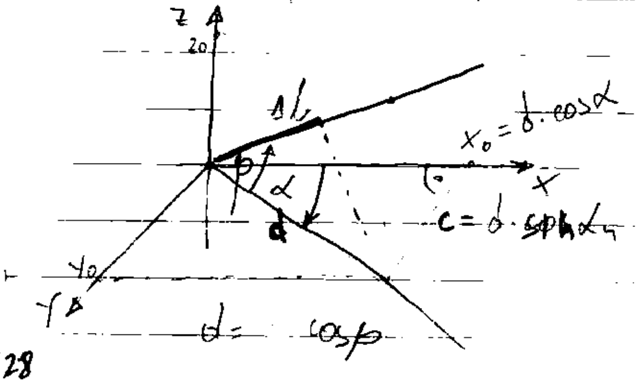
$\alpha_n = \pi \quad f_d = -f_{dmax}$

$$\boxed{f \in [f_c - f_{dmax}, f_c + f_{dmax}]} \qquad \boxed{\omega_{dn} = \frac{2\pi v}{\lambda}\cdot\cos\alpha_n}$$

$$\epsilon_z = \sum_{n=1}^{N}\vec{\epsilon_n} = \epsilon_0\sum_{n=1}^{n} C_n\cos(\omega_c t + \theta_n)$$

SHORT TERM FADING (PARSON)

- n-th incoming wave has amplitude C_n, PHASE ϕ_n AND SPACIAL ANGLES OF ARRIVAL: α_n, β_n



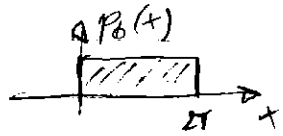
$$E\{C_n^2\} = \frac{\epsilon_0}{N}$$

ϵ_0 - POSITIVE CONSTANT

Clarke's Model

$\beta_n = 0$

$$E(t) = \sum_{n=1}^{N} E_n(t) \qquad \theta_n \in [0, 2\pi]$$



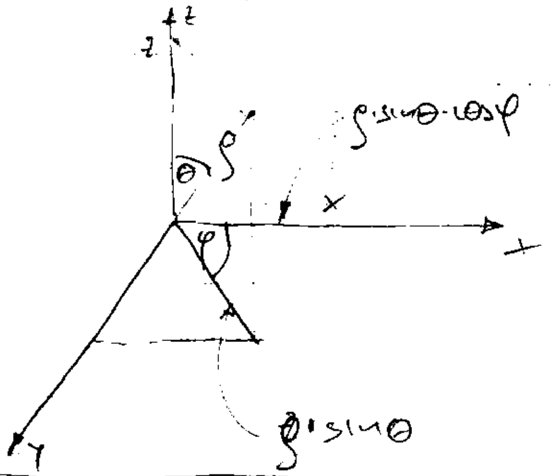
$\{x_0, y_0, z_0\}$

$$E_n(t) = C_n \cos\left(\omega_0 t - \frac{2\pi \Delta l}{\lambda} + \varphi_n\right)$$

$$\Delta l = x_0 \cos\alpha_n \cos\beta_n + y_0 \sin\alpha_n \cdot \cos\beta_n + z_0 \sin\beta_n$$

$$\cos\alpha_n = \frac{x_0}{d}; \quad x_0 = d \cdot \cos\alpha \qquad \Delta l = d \cdot \cos\beta$$

SFERIČNI KOORDINATI



$$x = \rho\sin\theta \cdot \cos\varphi$$
$$y = \rho\sin\theta \cdot \sin\varphi$$
$$z = \rho \cdot \cos\theta$$

$$E(t) = I(t) \cdot \cos(\omega_c t) - Q(t) \cdot \sin(\omega_c t)$$

$$\underset{T_c(t)}{I(t)} = \sum_{n=1}^{N} C_n \cos(\omega_n t + \theta_n) \qquad \underset{T_s(t)}{Q(t)} = \sum_{n=1}^{N} C_n \sin(\omega_n t + \theta_n)$$

$$p_x(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \qquad * \; T(t), \text{ t.e. } x = Q(t)$$

$$E\{I(t)\} = E\{Q(t)\} = 0 \qquad E\{C_n^2\} = \frac{E_0}{N}$$

$$r(t) = R(t)\cos[\omega_c t - \varphi(t)] = T_c(t)\cos(\omega_c t) - T_s(t)\sin(\omega_c t)$$

$$\overline{r(t) \cdot r(t+\tau)} = \overline{T_c(t) T_c(t+\tau)} \cos(\omega_c \tau) - \overline{T_s(t) T_s(t+\tau)} \sin(\omega_c \tau))$$

$$\overline{T_c(t)\cos(\omega_c t) \cdot T_c(t+\tau)\cos(\omega_c(t+\tau))} = \int_{-\infty}^{\infty} \left(\sum_{n=1}^{N} C_n \cos(\omega_n t + \theta_n)\right) \cdot \left(\sum_{n=1}^{N} C_n \cos(\omega_n t + \omega_n \tau + \theta_n)\right)$$

$$\cos(\omega_c t) \cdot \cos(\omega_c t + \tau)\, dt$$

NEPARNA FUNKCIJA

$$\cos(\omega_c t + \tau) = \cos(\omega_c t) \cdot \cos(\omega_c \tau) - \sin(\omega_c t) \cdot \sin(\omega_c \tau)$$

$$= \int_{-\infty}^{\infty} T_c(t) \cdot \cos(\omega_c t) \cdot T_c(t+\tau) \cdot \cos(\omega_c t) \cos(\omega_c \tau))\, dt$$

$$\overline{r(t)\,r(t+\tau)} = \overline{T_c(t)T_c(t+\tau)}\cos(\omega_c\tau) - \overline{T_s(t)\cdot T_s(t+\tau)}\sin(\omega_c\tau)$$
$$= \Phi_{T_cT_c}\cos(\omega_c\tau) - \Phi_{T_sT_c}\sin(\omega_c\tau)$$

$$\Phi_{T_cT_c}(\tau) = \overline{T_c(t)\,T_c(t+\tau)} = \frac{E_0^2}{2}\,E_{\alpha_n}\left\{\cos\left[\omega_{d\,max}\tau\cdot\cos(\alpha_n)\right]\right\}$$

PARSON: $E\{E(t)\,E(t+\tau)\} = E\{I(t)\cdot I(t+\tau)\}\cos\omega_c\tau - E\{Q(t)\,Q(t+\tau)\}\sin\omega_c\tau$

$$\overline{E(t)\,E(t+\tau)} = a(\tau)\cdot\cos(\omega_c\tau) - c(\tau)\sin(\omega_c\tau)$$
$$a(\tau) = \frac{E_0}{2}E\{\cos\omega\tau\} \qquad c(t) = \frac{E_0}{2}E\{\sin\omega\tau\}$$

$p_\alpha(\alpha) = \frac{1}{2\pi}$ UNIFORM PDF for incoming waves

$$a(\tau) = \frac{E_0}{2}\int_{-\pi}^{\pi} J_0(2\pi f_m\tau\cos\beta)\,p_\beta(\beta)\,d\beta \qquad c(\tau) = \varnothing$$

$p_\beta(\beta) = \delta(\beta)$ $\beta = \varnothing$ $\qquad a_0(\tau) = \frac{E_0}{2}J_0(2\pi f_m\tau)$

$$J(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{+x\cos\mu}\,d\mu$$

$$a_0(\tau) = \frac{E_0}{4\pi}\int_0^{2\pi} e^{2\pi f_m\tau\cdot\cos\mu}\,d\mu \qquad f_m = \frac{v}{\lambda}$$

EXAMPLE:

$$\lambda = \frac{300}{f[MHz]} = \frac{300}{900} = 0{,}33\,m, \qquad f_m = \frac{10^5\,m/3600\,sec}{0.33}$$

$$f_m = \frac{27{,}7}{0.33} = 83.33\,Hz \qquad \tau = \frac{1}{3}\cdot 10^{-5}\,sec$$

$$a_0(\tau) = 1.000000085\cdot\frac{E_0}{2} \doteq 0.5E_0$$

$$A_0(f) = \mathcal{F}[a_0(\tau)] = \int_{-\infty}^{\infty} a_0(\tau)\,e^{-j\omega\tau}\,d\tau$$

WIKI

$$\mathcal{F}\{J_0(t)\} = \sqrt{\frac{2}{\pi}}\,\frac{\mathrm{rect}\left(\frac{\omega}{2}\right)}{\sqrt{1-\omega^2}} \qquad A_0(f) = \frac{E_0}{2}\,\frac{2\cdot\mathrm{rect}(f)}{\sqrt{1-4\pi^2f^2}}$$

$$A_0(f) = \mathcal{F}\{a_0(\tau)\} = \begin{cases}\frac{E_0}{4\pi f_m}\frac{1}{\sqrt{1-(f/f_m)^2}} & |f| \le f_m\\ \varnothing & \text{elsewhere}\end{cases}$$

wiki: $A_0(f) = \epsilon_0 \frac{1}{\sqrt{1 - 4\pi^2 f^2}}$ $\quad \epsilon_0 \frac{2\pi f_m}{2\pi f_m} \frac{1}{\sqrt{1-4\pi^2 f^2}}$?

$a_0(\tau) = \frac{\epsilon_0}{2} J_0(2\pi f_m \tau)$ $\quad \left(\mathcal{F}[J_0(\tau)] = \frac{2}{\sqrt{1-4\pi^2 f^2}} \right)$

$$\int_{-\infty}^{\infty} J_0(2\pi f_m \tau) e^{-j\omega\tau} d\tau = \left| \begin{array}{l} u = 2\pi f_m \tau \\ du = 2\pi f_m d\tau \\ d\tau = \frac{du}{2\pi f_m} \end{array} \right|$$

$$= \int_{-\infty}^{\infty} J_0(u) e^{-j\omega \frac{u}{2\pi f_m}} du = \frac{1}{2\pi f_m} \int_{-\infty}^{\infty} J_0(u) e^{-j\frac{2\pi f}{2\pi f_m} u} du$$

$$\mathcal{J}(\xi) = \frac{1}{2\pi f_m} \int_{-\infty}^{\infty} J_0(\xi) e^{-j\xi u} du = \frac{1}{2\pi f_m} \frac{2}{\sqrt{1-\xi^2}} = \frac{1}{2\pi f_m} \frac{2}{\sqrt{1-\left(\frac{f}{f_m}\right)^2}}$$

$$A_0(f) = \frac{\epsilon_0}{2} \frac{1}{2\pi f_m} \frac{2}{\sqrt{1-\left(\frac{f}{f_m}\right)^2}}$$

$$\bar{F}_n = F(n\omega) = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jn\omega t} dt \qquad f(t) = \sum_{-\infty}^{\infty} \bar{F}_n e^{jn\omega t}$$

$$R_{11}(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} f(t) f(t+\tau) dt = \frac{1}{T} \int_{-T/2}^{T/2} f(t) \left(\sum_{n=-\infty}^{\infty} \bar{F}_n e^{jn\omega t} \right) e^{jn\omega\tau} dt$$

$$= e^{jn\omega\tau} \sum_{n=-\infty}^{\infty} \bar{F}_n \underbrace{\frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{jn\omega t} dt}_{\bar{F}_n^*} = \sum_{n=-\infty}^{\infty} |\bar{F}_n|^2 e^{jn\omega\tau}$$

$$R_{11}(\tau) = \sum_{n=-\infty}^{\infty} |\bar{F}_n|^2 e^{jn\omega\tau}$$

FUR. IZ. PAR

THE RECEIVED SIGNAL ENVELOPE:

BINOMNA RASPODJELA:

$P_\xi(k, n) \qquad \xi = k; \quad k = 0, 1, 2, 3 \ldots n$

$P(A) = p \qquad$ VEROJATNOST DA SE SLUČI NASTANAK A

$P(\bar{A}) = 1 - p = q \qquad$ VEROJATNOST DA SE SLUČI $\bar{A}$

$p^k q^{n-k}$ } VEROJATNOST DA „n" OPITI A SE REALIZIRAO k - PUTI, A $\bar{A}$ SE REALIZIRAO $(n-k)$ PUTI

$$P_\xi(k,n) = C_n^k p^k q^{n-k} = \frac{n!}{k!(n-k)!} p^k q^{n-k}$$

$n=3 \quad k=1 \qquad P_\xi(k,n) = \frac{3!}{1!\,2!} \cdot p^k \cdot q^{(n-k)} = 3 \cdot p^1 q^2 =$

$= |p = q = 0.5| = 3 \cdot 0.5 \cdot 0.25 = 1.5 \cdot 0.25 = \underline{0.375}$

$$(pv+q)^n = \sum_{k=0}^{n} C_n^k p^k v^k q^{n-k}$$ ← v - POMOĆNA PROMENLJIVA

$v=1: \quad (p+q)^n = \sum_{k=0}^{n} C_n^k p^k \cdot q^{n-k} = 1 = \sum_{k=0}^{n} P_\xi(k,n)$

$\boxed{\bar{\xi} = ?}$ $\quad (pv+q)^n = \sum_{k=0}^{n} C_n^k p^k v^k q^{n-k} \quad / \frac{d}{dv}$

$$n \cdot p (pv+q)^{n-1} = \sum_{k=0}^{n} k C_n^k p^k \cdot v^{k-1} q^{n-k} \quad (**)$$

$\boxed{v=1}$ $\quad \underbrace{n \cdot p (p+q)^{n-1}}_{1} = \sum_{k=0}^{n} k \underbrace{C_n^k p^k \cdot q^{n-k}}_{P_\xi(k,n)}$

$$n \cdot p = \sum_{k=0}^{n} k \cdot P_\xi(k,n) = \bar{\xi} = \bar{k} \qquad \boxed{\bar{\xi} = n \cdot p}$$

$(**) \cdot v \qquad n \cdot p \cdot v (pv+q)^{n-1} = \sum_{k=0}^{n} k \cdot C_n^k p^k v^k q^{n-k} \quad / \frac{d}{dv}$

$$n \cdot p \cdot (pv+q)^{n-1} + n(n-1) p^2 v \cdot (pv+q)^{n-2} = \sum_{k=0}^{n} k^2 C_n^k p^k v^{k-1} q^{n-k}$$

$v=1: \quad \underbrace{n \cdot p (p+q)^{n-1} + n(n-1) p^2 (p+q)^{n-2}}_{①} = \sum_{k=0}^{n} k^2 \underbrace{C_n^k p^k q^{n-k}}_{P_\xi(k,n)}$

$① = |\underbrace{p+q}_{=1}| = n \cdot p + n(n-1) p^2 = \sum_{k=0}^{n} k^2 P_\xi(k,n) = \overline{\xi^2}$

$$\overline{\xi^2} = np(1 + (n-1)p) = np + n^2p^2 - np^2$$

$$\sigma_\xi^2 = \overline{\xi^2} - \bar{\xi}^2 = np + n^2p^2 - np^2 - n^2p^2 = np - np^2$$

$\boxed{\sigma_\xi^2 = np(1-p) = npq}$ $\qquad \boxed{\sigma_\xi = \sqrt{npq}}$

$$\frac{\sigma_\xi}{\bar{\xi}} = \sigma_{\xi rel} = \frac{\sqrt{npq}}{np} = \sqrt{\frac{q}{np}} = \sigma_{\xi rel}$$

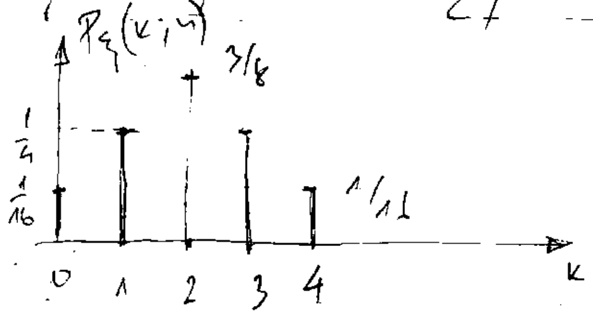
$\sigma_{\xi rel} \downarrow \quad n \uparrow$

PRIMJER: $n = 4$, $p = q = 0.5$, $k = 0, 1, 2, 3, 4$, $P_\xi(k;4) = C_k^4 p^k \cdot q^{4-k}$

$$\bar{\xi} = n \cdot p = 4 \cdot 0.5 = 2$$

$$P_\xi(1;4) = C_1^4 \frac{1}{2} \cdot \left(\frac{1}{2}\right)^3 = \frac{\overset{24}{4!}}{\underset{6}{3! \cdot 1!}} \frac{1}{2} \cdot \frac{1}{8} = 4 \cdot \frac{1}{2} \cdot \frac{1}{8} = \frac{1}{4}$$

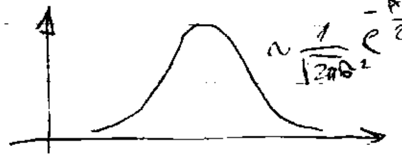
$$P_\xi(0;4) = 1 \cdot 1 \cdot \left(\frac{1}{2}\right)^4 = \frac{1}{16}$$



$$P(3;4) = C_3^4 \cdot \frac{1}{8} \cdot \frac{1}{2} = \frac{4!}{1! \cdot 3!} \frac{1}{16} = 4 \cdot \frac{1}{16} = \frac{1}{4}$$

$$P(2;4) = C_2^4 \frac{1}{4} \cdot \frac{1}{4} = \frac{4!}{2!2!} \frac{1}{16}$$

$$P(2;4) = \frac{24}{2 \cdot 2} \frac{1}{16} = \frac{6}{16} = \frac{3}{8}$$



(1) $n \to \infty \quad p = const \qquad \bar{\xi} = n \cdot p \to \infty$

(2) $n \to \infty \quad p \to 0$

$\bar{\xi} = n \cdot p = const$

$\bar{\xi} = n p = \lambda \qquad \boxed{p = \frac{\lambda}{n}} \qquad \lambda = const$

$$\lim_{n\to\infty} P_\xi(k;n) = \lim_{n\to\infty} C_k^n p^k (1-p)^{n-k} = \lim_{n\to\infty} \frac{n!}{k!(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$\lim_{n\to\infty} \frac{n(n-1)\cdots(n-k+1)}{k!} \frac{\lambda^k}{n^k} \left(1 - \frac{\lambda}{n}\right)^{n-k} = \frac{\lambda^k}{k!} \lim_{n\to\infty} \frac{1 \cdot \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right)}{\left(1 - \frac{\lambda}{n}\right)^k} \cdot \left(1 - \frac{\lambda}{n}\right)^n$$

$$= \frac{\lambda^k}{k!} \underbrace{\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n}_{e^{-\lambda}} = \frac{\lambda^k}{k!} e^{-\lambda}$$

$$\left(1 - \frac{\lambda}{n}\right)^n = C_0^n \left(-\frac{\lambda}{n}\right)^0 + C_1^n \left(-\frac{\lambda}{n}\right)^{n-1} \ldots$$

$$f(x) = 1 + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^k}{k!} f^{(k)}(0) + \ldots$$

$$f(x) = e^x = 1 + \frac{x}{1!} \cdot 1 + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots \frac{x^4}{4!} +$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} \ldots \frac{x^4}{4!} +$$

$$(p+q)^n = \sum_{k=0}^{n} C_k^n p^n q^{n-k}$$

$$\left(1-\frac{\lambda}{n}\right)^n = (-1)^n\left(\frac{\lambda}{n}-1\right)^n = \left[1+\left(\frac{-\lambda}{n}\right)\right]^n = \left[\left(-\frac{\lambda}{n}\right)+1\right]^n$$

$$= \sum_{k=0}^{n} C_k^n \left(-\frac{\lambda}{n}\right)^k \cdot (1)^{n-k} = \sum_{k=0}^{n} C_k^n (-1)^k \cdot \left(\frac{\lambda}{n}\right)^k = C_0^n \cdot 1 + C_1^n \frac{(-\lambda)}{n}$$

$$+ C_2^n \left(\frac{-\lambda}{n}\right)^2 + C_3^n \left(\frac{-\lambda}{n}\right)^3 + \dots + C_k^n \left(\frac{-\lambda}{n}\right)^k + \dots$$

$$(1)\quad = 1 + \frac{n!}{(n-1)!\,1!}\frac{(-\lambda)}{n} + \frac{n!}{(n-2)!\,2!}\left(\frac{-\lambda}{n}\right)^2 + \dots + \underbrace{\frac{n!}{(n-k)!\cdot k!}\left(\frac{-\lambda}{n}\right)^k}_{(*)} + \dots$$

$$(*) = \frac{(n-k+1)!}{k!}\frac{\lambda^k (-1)^k}{n^k} = \frac{(-\lambda)^k}{k!}\, 1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)$$

$$\lim_{n\to\infty} (*) = (-1)^k \frac{\lambda^k}{k!} \qquad \lim_{n\to\infty} (1) = \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n$$

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = 1 - \lambda + \frac{\lambda^2}{2!} - \frac{\lambda^3}{3!} + \dots = e^{-\lambda}$$

$$\lim_{n\to\infty} P_\xi(k;n) = \lim_{n\to\infty} C_k^n p^k (1-p)^{n-k} = \frac{\lambda^k}{k!} e^{-\lambda}$$

$$\boxed{\bar{\xi} = \lambda} \qquad \boxed{P_\xi(k,\lambda) = \frac{\lambda^k}{k!} e^{-\lambda}}$$

$$\bar{\xi} = \sum_{k=0}^{\infty} k\, P_\xi(k,\lambda) = \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda} = \sum_{k=1}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda} =$$

$$= e^{-\lambda}\lambda \sum_{k=1}^{\infty} k\frac{\lambda^{k-1}}{k!} = e^{-\lambda}\lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = e^{-\lambda}\lambda \underbrace{\sum_{\mu=0}^{\infty} \frac{\lambda^\mu}{\mu!}}_{e^{\lambda}}$$

$$\boxed{\bar{\xi} = e^{-\lambda}\cdot\lambda\cdot e^{\lambda} = \lambda}$$

$$\sigma_\xi^2 = ? \qquad \sigma_\xi^2 = \overline{(\xi-\bar{\xi})^2} = \int_{-\infty}^{\infty} (x-\bar{\xi})^2 p_\xi(x)\,dx =$$

$$= \underbrace{\int_{-\infty}^{\infty} x^2 p_\xi(x)\,dx}_{\overline{\xi^2}} - 2\bar{\xi}\underbrace{\int_{-\infty}^{\infty} x\, p_\xi(x)\,dx}_{\bar{\xi}} + \bar{\xi}^2\underbrace{\int_{-\infty}^{\infty} p_\xi(x)\,dx}_{1} = \overline{\xi^2} - 2\bar{\xi}\bar{\xi} + \bar{\xi}^2$$

$$= \overline{\xi^2} - \bar{\xi}^2$$

$\sigma_\xi^2 = \overline{\xi^2} - \bar{\xi}^2 ; \quad \overline{\xi^2} = \sum_{k=0}^{\infty} k^2 \frac{\lambda^k e^{-\lambda}}{k!} = \sum_{k=1}^{\infty} k \frac{\lambda^{k-1}}{(k-1)!}\lambda e^{-\lambda} =$

$= \lambda e^{-\lambda}\sum_{k=1}^{\infty} k \frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda}\left[\sum_{k=1}^{\infty} (k-1)\frac{\lambda^{k-1}}{(k-1)!} + \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}\right]$

$\mu = k-1 \quad k=1 \quad \mu = 0 \quad k\to\infty \quad \mu\to\infty$

$$\overline{\xi^2} = \Bigg[\underbrace{\sum_{\mu=0}^{\infty} \mu \frac{\lambda^\mu}{\mu!}}_{\lambda e^{\lambda}} + \underbrace{\sum_{\mu=0}^{\infty} \frac{\lambda^\mu}{\mu!}}_{e^{\lambda}}\Bigg]\lambda e^{-\lambda} = (\lambda e^{\lambda} + e^{\lambda})\lambda e^{-\lambda} = \lambda^2 + \lambda$$

$\overline{\xi^2} = \lambda^2 + \lambda ; \quad \sigma_\xi^2 = \overline{\xi^2} - \bar{\xi}^2 = \lambda^2 + \lambda - \lambda^2 = \lambda ,$

$\boxed{\sigma_\xi^2 = \lambda} \qquad P_\xi(k,\lambda) = \frac{\lambda^k}{k!}e^{-\lambda} \qquad \boxed{n\cdot p = \lambda \quad p = \frac{\lambda}{n}}$

$$p_\xi(x) = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}e^{-\lambda}\,\delta(x-k) = \sum_{k=0}^{\infty} P_\xi(k,\lambda)\,\delta(x-k)$$

$$F_\xi(x) = P(\xi \le x) = \sum_{k=0}^{\lfloor x\rfloor} P_\xi(k;\lambda)$$

$\lambda = c\cdot\tau \qquad P_\xi(k;\tau) = \frac{(c\tau)^k}{k!}e^{-c\tau}$

TELEGRAFSKI SIGNAL VEROJATNOSTA DA VO INTERVAL τ IMAME K PROMENI NA NIVOTO NA SIGNALOT E RASPREDELENA PO PUASONOVIOT ZAKON !!!

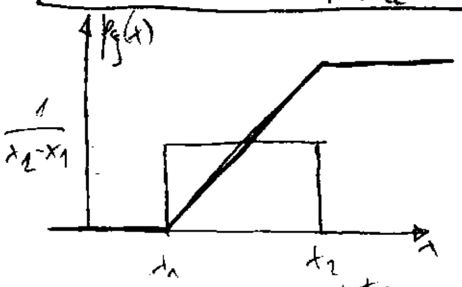
$\xi ; \quad x \in \{-A, A\}$ MOMENTALNA VREDNOST NA SIGNALOT

$\eta ; \quad y \in \{0, 1\}$ NASTAN DA VO DADEN MOMENT IMA PROMENA NA NIVOTO

$P_\eta(k;\tau) = \frac{(c\tau)^k}{k!}e^{-c\tau} \qquad P(\eta = 1) = c\cdot d\tau \quad P(\eta = 0) = 1 - c\,d\tau$

c – PROSEČEN BROJ NASTANI VO EDINICA VREME

UNIFORMNA RASPREDELBA



$$p_\xi(x) = \begin{cases} \frac{1}{x_2-x_1} & x_1 \le x \le x_2 \\ 0 & \text{OTHER} \end{cases}$$

$$\bar{\xi} = \int_{-\infty}^{\infty} x\frac{1}{x_2-x_1}\,dx = \frac{1}{x_2-x_1}\int_{x_1}^{x_2} x\,dx =$$

$$= \frac{1}{x_2-x_1}\,\frac{x^2}{2}\Big|_{x_1}^{x_2} = \frac{1}{x_2-x_1}\,\frac{1}{2}(x_2^2 - x_1^2) = \frac{x_2+x_1}{2}$$

$$\sigma_\xi^2 = \overline{\xi^2} - \bar{\xi}^2 = \frac{1}{x_2-x_1}\int_{x_1}^{x_2} x^2\,dx - \bar{\xi}^2 = \frac{1}{x_2-x_1}\,\frac{x^3}{3}\Big|_{x_1}^{x_2} - \bar{\xi}^2 = \frac{1}{x_2-x_1}\,\frac{x_2^3-x_1^3}{3} - \bar{\xi}^2$$

$$\sigma_\xi^2 = \frac{1}{x_2-x_1}\,\frac{x_2^3-x_1^3}{3} - \frac{(x_1+x_2)^2}{4}$$

$$a^3-b^3 = (a-b)(a^2+ab+b^2) = a^3+a^2b - ba^2 - ab^2 - b^3 = a^3-b^3$$

$$\sigma_\xi^2 = \frac{1}{3(x_2-x_1)}\,\frac{(x_2-x_1)(x_2^2+x_1x_2+x_1^2)}{1} - \frac{x_1^2+2x_1x_2+x_2^2}{4}$$

$$= \frac{4x_2^2+4x_1x_2+4x_1^2-3x_1^2-6x_1x_2-3x_2^2}{12} = \frac{x_1^2-2x_1x_2+x_2^2}{12}$$

$$\boxed{\sigma_\xi^2 = \frac{(x_1-x_2)^2}{12}}$$

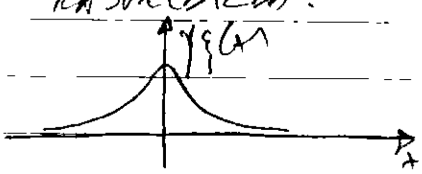
$$F_\xi(x) = \int_{x_1}^{x} \frac{1}{x_2-x_1}\,dx = \frac{1}{x_2-x_1}\,x\Big|_{x_1}^{x} = \frac{x-x_1}{x_2-x_1}$$

$$F_\xi(x) = \begin{cases} \dfrac{x-x_1}{x_2-x_1} & x_1\le x\le x_2 \\ 0 & x<x_1 \\ 1 & x>x_2 \end{cases}$$

$$P(x\le\xi\le x+dx) = P(y\le\eta\le y+dy)$$

$$p_\xi(x)\,dx = p_\eta(y)\,dy \qquad \boxed{p_\eta(y) = \frac{p_\xi(x)}{\left|\frac{dy}{dx}\right|}\Bigg|_{x=g(y)}}$$

- БИЛО КОЈА РАСПРЕДЕЛБА НА ГУСТИНА НА ВЕРОЈАТНОСТ МОЖЕ ДА СЕ ПРЕТВОРИ СО СООДВЕТНА ТРАНСФОРМАЦИЈА ВО УНИФОРМ. РАСПРЕДЕЛБА.



$\eta = f(\xi)$

$y = f(x)$

$F_\xi(x)\in 0..1$

$\boxed{y\in 0\div a}$

$$p_\eta(y) = \frac{p_\xi(x)}{\left|\frac{dy}{dx}\right|}\Bigg|_{x=f^{-1}(y)}$$

$$y = a\cdot F_\xi(x)$$

$$\frac{dy}{dx} = a\,\frac{d}{dx}\left(F_\xi(x)\right) = a\cdot p_\xi(x)$$

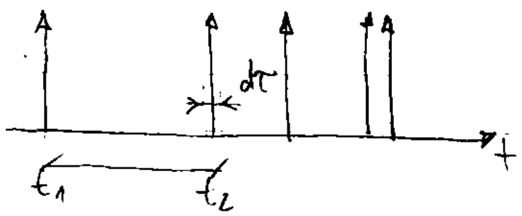
$$\boxed{p_\eta(y) = \frac{p_\xi(x)}{a\,p_\xi(x)} = \frac{1}{a}} \quad y\in(0,a)$$

ЛАПЛАСОВА: $p_\xi(x) = \frac{a}{2}e^{-a|x|}$ РЕЛЕЕВА: $p_\xi(x) = \frac{x}{a^2}e^{-\frac{x^2}{2a^2}}\cdot h(x)$

· КОШИЕВА: $p_\xi(x) = \frac{1}{\pi}\frac{a}{a^2+x^2}$

ПОАСОНОВИ НИЗИ ОД ДИРАКОВИ ИМПУЛСИ

1) УНИПОЛАРНА НИЗА



$P_\eta(k;\tau) = \frac{(c\tau)^k}{k!} e^{-c\tau}$

C – ПРОСЕЧЕН БРОЈ НА ИМПУЛСИ ВО ЕДИНИЦА ВРЕМЕ

$P_\eta(k,\tau)$ ВЕРОЈАТНОСТ ВО ИНТЕРВАЛОТ τ ИМА „K" ИМПУЛСИ

$\lim_{A\to\infty} A\,d\tau = 1$

$P(\eta=1) = c\,d\tau = P(A)$ → ИМА ИМПУЛС

$P(\eta=0) = 1 - c\,d\tau = P(0)$

$R_\xi(\tau) = \sum_{i=1}^{2}\sum_{j=1}^{2} x_{1i}x_{2j}\,P_{\xi_1\xi_2}(x_{2j}; x_{1i}; \tau) =$

$= \sum_{i=1}^{2}\sum_{j=1}^{2} x_{1i}x_{2j}\,P_{\xi_1}(x_{1i})\,P_{\xi_1\xi_2}(x_{2j}/x_{1i}; \tau)$

x_{1i}	x_{2j}
0	0
0	A
A	0
A	A

$R_\xi(\tau) = A\cdot A\cdot P(A)\,P(A/A;\tau)$

$R_{\xi\xi} = \lim_{T\to\infty}\frac{1}{2T}\int f(t)\,f(t+\tau)\,dt = \overline{f(t)\,f(t+\tau)}$

$R_{\xi\xi} = \overline{\xi_1\xi_2} = \iint_{-T}^{T} x_1x_2\,p_{\xi_1\xi_2}(x_1,x_2,\tau)\,dx_1\,dx_2$

$P(A/A;\tau) = P(A) = c\cdot d\tau$

$R_\xi(\tau) = A^2\cdot P(A)\,P(A)$

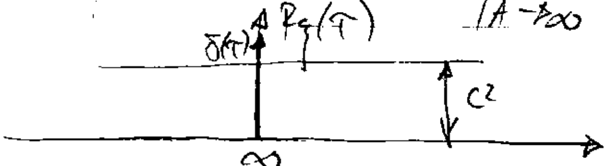
$\tau \neq 0$: $R_\xi(\tau) = A^2\cdot c\,d\tau\cdot c\,d\tau = A^2\cdot c^2\cdot d\tau^2 = c^2$

$\lim_{A\to\infty} A\cdot d\tau = 1$

$\tau = 0$: $P(A/A, 0) = 1$

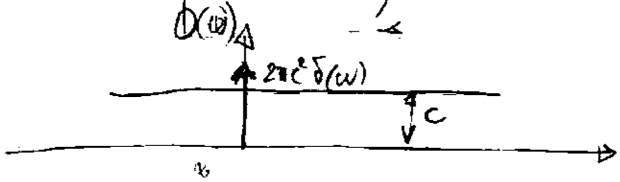
$R_\xi(\tau) = A^2\cdot c\,d\tau \Big|_{A\to\infty} = c\cdot\delta(\tau)$

$R_\xi(\tau) = c^2 + c\,\delta(\tau)$



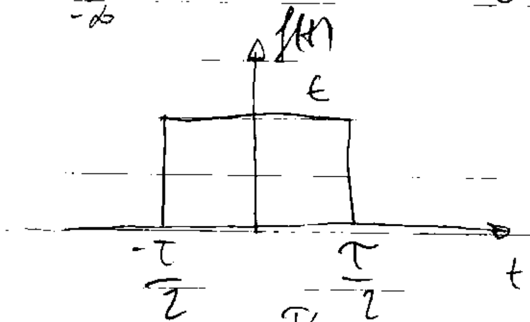
$\Phi(\omega) = \int_{-\infty}^{\infty} R_\xi(\tau)\cdot e^{-j\omega\tau}\,d\tau = \int_{-\infty}^{\infty}(c^2 + c\,\delta(\tau))\,e^{-j\omega\tau}\,d\tau$

$= 2\pi c^2\delta(\omega) + c$



$\int_{0}^{\infty} c^2 e^{-j\omega\tau}\,d\tau = c^2\int_{-\infty}^{\infty}(\cos\omega\tau - j\sin\omega\tau)\,d\tau = c^2\int_{-\infty}^{\infty}\cos(\omega\tau)\,d\tau$

$\epsilon^2 \int_{-\infty}^{\infty} \cos(\omega\tau)\, d\tau = \frac{\epsilon^2}{\omega}\int_{-\infty}^{\infty}\cos\omega\tau \, d(\omega\tau) = \frac{2\epsilon^2}{\omega} \sin\omega\tau \Big|_0^{\infty}$



$$F(j\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$F(j\omega) = \int_{-\tau/2}^{\tau/2} \epsilon \cdot e^{-j\omega t}\, dt = \epsilon \int_{-\tau/2}^{\tau/2} e^{-j\omega t}\, dt$$

$u = -j\omega t$

$du = -j\omega\, dt$

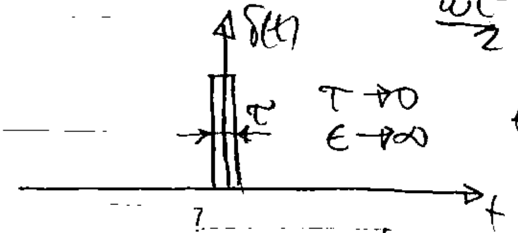
$t = -\tau/2 \quad u = +j\omega\tau/2$

$t = \tau/2 \quad u = -j\omega\tau/2$

$$F(j\omega) = \epsilon \, \frac{-1}{j\omega} \int_{j\omega\tau/2}^{-j\omega\tau/2} e^{u}\, du = \frac{-\epsilon}{j\omega}\left(e^{-j\omega\tau/2} - e^{j\omega\tau/2}\right)$$

$$= \frac{2\epsilon}{2j\omega}\left(e^{j\omega\tau/2} - e^{-j\omega\tau/2}\right) = \frac{2\epsilon}{\omega} \sin\omega\tau/2 = \frac{\epsilon\tau}{\frac{\omega\tau}{2}} \cdot \sin\omega\tau/2$$

$$\boxed{F(j\omega) = \epsilon\tau \, \frac{\sin \omega\tau/2}{\frac{\omega\tau}{2}}}$$



$\tau \to 0$

$\epsilon \to \infty$

$\epsilon\tau = 1$

$$\boxed{\Delta(j\omega) = \mathcal{F}\{\delta(t)\} = \lim_{\tau \to 0} F(j\omega) = 1}$$

$$\boxed{\mathcal{F}\{\delta(t-t_0)\} = 1 \cdot e^{-j\omega t_0}}$$

$$\int_{-\infty}^{\infty} \delta(t-t_0) f(t)\, dt = f(t_0)$$

$$\Delta(j\omega) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt \qquad \boxed{\Delta(j\omega) = 1}$$

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Delta(j\omega)\, e^{+j\omega t}\, d\omega \qquad \boxed{\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\, d\omega}$$

$$\mathcal{F}\{K\} = \int_{-\infty}^{\infty} K\, e^{-j\omega t}\, dt = K \int_{-\infty}^{\infty} e^{-j\omega t}\, dt$$

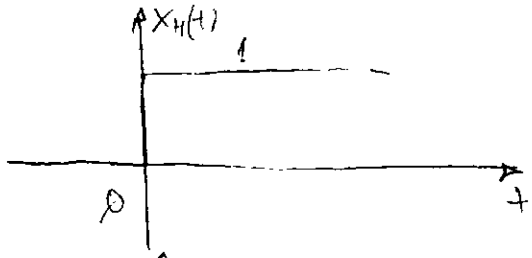
$t = -\omega$

$\omega = t$

$$\delta(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega t}\, dt$$

$$\int_{-\infty}^{\infty} e^{-j\omega t}\, dt = 2\pi\,\delta(-\omega) = 2\pi\,\delta(\omega) \qquad \boxed{\mathcal{F}\{K\} = K \cdot 2\pi\,\delta(\omega)}$$

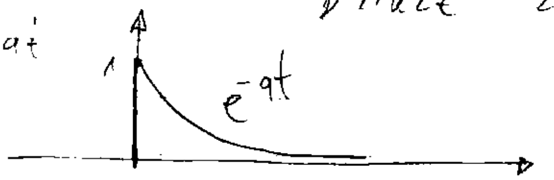
• СПЕКТРАЛНА ГУСТИНА НА АМПЛИТУДИ И ФАЗАТА НА ХЕВИСАЈДОВ СИГНАЛ



$$x_H(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

(*) $\int_{-\infty}^{\infty} |x_H(t)|\, dt \to \infty$ НЕ ГО ИСПОЛНУВА УСЛОВОТ НА ДИРИХЛЕ ЗА ФУРИЕВА ТРАНСФОРМАЦИЈА

$x_H(t)\, e^{-at}$



$$F(j\omega) = \int_{-\infty}^{\infty} e^{-at} \cdot e^{-j\omega t}\, dt = \int_{0^+}^{\infty} e^{-(at + j\omega t)}\, dt = \int_{0^+}^{\infty} e^{-(a+j\omega)t}\, dt =$$

$$= -\frac{1}{(a+j\omega)} \int_{0^+}^{\infty} e^{-(a+j\omega)t}\, d[(a+j\omega)t] = -\frac{1}{a+j\omega}\, e^{-(a+j\omega)t}\Big|_{0^+}^{\infty}$$

$$-\frac{1}{a+j\omega}\left[\frac{1}{e^{+\infty}} - 1\right] = \frac{1}{a+j\omega} = \frac{a - j\omega}{a^2+\omega^2}$$

$$F(j\omega) = \frac{1}{a^2+\omega^2}(a - j\omega) = \frac{1}{a^2+\omega^2}\sqrt{a^2+\omega^2}\, e^{-j\,\mathrm{arctg}\frac{\omega}{a}} = \frac{e^{-j\,\mathrm{arctg}\frac{\omega}{a}}}{\sqrt{a^2+\omega^2}}$$

$$\mathcal{F}\left[e^{-at} x_H(t)\right] = \frac{1}{a+j\omega}$$

$a \to 0$ $\quad \mathcal{F}[x_H(t)] = \frac{1}{j\omega} + K = \frac{1}{j\omega} + \mathcal{F}\{x_H(0)\}$

$$x_H(t)\Big|_{t=0} = \frac{1}{2}\left[x_H(0^-) + x_H(0^+)\right] = \frac{1}{2}$$

(где $x_H(0^-) = 0$, $x_H(0^+) = 1$)

$$\mathcal{F}\{K\} = 2\pi K\, \delta(\omega) \Rightarrow \mathcal{F}\left\{\frac{1}{2}\right\} = \pi\, \delta(\omega)$$

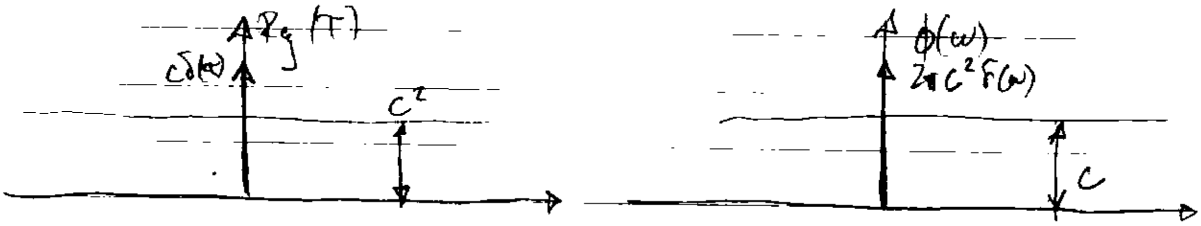
$$\boxed{\mathcal{F}\{x_H(t)\} = \frac{1}{j\omega} + \pi\, \delta(\omega)}$$

$R_q(\tau) = A^2\, P(A)\, P(A)$ $\quad \tau \neq 0 \quad R_q(\tau) = A^2 \cdot c\, d\tau\, c\, d\tau = c^2$

$(\tau = 0)$ $\quad R_q(\tau) = A^2\, P(A/A, 0)\, P(A) = A^2 \cdot c\, d\tau = c\, \delta(\tau)$ (где $P(A/A,0) = 1$)

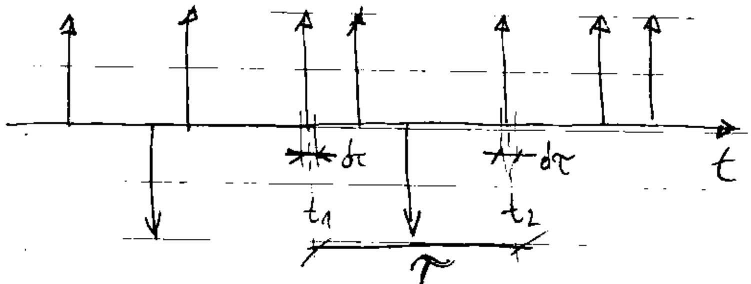
$$R_q(\tau) = c^2 + c\,\delta(\tau) \qquad \Phi(\omega) = \int_{-\infty}^{\infty} [c^2 + c\,\delta(\tau)]\, e^{-j\omega\tau}\, d\tau$$

$$\Phi(\omega) = 2\pi c^2 \delta(\omega) + c$$



$\overline{\xi}^2 = 2\pi c^2$ → ЕДНОСМЕРНА СНАГА

б. [illegible] БИПОЛАРНА НИЗА



$P(\eta=1) = c\cdot d\tau$

$P(A) = P(-A) = \dfrac{c\,d\tau}{2}$

$P(0) = 1 - c\,d\tau$ → ВЕРОЈАТНОСТ ДА НЕ СЕ ПОЈАВИ ИМПУЛС

$$R_\xi(\tau) = \sum_{i=1}^{3}\sum_{j=1}^{3} x_{1i}x_{2j}\,P_{\xi_1\xi_2}(x_{1i}, x_{2j}, \tau)$$

(A A)	(-A A)	0 A
A 0	-A 0	0 0
(A -A)	(-A -A)	0 -A

} МОЖНИ ВРЕДНОСТИ НА x_{1i}, x_{2j} ВО МОМЕНТИТЕ t_1 и t_2

$$R_\xi(\tau) = \sum_{i=1}^{3}\sum_{j=1}^{3} x_{1i}x_{2j}\,P_{\xi_1}(x_{1i})\,P_{\xi_1\xi_2}(x_{2j}/x_{1i};\tau)$$

$= A^2P(A)\cdot P(A/A;\tau) + A^2P(-A)P(-A/-A;\tau) - A^2P(-A)P(A/-A;\tau) -$
$- A^2P(A)P(-A/A;\tau) = 2A^2P(A)P(A/A;\tau) - 2A^2P(A)P(-A/A;\tau)$

$P(A/A;\tau) = P(-A/-A;\tau) = P(A) = \dfrac{c\,d\tau}{2}$

$P(-A/A;\tau) = P(A/-A;\tau) = P(A) = \dfrac{c\,d\tau}{2}$

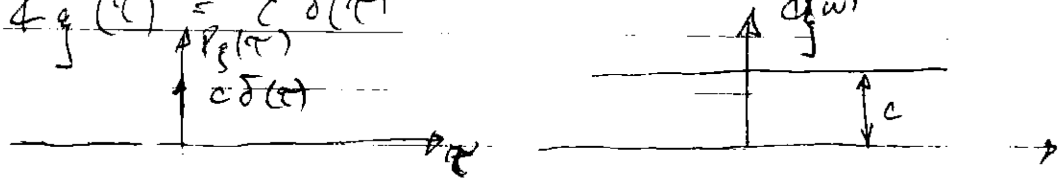
$R_\xi(\tau) = 2A^2\cdot\dfrac{c^2\,d\tau\,d\tau}{4} - 2A^2\dfrac{c^2\,d\tau\,d\tau}{4} = 0 \qquad \tau \neq 0$

$R_\xi(\tau) = 0$ за $\tau\neq 0$ — РАНДОМИЗИРАНАТА НИЗА Е НЕКОРЕЛИРАНА ЗА $\tau \neq 0$.

• за $\tau = 0$ $\quad P(A/A;0) = 1 \quad P(-A/A;0) = 0$.

$R_\xi(\tau) = A^2P(A)\cdot 1 + A^2P(-A)\cdot 1 = 2A^2P(A) = A^2\cdot c\,d\tau = c\delta(\tau)$

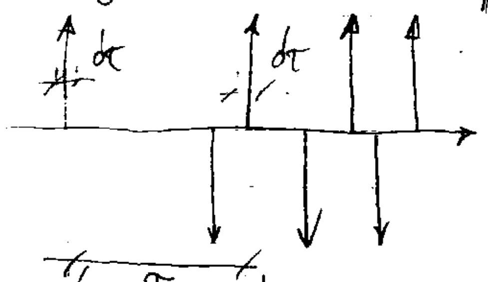
$R_\xi(\tau) = c\,\delta(\tau)$



6. АЛТЕРНАТИВНА ИМПУЛСНА НИЗА

$$p_{\xi\eta}(y/x) = \frac{p_{\xi\eta}(x,y)}{p_\xi(x)}, \quad p_\xi(x) > 0$$

$$p_{\xi\eta}(x/y) = \frac{p_{\xi\eta}(x,y)}{p_\eta(y)}, \quad p_\eta(y) > 0$$



$$R_\xi(\tau) = \sum_{i=1}^{2}\sum_{j=1}^{2} x_{1i}x_{2j} P_{\xi_1}(x_{1i}) P_{\xi_1\xi_2}(x_{2j}/x_{1i};\tau)$$

$$R_\xi(\tau) = 2A^2 P(A) P(A/A;\tau) - 2A^2 P(A) P(-A/A;\tau)$$

$$P(A) = P(-A) = \frac{c\,d\tau}{2}$$

$$P(A/A;\tau) = \underbrace{c\,d\tau}_{(*)} \underbrace{\sum_{k=1,3,5} P_\eta(k,\tau)}_{(**)} \quad \tau \neq 0$$

(*) – веројатност во t_2 да има импулс

(**) – веројатност во t_2 да има импулс „A" ≡ во τ има непарен број импулси

$$P(-A/A;\tau) = c\,d\tau \cdot \sum_{k=2,4,6} P_\eta(k;\tau)$$

$$R_\xi(\tau) = 2A^2 \frac{c\,d\tau}{2} c\,d\tau \left[\sum_{k=1,3,5} P_\eta(k,\tau) - \sum_{k=2,4,6} P_\eta(k,\tau)\right]$$

$$= A^2(c\,d\tau)^2 \left[\sum_{k=1,3,5} \frac{(c\tau)^k}{k!} e^{-c\tau} - \sum_{k=2,4,6} \frac{(c\tau)^k}{k!} e^{-c\tau}\right] =$$

$$= A^2 (c\,d\tau)^2 \frac{1}{2}\left[\sum_{k=0}^{\infty}\frac{(c\tau)^k}{k!}e^{-c\tau} - \sum_{k=0}^{\infty}\frac{(-c\tau)^k}{k!}e^{-c\tau} - \sum_{k=0}^{\infty}\frac{(c\tau)^k}{k!}e^{-c\tau} - \sum_{k=0}^{\infty}\frac{(-c\tau)^k}{k!}e^{-c\tau}\right]$$

$$= \frac{A^2(c\,d\tau)^2}{2}\left[-2\sum_{k=0}^{\infty}\frac{(-c\tau)^k}{k!}e^{-c\tau}\right] = -A^2(c\,d\tau)^2 \cdot e^{-c\tau}\sum_{k=0}^{\infty}\frac{(-c\tau)^k}{k!}$$

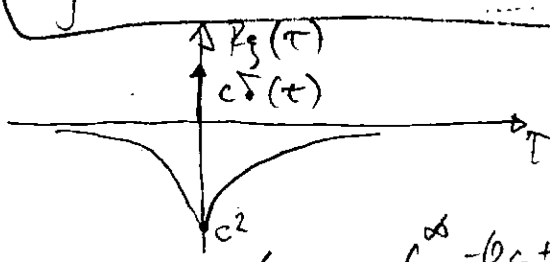
$$R_\xi(\tau) = -A^2c^2d\tau^2 e^{-(c\tau+c\tau)} = -c^2 \cdot e^{-2c\tau} = \underline{-c^2 e^{-2c|\tau|}}$$

$$\boxed{t \neq 0 \quad R_\xi(\tau) = -c^2 e^{-2c|\tau|}}$$

$\boxed{t = 0}$ $\quad P(A/A;0) = 1 \qquad P(-A/A;0) = 0$

$$R_\xi(0) = 2A^2 \cdot \frac{c\,d\tau}{2} \cdot 1 = A^2 c\,d\tau = A \cdot c\Big|_{A\to\infty} = c\,\delta(\tau)$$

$$\boxed{R_\xi = c\,\delta(\tau) - c^2 e^{-2c|\tau|}}$$



$t \geq 0$

$$\Phi(\omega) = \int_{-\infty}^{\infty} R_\xi(\tau) e^{-j\omega\tau} d\tau =$$

$$= c - c^2\, 2\int_0^{\infty} e^{-2c\tau} e^{-j\omega\tau} d\tau =$$

$$= c + 2c^2 \cdot \frac{1}{2c+j\omega}\int_0^{\infty} e^{-(2c+j\omega)\tau} d[-(2c+j\omega)\tau] = c + 2c^2 \frac{1}{2c+j\omega} e^{-(2c+j\omega)\tau}\Big|_0^{\infty} = c - \frac{2c^2}{2c+j\omega}$$

~~$\Phi_\xi(\omega) = c - \frac{2c^2}{2c+j\omega} = c - \frac{2c^2(2c-j\omega)}{4c^2+\omega^2} = c - \frac{4c^3}{4c^2+\omega^2} + \frac{j\omega}{4c^2+\omega^2}$~~ NO

$$\Phi_\xi(\omega) = c - \frac{4c^3}{4c^2+\omega^2}$$

$$\Phi_\xi(\omega) = c - c^2\int_{-\infty}^{\infty} e^{-2c|\tau|}e^{-j\omega\tau}\,d\tau = c - c^2\int_{-\infty}^{\infty} e^{-2c|\tau|}(\cos\omega\tau - j\sin(\omega\tau))\,d\tau$$

NEPARNA = PARNA · NEPARNA

$$\Phi_\xi(\omega) = c - c^2\int_{-\infty}^{\infty} e^{-2c|\tau|}\cos(\omega\tau)\,d\tau = c - c^2\cdot 2\int_0^{\infty} e^{-2c\tau}\cos\omega\tau\,d\tau$$

$$\boxed{I = \int_{-\infty}^{\infty} e^{-a|x|}e^{-j\omega x}\,dx = \frac{2a}{a^2+\omega^2}}$$

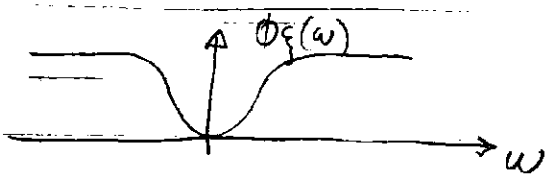
$$I = \int_{-\infty}^{0} e^{+ax}e^{j\omega x}\,dx + \int_{0}^{\infty} e^{-ax}e^{-j\omega x}\,dx =$$

$$= \int_{-\infty}^{0} e^{(a+j\omega)x}\,dx + \int_{0}^{\infty} e^{-(a+j\omega)x}\,dx = \frac{1}{a+j\omega}\,e^{(a+j\omega)x}\Big|_{-\infty}^{0} - \frac{1}{a+j\omega}\,e^{-(a+j\omega)x}\Big|_{0}^{\infty}$$

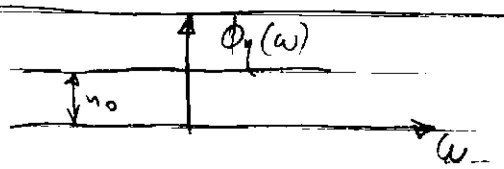
$$= \frac{1}{a+j\omega} + \frac{1}{a+j\omega} = \frac{2}{a+j\omega}\cdot\frac{a-j\omega}{a-j\omega} = \frac{2a-j2\omega}{a^2+\omega^2}$$

$$Re\{I\} = \frac{2a}{a^2+\omega^2} \qquad Im\{I\} = \frac{-2\omega}{a^2+\omega^2}$$

$$\Phi_\xi(\omega) = c - \frac{4c^3}{4c^2+\omega^2}$$



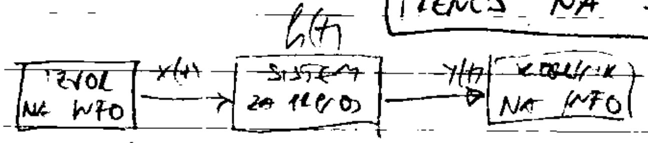
BEL ŠUM:



$$R_\nu(\tau) = \frac{1}{2\pi}\int n_0\, e^{j\omega\tau}\,d\omega$$

$$\boxed{R_\nu(\tau) = n_0\,\delta(\tau)}$$

PRENOS NA SLUČAJNI PROCESI NIZ LINEAREN SP.!



$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau =$$

$$= \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau$$

$$\overline{y(t)} = \int_{-\infty}^{\infty} \overline{h(\tau)x(t-\tau)}\,d\tau = \int_{-\infty}^{\infty} h(\tau)\,\overline{x(t-\tau)}\,d\tau = \overline{x(t)}\int_{-\infty}^{\infty} h(t)\,dt$$

$$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt \qquad H(0) = \int_{-\infty}^{\infty} h(t)\, dt$$

SR. VRED.

$$\eta(t) \quad H(0)\, \xi(t)$$

SP. GUST. NA IZLAZ $\quad \Phi_{\eta}(\omega)$

$$R_{\eta\eta}(\tau) = \overline{\eta(t)\,\eta(t+\tau)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \eta(t)\, \eta(t+\tau)\, dt =$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left\{ \int_{-\infty}^{\infty} h(\mu)\, x(t-\mu)\, d\mu \int_{-\infty}^{\infty} h(\sigma)\, x(t+\tau-\sigma)\, d\sigma \right\} dt =$$

$$= \int_{-\infty}^{\infty} h(\mu)\, d\mu \int_{-\infty}^{\infty} h(\sigma)\, d\sigma \underbrace{\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t-\mu)\, x(t+\tau-\sigma)\, dt}$$

$$R_{\xi\xi}(t+\tau-\sigma-t+\mu) = R_{\xi\xi}(\tau+\mu-\sigma)$$

$$R_{\eta\eta}(\tau) = \int_{-\infty}^{\infty} h(\mu)\, d\mu \int_{-\infty}^{\infty} h(\sigma)\, R_{\xi\xi}(\tau+\mu-\sigma)\, d\sigma$$

$$\Phi_{\eta\eta}(\omega) = \int_{-\infty}^{\infty} e^{-j\omega\tau} d\tau \int_{-\infty}^{\infty} h(\mu)\, d\mu \int_{-\infty}^{\infty} h(\sigma)\, R_{\xi\xi}(\tau+\mu-\sigma)\, d\sigma$$

$$\nu = \mu+\tau-\sigma \qquad \tau = \nu-\mu+\sigma \qquad d\tau = d\nu$$

$$\Phi_{\eta\eta}(\omega) = \int_{-\infty}^{\infty} e^{-j\omega(\nu-\mu+\sigma)} d\nu \int_{-\infty}^{\infty} h(\mu)\, d\mu \int_{-\infty}^{\infty} h(\sigma)\, R_{\xi\xi}(\nu)\, d\sigma =$$

$$= \int_{-\infty}^{\infty} h(\mu) e^{+j\omega\mu} d\mu \int_{-\infty}^{\infty} h(\sigma) e^{-j\omega\sigma} d\sigma \int_{-\infty}^{\infty} R_{\xi\xi}(\nu) e^{-j\omega\nu} d\nu$$

$$= H^*(j\omega) \cdot H(j\omega) \cdot \Phi_{\xi\xi}(\omega) = |H(j\omega)|^2\, \Phi_{\xi\xi}(\omega)$$

$$R_{\xi\eta}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, \eta(t+\tau)\, dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, dt \int_{-\infty}^{\infty} h(\mu)\, x(t+\tau-\mu)\, d\mu$$

$$= \left| \eta(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \right| = \int_{-\infty}^{\infty} h(\mu)\, d\mu \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, x(t+\tau-\mu)\, dt$$

$R_{\xi\eta}(\tau)=\int_{-\infty}^{\infty} h(\mu) R_{\xi\xi}(\tau-\mu)\, d\mu$ KONVOLUCIJA OD IMPULSNIOT ODZIV I AVTOKOREL. FUNKC. NA VLEZNIOT SIGNAL

$$\Phi_{\xi\eta}(\omega)= H(j\omega)\, \Phi_{\xi\xi}(\omega)$$

VLEZNO-IZLEZNA MEĐUKORELACIONA TEOREMA

2 IST

• AVTOKORELACIJA NA IZLEZNIOT ODZIV:

$$R_{\eta\eta}(\tau)=\int_{-\infty}^{\infty} h(\mu)\,d\mu \int_{-\infty}^{\infty} h(\sigma) R_{\xi\xi}(\tau+\mu-\sigma)\, d\sigma \qquad t=\sigma-\mu$$

$$\sigma=t-\mu \quad d\sigma=dt = \int_{-\infty}^{\infty} h(\mu)\,d\mu \int_{-\infty}^{\infty} h(t-\mu) R_{\xi\xi}(\tau-t)\, dt$$

$$=\int_{-\infty}^{\infty} \underbrace{h(\mu) h(t-\mu)\, d\mu}_{R_{hh}(t)} \int_{-\infty}^{\infty} R_{\xi\xi}(\tau-t)\, dt = \int_{-\infty}^{\infty} R_{hh}(t) R_{\xi\xi}(\tau-t)\, dt$$

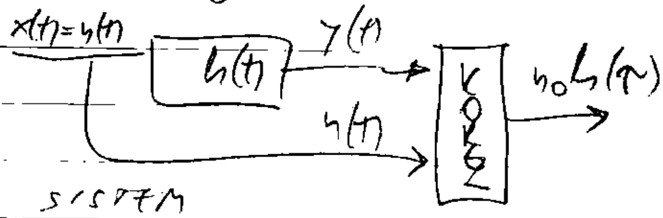
$$\Phi_{\eta}(\omega)=|H(j\omega)|^2\, \Phi_{\xi}(\omega) \qquad R_{hh}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}|H(j\omega)|^2 e^{j\omega t}\, d\omega$$

• ODREDUVANJE NA K-KI NA LINEAREN SISTEM SO MEĐUKORELACIJA

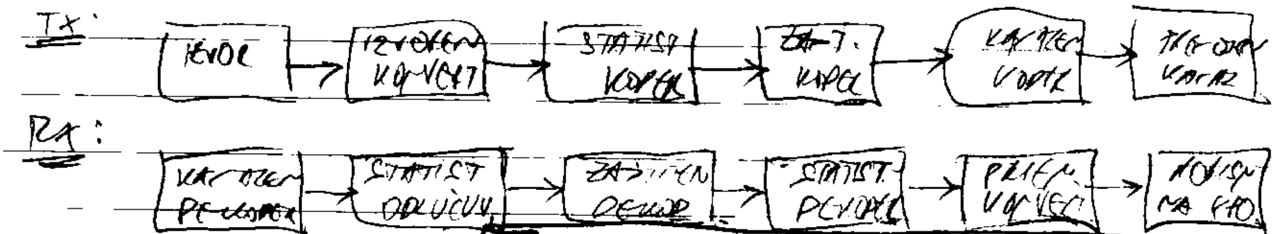
$x(t)=n(t)$ $\qquad R_{\xi\xi}=n_0\,\delta(\tau)$

$$R_{\xi\eta}(\tau)=\int_{-\infty}^{\infty} h(\mu) R_{\xi\xi}(\tau-\mu)\, d\mu = \int_{-\infty}^{\infty} h(\mu)\, n_0\, \delta(\tau-\mu)\, d\mu$$

$$R_{\xi\eta}(\tau)=n_0\, h(\tau)$$



• MODEL NA KOMUNIKACISKI SISTEM



TEORIJA NA INFORMACII

$J'=\{J_1, J_2, \dots, J_q\}$ $\quad$ q DISKRETNI SIMBOLI

$\{P(J_i)\}$; $i=1,2,\dots q$ $\quad$ VEROJATNOST NA POJAVUVANJE NA SIMBOLITE

$$\sum_{i=1}^{q} P(J_i)=1 \qquad I(J_i)\sim\frac{1}{P(J_i)}$$

s_i, s_j ; $P(s_i, s_j)$, $I(s_i, s_j) \sim \frac{1}{P(s_i, s_j)} = \frac{1}{P(s_i)P(s_j)}$

$I(s_i) = \log_a \frac{1}{P(s_i)}$ $\quad I(s_i, s_j) = \log_a \frac{1}{P(s_i)P(s_j)} = \log_a \frac{1}{P(s_i)} + \log_a \frac{1}{P(s_j)}$

$I(s_i, s_j) = I(s_i) + I(s_j)$

(1) $I(s_i) = \log_{10} \frac{1}{P(s_i)}$ [Hartley] , $I(s_i) = \ln \frac{1}{P(s_i)}$ [nat] ; '

(3) $I(s_i) = \log_2 \frac{1}{P(s_i)} = ld \frac{1}{P(s_i)} = - ld\, P(s_i)$ [Sh]

2.110

Дискретни извори на информации без меморија

ЕНТРОПИЈА - просечно количество на информација по симбол

$$\overline{I}(s_i) = \sum_{i=1}^{q} I(s_i) P(s_i) = \sum_{i=1}^{q} P(s_i)\, ld \frac{1}{P(s_i)} = H(S)$$

просечна неизвесност за појавување на одреден симбол

$S = \{s_1, s_2, \ldots s_q\}$

ОИЗ 22.07.07

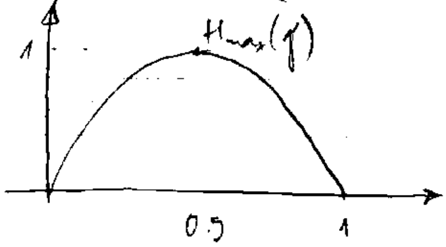
• Пример (бинарен извор)

$S = \{s_1, s_2\} = \{0, 1\}$

s_i	$P(s_i)$
s_1	p
s_2	$1-p$

$$H(S) = \sum_{i=1}^{2} P(s_i)\, ld \frac{1}{P(s_i)} = p\, ld \frac{1}{p} + (1-p)\, ld \frac{1}{1-p}$$

$H(S)$ (=) $\frac{Sh}{\text{симбол}}$ $\qquad H(p)$ (=) $\frac{Sh}{\text{BIT}}$



$H'(S) = \left(-p \frac{\ln(p)}{\ln(2)} - (1-p) \frac{\ln(1-p)}{\ln(2)} \right)'$

$= -\frac{\ln(p)}{\ln(2)} - p \frac{1}{p \ln(2)} + \frac{\ln(1-p)}{\ln(2)} + \frac{(1-p)}{\ln(2)} \frac{1}{1-p}$

$ld\, x = \frac{\ln(x)}{\ln(2)}$ $\qquad -\ln(p) - 1 + \ln(1-p) + 1 = 0$

$\ln(p) = \ln(1-p)$ $\qquad \ln\left(\frac{1-p}{p}\right) = 0 \Rightarrow$

$\frac{1-p}{p} = 1$ $\qquad 1 - p = p$ $\qquad 2p = 1$ $\qquad \boxed{p = \frac{1}{2}}$ $\qquad H(S)\big|_{p=\frac{1}{2}} = H_{max} = 1$

$H = \frac{1}{2}\, ld\, 2 + \frac{1}{2}\, ld\, 2 = \frac{1}{2} + \frac{1}{2}$

• $H(S) \geq 0$

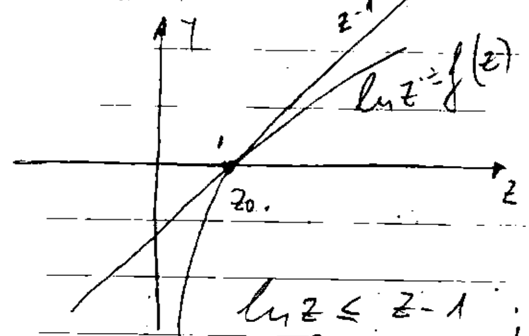
• $H(S)_{max} = ld\, q$ $\quad \therefore q = \frac{1}{P(s_i)}$ $\quad P(s_i) = \frac{1}{q}$

$H(S)_{max} = \sum_{i=1}^{q} \frac{1}{q}\, ld\, q = q \cdot \frac{1}{q} \cdot ld\, q = ld\, q$

DOKAZ: $ld\,q - H(S) = ld\,q - \sum_{i=1}^{q} P(s_i)\, ld \frac{1}{P(s_i)} =$

$= \sum_{i=1}^{q} \frac{1}{q} ld\,q - \sum_{i=1}^{q} P(s_i)\, ld \frac{1}{P(s_i)} = \sum_{i=1}^{q} P(s_i)\left[-ld \frac{1}{q} - ld \frac{1}{P(s_i)}\right]$

$= -\sum_{i=1}^{q} P(s_i)\, ld \frac{1}{q P(s_i)} = -\frac{1}{\ln 2} \sum_{i=1}^{q} P(s_i) \ln \frac{1}{q P(s_i)}$ (*)



$y = f'(z_0)(z - z_0)$

$y = \frac{1}{z_0}(z-1) = 1(z-1) = z-1$

$y = z - 1$ TANGENTA NA $f(z)$ VO z_0

$\ln z \le z - 1$; $-\ln z \ge 1 - z$

(*) $\ge \frac{1}{\ln 2} \sum_{i=1}^{q} P(s_i)\left[1 - \frac{1}{q P(s_i)}\right] = \frac{1}{\ln(2)} - \frac{1}{\ln(2)} = \emptyset$

$ld\,q - H(S) \ge \emptyset$ $\quad H(S) \le ld\,q \Rightarrow$ $H_{max}(S) = ld\,q$

INFORMACIONEN DEBIT $\quad \Phi(S) = H(S) \cdot v(S)$

$v(S)$ - BRZINA NA GENERIRANJE NA SIMBOLITE VO IZVOROT

$\Phi(S)(=) \frac{sh}{sim} \cdot \frac{sim}{sec} = \frac{sh}{sec} \qquad \Phi(S)(=)\left[\frac{sh}{sec}\right]$

PROŠIRUVANJE NA DISKRETEN IZVOR BEZ MEMORIJA

- PROŠIRUVANJE OD 2 RED (SEKVENC SO DOLŽ 2)

$S = \{s_1, s_2\} = \{\underbrace{0, 1}_{q}\}$

s_i	$P(s_i)$
s_1	$P(s_1)$
s_2	$P(s_2)$

$W = \{w_1, w_2, w_3, w_4\} = \{00, 01, 10, 11\}$ 43:17

PROŠIRUVANJE OD n-TI RED:

$S = \{s_1, s_2, \ldots, s_q\}$

$S^n = \{W_1, W_2, \ldots, W_{q^n}\}$ $\quad W_i = \{s_{i1}, s_{i2}, \ldots, s_{in}\}$, $i_k = 1, 2, \ldots, q$

$P(w_i) = P(s_{i1}, s_{i2}, \ldots, s_{in}) = \prod_{k=1}^{n} P(s_{ik})$

W_i	$P(w_i)$
W_1	$P(s_{i1}) \cdot P(s_{i2}) \cdots P(s_{in})$
W_2	
⋮	
W_{q^n}	

$H(w) = \sum_{i=1}^{q^n} P(w_i)\, ld \frac{1}{P(w_i)}$

$H(w) = n \cdot H(S)$

34:18

$$H(W) = \sum_{i=1}^{2^n} P(w_i)\, ld \frac{1}{P(w_i)} = \sum_{i=1}^{2^n} P(w_i)\, ld \frac{1}{P(s_{i1})P(s_{i2}) \cdots P(s_{in})} =$$

$$= \sum_{i=1}^{2^n} P(w_i)\, ld \frac{1}{P(s_{i1})} + \sum_{i=1}^{2^n} P(w_i)\, ld \frac{1}{P(s_{i2})} + \cdots + \sum_{i=1}^{2^n} P(w_i)\, ld \frac{1}{P(s_{in})} = (*)$$

$$\sum_{i=1}^{2^n} P(w_i)\, ld \frac{1}{P(s_{i1})} = \sum_{i_1=1}^{2^n} \left[P(s_{i1}) P(s_{i2}) \cdots P(s_{in})\right] ld \frac{1}{P(s_{i1})} =$$

$$= \sum_{i_1=1}^{2} P(s_{i1})\, ld \frac{1}{P(s_{i1})} \underbrace{\sum_{i_2=1}^{2} P(s_{i2})}_{1} \cdots \underbrace{\sum_{i_n=1}^{2} P(s_{in})}_{1} = H(S)$$

$$(*) = n \cdot H(S) \qquad \boxed{H(W) = n \cdot H(S)}$$

DISKRETNI IZVORI SO MEMORIJA

$$V_S = \{s_1, s_2, \ldots, s_q\}$$

$$P(s_i / s_{j1}, s_{j2}, \ldots, s_{jr}, \ldots, s_{jk}) \qquad i = 1,2,\ldots,q \quad j_r = 1,2,\ldots,q$$

VEROJATNOSTA NA EDNA SOSTOJBA

$$P(s_{j1}, s_{j2}, \ldots, s_{jr}, \ldots, s_{jk}), \qquad j_r = 1,2,\ldots,q$$

SKRATENO $P(S_p)$, $p = 1,2,\ldots,q^k$

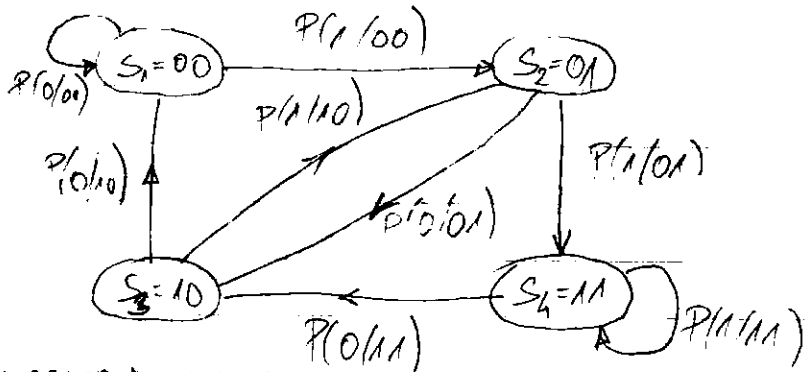
S - SOSTOJBA
s - SIMBOL

$$P(s_i / S_p) \qquad i = 1,2,\ldots,q \quad p = 1,2,\ldots,q^k$$

- MARKOV IZVOR OD VTOR RED:

$$P(s_i/S_p) \qquad s_i \in (0,1) \qquad S_p \in (00, 01, 10, 11)$$

$q^k = 2^2 = 4$ SOSTOJBI, 2 SIMBOLI!



$$\sum_{i=1}^{2} P(s_i/S_p) = 1 \qquad p = 1,2,3,4$$

$$\boxed{P(s_i/S_p) \neq 0 \qquad i = 1,2 \quad p = 1,2,3,4}$$

ERGODIČEN MARKOV IZVOR

- ERGODIČNOST NA MARKOV IZVOR OD K-TI RED

$$\boxed{P(s_i/S_p) \neq 0 \qquad i = 1,2,\ldots,q \quad p = 1,2,\ldots,q^K}$$

ЕНТРОПИЈА НА МАРКОВ ИЗВОР

$I(s_i/s_r) = ld \frac{1}{P(s_i/s_r)}$ $S_r = (s_{j1}, s_{j2}, s_{jT}, \ldots s_{jk})$ $j_T = 1, 2, \ldots q$

$$H({}^{k}S/s_r) = \sum_{i=1}^{q} P(s_i/s_r)\, ld \frac{1}{P(s_i/s_r)} = \overline{I(s_i/s_r)}$$

УСЛОВНА ЕНТРОПИЈА = просечно количество информација по симбол добиена (кога) изворот се наоѓа во една (определена) специф. состојба

$$H({}^{k}S) = H({}^{k}S/s_r) = \sum_{r=1}^{q^k} \sum_{i=1}^{q} P(s_i) \, ld \frac{1}{P(s_i/s_r)}$$

КОНТИНУИРАН ИЗВОР НА ИНФОРМ.

$$P(S_1) = P(S_1) \cdot P(0/S_1) + P(S_3) \cdot P(0/S_3)$$
$$P(S_2) = P(S_1) P(1/S_1) + P(S_3) \cdot P(1/S_3)$$
$$P(S_3) = P(S_2) P(0/S_2) + P(S_4) \cdot P(0/S_4)$$
$$P(S_4) = P(S_2) P(1/S_2) + P(S_4) P(1/S_4)$$

$s \in (-\infty, \infty)$ $p(s)$

$$H(s) = \int_{-\infty}^{\infty} I(s)\, p(s)\, ds = \int_{-\infty}^{\infty} p(s)\, ld \frac{1}{p(s)}\, ds \; (=) \frac{sh}{sample}$$

$$\Phi(s) = H(s) \cdot v(s) \; (=) \frac{sh}{\text{pr.}} \cdot \frac{\text{pr.}}{sec} \; (=) \frac{sh}{sec}$$

$T_s \leq \frac{1}{2f_g}$ $\frac{1}{f_s} \leq \frac{1}{2f_g}$ $f_s \geq 2f_g$

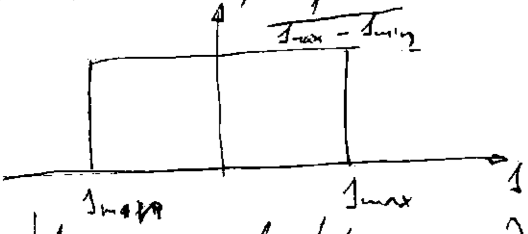
$v(s) = 2f_g$ $\boxed{\Phi(s) = 2f_g H(s)}$

• 2 ТИПА НА КОНТИНУАЛНИ ИЗВОРИ:

(a) $s \in \{s_{min}, s_{max}\}$ ОГРАНИЧЕН ПО ~~СНАГА~~ АМПЛИТУДА

(б) $s \in (-\infty, \infty)$ ОГРАНИЧЕН ПО СНАГА

(a) $$H(s) = \int_{-s_{min}}^{s_{max}} p(s)\, ld \frac{1}{p(s)}\, ds$$



$$H_{max}(s) = \frac{1}{s_{max} - s_{min}}\, ld(s_{max} - s_{min})\, s \Big|_{s_{min}}^{s_{max}} = ld(s_{max} - s_{min})$$

$s_{max} = s_g$
$s_{min} = -s_g$ $\boxed{H_{max}(s) = ld(2 s_g)}$

(δ) $\overline{s^2} = \int_{-\infty}^{\infty} s^2 p(s)\, ds < \infty$ [21:41] $p(s) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{s^2}{2\sigma^2}}$

$$H_m(s) = \int_{-\infty}^{\infty} p(s)\, ld \frac{1}{p(s)}\, ds = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{s^2}{2\sigma^2}} \underbrace{ld\left(\sqrt{2\pi\sigma^2}\, e^{\frac{s^2}{2\sigma^2}}\right)}_{(*)} ds$$

$ld\,x = \frac{\ln x}{\ln 2}$ 6G → 2.8G 80 → 28 : 0.9 +

$$(*) = \frac{1}{\ln 2}\left[\ln\sqrt{2\pi\sigma^2} + \ln e^{\frac{s^2}{2\sigma^2}}\right]$$

$$H_m(s) = \frac{1}{\ln 2\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{2\sigma^2}}\left[\ln\sqrt{2\pi\sigma^2} + \frac{s^2}{2\sigma^2}\right] ds =$$

$$= \frac{1}{\ln 2\sqrt{2\pi\sigma^2}}\left[\ln\sqrt{2\pi\sigma^2}\underbrace{\int_{-\infty}^{\infty} e^{-\frac{s^2}{2\sigma^2}} ds}_{I_1} + \underbrace{\int_{-\infty}^{\infty} \frac{s^2 e^{-\frac{s^2}{2\sigma^2}}}{2\sigma^2} ds}_{I_2}\right]$$

$$I_1 = \left| \int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi} \quad u = \frac{s}{\sqrt{2\sigma^2}} \quad du = \frac{ds}{\sqrt{2\sigma^2}} \right| = \sqrt{2\sigma^2}\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{2\pi\sigma^2}$$

$$I_2 = \int_{-\infty}^{\infty} \frac{s^2 e^{-\frac{s^2}{2\sigma^2}}}{2\sigma^2} ds = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \frac{\sqrt{\pi}}{\sqrt{2\sigma^2}} \int_{-\infty}^{\infty} s^2 e^{-\frac{s^2}{2\sigma^2}} ds \qquad \overline{s^2} = \sigma^2$$

$$I_2 = \frac{\sqrt{\pi}}{\sqrt{2\sigma^2}} \cdot \sigma^2 = \frac{\sqrt{\pi}\,\sigma}{\sqrt{2}} = \sqrt{\frac{\pi}{2}}\,\sigma$$

$$H_m(s) = \frac{1}{\ln 2\sqrt{2\pi\sigma^2}}\left(\ln\sqrt{2\pi\sigma^2}\cdot\sqrt{2\pi\sigma^2} + \sqrt{\frac{\pi}{2}}\,\sigma\right)$$

$$H_m(s) = ld\sqrt{2\pi\sigma^2} + \frac{1}{\ln 2\sqrt{2\pi\sigma^2}}\frac{\sqrt{\pi}}{\sqrt{2}}\sigma = ld\sqrt{2\pi\sigma^2} + \underbrace{\frac{1}{2\ln(2)}}_{= ld\sqrt{e}}$$

$$H_m(s) = ld\sqrt{2\pi\sigma^2} + \frac{\ln e}{2\ln(2)} = ld\sqrt{2\pi\sigma^2} + \frac{\ln\sqrt{e}}{\ln(2)} = \underline{ld\sqrt{2\pi e\sigma^2}}$$

$$\boxed{H_{max}(s) = ld\sqrt{2\pi e\sigma^2}} \qquad H(s) \le H_{max}(s)$$

STATISTIČKO KODIRANJE

21:36

$s_i \to X_i = \{x_{i1}, x_{i2}, \ldots, x_{jk}, \ldots, x_{jn}\}$

$x_{jk} \in \{x_1, x_2, \ldots, x_m\}$ KODNA LISTA (AZBUKA)

$i = 1, 2, \ldots, q$ — BROJ NA KODNI ZBOROVI

$m^n = q$ $\qquad$ $n = \log_m q$

$$L_n(x) = \sum_{i=1}^{q} l_i P(l_i) = \sum_{i=1}^{q} l_i P(s_i)$$

DOLŽINA ZA ZADRŠKA:

$$\left[\frac{K}{2}\right] l_{min} < l_d < \left[\frac{K+1}{2}\right] l_{max}$$

K – BROJ ZA VLEZ

s_i	S_0	S_1
s_1	0	1
s_2	11	
s_3	100	
s_4	101	

- $K=1$ $\quad \left[\frac{1}{2}\right] \cdot 1 < l_d < \left[\frac{1+1}{2}\right] \cdot 3$

$1 < l_d < [1] \cdot 3 \qquad 1 < l_d < 3$

- S_0, S_1, S_2 $(K=3)$

$\left[\frac{3}{2}\right] \cdot 2 < l_d < \left[\frac{4}{2}\right] \cdot 3 \qquad 4 < l_d < 6$

- KRAFTOVO NERAVENSTVO

$$\sum_{i=1}^{q} m^{-l_i} \le 1 \qquad l_i = \{l_1, l_2, \ldots, l_q\}$$

$N(l_i)$ → BROJ NA KODNI ZBOROVI SO DOLŽINA l_i

$$\sum_{l_i=1}^{l_{max}} N(l_i) \le q$$

- OSNOVNA TEOREMA NA STATISTIČKO KODIRANJE

$$L_n(x) \ge \frac{H(S)}{\operatorname{ld} m} \qquad L_n(x)_{min} = \frac{H(S)}{\operatorname{ld} m}$$

$S = \{s_1, s_2, \ldots, s_q\}$; $\{P(s_1), P(s_2), \ldots, P(s_q)\}$

KODNA LISTA: $X = \{x_1, x_2, \ldots, x_m\}$

$L = \{l_1, l_2, \ldots, l_q\}$

$x_i \ge 0 \quad \sum_{i=1}^{q} x_i = 1 \qquad y_i \ge 0 \quad \sum_{i=1}^{q} y_i = 1$

$$\sum_{i=1}^{q} x_i \operatorname{ld} \frac{y_i}{x_i} = \frac{1}{\ln 2} \sum_{i=1}^{q} x_i \ln \frac{y_i}{x_i}$$

$$\ln z \le z - 1$$

$$f(x) = 1 + x f'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{3!} f'''(0)$$

$$(\ln(x))' = \frac{1}{x} \qquad (\ln(x))'' = \left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

$$\frac{1}{\ln 2} \sum_{i=1}^{q} x_i \ln \frac{y_i}{x_i} \le \frac{1}{\ln 2} \sum_{i=1}^{q} x_i \left(\frac{y_i}{x_i} - 1\right) = \frac{1}{\ln 2}\left[\sum_{i=1}^{q} y_i - \sum_{i=1}^{q} x_i\right] = 0$$

$$\sum_{i=1}^{q} x_i \left(\mathrm{ld}\frac{1}{x_i} + \mathrm{ld}\, y_i\right) = \sum_{i=1}^{q} x_i\, \mathrm{ld}\frac{1}{x_i} - \sum_{i=1}^{q} x_i\, \mathrm{ld}\frac{1}{y_i} \le 0$$

$$\boxed{\sum_{i=1}^{q} x_i\, \mathrm{ld}\frac{1}{x_i} \le \sum_{i=1}^{q} x_i\, \mathrm{ld}\frac{1}{y_i}} \qquad x_i = P(s_i)$$

$$\sum_{i=1}^{q} x_i\, \mathrm{ld}\frac{1}{x_i} = \sum_{i=1}^{q} P(s_i)\, \mathrm{ld}\frac{1}{P(s_i)} = H(S)$$

$$y_i = \frac{m^{-l_i}}{\sum_{j=1}^{q} m^{-l_j}}, \quad i = 1, 2, \dots q$$

$$H(S) = \sum_{i=1}^{q} P(s_i)\, \mathrm{ld}\frac{1}{P(s_i)} \le \sum_{i=1}^{q} P(s_i)\, \mathrm{ld}\frac{\sum_{j=1}^{q} m^{-l_j}}{m^{-l_i}} =$$

$$= \sum_{i=1}^{q} P(s_i)\, \mathrm{ld} \sum_{i=1}^{q} m^{-l_i} - \sum_{i=1}^{q} P(s_i)\, \mathrm{ld}(m^{-l_i})$$

$$= \mathrm{ld} \sum_{i=1}^{q} m^{-l_i} + \sum_{i=1}^{q} P(s_i)\, l_i\, \mathrm{ld}\, m$$

$$H(S) \le \mathrm{ld} \sum_{i=1}^{q} m^{-l_i} + \sum_{i=1}^{q} P(s_i)\, l_i\, \mathrm{ld}\, m$$

$$H(S) \le \mathrm{ld} \sum_{i=1}^{q} m^{-l_i} + \mathrm{ld}(m) \cdot L_m(x)$$

$(*) \le 0$

$$\boxed{H(S) \le \mathrm{ld}(m)\, L_m(x)}$$

$$L_m(x) \ge \frac{H(S)}{\mathrm{ld}(m)} = H_m(S)$$

$$H_m(S) = \frac{1}{\mathrm{ld}(m)} \sum_{i=1}^{q} P(s_i)\, \mathrm{ld}\frac{1}{P(s_i)} = \frac{1}{\frac{\ln m}{\ln 2}} \sum_{i=1}^{q} P(s_i) \frac{\ln \frac{1}{P(s_i)}}{\ln 2}$$

$$= \sum_{i=1}^{q} P(s_i) \log_m \frac{1}{P(s_i)} \qquad \log_m x = \frac{\ln x}{\ln(m)}$$

$$i \Rightarrow \quad P(s_i) = \frac{m^{-l_i}}{\sum_{j=1}^{q} m^{-l_j}} \qquad i = 1, 2, \dots q$$

$$P(s_i) = m^{-l_i} \quad i = 1, 2, \dots q \qquad L_m(x)_{min} = \frac{H(S)}{ld(m)}$$

OSNOVNA TEOREMA NA STAT. KODIRANJE

$$L_m(x) = \sum_{i=1}^{q} l_i P(s_i)$$

$$L_m(x) \geq \frac{H(S)}{ld\, m} \; ; \; L_m(x) = \frac{H(S)}{ld(m)} \quad \text{ZA } P(s_i) = m^{-l_i}$$

PRIMER: $L_2(x) = L_m(x) \geq \frac{H(S)}{ld(2)} = H(S)$ $\quad L_2(x) \geq H(S)$

- $P(s_i) = m^{-l_i} \;/\; ld \qquad ld\, P(s_i) = -l_i\, ld\, m$

$$l_i = \frac{ld \frac{1}{P(s_i)}}{ld\, m}$$

- $\frac{ld \frac{1}{P(s_i)}}{ld\, m} \leq l_i \leq \frac{ld \frac{1}{P(s_i)}}{ld\, m} + 1 \quad \Big/ \sum_{i=1}^{q} P(s_i)$

$$l_i = \left\lceil \log_m \frac{1}{P(s_i)} \right\rceil = \left\lceil \frac{ld \frac{1}{P(s_i)}}{ld(m)} \right\rceil \qquad i = 1, 2, \dots q$$

$$\frac{H(S)}{ld\, m} \leq L_m(x) \leq \frac{H(S)}{ld\, m} + 1$$

PRVA ŠENONOVA TEOREMA

$$\frac{H(S^n)}{ld\, m} \leq L_m(x^n) \leq \frac{H(S^n)}{ld(m)} + 1$$

$$\frac{n \cdot H(S)}{ld\, m} \leq L_m(x^n) \leq \frac{n\, H(S)}{ld(m)} + 1$$

$$\frac{H(S)}{ld\, m} \leq \frac{L_m(x^n)}{n} \leq \frac{H(S)}{ld(m)} + \frac{1}{n}$$

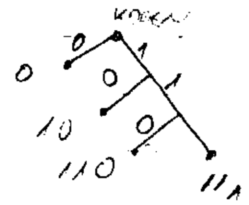
$$\lim_{n \to \infty} \frac{L_m(x^n)}{n} = \frac{H(S)}{ld\, m}$$

MOŽE DA BIDE NA MAKS. ZEMAČ(?) PO SIMBOL OD ORIG. LISTA

МЕТОДИ ЗА ОПТИМАЛНО КОДИРАЊЕ

• ФАНОВА ПОСТАПКА: $l_i = -\frac{ld\,P(s_i)}{ld\,m}$ $\quad P(s_i) = m^{-l_i}$, $\quad P(s_i) = \left(\frac{1}{m}\right)^{l_i}$

s_i	$P(s_i)$				x_i
s_1	$\frac{1}{2}$	I			0
s_2	$\frac{1}{4}$	II	I		10
s_3	$\frac{1}{8}$		II	I	110
s_4	$\frac{1}{8}$			II	111



$m = 2$

$$L_2(x) = \sum_{i=1}^{q} l_i P(s_i) = 1\cdot\frac{1}{2} + 2\cdot\frac{1}{4} + 3\cdot\frac{1}{8} + 3\frac{1}{8} = \frac{4+4+3+3}{8} = \frac{14}{8}$$

$$\boxed{L_2(x) = \frac{7}{4}}$$

$$H(S) = \sum_{i=1}^{q} P(s_i)\, ld\frac{1}{P(s_i)} = \frac{1}{2}\cdot ld2 + \frac{1}{4}ld4 + \frac{1}{8}ld8 + \frac{1}{8}ld8$$

$$H(S) = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{3}{8} = \frac{4+4+3+3}{8} = \frac{14}{8} = \frac{7}{4} = 1.75$$

$L(S) \doteq H(S)$ НУМЕРИЧКА ЕДНАКВОСТ

$L(S) = 1.75\ \frac{b}{симбол}$ $\quad H(S) = 1.75\ \frac{sh}{симбол}$

ЕФИКАСНОСТ И РЕДУНДАНСА НА СТАТИСТИЧКИТЕ КОДОВИ

$$\eta = \frac{H_m(S)}{L_m(x)} = \frac{H(S)}{ld(m)\, L_m(x)} \qquad \boxed{\eta \to 1} \qquad L_m(x) \doteq H_m(S)$$

ПРОШИРЕН ИЗВОР: $\hat{\eta} = \frac{H_m(S)}{H_m(S) + \frac{1}{n}} = \frac{H(S)}{H(S) + \frac{ld\,m}{n}}$

$$\boxed{q = m^n \qquad n = \log_m q}$$

$$\rho_{teor} = \frac{n}{L_m(x)_{min}} = \frac{\lceil \log_m q \rceil}{\sum_{i=1}^{q} P(s_i) \left\lceil \log_m \frac{1}{P(s_i)} \right\rceil} = \left| \begin{matrix} m=2 \\ \text{бинарно} \\ \text{кодирање} \end{matrix} \right| =$$

$$= \frac{\lceil ld\, q \rceil}{\sum_{i=1}^{q} P(s_i) \left\lceil ld \frac{1}{P(s_i)} \right\rceil}$$

и $\rho_{real} = \rho_r = \frac{\lceil \log_m q \rceil}{L_m(x)} = \frac{\lceil \log_m q \rceil}{\sum_{i=1}^{q} l_i P(s_i)}$ $\qquad \boxed{\rho_r < \rho_T}$

• Редунданса: $Red = 1 - \eta = 1 - \frac{H_m(S)}{L_m(x)} = \frac{L_m(x) - H_m(S)}{L_m(x)}$

ПРОЦЕНТУАЛНО КОЛИЧЕСТВО НА НЕКОРИСНА ИНФОРМАЦИЈА ШТО ЈА СОДРЖАТ КОДНИТЕ ЗБОРОВИ (СИМБОЛИ)

$$\bar{\nu}_j = \frac{\sum_{i=1}^{2} b(x_{ji})\, P(x_i)}{L_m(\lambda)} \qquad j = 1, 2, \dots, m$$

$b(x_{ji})$ – број на влезни знаци x_j во зборот x_i

$x_i = \{x_1, x_2, \dots, x_n\}$ – влезна листа

⑧ Преносен канал

• Статистички модел на канал без меморија.

$x = \{x_1, x_2, \dots, x_m\}$ —— $P(y_j/x_i)$ —— $y = \{y_1, y_2, \dots, y_r\}$

влезна листа на симболи (фазен простор)

излезна листа на симб. (фазен простор)

	y_1	y_2	$\dots$	y_r
x_1	$P(y_1/x_1)$	$P(y_2/x_1)$	$\dots$	$P(y_r/x_1)$
x_2	$P(y_1/x_2)$	$P(y_2/x_2)$	$\dots$	$P(y_r/x_2)$
x_m	$P(y_1/x_m)$	$P(y_2/x_m)$	$\dots$	$P(y_r/x_m)$

$$P(j) = \sum_{i=1}^{m} P(x_i) \cdot P(y_j/x_i) \qquad j = 1, 2, \dots, r$$

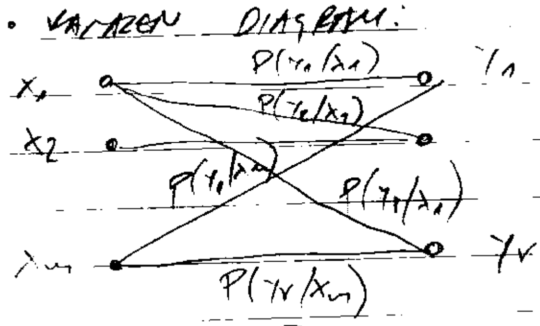
$$P_{ij} = P(y_j/x_i)$$

$$\sum_{j=1}^{r} P(y_j/x_i) = 1 \qquad i = 1, 2, \dots, m$$

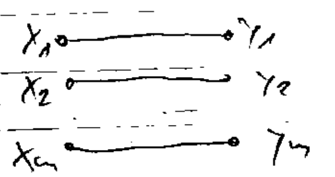
$$\Pi_C = \begin{bmatrix} P_{11} & P_{12} & \dots & P_{1r} \\ P_{21} & P_{22} & \dots & P_{2r} \\ \vdots & & & \\ P_{m1} & P_{m2} & \dots & P_{mr} \end{bmatrix} \qquad \Pi_x = \begin{bmatrix} P(x_1) \\ P(x_2) \\ \vdots \\ P(x_m) \end{bmatrix} \qquad \Pi_y = \begin{bmatrix} P(y_1) \\ P(y_2) \\ \vdots \\ P(y_r) \end{bmatrix}$$

$$\Pi_y = \Pi_C^T \cdot \Pi_x = \Pi_x^T \cdot \Pi_C$$

• Канален дијаграм:



$$\Pi_{C(ideal)} = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

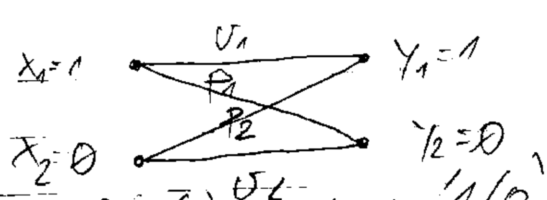


идеален бешумен канал

$$\Pi_y = \Pi_C' \cdot \Pi_x = \Pi_x$$

• Бинарен канал:

$$\Pi_{BC} = \begin{array}{c} \\ 1 \\ 0 \end{array}\begin{bmatrix} \upsilon_1 & p_1 \\ p_2 & \upsilon_2 \end{bmatrix}$$ (колони: 1, 0)



$\upsilon_1 = P(1/1)$ $\upsilon_2 = P(0/0)$ $p_1 = P(0/1)$ p_2: $1/0$

U_i – TRANSVEROJATNOST NA VEREN PRENOS

p_i – " – NA POGREŠEN – " –

SIMETRIČEN BINAREN KANAL: $\Pi_{BSC} = \begin{bmatrix} \sigma & p \\ p & \sigma \end{bmatrix}$

(8.2) TRANSINFORMACIJA

$X = \{x_1, x_2, \dots x_m\}$ $\quad Y \; \{y_1, y_2, \dots y_r\}$

- APRIORNA ENTROPIJA: $H(X) = \sum_{i=1}^{m} P(x_i)\, ld \frac{1}{P(x_i)}$
- APOSTERIORNA – " – : $H(X/y_j) = \sum_{i=1}^{m} P(x_i/y_j)\, ld \frac{1}{P(x_i/y_j)}$

EKVIVOKACIJA

$$H(X/Y) = \overline{H(X/y_j)} = \sum_{j=1}^{r} H(X/y_j) P(y_j) = \sum_{i=1}^{m}\sum_{j=1}^{r} P(x_i/y_j)\, ld \frac{1}{P(x_i/y_j)}$$

$$P(x_i; y_j) = P(y_j) P(x_i/y_j) = P(x_i) \cdot P(y_j/x_i)$$

$$\boxed{H(X/Y) = \sum_{i=1}^{m}\sum_{j=1}^{r} P(x_i; y_j)\, ld \frac{1}{P(x_i/y_j)}}$$

$$I(X;Y) = H(X) - H(X/Y) \quad \left[\frac{sh}{simb}\right] \qquad P(x_i; y_j)$$

TRANSINFORMACIJA:

$$I(X;Y) = \sum_{i=1}^{m} P(x_i)\, ld \frac{1}{P(x_i)} \sum_{j=1}^{r} P(y_j/x_i) - H(X/Y) = \sum_{i=1}^{m}\sum_{j=1}^{r} P(x_i) P(y_j/x_i)\, ld \frac{1}{P(x_i)} - H(X/Y)$$

$$I(X;Y) = \sum_{i=1}^{m}\sum_{j=1}^{r} P(x_i; y_j)\, ld \frac{1}{P(x_i)} - \sum_{i=1}^{m}\sum_{j=1}^{r} P(x_i, y_j)\, ld \frac{1}{P(x_i/y_j)}$$

$$I(X,Y) = \sum_{i=1}^{m}\sum_{j=1}^{r} P(x_i; y_j)\, ld \frac{P(x_i/y_j)}{P(x_i)} = \sum_{i=1}^{m}\sum_{j=1}^{r} P(x_i; y_j)\, ld \frac{P(x_i; y_j)}{P(x_i)P(y_j)}$$

TRANSINFORMACIJATA ZAVISI OD RASPREDELBATA NA VLEZNITE SIMBOLI I OD Π_C (KANALNATA MATRICA)

- OSOBINI NA $I(X;Y)$ $\quad$ → KANALOT E VO PREKIN

(1°) $I(X;Y) \geq 0 \qquad I(X;Y) = 0 \quad P(x_i; y_j) = P(x_i) P(y_j)$

(2°) $I(X;Y) = I(Y;X) \qquad \frac{P(x_i;y_j)}{P(x_i)} = P(y_j/x_i)$

$$I(Y;X) = \sum_{i=1}^{m}\sum_{j=1}^{r} P(x_i; y_j)\, ld \frac{P(x_i; y_j)}{P(x_i)P(y_j)} = \sum_{i=1}^{m}\sum_{j=1}^{r} P(x_i; y_j)\, ld \frac{P(y_j/x_i)}{P(y_j)}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{r} P(y_j) P(x_i/y_j) \cdot ld \frac{1}{P(y_j)} - \sum_{i=1}^{m}\sum_{j=1}^{r} P(x_i; y_j)\, ld \frac{1}{P(y_j/x_i)}$$

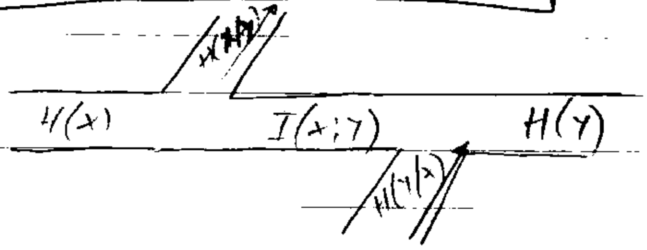
$$= \sum_{j=1}^{r} P(y_j)\, ld \frac{1}{P(y_j)} - H(Y/X) = H(Y) - H(Y/X)$$

$$I(Y;X) = H(Y) - H(Y/X) \quad \text{II}$$

$$I(X;Y) = H(X) - H(X/Y) \quad \text{I}$$

$H(Y/X)$ ИРЕЛЕВАНЦИЈА – ентропија на шумот на каналот

$I(X/Y)$ ТРАНСИНФОРМАЦИЈА



$$H(Y) = I(X;Y) + H(Y/X)$$
$$I(X;Y) = H(X) - H(X/Y)$$

- ЗДРУЖЕНА ЕНТРОПИЈА:

$$H(X;Y) = \sum_{i=1}^{m} \sum_{j=1}^{r} P(x_i, y_j) \operatorname{ld} \frac{1}{P(x_i, y_j)}$$

$$H(X;Y) = \sum_{i=1}^{m} \sum_{j=1}^{r} P(x_i, y_j) \operatorname{ld} \frac{P(x_i) P(y_j)}{P(x_i, y_j) P(x_i) P(y_j)} \quad \nearrow 1$$

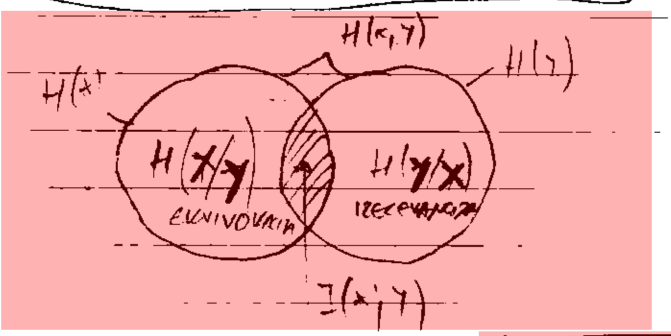
$$= \sum_{i=1}^{m} \sum_{j=1}^{r} P(x_i; y_j) \operatorname{ld} \frac{P(x_i) P(y_j)}{P(x_i, y_j)} + \sum_{i=1}^{m} \sum_{j=1}^{r} P(x_i) P(y_j/x_i) \operatorname{ld} \frac{1}{P(x_i)}$$

$$+ \sum_{i=1}^{m} \sum_{j=1}^{r} P(y_j) P(x_i/y_j) \operatorname{ld} \frac{1}{P(y_j)} = -I(X,Y) + H(X) + H(Y)$$

$$H(X;Y) = H(X) + H(Y) - I(X;Y) \qquad I(X;Y) = \sum\sum P(x_i y_j) \operatorname{ld} \frac{P(x_i y_j)}{P(x_i) P(y_j)}$$

I, II $\Rightarrow$

$$H(X;Y) = H(X) - H(Y/X) \quad ; \quad H(X;Y) = H(Y) - H(X/Y)$$



$$H(X,Y) = H(X) + H(Y) - I(X;Y)$$
$$I(X;Y) = H(Y) - H(Y/X)$$
$$I(X;Y) = H(X) - H(X/Y)$$

КАПАЦИТЕТ НА ДИСКРЕТЕН КАНАЛ

$$I(X;Y) = \sum\sum P(x_i, y_j) \operatorname{ld} \frac{P(x_i, y_j)}{P(x_i) P(y_j)} = \sum\sum P(x_i y_j) \operatorname{ld} \frac{P(y_j/x_i)}{P(y_j)}$$

$$P(y_j) = \sum_{i=1}^{m} P(x_i, y_j) = \sum_{i=1}^{m} P(x_i) P(y_j/x_i)$$

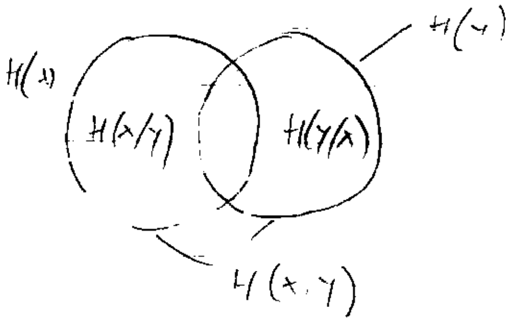
$$p(y) = \int_{-\infty}^{\infty} p(x,y)\, dx$$

$$I(X;Y) = \sum_{i=1}^{m} \sum_{j=1}^{r} P(x_i) P(y_j/x_i) \cdot ld \frac{P(y_j/x_i)}{\sum_{i=1}^{m} P(x_i) \cdot P(y_j/x_i)}$$

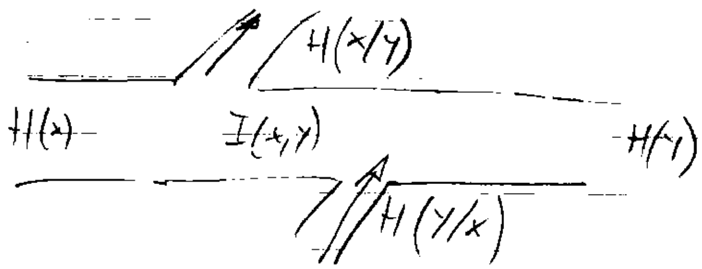
Max. VREDNOST NA $I(X;Y)$ ZAVISI SAMO OD $P(x_i)$ ZOŠTO $P(y_j/x_i)$ SE DEFINIRAAT OD KANALOT MATRICA E ŠUMOT NA KANALOT.

10PERIL 0.05, detomelasou, awilaciu | REPLEK
ANTIPODE

5den 3+2 casovi



$$H(X,Y) = H(Y) + H(X/Y)$$
$$H(X,Y) = H(X) + H(Y/X)$$



$$H(X) = I(X,Y) + H(X/Y)$$
$$H(Y) = I(X,Y) + H(Y/X)$$
$$I(X,Y) = H(X) - H(X/Y) \quad \text{(I)}$$
$$I(X,Y) = H(Y) - H(Y/X) \quad \text{(II)}$$

$$I(X,Y) = \sum_{i=1}^{m} \sum_{j=1}^{r} P(x_i, y_j)\, ld \frac{P(x_i, y_j)}{P(x_i) P(y_j)}$$

$$P(x_i, y_j) = P(x_i) \cdot P(y_j/x_i) \qquad \sum_{i=1}^{m} P(x_i) P(y_j/x_i) = P(y_j)$$

$$I(X,Y) = \sum_{i=1}^{m} \sum_{j=1}^{r} P(x_i) P(y_j/x_i)\, ld \frac{P(y_j/x_i)}{\sum_{i=1}^{m} P(x_i) P(y_j/x_i)}$$

$$C = v(X,Y) \max_{P(x_i)} \left[I(X;Y) \right]$$

• ISPIT $\tau = \frac{1}{2f_g}$ $\quad v_{max} = 2f_g = v(X,Y) = v_0$

$$\Phi(X,Y) = v(X,Y) \cdot I(X,Y) \quad \left[\frac{sh}{s}\right]$$

$$\eta_c = \frac{\Phi(X,Y)}{C}$$ KOEFICIENT NA ISKORISTENOST NA KANALOT !!

$$\eta_{CTEOR} = \frac{I(X,Y)}{\max_{P(x_i)} [I(X,Y)]} \qquad \eta_{CREAL} = \frac{v(X,Y) I(X,Y)}{v_0 \max_{P(x_i)} [I(X,Y)]}$$

$C \geq v(x) \cdot H(x) = \Phi(x)$ ИНФОРМАЦИОНЕН ФЛУКС НА ИЗЛОЗОТ

- ИДЕАЛЕН БЕЗШУМЕН КАНАЛ

$$I(X;Y) = H(X) - H(X/Y) = H(Y) - H(Y/X)$$

$$H(X/Y) = H(Y/X) = 0 \qquad I(X,Y) = H(X) = H(Y)$$

- БЕСКОРИСЕН КАНАЛ $(C=0)$

$$I(X,Y) = 0 \qquad H(X) = H(X/Y) \qquad H(Y) = H(Y/X)$$

$$I(X,Y) = \sum_{i=1}^{m} \sum_{j=1}^{r} P(x_i, y_j) \, ld \, \frac{P(y_j/x_i)}{P(y_j)}$$

$$P(y_j) = P(y_j/x_i) \quad \forall i,j$$

$$\Pi_Y = \Pi_C^T \cdot \Pi_X \qquad P_{ij} = P(y_j/x_i)$$

$$\begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1r} \\ P_{21} & P_{22} & \cdots & P_{2r} \\ \vdots & & & \\ P_{m1} & P_{m2} & \cdots & P_{mr} \end{bmatrix} \qquad \begin{bmatrix} 0,2 & 0,3 & 0,5 \\ 0,2 & 0,3 & 0,5 \\ 0,2 & 0,3 & 0,5 \end{bmatrix}$$

(ПЕРМУТАЦИЈА ПО РЕДИЦИ И ПО КОЛОНИ)

- СИМЕТРИЧЕН КАНАЛ

- [IN]

$$H(Y/x_i) = \sum_{j=1}^{r} P(y_j/x_i) \, ld \, \frac{1}{P(y_j/x_i)} = const \qquad i=1,2,\dots m \quad \forall i$$

$$H(Y/x_i) = const \quad \forall i \qquad H(Y/x_i) = \sum_{j=1}^{r} P_{ij} \, ld \, \frac{1}{P_{ij}}$$

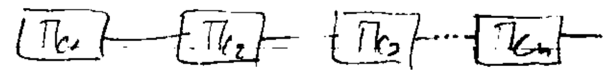
- [OUT] $P(x_i) = \frac{1}{m} \quad i=1,2,\dots m$

$$P(y_j) = \sum_{i=1}^{m} P(x_i) P(y_j/x_i) = \frac{1}{m} \underbrace{\sum_{i=1}^{m} P(y_j/x_i)}_{const \text{ за } \forall j} = \frac{1}{m} \cdot const$$

$j=1,2,\dots r$

$$\begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 1/6 & 1/2 & 1/3 \\ 1/3 & 1/6 & 1/2 \end{bmatrix} \qquad \begin{bmatrix} 1/3 & 1/3 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/3 & 1/3 \end{bmatrix}$$

• Каскада од преносни канали



$$\Pi_C = \Pi_{C_1} \cdot \Pi_{C_2} \cdots \Pi_{C_n}$$

• Пресметка на C кај симетрични канали:

$$c = v(x,y) \max_{P(x_i)} \left[I(x,y) \right] \qquad \boxed{I(x,y) = H(Y) - H(Y/X)}$$

$$\max_{P(x_i)} \left[I(x,y) \right] = ld(r) - \sum_{j=1}^{r} P(y_j/x_i)\, ld \frac{1}{P(y_j/x_i)}$$

$$\max \left[H(Y) \right] = \sum_{j=1}^{r} P(y_j)\, ld \frac{1}{P(y_j)} = \sum_{j=1}^{r} \frac{1}{r}\, ld(r) = r \cdot \frac{1}{r}\, ld(r) = ld(r)$$

$$H(Y/X) = \sum_{i=1}^{m} \sum_{j=1}^{r} P(x_i, y_j)\, ld \frac{1}{P(y_j/x_i)} =$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{r} P(x_i) P(y_j/x_i)\, ld \frac{1}{P(y_j/x_i)} = \sum_{i=1}^{m} P(x_i) \underbrace{\sum_{j=1}^{r} P(y_j/x_i)\, ld \frac{1}{P(y_j/x_i)}}_{const}$$

$$H(Y/X) = H(Y/x_i) = \sum_{j=1}^{r} P(y_j/x_i)\, ld \frac{1}{P(y_j/x_i)}$$

$$H(Y/x_i) = H(Y/X) = H(Y/x_i)$$

$$\boxed{\max_{P(x_i)} \left[I(x,y) \right] = ld(r) - \sum_{j=1}^{r} P(y_j/x_i)\, ld \frac{1}{P(y_j/x_i)}}$$

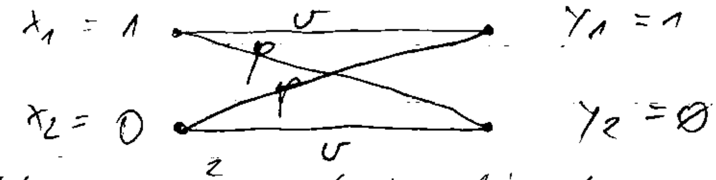
- Капацитет на сим. канал

$$C = v(x,y) \left[ld(r) - \sum_{j=1}^{r} P(y_j/x_i)\, ld \frac{1}{P(y_j/x_i)} \right]$$

(сума на ред на канал. матрица по било која редица)

• Бинарен симетричен канал

$C_{BSC} = ?$



$$C_{BSC} = v(x,y) \Big[ld 2 - \underbrace{\sum_{j=1}^{2} P(y_j/x_i)\, ld \frac{1}{P(y_j/x_i)}}_{\circledast} \Big] = v(x,y) \left[1 - H(p) \right]$$

$$\begin{matrix} & 1 & 2 \\ 1 & v & p \\ 2 & p & v \end{matrix} \qquad \circledast = P_{11}\, ld \frac{1}{P_{11}} + P_{12}\, ld \frac{1}{P_{21}} = v\, ld \frac{1}{v} + p\, ld \frac{1}{p}$$

$$\circledast = (1-p)\, ld \frac{1}{1-p} + p\, ld \frac{1}{p} = ld \frac{1}{1-p} + p \left(ld(1-p) + ld \frac{1}{p} \right)$$

$$\circledast = ld \frac{1}{1-p} + p\, ld \frac{1-p}{p} = H(p)$$ → ентрописка функција на бинарен извор

$$H(S) = \sum_{i=1}^{q} P(s_i)\, ld \frac{1}{P(s_i)} \qquad S = \{s_1, s_2, \dots, s_q\}$$

$$q = 2 \Rightarrow S = \{0, 1\} \qquad H(S) = P(s_1)\cdot ld\frac{1}{P(s_1)} + P(s_2)\, ld\frac{1}{P(s_2)}$$

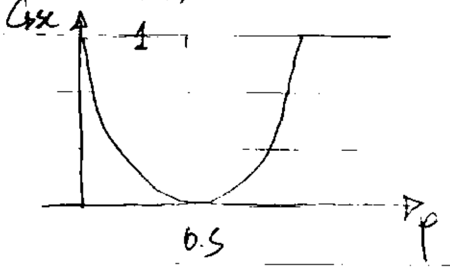
$$P(0) = p$$

$$P(1) = \sigma = 1 - p \qquad H(S) = p\, ld\frac{1}{p} + (1-p)\, ld\frac{1}{1-p}$$

$$C_{BSC} = U(x,y)\left[1 - H(p)\right] \qquad \Phi(x) = U(x)\cdot H(x)$$

$$(=) \frac{SIMB}{SEC}\cdot\frac{SH}{SIMB} = \frac{SH}{SEC}$$

IF: $U(x,y) = 1$



$$\Pi_{BSC} = \begin{bmatrix} p & \sigma \\ \sigma & p \end{bmatrix}$$

$p = 0.5 \Rightarrow \sigma = 1 - p = 0.5 \qquad C_{BSC} = 0$

KANALOT E VO PREKIN

$$\Pi_{BSC} = \begin{bmatrix} \sigma & p \\ p & \sigma \end{bmatrix} \qquad \Pi_{BSC}^{(2)} = \begin{bmatrix} \sigma & p \\ p & \sigma \end{bmatrix}\begin{bmatrix} \sigma & p \\ p & \sigma \end{bmatrix} = \begin{bmatrix} \sigma^2+p^2 ; 2\sigma p \\ 2\sigma p ; \sigma^2+p^2 \end{bmatrix}$$

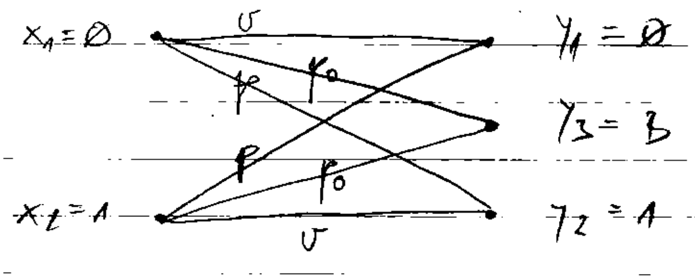
$$\Pi_{BSC}^{(3)} = \begin{bmatrix} \sigma^2+p^2 ; 2\sigma p \\ 2\sigma p ; \sigma^2+p^2 \end{bmatrix}\begin{bmatrix} \sigma & p \\ p & \sigma \end{bmatrix} = \begin{bmatrix} \sigma^3+p^2\sigma+2\sigma p^2 ; \sigma^2 p+p^3+2\sigma^2 p \\ 2\sigma^2 p+\sigma^2 p+p^3 ; \sigma^3+3\sigma p^2 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma^3+3\sigma p^2 ; p^3+3\sigma^2 p \\ p^3+3\sigma^2 p ; \sigma^3+3\sigma p^2 \end{bmatrix}$$

$$C_{BSC}^{(2)} = U(x,y)\left[1 - H(p_{ekv})\right] = U(x,y)\left[1 - H(2\sigma p)\right]$$

$$C_{BSC}^{(3)} = U(x,y)\left[1 - H(p^3+3\sigma^2 p)\right]$$

- BINAREN KANAL SO BRIŠENJE



$$\Pi_C = \begin{bmatrix} \sigma & p & p_0 \\ p & \sigma & p_0 \end{bmatrix} \begin{matrix} P(x_1) \\ P(x_2) \end{matrix}$$

$$y_1 = \sigma P(x_1) + p P(x_2)$$

$$y_2 = p P(x_1) + \sigma P(x_2)$$

$$y_3 = p_0 P(x_1) + p_0 P(x_2)$$

$$C = U(x,y)\left[\max_{P(x_i)}\left[H(y)\right] - \sum_{j=1}^{r} P(y_j/x_i)\, ld\frac{1}{P(y_j/x_i)}\right]$$

$H(y) = H(y)_{max}$ IF $P(x_1) = P(x_2) = 0.5$

$$P(y_1) = P(y_1/x_1)\cdot P(x_1) + P(y_1/x_2)P(x_2) = \frac{1}{2}(p+\sigma)$$

$$P(y_2) = P(y_2/x_1)P(x_1) + P(y_2/x_2)P(x_2) = \frac{1}{2}(p+\sigma)$$

$P(y_1) = P(y_2)$

$$P(y_1) + P(y_2) + P(y_3) = 1 \qquad P(y_3) = 1 - P(y_1) - P(y_2) = 1 - (\sigma + p) = p_0$$

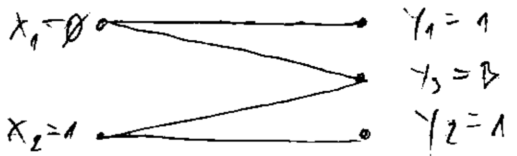
$$\max_{P(x_1)=0.5}[H(Y)] = \sum_{j=1}^{3} P(y_j)\, ld \frac{1}{P(y_j)} = 2\cdot\frac{1}{2}(v+p)\, ld \frac{2}{(v+p)} + \underbrace{(1-v+p)}_{p_0}\, ld \frac{1}{1-(v+p)}$$

$$\sum_{j=1}^{3} P(y_j/x_i)\, ld \frac{1}{P(y_j/x_i)} = v\, ld \frac{1}{v} + p\, ld \frac{1}{p} + p_0\, ld \frac{1}{p_0}$$

$$C = \sigma(x,y)\left[(v+p)\, ld \frac{2}{(v+p)} + p_0\, ld \frac{1}{p_0} - v\, ld \frac{1}{v} - p\, ld \frac{1}{p} - p_0\, ld \frac{1}{p_0}\right]$$

$$C = \sigma(x,y)\left[(v+p)\left[1 - ld(v+p)\right] + v\, ld\, v + p\, ld\, p\right]$$

- IDEALEN KANAL SO BRIŠENJE $(p = 0)$



$$C = \sigma(x,y)\left[v(1 - ld\, v) + v\, ld\, v\right]$$

$$C = \sigma(x,y)\left[(v+p)\left[1 - ld(v+p)\right] + v\, ld\, v - p\, ld \frac{1}{p}\right]$$

$$(*) = \lim_{p\to 0} p\, ld \frac{1}{p} \qquad \ln x = y \quad x = e^{y} \quad x = e^{\ln x}$$

$$(*) = -\lim_{p\to 0} p\, ld\, p = -\lim_{p\to 0} p \frac{\ln p}{\ln 2} = +\frac{1}{\ln 2} \underbrace{\lim_{p\to 0}(-p)\cdot \ln p}_{(1)}$$

$$(1) = \lim_{p\to 0} \frac{\ln p}{-\frac{1}{p}} = \lim_{z\to\infty} \frac{\ln \frac{1}{z}}{-z} = \left| \frac{+-\infty}{-\infty} = \frac{\infty}{\infty} \right| = \lim_{z\to\infty} \frac{\frac{1}{\frac{1}{z}}\cdot\left(-\frac{1}{z^2}\right)}{-1}$$

$$(1) = \lim_{z\to\infty} z\cdot\frac{1}{z^2} = \lim_{z\to\infty} \frac{1}{z} = 0 \qquad \boxed{(*) = \frac{1}{\ln(2)}\cdot(1) = 0}$$

$$C = \sigma(x,y)\left[v(1 - ld(v)) + v\, ld\, v\right] = \sigma(x,y)[v]$$

$$v + p_0 = 1 \qquad v = 1 - p_0 \qquad \boxed{C = \sigma(x,y)(1 - p_0)}$$

- DOVERLIVOST NA PRENESENITE PORAKI

$$d(y_j) = x_i \qquad j = 1, 2, \dots r$$

$$x_i \in \{x_1, x_2, \dots x_n\}$$

$$\boxed{\underbrace{P(x_i/y_j)}_{(*)} + \underbrace{P(E/y_j)}_{(**)} = 1}$$

(*) - VEROJATNOST NA TOČNO ODLUČUVANJE

(**) - VER. NA GREŠNO ODL.

VKUPNA VER. NA GREŠKA $P(E) = \overline{P(E/y_j)} = \sum_{j=1}^{r} P(E/y_j)P(y_j)$

$P(\varepsilon/y_j) = 1 - P(d(y_j)/y_j) \qquad d(y_j) = x_0 \quad \text{TŠ} \quad P(x_0/y_j) \geq P(x_i/y_j)\ \forall i$

$P(x_0/y_j) \geq P(x_i/y_j) \quad \forall i$

$P(x_0)\, P(y_j/x_0) \geq P(x_i)\, P(y_j/x_i) \quad \forall i$

$P(x_i) = \frac{1}{m} \quad i = 1 \ldots m \quad \Rightarrow \quad P(y_j/x_0) \geq P(y_j/x_i) \quad \forall i$

$P(\varepsilon) = \sum_{j=1}^{r} P(\varepsilon/y_j)\, P(y_j) = \sum_{j=1}^{r} \left[1 - P(x_0/y_j)\right] P(y_j) =$

$$\boxed{P(\varepsilon) = 1 - \sum_{j=1}^{r} P(x_0, y_j)}$$

$\sum_{X} \sum_{Y} P(x_i, y_j) = 1 \qquad \sum_{X - x_0} \sum_{Y} P(x_i, y_j) + \sum_{j=1}^{r} P(x_0, y_j) = 1$

$$\boxed{P(\varepsilon) = \sum_{X - x_0} \sum_{Y} P(x_i, y_j)} = \sum_{X - x_0} \sum_{Y} P(x_i)\, P(y_j/x_i)$$

$P(\varepsilon) = \sum_{\substack{k=1 \\ x_i \neq x_{0j}}}^{m} \sum_{j=1}^{r} \underbrace{P(x_i)\, P(y_j/x_i)}$ site minimalni proizvodi po redica

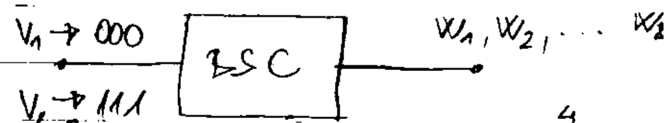
$P(\omega) = 1 - P(\varepsilon) = 1 - \sum_{X - x_0} \sum_{Y} P(x_i, y_j) = \sum_{Y} P(x_0, y_j)$

$P(\omega) = \sum_{j=1}^{r} \underbrace{P(x_0)\, P(y_j/x_0)}$

maksimalni proizvodi po kolona

ZAŠTITNO KODIRANJE

$P_\varepsilon(n, e) = C_n^e\, p^e\, \omega^{n-e}$



w_1	000	V_1
w_2	001	V_1
w_3	010	V_1
w_4	100	V_1
w_5	011	V_2
w_6	101	V_2
w_7	110	V_2
w_8	111	V_2

$P(\varepsilon) = P(V_1) \sum_{i=5}^{8} P(w_i/V_1) + P(V_2) \cdot \sum_{i=1}^{4} P(w_i/V_2)$

	w_1	w_2	$\ldots$	w_8
V_1	$P(w_1/V_1)$	$P(w_2/V_1)$	$\ldots$	$P(w_8/V_1)$
V_2	$P(w_1/V_2)$	$P(w_2/V_2)$	$\ldots$	$P(w_8/V_2)$

$P(\varepsilon) = P(3,2) + P(3,3)$

$P(\varepsilon) = C_3^2\, p^2 \omega^1 + C_3^3\, p^3 \omega^0 = \frac{3!}{1! \cdot 2!} p^2 \omega + \frac{3!}{0!\,3!} p^3 = 3p^2\omega + p^3$

$P(\varepsilon) = 3 \cdot (0.1)^2 \cdot 0.9 + (0.1)^3 = 2.7 \cdot 0.01 + 0.001 = 0.027 + 0.001 = 0.028$

$\boxed{n=5} \quad P(\varepsilon) = C_n^e\, p^e\, \omega^{n-e} = P(5,3) + P(5,4) + P(5,5) = C_5^3 p^3 \omega^2 + C_5^4 p^4 \omega^1$

162 $+ C_5^5\, p^5 \omega^0 = \frac{5!}{2!\,3!} p^3 \omega^2 + \frac{5!}{1!\,4!} p^4 \omega^1 + p^5 = 10 p^3 \omega^2 + 5 p^4 \omega + p^5$

$$P(e) = \sum_{e=\lceil \frac{n}{2} \rceil}^{n} P_E(n,e) = \sum_{e=\lceil \frac{n}{2} \rceil}^{n} C_n^e \, p^e q^{n-e}$$

• Брзина на пренесување на пораките

$R = U(x,y) = \frac{ld\, M}{n} \left[\frac{sh}{s}\right]$ n - број на бити по кодирана порака

$\Phi = U(x,y) \cdot H(x) (=) \frac{порака}{sec} \cdot \frac{sh}{порака} (=) \frac{sh}{sec}$

$H(x) = \sum_{i=1}^{n} P(x_i)\, ld \frac{1}{P(x_i)}$ → $I(x)$

$R (=) \frac{\text{бинарен број на пораки}}{sec}$

II Шенонова теорема: $R \leq C$

• Хемингово растојание:

V_i, V_j $D(V_i, V_j) = D(i,j)$ - Хемингово растојан.

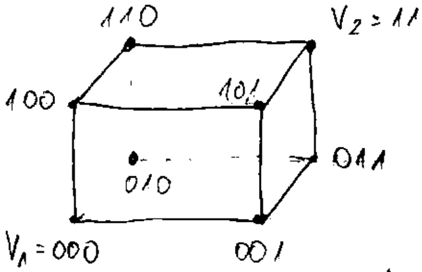
• Својства:

1° $D(i,j) \geq 0$ 2° $D(i,j) = D(j,i)$

3° $D(i,k) + D(j,k) \geq D(i,j)$ правило на триаголник

$D(v_i, w_j)$ v_i - влезен бинарен код. збор
w_j - излезен -||- -||-

$\min[D(v_i, w_j)]$ → правило на одлучување



$W_j = 100$

$\underbrace{D(V_1, W_j)}_{V_1} = 1 \leq D(V_2, W_j) = 2$

• Општ случај: $d(w_j) = V_0$ ако $D(w_j, V_0) \leq D(w_j, V_i)$ $\forall i$

$\underbrace{D(V_i, V_j)}$ $\geq e_d + 1$ e_d → број на погрешни бити што може да ги детектира кодот

минимално Хемингово растојание на влезните кодирани пораки

$D(V_i, V_j) \geq 2e_c + 1$ e_c → број на погрешно пренесени бити кои кодот може да ги корегира

$D(V_i, V_j) \geq 1 + 2e_c + (e_d - e_c) = 1 + e_d + e_c$ $e_d \geq e_c$

• ZAŠTITNO KODIRANJE SO KONTROLA NA PARNOST

n - informaciski biti $\quad \boxed{n+k=N}$

k - ZAŠTITNI BITI

- KOD ZA DETEKCIJA NA EDNA GREŠKA

$ed=1 \qquad D(V_i, V_j)=2$ } min Hemingovo rastojanie pomeđu vektorite točki

$n+1=N$

0 0 1 0 1 0 0 1	1
0 0 0 1 1 0 1 1	0

$\underbrace{}_{n} \quad \underbrace{}_{k}$

• KOD ZA KOREKCIJA NA EDNA GREŠKA

$$k \geq \lceil ld(k+n+1) \rceil$$

$$V_i = (x_1 x_2 x_3, \dots x_j, \dots x_N) \qquad \frac{i=1,2,\dots M}{x_i \in \{0,1\}}$$

$$\sum_{i=1}^{n} a_i 2^{i-1} = a_n \cdot 2^{n-1} + \dots + a_i 2^{i-1} + \dots + a_2 2^1 + a_1 2^0$$

$n = ld(M)$ $\quad$ HAMINGOV KOD SO K KOREKCIONI BITI

$V_{11} = 1011 \qquad ld\,M = ld\,12 = 4 = n$ broj na informac. biti

$k = \lceil ld(k+n+1) \rceil \qquad k=3 \quad ld(8)=3$

$x_1\ x_2\ x_4$

$V_{11} = |x_1|x_2|1\ x_4|0\ 1\ 1|$
$\quad 1\ 2\ 3\ 4\ 5\ 6\ 7$

"x x 1" 1 3 5 7
"x 1 x⁴" 2 3 6 7
"1 x x⁴" 4 5 6 7

$S_1 = x_1 \oplus 1 \oplus 0 \oplus 1 = 0 \qquad x_1 = 0$

$S_2 = x_2 \oplus 1 \oplus 1 \oplus 1 = 0 \qquad x_2 = 1$

$S_3 = x_4 \oplus 0 \oplus 1 \oplus 1 = 0 \qquad x_4 = 0$

$V_{11} = 0110011$

ex.: ERROR AT BIT (5) $\Rightarrow V_{11} = 0110111$ (1 2 3 4 5 6 7)

$S_1 = 0 \oplus 1 \oplus 1 \oplus 1 = 1$

$S_2 = 1 \oplus 1 \oplus 1 \oplus 1 = 0 \qquad N_k = S_1 S_2 S_3 = 101 \qquad$ 5-ot bit e grešen!!

$S_3 = 0 \oplus 1 \oplus 1 \oplus 1 = 1$

DETEKCIJA (IZDVOJUVANJE) NA PERIODIČNI SIGNALI

• AVTOKORELACIONA POSTAPKA

$f(t) \rightarrow$ [KORELATOR] $\rightarrow R_f(\tau)$

$$R_f(\tau) = \int_{-\infty}^{\infty} f(t) f(t+\tau)\, dt$$

$f(t) = s(t) + n(t) \qquad R_f(\tau) = \overline{f(t) \cdot f(t+\tau)} = \overline{(s(t)+n(t))(s(t+\tau)+n(t+\tau))}$

$$R_{11}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} f_1(t)\, f_1(t+\tau)\, dt \qquad f_1(t) = \sum_{-\infty}^{\infty} F_1(jn\omega_0)\, e^{jn\omega_0 t}$$

$$R_{11}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} f_1(t) \cdot \left[\sum_{n=-\infty}^{\infty} F_1(jn\omega_0)\, e^{jn\omega_0\tau}\, e^{jn\omega_0 t}\right] dt =$$

$$= \sum_{n=-\infty}^{\infty} F_1(jn\omega_0)\, e^{jn\omega_0\tau} \underbrace{\left[\frac{1}{T}\int_{-T/2}^{T/2} f_1(t)\, e^{jn\omega_0 t}\, dt\right]}_{F_1^*(jn\omega_0)} = \sum_{n=-\infty}^{\infty} |F_1(jn\omega_0)|^2 e^{jn\omega_0\tau}$$

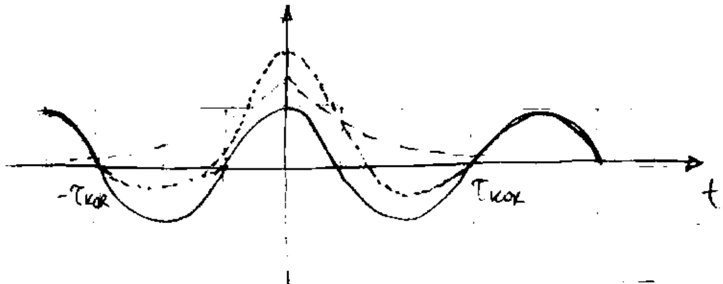
$$R_{11}(\tau) = \sum_{n=-\infty}^{\infty} |F_1(jn\omega_0)|^2 e^{jn\omega_0\tau}$$

FUR. TRANS. PAR

$$R_f(\tau) = \overline{(s(t)+n(t))(s(t+\tau)+n(t+\tau))} = \overline{s(t)\cdot s(t+\tau)} + \overline{s(t)\cdot n(t+\tau)} +$$
$$+ \overline{n(t)\, s(t+\tau)} + \overline{n(t)\cdot n(t+\tau)} = R_s(\tau) + R_n(\tau) + R_{sn}(\tau) + R_{ns}(\tau)$$

$R_{sn}(\tau) = R_{ns}(\tau) = 0$ STATISTIČKI NEOVISNI $\overline{n(t)} = 0$

$R_f(\tau) = R_s(\tau) + R_n(\tau)$ $\qquad R_n(\tau > \tau_{kor}) = 0$

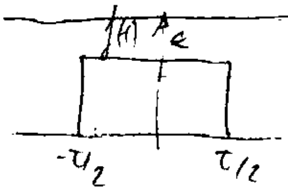


$s(t) = A\cos(\omega t)$ $\qquad S(jn\omega_0) = \frac{1}{T}\int_{-T/2}^{T/2} A\cos(\omega t)\cdot e^{-jn\omega_0 t}\, dt$

$$S(jn\omega_0) = \frac{A}{2}\delta(\omega-\omega_0) + \frac{A}{2}\delta(\omega+\omega_0)$$

$$S(jn\omega_0) = \frac{1}{T}\int_{-T/2}^{T/2} \frac{A}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) e^{-jn\omega_0 t}\, dt = \frac{A}{2}\left[\frac{1}{T}\int_{-T/2}^{T/2} e^{j(\omega-n\omega_0)t}\, dt + \frac{1}{T}\int_{-T/2}^{T/2} e^{-j(\omega+n\omega_0)t}\, dt\right]$$

$$= \frac{A}{2}\delta(\omega - n\omega_0) + \frac{A}{2}\delta(\omega + n\omega_0)$$

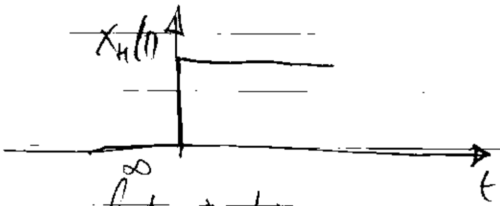


$$\mathcal{F}\{f(t)\} = \int_{\tau/2} E\, e^{j\omega t}\, dt = E\tau \frac{\sin \omega\tau/2}{\omega\tau/2}$$

$\tau \to 0$ $\qquad \lim_{\tau\to 0} E\tau \frac{\sin \omega\tau/2}{\omega\tau/2} = 1 = \Delta(j\omega)$

$E = \frac{1}{\tau}$ $\qquad \Delta(j\omega) = \mathcal{F}\{\delta(t)\}$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} 1\, e^{-j\omega t}\, d\omega$$



$$x_H(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

$$x_H(0) = \frac{1}{2}\left\{x_H(0_-) + x_H(0_+)\right\} = \frac{1}{2}$$

$$\int_{-\infty}^{\infty} |x_H|\, dt \to \infty$$ NE E ISPOLNET USLOVOT DIRIHLE

$(e^{-\infty} \to 0 - e^{0} \to 1)$

$$X_H(j\omega) = \frac{1}{j\omega} + \mathcal{F}\left\{\frac{1}{2}\right\}$$

$$\int_0^{\infty} e^{-j\omega t} dt = \frac{1}{j\omega}\int_0^{\infty} e^{-j\omega t} dj\omega t = \frac{1}{j\omega}\left(e^{-j\omega t}\Big|_0^{\infty}\right) = \frac{1}{j\omega}$$

$$\mathcal{F}\{K\} = \int_{-\infty}^{\infty} K e^{-j\omega t} dt = K \int_{-\infty}^{\infty} e^{-j\omega t} dt = 2\pi K \delta(\omega)$$

($\int_{-\infty}^{\infty} e^{-j\omega t} dt = 2\pi\delta(t)$)

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t} d\omega = \left| \begin{matrix} \omega = t \\ t = \omega \end{matrix} \right| = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t} dt$$

$$X_H(j\omega) = \frac{1}{j\omega} + \mathcal{F}\left\{\frac{1}{2}\right\} = \frac{1}{j\omega} + 2\pi \frac{1}{2}\delta(\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

$$\boxed{\mathcal{F}\{1\} = 2\pi\delta(\omega)}$$

- FUR. TRANSF. OD $\cos(\omega_0 t)$, $\sin(\omega_0 t)$

$$\mathcal{F}\{e^{j\omega_0 t}\} = \mathcal{F}\{1 \cdot e^{j\omega_0 t}\} = \left| \mathcal{F}\{1\} = 2\pi\delta(\omega) \right|$$

$$= 2\pi\delta(\omega - \omega_0)$$

$$\boxed{\mathcal{F}\{\cos(\omega_0 t)\} = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)}$$

$$\mathcal{F}\{\sin(\omega_0 t)\} = \mathcal{F}\left\{\frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)\right\}$$

$$= -j\pi\delta(\omega - \omega_0) + j\pi\delta(\omega + \omega_0)$$

$$s(t) = A\cos(\omega_0 t) \qquad R_s(\tau) = \int_{-\infty}^{\infty} s(t) \cdot s(t+\tau)\, dt$$

$$R_s(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |S(j\omega)|^2 e^{j\omega\tau} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} A^2\pi^2 \left(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right)^2 e^{j\omega\tau} d\omega$$

$$= \frac{A^2\pi^2}{2\pi}\left[\int_{-\infty}^{\infty} \delta^2(\omega-\omega_0) e^{j\omega\tau} d\omega + \int_{-\infty}^{\infty} 2\delta(\omega-\omega_0)\delta(\omega+\omega_0) e^{j\omega\tau} d\omega + \int_{-\infty}^{\infty} \delta^2(\omega+\omega_0) e^{j\omega\tau} d\omega\right]$$

($\to e^{j\omega_0\tau}$, $\to 0$, $\to e^{-j\omega_0\tau}$)

$$R_s(\tau) = \frac{A^2\pi^2}{2T}\left[e^{j\omega_0\tau} + e^{-j\omega_0\tau}\right] = A^2\pi \cos(\omega_0\tau)$$

$f(t) = A\cos(\omega_0 t + \varphi)$ $R_f(\tau) = ?$

$$R_f(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t)\cdot f(t+\tau)\,dt \qquad \boxed{\omega_0 = \frac{2\pi}{T_0} \quad T_0 \neq T}$$

$$\lim_{T\to\infty} \frac{A^2}{2T}\int_{-T}^{T} \underbrace{\cos(\omega_0 t+\varphi)}_{\beta}\cdot \cos(\omega_0 t + \underbrace{\omega_0\tau}_{\tau} + \varphi)\,dt = \circledast$$

$$\left.\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\\ \cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta\end{aligned}\right\} \quad \cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$$

$$\circledast = \lim_{T\to\infty}\frac{A^2}{2T}\left[\int_{-T}^{T}\frac{1}{2}\cos(\omega_0\tau)\,dt + \int_{-T}^{T}\frac{1}{2}\cos(2\omega_0 t + 2\omega_0\tau + \varphi)\,dt\right]$$

$$= \lim_{T\to\infty}\frac{A^2}{2T}\cdot\frac{1}{2}(\cos\omega_0\tau)\cdot 2T + \underbrace{\lim_{T\to\infty}\frac{A^2}{2T}\int_{-T}^{T}\frac{1}{2}\cos(2\omega_0 t)\cos(2\omega_0\tau+\varphi)\,dt}_{(*)}$$

$$- \underbrace{\lim_{T\to\infty}\frac{A^2}{2T}\int_{-T}^{T}\frac{1}{2}\sin(2\omega_0 t)\sin(2\omega_0\tau+\varphi)\,dt}_{(**)}$$

$$(*) = \lim_{T\to\infty}\frac{A^2}{2T}\cdot\frac{1}{2}\cos(2\omega_0\tau+\varphi)\int_{-T}^{T}\cos(2\omega_0 t)\,dt =$$

$$= \frac{A^2}{4T}\cdot\cos(2\omega_0\tau+\varphi)\lim_{T\to\infty}\left[\frac{\sin(2\omega_0 t)}{2\omega_0}\Bigg|_{-T}^{T}\right] =$$

$$= \frac{A^2}{4T}\cos(2\omega_0\tau+\varphi)\lim_{T\to\infty}\frac{2\sin(2\omega_0 T)}{2\omega_0}$$

$$= \frac{A^2}{2}\cos(2\omega_0\tau+\varphi)\lim_{T\to\infty}\frac{\sin(2\omega_0 T)}{2\omega_0 T}$$

$$\boxed{\lim_{x\to\infty}\frac{\sin x}{x} = 0} \qquad (**) = K\cdot\int_{-T}^{T}\sin(2\omega_0 t)\,dt = 0 \quad \text{NEPARNA FUNKCIJA}$$

$$\boxed{R_f(\tau) = \frac{A^2}{2}\cos(\omega_0\tau)}$$

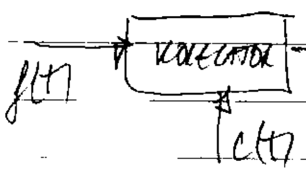
$$\Phi_f(\omega) = \int_{-\infty}^{\infty} R_f(\tau)e^{-j\omega\tau}\,d\tau = \int_{-\infty}^{\infty}\frac{A^2}{2}\frac{1}{2}\left(e^{j\omega_0\tau} + e^{-j\omega_0\tau}\right)e^{-j\omega\tau}\,d\tau$$

$$= \frac{A^2}{4}\left[2\pi\delta(\omega-\omega_0) + 2\pi\delta(\omega+\omega_0)\right] = \frac{A^2\pi}{2}\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$$

$$s(t) = A\cos(\omega_0 t) \rightarrow R_s(\tau) = \frac{A^2}{2}\cos(\omega_0 \tau)$$

НА ИЗЛЕЗ ОД КОРЕЛАТОРОТ НЕМА ШУМ Т.Е ТОЈ Е ЕЛИМИНИРАН !!

- Меѓукорелациона постапка



$$\boxed{f(t) = s(t) + n(t)}$$

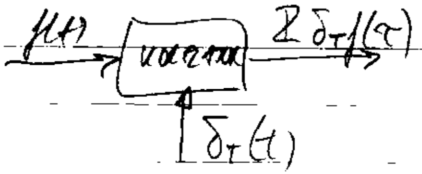
$$R_{fc}(\tau) = \overline{[s(t) + n(t)]\, c(t+\tau)} = R_{sc}(\tau) + R_{nc}(\tau)$$

($R_{nc}(\tau) \rightarrow 0$)

$$c(t) = A_c \cos(\omega_0 t + \varphi_c)$$
$$s(t) = A \cos(\omega_0 t + \varphi_s)$$

$$R_{sc} = \frac{A \cdot A_c}{2}\cos(\omega_0 \tau + \varphi) \qquad \boxed{\varphi = \varphi_c - \varphi_s}$$

- Меѓукорелац. постапка каде помошниот сиг. е $\delta(t)$



$$f(t) = n(t) + s(t)$$

$$R_{\delta_T f}(\tau) = \overline{\delta_T(t) \cdot \left(n(t+\tau) + s(t+\tau)\right)} =$$

$$= R_{\delta_T s}(\tau) + R_{\delta_T n}(\tau) = \frac{1}{T} s(\tau)$$

($R_{\delta_T n}(\tau) \rightarrow 0$)

$$R_{\delta_T s}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \delta_T(t)\, A \cos(\omega_0 t + \omega_0 \tau)\, dt$$

$$s(t) = \sum_{n=-\infty}^{\infty} S(jn\omega)\, e^{jn\omega t} = \sum_{n=-\infty}^{\infty} e^{jn\omega t} \frac{1}{T} \int_{-T/2}^{T/2} s(\tau)\, e^{-jn\omega \tau} d\tau =$$

$$= \int_{-T/2}^{T/2} s(\tau) \left[\sum_{n=-\infty}^{\infty} \frac{1}{T} e^{+jn\omega(t-\tau)}\right] d\tau = \int_{-T/2}^{T/2} s(\tau) \cdot \delta(t-\tau)\, d\tau$$

$$\frac{1}{T} s(t) = \frac{1}{T} \int_{-T/2}^{T/2} s(\tau)\, \delta(t-\tau)\, d\tau = \rho_{\delta_T s}(\tau) = \rho_{\delta_T s}(\tau)$$

СТАТИСТИЧКА ТЕОРИЈА НА ОДЛУЧУВАЊЕ

- БАЈЕСОВ КРИТЕРИУМ ЗА ОДЛУЧУВАЊЕ

$H_0, H_1 \rightarrow$ ДВЕ ХИПОТЕЗИ

1.) $P(H_0)$, $P(H_1)$

2.) $p_0(v)$, $p_1(v)$

3.) МАТРИЦА НА ЦЕНЕЊЕ

$$K = \begin{bmatrix} K_{0,0} & K_{0,1} \\ K_{1,0} & K_{1,1} \end{bmatrix}$$

$K_{0,0}$ $K_{1,1}$ НАГРАДА ЗА ТОЧНА ОДЛУКА

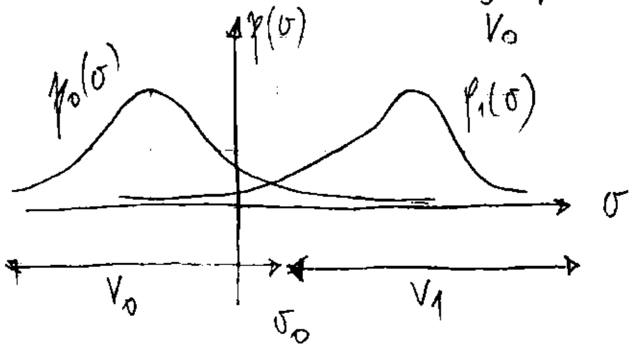
$K_{0,1}$ $K_{1,0}$ КАЗНА

- ВЕРОЈАТНОСТ НА ПОГРЕШНА ОДЛУКА Т.Е. КУМУЛАТИВНА ВЕР. НА ГРЕШКА

$$P(H_1/H_0) = Q_{1,0} = \int_{V_1} p_0(v)\,dv = \int_{v_0}^{\infty} p_0(v)\,dv$$

ДЕТЕКТИРАНО Е H_1
ПРАТЕНО Е H_0

$$P(H_0/H_1) = Q_{0,1} = \int_{V_0} p_1(v)\,dv = \int_{-\infty}^{v_0} p_1(v)\,dv$$



$$Q_{0,0} = \int_{-\infty}^{\infty} p_0(v)\,dv = \int_{V_0} p_0(v)\,dv$$

$$Q_{1,1} = \int_{v_0}^{\infty} p_1(v)\,dv = \int_{V_1} p_1(v)\,dv$$

$$P(e) = P(H_0)\,P(H_1/H_0) + P(H_1)\,P(H_0/H_1)$$

- ВКУПЕН СРЕДЕН РИЗИК:

$$\bar{K}_0 = P(H_0)\left(K_{1,0}Q_{1,0} + K_{0,0}Q_{0,0}\right)$$

$$\bar{K}_1 = P(H_1)\left(K_{0,1}Q_{0,1} + K_{1,1}Q_{1,1}\right)$$

СРЕДЕН РИЗИК ПРИ РЕАЛИЗАЦИЈА НА H_0
СРЕДЕН РИЗИК ПРИ РЕАЛИЗАЦИЈА НА H_1

$$\bar{K} = \bar{K}_0 + \bar{K}_1 = \bar{K}(v_0) = P(H_0)\left(K_{1,0}\int_{v_0}^{\infty} p_0(v)\,dv + K_{0,0}\int_{-\infty}^{v_0} p_0(v)\,dv\right) +$$

$$P(H_1)\left(K_{0,1}\int_{-\infty}^{v_0} p_1(v)\,dv + K_{1,1}\int_{v_0}^{\infty} p_1(v)\,dv\right)$$

$$\boxed{\frac{\partial K}{\partial v_0} = 0 \rightarrow v_0}$$

$$\frac{\partial K}{\partial v_0} = P(H_0)\left(-K_{1,0}\,p_0(v_0) + K_{0,0}\,p_0(v_0)\right) + P(H_1)\left(K_{0,1}\,p_1(v_0) - K_{1,1}\,p_1(v_0)\right)$$

$$\int_{v_0}^{\infty} p_0(v)\,dv + \int_{-\infty}^{v_0} p_0(v)\,dv = 1 \qquad \int_{v_0}^{\infty} p_0(v)\,dv = 1 - \int_{-\infty}^{v_0} p_0(v)\,dv$$

$$\frac{\partial}{\partial v}\left[1 - \int_{-\infty}^{v_0} p_0(v)\,dv\right] = -p_0(v)$$

$$P(H_0)\left[-K_{1,0}\,\varphi_0(v_0) + K_{0,0}\,\varphi_0(v_0)\right] + P(H_1)\left[K_{0,1}\,\varphi_1(v_0) - K_{1,1}\,\varphi_1(v_0)\right] = 0$$

$$P(H_0)\left[K_{0,0} - K_{1,0}\right]\varphi_0(v_0) = P(H_1)\left[K_{1,1} - K_{0,1}\right]\varphi_1(v_0)$$

$$\boxed{\frac{\varphi_1(v_0)}{\varphi_0(v_0)} = \frac{P(H_0)}{P(H_1)}\cdot\frac{K_{0,0}-K_{1,0}}{K_{1,1}-K_{0,1}} = \lambda_0}$$

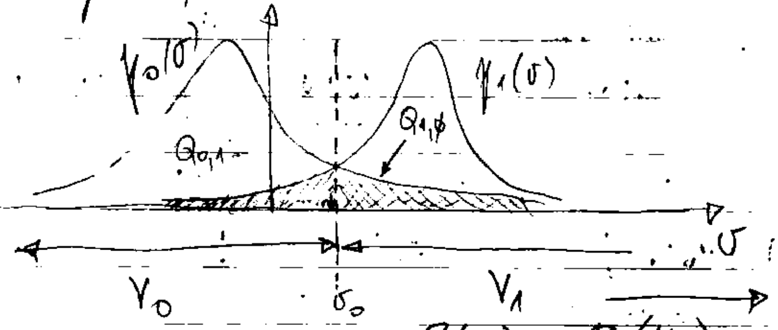
$v_0 = v_{0opt}$ — λ_0 - КОЕФИЦИЕНТ НА ВЕРОЈАТНОСТ

$\overline{K}(v_0) = \overline{K}(v)_{min}$

$$\frac{\varphi_1(v)}{\varphi_0(v)} \gtrless \lambda_0 \qquad \begin{cases} \dfrac{\varphi_1(v)}{\varphi_0(v)} > \lambda_0 \Rightarrow \text{ОДЛУЧИ } H_1,\ v \in V_1 \\[2ex] \dfrac{\varphi_1(v)}{\varphi_0(v)} < \lambda_0 \Rightarrow \text{ОДЛУЧИ } H_0,\ v \in V_0 \end{cases}$$

- Ако $K_{0,0} = K_{1,1}$, $K_{0,1} = K_{1,0}$: $\quad \dfrac{\varphi_1(v)}{\varphi_0(v)} \gtrless \dfrac{P(H_0)}{P(H_1)} = \lambda_0$
- Ако хипотезите се еднакво веројатни т.е. $P(H_0) = P(H_1)$

$$\frac{\varphi_1(v)}{\varphi_0(v)} \gtrless 1 \qquad \boxed{\varphi_1(v) \gtrless \varphi_0(v)} \qquad \begin{matrix} > H_1 \\ < H_0 \end{matrix}$$



→ ПРАГОТ ОДИ КОН ПОМАЛКУ ВЕРОЈАТНАТА ХИПОТЕЗА

$P(H_0) > P(H_1)$

- МИНИМАКС КРИТЕРИУМ ЗА ОДЛУЧУВАЊЕ (КОГА НЕ СЕ ПОЗНАТИ $P(H_0)$, $P(H_1)$)

$P(H_0) = \xi \qquad P(H_1) = 1 - \xi \qquad 0 \le \xi \le 1$

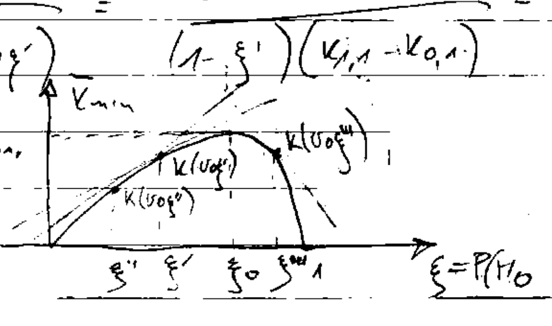
$$\overline{K} = \xi\left(K_{0,0}Q_{0,0} + K_{1,0}Q_{1,0}\right) + (1-\xi)\left(K_{1,1}Q_{1,1} + K_{0,1}Q_{0,1}\right) =$$

$$= \xi\underbrace{\left[\left(K_{0,0}Q_{0,0} + K_{1,0}Q_{1,0}\right) - \left(K_{1,1}Q_{1,1} + K_{0,1}Q_{0,1}\right)\right]}_{m} + \underbrace{\left(K_{1,1}Q_{1,1} + K_{0,1}Q_{0,1}\right)}_{C}$$

$\boxed{\overline{K} = m\cdot\xi + C}$ ВКУПЕН СРЕДЕН РИЗИК

$$\frac{\varphi_1(v_{0\xi'})}{\varphi_0(v_{0\xi'})} = \frac{\xi'(K_{0,0}-K_{1,0})}{(1-\xi')(K_{1,1}-K_{0,1})} = \lambda_0 \Rightarrow v_{0\xi'} = \dots$$

$\overline{K}(v_0) = \overline{K}_{min/max}$



МИНИМАКС КРИТЕРИУМ ЗА РАЗЛИЧНИ „ξ"

$\overline{K}_{min/max} = C \quad (m = 0)$

$m=0 \Rightarrow K_{0,0}\int_{-\infty}^{v_0} p_0(v)\,dv + K_{1,0}\int_{v_0}^{\infty} p_0(v)\,dv = K_{1,1}\int_{v_0}^{\infty} p_1(v)\,dv + K_{0,1}\int_{-\infty}^{v_0} p_1(v)\,dv$

$$\frac{p_1(v)}{p_0(v)} \gtrless \frac{P(H_0)_m\,(K_{0,0}-K_{1,0})}{[1-P(H_0)_m]\,[K_{1,1}-K_{0,1}]}$$

$P(H_0)_m \Rightarrow$ ~~ANALOGNA~~ VEROJATNOST ŠTO GO MAXIMIZIRA IZRAZOT ~~KRITERIUM~~

COOPERATIVE COMMUNICATIONS

$x_1(t) = [a_{11} b_1^{(1)} c_1(t),\ a_{12} b_1^{(2)} c_1(t),\ a_{13} b_1^{(2)} c_1(t) + a_{14} b_2^{(2)} c_2(t)]$

$x_2(t) = [a_{21} b_2^{(1)} c_2(t),\ a_{22} b_2^{(2)} c_2(t),\ a_{23} b_1^{(2)} c_1(t) + a_{24} b_2^{(2)} c_2(t)]$

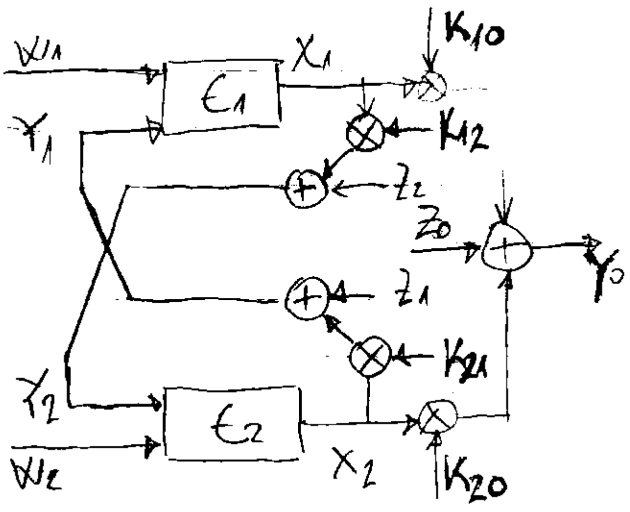
DVD 1/4 + com jsac, ucl...
DVD 2/4 + com

PLOMOL PENŠKI 13
3124340
3125105 } CAKAN

CONVOLUTIONAL CODING (MATLAB)

2.032, 2.035, ~~2.047~~ 2.081, 2.082, 2.088, 2.132, 2.134, 2.143

USER COOPERATION DIVERSITY | PART 1



FADING COEFICIENTS

- W_i $i=1,2$ – user's information
  $Y_0(t) = K_{10}X_1(t) + K_{20}X_2(t) + Z_0(t)$
  $Y_1(t) = K_{21}X_2(t) + Z_1(t)$
  $Y_2(t) = K_{12}X_1(t) + Z_2(t)$
- X_i HAVE AVERAGE POWER CONSTRAINT P_i, $i=1,2$
  - $Z_i(t)$ ZERO-MEAN COMPLEX GAUSSIAN RANDOM PROCES WITH SPECTRAL HEIGHT $N_i/2$, $i=0,1,2$

- K_{ij} ZERO-MEAN COMPLEX GAUSSIAN RANDOM VAR. WITH VARIANCE ξ_{ij}^2 (CORESPONDS TO RAYLEIGH FADING)

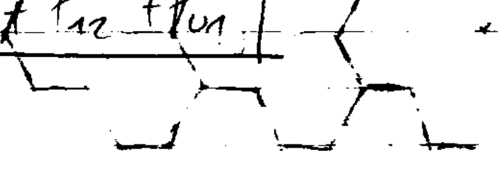
$Y_0 = K_{10}X_1 + K_{20}X_2 + Z_0$
$Y_1 = K_{21}X_2 + Z_1$
$Y_2 = K_{12}X_1 + Z_2$

$Z_0 \sim N(0,\Theta_0)$; $Z_1 \sim N(0,\Theta_1)$; $Z_2 \sim N(0,\Theta_2)$
$\Theta_1 = \Theta_2$

TRANSMIT B BLOCKS OF LENGHT (4)

- the signal of user 1 at time j: $j = 1, \dots 4$
  $x_1(W_1, Y_1(j-1), Y_1(j-2), \dots Y_1(1))$
- for user 2:
  $x_2(W_2, Y_2(j-1), Y_2(j-2), \dots Y_2(1))$

$x_1 = x_{10} + x_{12} + U_1 \qquad P_1 = P_{10} + P_{12} + P_{U1}$



Theorem 1: Achievable rate region of the system is the closure of the convex hull of all rate pairs (R_1, R_2)

$R_1 = R_{10} + R_{12}$ $\quad R_2 = R_{20} + R_{21}$

$$R_{12} < E\left\{C\left(\frac{K_{12}^2 P_{12}}{K_{12}^2 P_{10} + \Theta_1}\right)\right\} \qquad R_{21} < E\left\{C\left(\frac{K_{21}^2 P_{21}}{K_{21}^2 P_{20} + \Theta_2}\right)\right\}$$

$$R_{10} < E\left\{C\left(\frac{K_{10}^2 P_{10}}{\Theta_0}\right)\right\} \qquad R_{20} < E\left\{C\left(\frac{K_{20}^2 P_{20}}{\Theta_0}\right)\right\}$$

$$R_{10} + R_{20} < E\left\{C\left(\frac{K_{10}^2 P_{10} + K_{20}^2 P_{20}}{\Theta_0}\right)\right\}$$

$$R_{10} + R_{20} + R_{12} + R_{21} < E\left\{C\left(\frac{K_{10}^2 P_{10} + K_{20}^2 P_{20} + 2K_{10}K_{20}\sqrt{P_{U1}P_{U2}}}{\Theta_0}\right)\right\}$$

$P_1 = P_{10} + P_{12} + P_{U1}$ $\quad P_2 = P_{20} + P_{21} + P_{U2}$

$C(x) = \frac{1}{2}\log(1+x)$ CAPACITY OF AWGN channel

+381 38231300

MIRKO

W3 W1

MEHDI

WIRELESS COMMUNICATIONS PRINCIPLES & PRACTICE

- S DUPLEX CHANNELS
- k - NUMBER OF CHANNELS PER CELL

$S = k \cdot N$ $\quad$ N - NUMBER OF CELLS IN A CLUSTER (i.e. CLUSTER SIZE)

$C = M \cdot k \cdot N$ → CLUSTER CAPACITY

$N = 4, 7, 12$

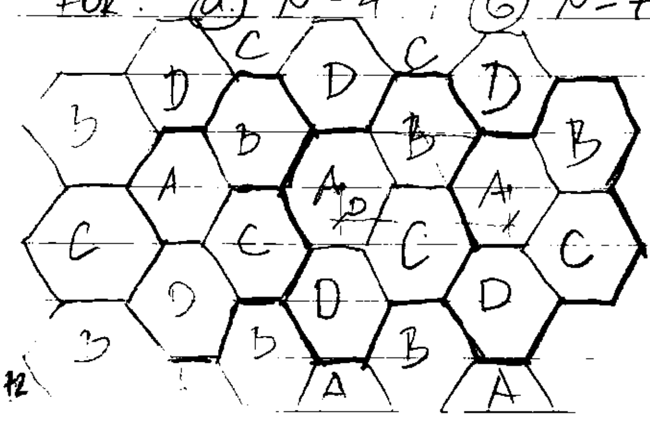
$N = i^2 + ij + j^2$ $\quad i=3, j=2$ $\quad N = 9 + 6 + 4 = 19$

FREQ REUSE $= 1/N = 1/19$

EXAMPLE 2.1 33MHz TOTAL BANDWIDTH FDD, 25kHz channel to provide voice & CC

- NUMBER OF CHANNELS AVAILABLE PER CELL = ?

FOR: (a) N=4 ; (b) N=7 ; (c) N=12



(a) N=4 $\quad k = ?$

$$S = \frac{33 \cdot 10^6}{25 \cdot 10^3} = \frac{33}{25} \cdot 10^3 = 1320 \text{ ch} = 1{,}32k$$

$1320 = k \cdot N$

$$k = \frac{1320}{4} = 330 \text{ ch/cell}$$

(b) → $k = \frac{1320}{7} = 188$ $\quad$ (c) $k = \frac{1320}{12} = 110$

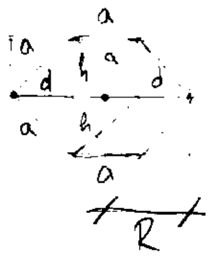
• CO-CHANNEL INTERFERENCE

R - RADIUS OF THE CELL

D - DISTANCE BETWEEN NEAREST CO-CHANNEL CELLS

$\frac{D}{R} \uparrow \rightarrow$ BETTER SPATIAL SEPARATION

$$Q = \frac{D}{R} = \sqrt{3N}$$ CO-CHANNEL REUSE RATIO



$s^2 = h^2 + d^2 \quad h = d$

$2d = \sqrt{a^2 + a^2} \qquad 2d = a\sqrt{2}$

$d = \frac{a}{2}\sqrt{2}$

$R = \frac{a}{2} + \frac{a}{2}\sqrt{2}$

$R = \frac{a}{2}(1+\sqrt{2})$

$N = 4 \Rightarrow D = 2R + a = \frac{a}{2}(1+\sqrt{2}) + a$

$D = \frac{a}{2}(1+1+\sqrt{2}) = \frac{a}{2}(2+\sqrt{2})$

$Q = \frac{D}{R} = 2\,\frac{2+\sqrt{2}}{1+\sqrt{2}} \cdot \frac{1-\sqrt{2}}{1-\sqrt{2}} = 2\,\frac{2-2\sqrt{2}+\sqrt{2}-2}{1-2} = -2(0-\sqrt{2}) = 2\sqrt{2}$

$Q = 2\sqrt{2}$ (*) $\Rightarrow Q = \sqrt{3 \cdot 4} = 2\sqrt{3}$

• SIGNAL TO INTERFERENCE

$$\frac{S}{I} = \frac{S}{\sum_{i=1}^{i_0} I_i}$$

i_0 - NUMBER OF INTERFERING CELLS

• AVERAGE POWER AT DISTANCE „d"

$$P_r = P_0 \left(\frac{d}{d_0}\right)^{-n} \qquad P_r(dBm) = P_0(dBm) - 10\,n \log\left(\frac{d}{d_0}\right)$$

P_0 - POWER AT A CLOSE-IN REFERENCE POINT

d_0 - DISTANCE BETWEEN REFERENCE POINT AND ANTENNA

n - PATH LOSS EXPONENT (2-4 FOR URBAN CELLULAR SYSTEMS)

$$\frac{S}{I} = \frac{R^{-4}}{\sum_{i=1}^{i_0} D_i^{-4}}$$

$D_1 \approx D_2 \approx \dots \approx D_{i_0}$

$$\frac{S}{I} = \frac{R^{-4}}{i_0 D^{-4}} = \frac{(D/R)^4}{i_0} = \frac{D^4}{i_0 R^4}$$

$$\frac{S}{I} = \frac{(\sqrt{3N})^4}{i_0}$$

$63 = \frac{(\sqrt{3N})^4}{6} \qquad 378 = (3N)^2$

$N^2 = \frac{108}{9} \Rightarrow N = \frac{1}{3}\sqrt{378}$ $\quad N = 6,48$

$18 = 10\log\left(\frac{S}{I}\right) \qquad \left(\frac{S}{I}\right)^{10} = 10^{18} \qquad \frac{S}{I} = 10^{\frac{18}{10}} = 10^{1,8} = 63$

↳ MINIMUM REQUIRED INTERFERENCE

$$\frac{S}{I} = \frac{R^{-n}}{\sum_{i=1}^{i_0} D^{-n}} = \frac{R^{-4}}{2D^{-4} + 2(D+R)^{-4} + 2(D-R)^{-4}} = \frac{1}{2Q^{-4} + 2(Q+1)^{-4} + 2(Q-1)^{-4}}$$

$Q = \sqrt{3 \cdot 7} = \sqrt{21} \doteq 4.6 \Rightarrow$ WORST CASE $\frac{S}{I} = 17.8$ dBm

EXAMPLE 2.2 $\frac{S}{I} = 15$ dB $\quad i_0 = 6 \quad Q = ?$ (a) $n = 4$ (b) $n = 3$

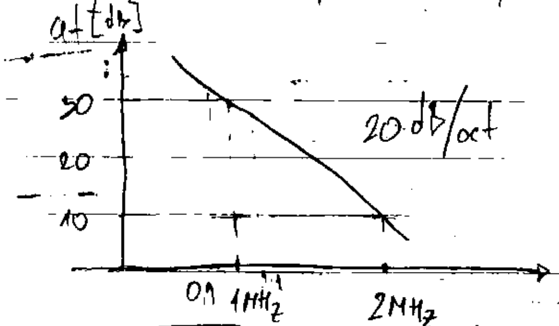
$\frac{S}{I} = \frac{(\sqrt{3N})^n}{i_0}$ ~~$15\,dB = 10 \log\left(\frac{S}{I}\right)$~~

$\left(\frac{S}{I}\right)^{10} = 10^{15}$ $\quad \boxed{\frac{S}{I} = 10^{1.5}} \quad \frac{S}{I} = 31.62$

$31.62 \cdot 6 = (\sqrt{3N})^n \qquad Q^n = 31.62 \cdot 6 = 189.72$

(a) $Q = \sqrt[4]{189.72} = 3.71 \qquad Q = \sqrt{3N} \qquad N = \frac{Q^2}{3} = 4.59$

(b) $Q = \sqrt[3]{189.72} = 5.74 \qquad N = \frac{Q^2}{3} = 11.01$



$\frac{S}{I} = (20)^{-4}$

$(-4) 10 \log 20 = -40(\log 2 + 1)$

$= -40(0.3 + 1) = -52$ dB

Example 2.3 666 duplex channels

- 1989 FCC ALOCATED ADDITIONAL 10 MHz i.e. 166 new ch.

TOTAL = 666 + 166 = 832 ch.

FC 870,030 MHz, RC 825,030 MHz } CH. 1. $\qquad$ FC 889,98 MHz, RC 844,98 MHz } CH. 666

EXTENDED BAND 667 – 799

990 – 1023

TRUNKING

$\boxed{A_u = \lambda \cdot H}$ [Er] $\quad H$ AVERAGE DURATION OF A CALL

λ – AVERAGE NUMBER OF CALL REQUESTS PER UNIT TIME

A_u – TRAFIC INTENSITY

- FOR SYSTEM CONTAINING U USERS TOTAL OFERED TRAFIC IS: $\boxed{A = U \cdot A_u}$ [Er]

$A_c = A/C = \frac{U \cdot A_u}{C}$ traffic intensity PER CHANNEL

$$Pr[blocking] = \frac{\frac{A^C}{C!}}{\sum_{k=0}^{C} \frac{A^k}{k!}}$$

ERLANG B
FOR INFINITE NUMBER OF USERS
BLOCK CALLS CLEARED

C - NUMBER OF TRUNKED CHANNELS
A - TOTAL OFFERED TRAFIC
$C \to \infty$

$$P_R = \frac{A^C}{C!} e^{-A}$$

• ERLANG C FORMULA

$$Pr[delay > 0] = \frac{A^C}{A^C + C!\left(1 - \frac{A}{C}\right) \sum_{k=0}^{C-1} \frac{A^k}{k!}}$$

ERLANG C
BLOCK CALLS DELAYED

$P(A,B) = P(A) \, P(B/A)$

$GOS = Pr[delay > t] = Pr[delay > 0] \cdot Pr[delay > t / delay > 0]$
$= Pr[delay > 0] \cdot \exp(-(C-A) \cdot t/H)$

AVERAGE DELAY OF A CALL:

$$D = Pr[delay > 0] \cdot \frac{H}{C-A}$$

EXAMPLE 2.4 HOW MANY USERS CAN BE SUPPORTED FOR $P_R = 0.5\%$ (0,005) BLOCK CALLS CLEARED

(a) $C=1$ (b) $C=5$ (c) $C=10$ (d) $C=20$ (e) $C=100$

$A_U = 0.1 [Er]$ $A = U \cdot A_U$

$$P_R = \frac{\frac{A^C}{C!}}{\sum_{k=1}^{C} \frac{A^k}{k!}} \Rightarrow$$

(a) $A = \left| \begin{matrix} P_R = 0.005 \\ C = 1 \end{matrix} \right| = 0.005025$

$U = \frac{A}{A_U} = \frac{0.005025}{0.1} = 0.05$

(b) $U = 1,13$ (c) $U = 39,6$ (d) $U = 110,9$ (e) $U = 809,1$

EXAMPLE 2.5 - 2M RESIDENTS ; A, B, C COMPETING MOBILE PROVIDERS

• A - 394 CELLS WITH 19 CH
• B - 98 CELLS WITH 57 CH
• C - 49 CELLS WITH 100 CH

$P_R = 2\%$ EACH USER MAKES 2 CALLS / HOUR DURATION 3 MIN

$A_U = 2 \cdot 3 \, min / 60 \, min = 6/60 = 0,1 \, [Er]$

$A = U \cdot A_U = 2 \cdot 10^6 \cdot 0,1 = 200.000,00 \, [Er]$

• ALL THREE OPERATE AT MAXIMUM CAPACITY. MARKET PENETRATION OF EACH PROVIDER = ?

(a.) $C_A = 394 \cdot 19 = 7486$

$$0,02 = \frac{A_A^{C_A} / C_A!}{\sum_{k=0}^{C_A} \frac{A_A^k}{k!}}$$

$C_A = 19$
$A_A = 12,333 \, [Er]$
$U_A = 394 \cdot U = 48591$

$A_A = U \cdot A_U$
$U = \frac{12,333}{0.1} = 123,33$ USERS

(b.) $c = 57$ $A = 468\,[Er] = U \cdot A_U$ $U = \frac{A}{A_U} = 468$

$U_B = 98 \cdot U = 45879{,}71$

(c.) $C = 100$ $A = 87{,}77$ $U = 879{,}7$

$U_C = 49 \cdot 877{,}7 = 43106{,}27$

$P_A = 2{,}48\,\%$ $P_B = 2{,}27\,\%$ $P_C = 2{,}16\,\%$ $P = (U_A + U_B + U_C)/2 \cdot 10^6 = 6.9\%$

EXAMPLE 2.6

CITY AREA = $1300\ mi^2$ $N = 7$

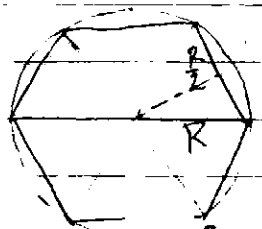
CELL RADIUS $R = 4\ mi$ $B = 40\ MHz$

CHANEL BANDWIDTH $B_C = 60\ kHz$ (FULL DUPLEX)

$GOS = 0{,}02 = 2\%$ $A_U = 0{,}03\ Er$

(a.) number of cells in area
(b.) number of ch/cell
(c) traffic intensity/cell
(d) maximum carried traffic
(e) total number of user that can be served
(f) mobiles/channel = ?
(g) theoretical max number of users served in one time

(a.) Cell Number $= \frac{1300}{R^2\pi} = \frac{1300}{16\pi} \doteq 25{,}86$



$P = 6 \cdot R \cdot \frac{h}{2}$ $h = \sqrt{R^2 - \frac{R^2}{4}} = \frac{R}{2}\sqrt{3}$

$P = 3 \cdot R \cdot \frac{R}{2}\sqrt{3} = \frac{R^2}{2} 3\sqrt{3} = \frac{16}{2} \cdot 3\sqrt{3}$

$P = \frac{3R^2}{2}\sqrt{3}$ Cell Number $= \frac{1300}{8 \cdot 3\sqrt{3}} = 31{,}273 \doteq 31$

(b.) $S = k \cdot N = \frac{40 \cdot 10^6}{60 \cdot 10^3} = 0{,}666\ 10^3 = 666$ ch

$k = C = \frac{666}{N} = \frac{666}{7} = 95{,}23$ $C = 95$/cell

(c) $0{,}02 = \frac{A^C/C!}{\sum_{k=0}^{C} \frac{A^k}{k!}} \Rightarrow A = 83{,}13\ [Er]$

(d) Max. CARRIED TRAFFIC $= A \cdot 31 = 2582\ [Er]$

(e) $A = A_U \cdot U \Rightarrow U = \frac{A}{A_U} = \frac{83{,}13}{0{,}03} = 2771$ users

TOTAL NUMBER OF USER $= 2771 \cdot 31 = 85901$

(f) mobile/ch $= \frac{85901}{666} = 129$ MOBILES/channel

(g) MaxUser $= 31 \cdot 95 = 2945$ users $\frac{2945}{85901} = 3{,}4\,\%$

EXAMPLE 2.7

- 4 CELL SYSTEM $R = 1{,}387$ km
- TOTAL CHANNELS = 60

- LOAD PER USER $A_U = 0{,}029$ [Er] $\lambda = 1$ call/hour
- ERLANG C SYSTEM $P_r[\text{delayed}] = 0{,}05$

$$P_r[\text{delay}>0] = \frac{A^C}{A^C + \left(1-\frac{A}{C}\right)\sum_{k=0}^{C-1}\frac{A^k}{k!}}$$

FIND: (a) $U = ?$ (users per square km) (c) $P_r[\text{delay} > 10s] = ?$
(b) $P_r[\text{delay} > 10 \mid \text{delay}] = ?$

(a) $P = \frac{3R^2}{2}\sqrt{3} = 5 \cdot 10^6\ m^2$

CHANNELS PER CELL $= \frac{60}{4} = 15$ ch/cell

$A = U \cdot A_U$ $U = ?$ $P_r = 0{,}05 \Rightarrow A = 10{,}745$

$U = \frac{A}{A_U} = \frac{10{,}745}{0{,}029} = 370$ users $U_{sqkm} = \frac{370}{5} = 74\ \frac{\text{users}}{km^2}$

(c) $P_r[\text{delay} > t] = P_r[\text{delay} > 0] \cdot e^{-\frac{(C-A)t}{H}}$

VEROJATNOST DEKA ~~POVIEOT E ZADOCNET~~ POVIKOT E ZADOCNET POVEKE OD 10 sec

H - holding time $\boxed{A_U = H \cdot \lambda}$ $0{,}029 = H \cdot 1/60$

$H = 0{,}029 \cdot 60 = 1{,}74$ min/call $- 0{,}029$ hours/call $= 104$ sec/call

$P_r[\text{delay} > 10s] = 0{,}05 \cdot e^{-\frac{(15-10{,}745)\cdot 10}{104{,}4}} = 0{,}033 = 3{,}3\%$

(b) $P_r[\text{delay} > 10 \mid \text{delay}] = e^{-\frac{(C-A)t}{H}} = e^{-\frac{(15-10{,}745)}{10{,}44}} = 0{,}665 = 66{,}5\%$

VEROJATNOST DEKA ODLOZEN POVIK CE CEKA POVEKE OD 10 sec

DISCUSSION

$P_R(10, A) = 0{,}01$ $A = 4{,}46$ Er } 1 CELL WITH 10 CHANNELS

$P_R(5, A) = 0{,}01$ $A = 1{,}36$ Er } 2 CELLS WITH 5 CHANNELS SUPPORT TRAFFIC OF $2 \times 1{,}36 = 2{,}72$ Er

- ~~CELL~~ SPLITTING

-- $P_r[\text{old cell bound}] \sim P_{t1} \cdot R^{-n}$

$P_r[\text{new cell bound}] \sim P_{t2}\left(\frac{R}{2}\right)^{-n}$

- IF $\boxed{n=4}$ $P_{t1} R^{-4} = P_{t2}\left(\frac{R}{2}\right)^{-4}$ $\frac{P_{t1}}{R^4} = P_{t2}\frac{16}{R^4}$ $\boxed{P_{t2} = \frac{P_{t1}}{16}}$

$10\log P_{t2} = 10\log P_{t1} - 10\log(16) = 10\log P_{t1} - 16\text{dB}$

EXAMPLE 28 BASE STATION USES 60CH

$R_{MACRO} = 1km$ $R_{\mu} = 0,5km$

NUMBER OF CHANNELS IN 3x3km SQUARE

(a) WITHOUT USE OF μ-cell (c) μ-cell used

(b) μ-cell used & M-cell used

(a) $5 \cdot 60 = 300ch$ (b) $11 \cdot 60 = 660CH$ (c) $17 \cdot 60 = 1020CH$

SECTORING

$$\frac{S}{I} = \frac{R^{-n}}{\sum_{k=1}^{10} D^{-n}} = \frac{R^{-n}}{10 D^{-n}} = \frac{(D/R)^n}{10} = \frac{(\sqrt{3N})^n}{2}$$

$N = 7 \Rightarrow$ $n = 4$ $\frac{(\sqrt{3 \cdot 7})^4}{2} = \frac{(21)^2}{2} = 220,5$ $10\log 220,5 = 23,4dB$

• UNSECTORED $N=7$, $n=4$: $10\log\frac{S}{I} = 10\log\frac{21^2}{6} = 18,66 dB$

$N=12$: $10\log\frac{S}{I} = 10\log\frac{(12\cdot 3)^2}{11} = 20,71 dB$

EXAMPLE 29 AVERAGE CALL LASTS 2 min $P_R = 0,01$

$H = 2 min$ $\lambda = 1$ CALL/HOUR

$C = 395$ $N = 7$ $C_{CELL} = 56,42 \doteq 57$, ERLANG B

$$P_R = \frac{\frac{A^C}{C!}}{\sum_{k=1}^{C} \frac{A^k}{k!}};$$ $P_R = 0,01 \Rightarrow A = 44,22 Er$

$A_U = \lambda \cdot H = \frac{1}{60} \cdot 2 = \frac{1}{30} = 0,033$

$A = U \cdot A_U \Rightarrow$ $U = \frac{A}{A_U} = \frac{44}{0,033} = 1340$ users (CELL) i.e. CALLS/HOUR

120° SECTORING $\Rightarrow$ **57/3 = 19** $P_R(19, A) = 0,01$ $\Rightarrow A = 11,23 Er$

$U = \frac{A}{A_U} = 336,93$ CALLS/HOUR (SECT) $3 \cdot 336 = 1008$ CALLS/HOUR (CELL)

$\frac{1340}{1008} = 1,33$ $1340(1-x) = 1008$ $x = \frac{1340-1008}{1340}$

$x = 24,8\%$ DECREASE OF TRUNKING EFFICIENCY

$1008 \cdot x + 1008 = 1340$ $x = \frac{1340-1008}{1008} = 33\%$ INCREASE

60° SECTORING $N = 7$ $10\log\left(\frac{S}{I}\right) = 10\log\frac{(\sqrt{3\cdot 7})^4}{1} = 10\log 21^2 = 26dB$

$57/6 = 9,5 \doteq 9$ CH $P_R(9, A) = 0,01 \rightarrow A = 3,78$

$U = \frac{A}{A_U} = \frac{3,78}{0,033} = 114,62$ USERS i.e CALLS/HOUR

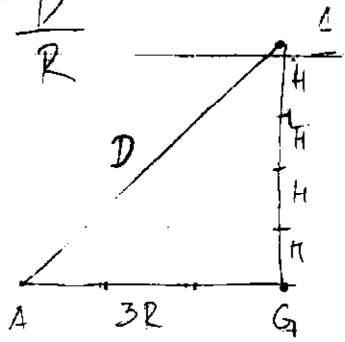
$114,62 \cdot 6 = 687$ $x = \frac{1340-687}{1340} = 48\%$ DECREASE

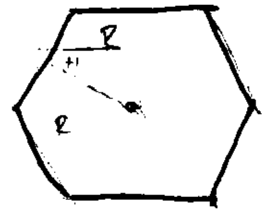
$$q = \frac{D}{R} = \sqrt{3N} \Rightarrow N = \frac{1}{3}\left(\frac{D}{R}\right)^2 = \left|\frac{D}{R} = 3\right| = \frac{9}{3} = 3$$

- PROBLEMS

PR.2.1

$q = \frac{D}{R}$





$H = \sqrt{R^2 - \frac{R^2}{4}}$

$H = \frac{R}{2}\sqrt{3}$

$D^2 = 9R^2 + \left(4\frac{R}{2}\sqrt{3}\right)^2$

$D^2 = 9R^2 + 4R^2 \cdot 3 = 9R^2 + 12R^2$

$D^2 = 21R^2 \qquad \frac{D}{R} = \sqrt{21} = \sqrt{3N}$

MOBILE RADIO PROPAGATION

$P_r = P_0 \left(\frac{d}{d_0}\right)^{-n} \qquad 10\log P_r = 10\log P_0 + 10\log\left(\frac{d}{d_0}\right)^{-n}$

$10\log P_r = 10\log P_0 - 10n\log\frac{d}{d_0}$

$10\log\frac{P_r}{P_0} = -10n\log\frac{d}{d_0} = $ IF $|P_0 = 1mW|$ $P_r[dBm] = -10n\log\frac{d}{d_0}$

FREE SPACE PROPAGATION MODEL

- FRIIS FREE SPACE EQUATION:

$$P_R(d) = \frac{P_t G_t G_r \lambda^2}{(4\pi)^2 d^2 L}$$

P_t TRANSMITTED POWER
P_R - RECEIVED POWER

G_t TRANSMITTER ANTENNA GAIN $\quad G_R$ RECEIVER ANTENNA GAIN
d - T-R SEPARATION DISTANCE (m); L SYSTEM LOSS FACTOR
$L \geq 1 \quad \lambda$ - WAVELENGTH (m)

$$G = \frac{4\pi A_e}{\lambda^2}$$ A_e - RELATIVE APERTURE $\quad \lambda = \frac{c}{f} = \frac{2\pi c}{\omega}$

$d_2 = 10 d_1$

$\frac{P_{R1}}{P_{T1}} = \frac{G_T G_R \lambda^2}{(4\pi)^2 d^2 L} \qquad \frac{P_{R2}}{P_{R2}} = \frac{G_T G_R \lambda^2}{(4\pi)^2 10^2 d^2 L}$

$\frac{P_{R1}/P_{T1}}{P_{R1}/P_{T2}} = \frac{1/d^2}{1/100d^2} = 100 \qquad 10\log\frac{P_{R1}}{P_{T1}} - 10\log\frac{P_{R2}}{P_{T2}} = 20dB$

$$\alpha_1 - \alpha_2 = 20dB$$

isotropic radiator is ideal antenna which radiates power with unit gain uniformly in all directions.

$$EIRP = P_t G_t$$ effective isotropic radiated power

- half wave dipole antenna $G = 1,64$ i.e. 2.15 dB above isotropic antenna

$$ERP = EIRP + 2,15dB$$ ERP - effective RADIATED POWER

$10\log 1.64 = 2.15$ dB

dBi – GAIN OF THE ANTENNA COMPARED TO ISOTROPIC ANT

dBd – ANTENNA GAIN WITH RESPECT TO HALF-WAVE DIPOLE

- PATH LOSS

$$PL = 10\log\frac{P_T}{P_R} = 10\log\frac{P_T}{\frac{P_T G_T G_R \cdot \lambda^2}{(4\pi)^2 d^2}} = -10\log\frac{G_T G_R \lambda^2}{(4\pi)^2 d^2}$$

- $G_T, G_R = 1$ $\quad PL = -10\log\frac{\lambda^2}{(4\pi)^2 d^2}$
- d_f – FAR FIELD DISTANCE $\quad \boxed{d_f = \frac{2D^2}{\lambda}}$ $\quad d_f \gg D$, $d_f \gg \lambda$

D – LARGEST PHYSICAL LINEAR DIMENSION OF THE ANTENNA

- d_0 – REFERENCE DISTANCE LIES IN THE FAR FIELD REGION

I.E. $\boxed{d_0 \geq d_f}$

$$P_r(d) = \frac{P_t \cdot G_t G_r \cdot \lambda^2}{(4\pi)^2 d^2 L} \qquad P_r(d_0) = \frac{P_t G_t G_r \lambda^2}{(4\pi)^2 d_0^2 L}$$

$$\frac{P_r(d)}{P_r(d_0)} = \frac{1/d^2}{1/d_0^2} = \left(\frac{d_0}{d}\right)^2 \qquad P_r(d) = P_r(d_0)\frac{d_0^2}{d^2} = P_r(d_0)\left(\frac{d_0}{d}\right)^2$$

$$\boxed{P_r(d)\,[dBm] = 10\log\frac{P_r(d_0)}{1\,mW} + 20\log\left(\frac{d_0}{d}\right)} \qquad d \geq d_0 \geq d_f$$

Ex 3.1 far field distance = ? $\quad D = 1\,m \quad f = 900\,MHz$

$$d_f = \frac{2D^2}{\lambda} \qquad \lambda = \frac{300}{f[MHz]} = \frac{300}{900} = \frac{1}{3} = 0{,}333$$

$$\boxed{d_f = \frac{2 \cdot 1^2}{\frac{1}{3}} = 6\,m}$$

Ex 3.2 $P_t = 50\,W \quad G_T = 1 \quad f = 900\,MHz \quad P_t[dBm] = ?$

$d = 100\,m \quad P_r = ?$

$d = 10\,km \quad P_r = ?$

$$P_t[dBm] = 10\log\frac{50}{10^{-3}} = 46\,dBm \qquad P_t[dBW] = 17\,dBW$$

$$P_r = \frac{P_t \cdot G_T \cdot G_R\,\lambda^2}{(4\pi)^2 \cdot d^2\,L} = \frac{50 \cdot 1 \cdot 1 \cdot \left(\frac{1}{3}\right)^2}{(4\pi)^2\,d^2} = \frac{50}{16 \cdot 9\,\pi^2\,10^4}$$

$$P_r = \frac{5}{144\,\pi^2\,10^3} = 3{,}518 \cdot 10^{-6} \qquad P_r[dBm] = -24{,}5\,dBm$$

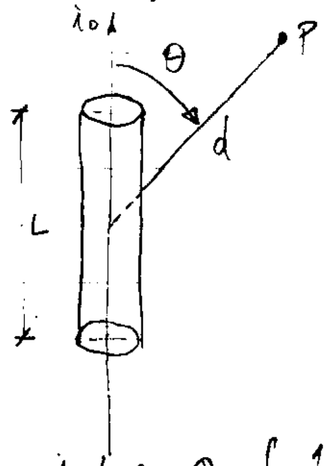
$$P_r[10\,km] = P_r[100]\,dBm + 20\log\left(\frac{d_0}{d}\right) = -24{,}5 + 20(-2) = -64{,}5\,dBm$$

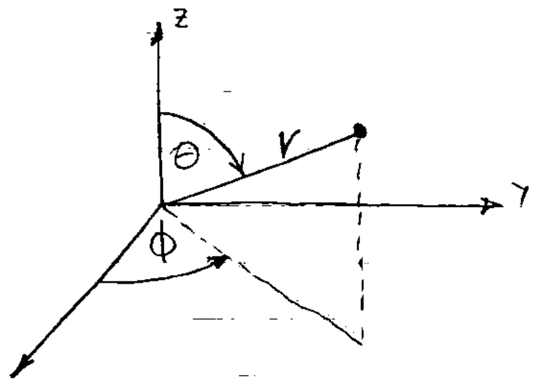
180 $P_r[10\,km](W) = ?$

$$-64{,}5 = 10 \log \frac{P_r[10\,km]}{0{,}001} \Rightarrow \left[\frac{P_r(10\,km)}{0{,}001}\right] = 10^{-\frac{64{,}5}{10}}$$

$$P_r(10\,km) = 0{,}001 \cdot 10^{-6{,}45} = 35 \cdot 10^{-9}\ W = \underline{35\ pW}$$

- RELATING POWER TO ELECTRIC FIELD





$$E_r = \frac{i_0 L \cos\theta}{2\pi \varepsilon_0 c}\left\{\frac{1}{d^2} + \frac{c}{j\omega_c d^3}\right\} e^{j\omega_c(t-dc)}$$

ELECTROSTATIC FIELD ($1/d^3$)

$$E_\theta = \frac{i_0 L \sin\theta}{4\pi \varepsilon_0 c^2}\left\{\frac{j\omega_c}{d} + \frac{c}{d^2} + \frac{c^2}{j\omega_c d^3}\right\} e^{-j\omega_c(t-dc)}$$

RADIATION FIELD ($1/d$)

$$H_\phi = \frac{i_0 L \sin\theta}{4\pi c}\left\{\frac{j\omega_c}{d} + \frac{c}{d^2}\right\} e^{j\omega_c(t-dc)}$$

INDUCTIVE FIELD ($1/d^2$)

$$E_\phi = H_r = H_\theta = \varnothing$$

TERM: $\frac{1}{d^2}$ - REPRESENT <u>INDUCTION</u> FIELD COMPONENT

TERM: $\frac{1}{d^3}$ - REPRESENT <u>ELECTROSTATIC</u> — " — "

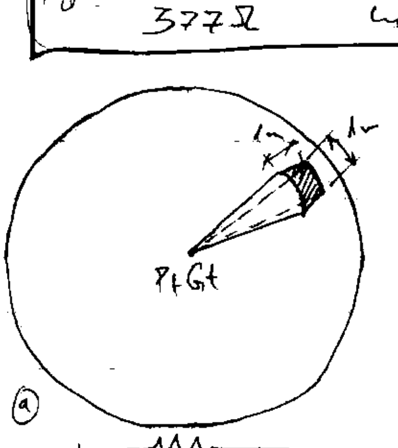
TERM: $1/d$ - REPRESENT <u>RADIATION</u> FIELD COMPONENT

- POWER FLUX DENSITY

$$P_d = \frac{EIRP}{4\pi d^2} = \frac{P_t G_t}{4\pi d^2} = \frac{E^2}{R_{fs}} = \frac{E^2}{\eta}\left[\frac{W}{m^2}\right]$$

R_{fs} - INTRINSIC <u>IMPEDANCE</u> OF FREE SPACE $\eta = 120\pi\,\Omega\ (377\,\Omega)$

$$\boxed{P_d = \frac{|E|^2}{377\,\Omega} = \frac{W}{m^2}}$$

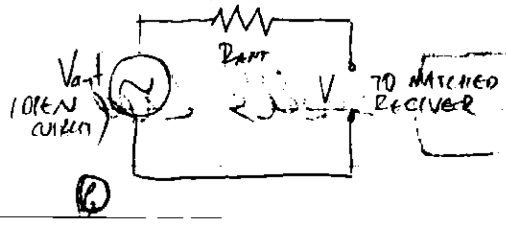


$$P_d = \frac{P_t G_t}{4\pi d^2} \quad \frac{EIRP}{4\pi d^2} \qquad \frac{P_d}{4\pi}$$

$$\boxed{P_r(d) = P_d A_e = \frac{|E|^2}{120\pi} A_e} = \frac{P_t G_t G_r \lambda^2}{(4\pi)^2 d^2}\ [\text{Watts}]$$

$$P_r(d) = \frac{V^2}{Z_{ant}} = \frac{(V_{ant}/2)^2}{R_{ant}} = \frac{V_{ant}^2}{4 R_{ant}}$$

$$\Rightarrow \boxed{A_e = \frac{G_r \lambda^2}{4\pi}}$$



Ex 3.3 Reciever located at $d = 10\,km$ $P_t = 50W$ $f_c = 900MHz$

$G_t = 1$ $G_r = 2$

ⓐ $P_r = ?$ ⓑ $|E| = ?$ ⓒ rms voltage? $R_{ant} = 50\Omega$

ⓐ $P_r = \frac{P_t \cdot G_t \cdot G_r \cdot \lambda^2}{(4\pi d)^2 \cdot L} = \frac{50 \cdot 1 \cdot 2 \cdot \left(\frac{1}{3}\right)^2}{(4\pi)^2 \cdot 10^8} = \frac{10^2}{16\pi^2 \cdot 10^8 \cdot 9} = \frac{1}{144\pi^2 10^6}$

$P_r = 0{,}7 \cdot 10^{-9} = 0{,}7\, nW = 700\, pW = -61\, dBm$

ⓑ $P_r(d) = P_d A_e = \frac{|E|^2}{120\pi} A_e$ $|E|^2 = \frac{120\pi P_r(d)}{A_e}$

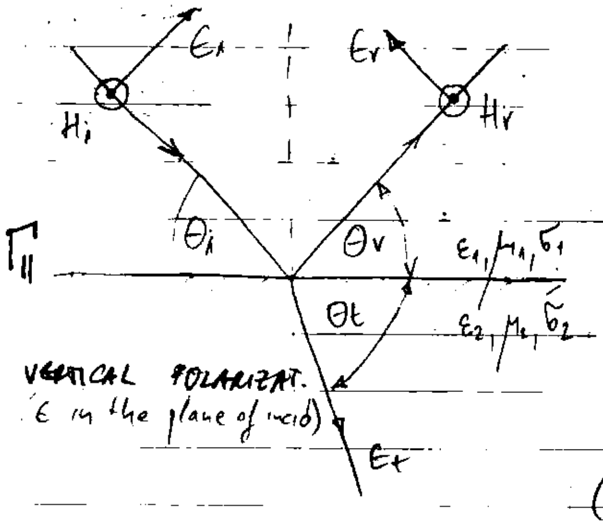
$|E| = \sqrt{\frac{120\pi P_r(d)}{A_e}} = \frac{0{,}51 \cdot 10^{-7}}{\sqrt{A_e}} = \left| A_e = \frac{G_r \lambda^2}{4\pi} \right| = \frac{0{,}51 \cdot 10^{-3}}{0{,}133}\ 3{,}8 \cdot 10^{-3}\ V/m$

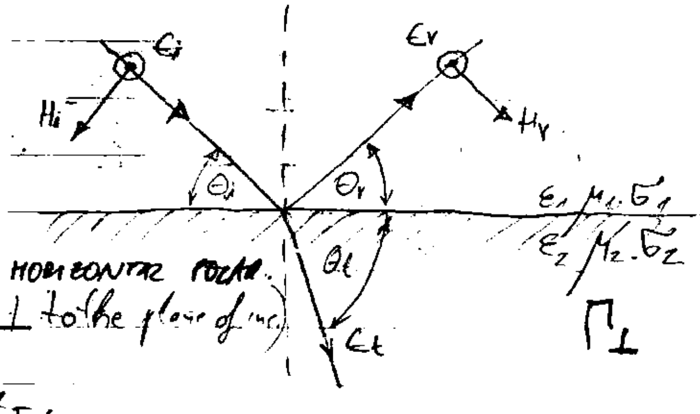
ⓒ $P_r = \frac{V^2}{R_{ant}}$ $V = \sqrt{P_r \cdot R_{ant}} = 0{,}18\, mV = 0{,}18 \cdot 10^{-3}$

$V_{ant} = 2V > 0{,}36\, mV$

- Reflection from dielectrics

ε - permittivity (dialectric constant)
μ - permeability
σ - conductance





- Perfects dielectric

$\varepsilon = \varepsilon_0 \cdot \varepsilon_r$ $\varepsilon_0 = 8.85 \cdot 10^{-12}\ F/m$

- Lossy dielectric

$\varepsilon = \varepsilon_0 \varepsilon_r - j\varepsilon'$ $\varepsilon' = \frac{\sigma}{2\pi f}$

$\sigma = f(f)$

σ - provodnost
σ [Siemens/meter]

$$\Gamma_{\parallel} = \frac{E_r}{E_i} = \frac{\eta_2 \sin\theta_t - \eta_1 \sin\theta_i}{\eta_2 \sin\theta_t + \eta_1 \sin\theta_i}$$

$$\Gamma_{\perp} = \frac{E_r}{E_i} = \frac{\eta_2 \sin\theta_i - \eta_1 \sin\theta_t}{\eta_2 \sin\theta_i + \eta_1 \sin\theta_t}$$

$\eta_i = \sqrt{\frac{\mu_i}{\varepsilon_i}}$ intrinsicna impedansa

$v = \frac{1}{\sqrt{\mu\varepsilon}}$ brzina na EM bran[illegible]

- Boundary conditions at the surface of incidence

$$\sqrt{\mu_1\varepsilon_1}\, \sin(90 - \theta_i) = \sqrt{\mu_2\varepsilon_2}\, \sin(90 - \theta_t)$$ Snell's Law

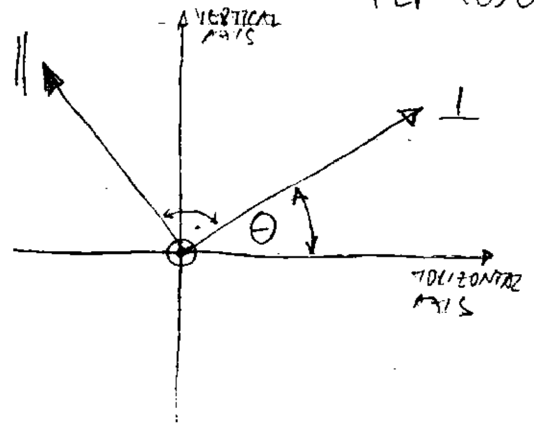
• FROM THE BOUNDARY CONDITIONS OF MAXWELL EQUATION ⇒

$\theta_i = \theta_r \quad E_r = \Gamma E_i \quad E_t = (1+\Gamma) E_i$ $\quad \Gamma \ (\Gamma_{\parallel} \text{ OR } \Gamma_{\perp})$

• FREE SPACE (FIRST MEDIUM) & $\mu_1 = \mu_2 = \mu$ $\quad \varepsilon = \varepsilon_0 \varepsilon_r$

$$\Gamma_{\parallel} = \frac{n_2 \sin\theta_t - n_1 \sin\theta_i}{n_2 \sin\theta_t + n_1 \sin\theta_i} = \frac{-\varepsilon_r \sin\theta_i + \sqrt{\varepsilon_r - \cos^2\theta_i}}{\varepsilon_r \sin\theta_i + \sqrt{\varepsilon_r - \cos^2\theta_i}}$$

$$\Gamma_{\perp} = \frac{\sin\theta_i - \sqrt{\varepsilon_r - \cos^2\theta_i}}{\sin\theta_i + \sqrt{\varepsilon_r - \cos^2\theta_i}}$$

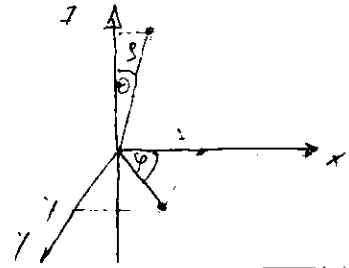


$$\begin{bmatrix} E_H^d \\ E_V^d \end{bmatrix} = R^T D_C R \begin{bmatrix} E_H^i \\ E_V^i \end{bmatrix}$$

$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$D_C = \begin{bmatrix} D_{\perp\perp} & \emptyset \\ \emptyset & D_{\parallel\parallel} \end{bmatrix} \rightarrow \text{DEPOLARIZATION MATRIX}$$

$D_{xx} = \Gamma_x$ (REFLECTION) $\quad D_{xx} = T_x = 1 + \Gamma_x$ (REFRACTION I.E. TRANSMISSION)

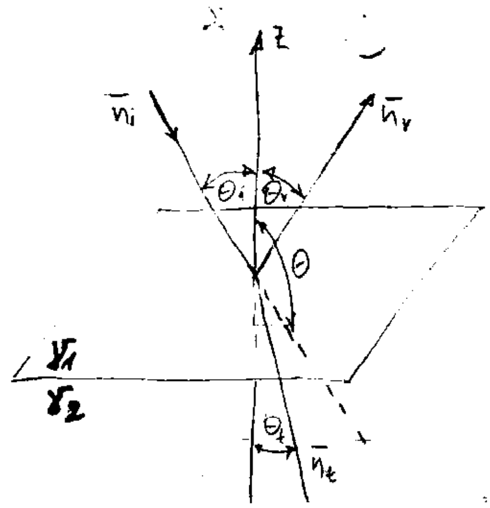


$z = \rho \cdot \cos\theta$

$x = \rho \sin\theta \cdot \cos\varphi$

$y = \rho \sin\theta \cdot \sin\varphi$

• ODBIVANJE I REFRAKCIJA NA E.M.B



$\theta = \pi - \theta_i \quad \gamma^2 = -\omega^2 \varepsilon\mu \quad \gamma = j\omega\sqrt{\varepsilon\mu}$

$$U = U_0 e^{-\gamma(x \sin\theta\cos\varphi + y\sin\theta\sin\varphi + z\cos\theta)}$$

$\gamma_1(x \sin\theta_i \cos\varphi_i + y \sin\theta_i \sin\varphi_i + z\cos\theta_i) =$ (z → ∅)

$\gamma_1(x \sin\theta_r \cos\varphi_r + y \sin\theta_r \sin\varphi_r + z\cos\theta_r) =$

$\gamma_2(x \sin\theta_t \cos\varphi_t + y \sin\theta_t \sin\varphi_t + z\cos\theta_t)$

$\gamma_1 \sin\theta_i \cos\varphi_i = \gamma_1 \sin\theta_r \cos\varphi_r$ I

$\gamma_1 \sin\theta_i \cos\varphi_i = \gamma_2 \sin\theta_t \cos\varphi_t$ II

$\gamma_1 \sin\theta_i \sin\varphi_i = \gamma_1 \sin\theta_r \sin\varphi_r$ III

$\gamma_1 \sin\theta_i \sin\varphi_i = \gamma_2 \sin\theta_t \sin\varphi_t$ IV

$\frac{III}{I}$ $\quad tg\varphi_i = tg\varphi_r \Rightarrow \varphi_i = \varphi_r$ (*)

(*), I $\Rightarrow \sin\theta_i = \sin\theta_r \Rightarrow \theta_i = \theta_r$ I SNELOV ZAKON

$\frac{IV}{II}$ $\quad tg\varphi_i = tg\varphi_t \Rightarrow \varphi_i = \varphi_t$ (1)

(1), II $\quad \gamma_1 \sin\theta_i = \gamma_2 \sin\theta_t \Rightarrow \sqrt{\varepsilon_1\mu_1} \sin\theta_i = \sqrt{\varepsilon_2\mu_2} \sin\theta_t$ $\quad \frac{\sin\theta_i}{\sin\theta_t} = \frac{n_2}{n_1}$ II SNELL'S LAW

(8)

• ИНДЕКС НА ПРЕКРШУВАЊЕ: $n_1 = \frac{c}{v_1}$ $v_1 = \frac{1}{\sqrt{\varepsilon_1\mu_1}}$; $n_2 = \frac{c}{v_2}$

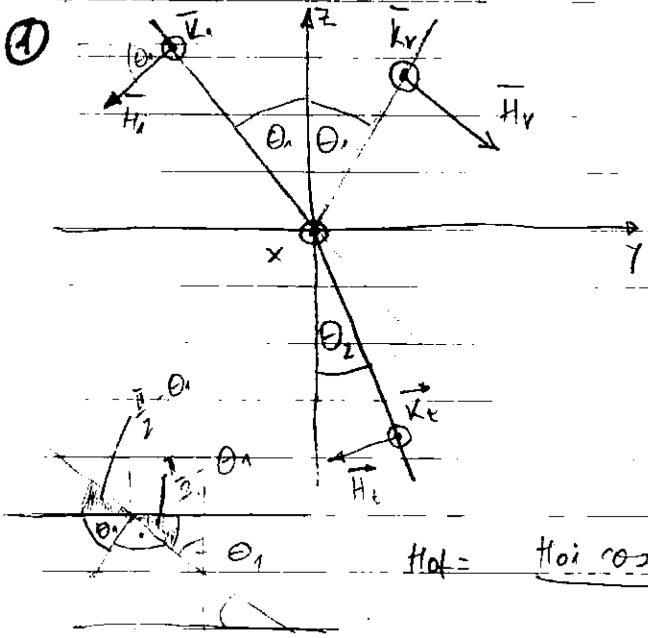
$v_2 = \frac{1}{\sqrt{\varepsilon_2\mu_2}}$ ⇒ $n_1 = c\sqrt{\varepsilon_1\mu_1}$. $n_2 = c\sqrt{\varepsilon_2\mu_2}$

II СНЕЛОВ ЗАКОН

$c\sqrt{\varepsilon_1\mu_1}\sin\theta_i = c\sqrt{\varepsilon_2\mu_2}\sin\theta_t$ $\sqrt{\varepsilon_1\mu_1}\sin\theta_i = \sqrt{\varepsilon_2\mu_2}\sin\theta_t$

• ФРЕНЕЛОВИ КОЕФИЦИЕНТИ

①



$z = 0$

$V_{0i} + V_{0r} = V_{0t}$ (*)

$H_{0i}\cos\theta_1 - H_{0r}\cos\theta_1 = H_{0t}\cos\theta_2$

$H_{0i} = \frac{V_{0i}}{Z_1}$ $H_{0r} = \frac{V_{0r}}{Z_1}$

$Z_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$, $H_{0t} = \frac{V_{0t}}{Z_2}$

$Z_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}$ (**)

$V_{0r} = V_{0t} - V_{0i} = Z_2 H_{0t} - V_{0i}$

$H_{0t} = \frac{H_{0i}\cos\theta_1 - H_{0r}\cos\theta_1}{\cos\theta_2}$

$V_{0r} = Z_2 \frac{\left(\frac{V_{0i}}{Z_1}\cos\theta_1 - \frac{V_{0r}}{Z_1}\cos\theta_1\right)}{\cos\theta_2} - V_{0i}$

$V_{0r}\cos\theta_2 = \frac{Z_2}{Z_1}V_{0i}\cos\theta_1 - \frac{Z_2}{Z_1}V_{0r}\cos\theta_1 - V_{0i}\cos\theta_2$

$V_{0r}\left(\cos\theta_2 + \frac{Z_2}{Z_1}\cos\theta_1\right) = V_{0i}\left(\frac{Z_2}{Z_1}\cos\theta_1 - \cos\theta_2\right)$

$$V_{0r} = \frac{Z_2\cos\theta_1 - Z_1\cos\theta_2}{Z_1\cos\theta_2 + Z_2\cos\theta_1} V_{0i}$$

⊙ → $\vec{K}$ е НОРМАЛНО НА ИНЦИДЕНТНАТА РАМНИНА

$V_{0t} = V_{0r} + V_{0i} = H_{0r} \cdot Z_1 + V_{0i}$ $H_{0r} = \frac{H_{0i}\cos\theta_1 - H_{0t}\cos\theta_2}{\cos\theta_1}$

$V_{0t} = Z_1 \frac{\frac{V_{0i}}{Z_1}\cos\theta_1 - \frac{V_{0t}}{Z_2}\cos\theta_2}{\cos\theta_1} + V_{0i}$

$V_{0t}\cos\theta_1 = V_{0i}\cos\theta_1 - \frac{Z_1 V_{0t}\cos\theta_2}{Z_2} + V_{0i}\cos\theta_1$

$V_{0t}\left(\cos\theta_1 + \frac{Z_1}{Z_2}\cos\theta_2\right) = 2V_{0i}\cos\theta_1$

$$V_{0t} = \frac{2Z_2\cos\theta_1}{Z_1\cos\theta_2 + Z_2\cos\theta_1} V_{0i}$$

T_n

$V_{0r} = R_n \cdot V_{0i}$ $R_n = \frac{Z_2\cos\theta_1 + Z_1\cos\theta_2}{Z_2\cos\theta_1 + Z_1\cos\theta_2}$

$V_{0t} = T_n \cdot V_{0i}$ $T_n = \frac{2Z_2\cos\theta_1}{Z_2\cos\theta_1 + Z_1\cos\theta_2}$

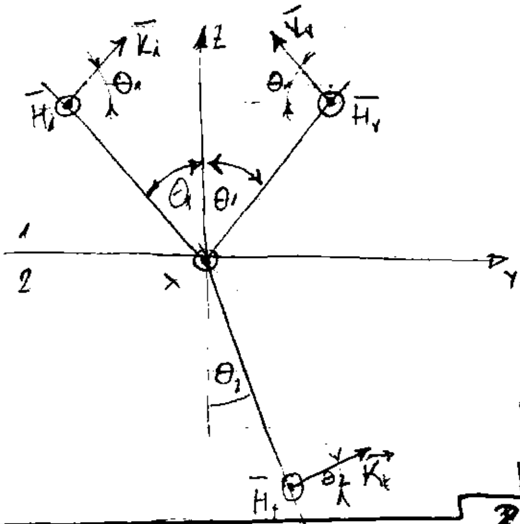
$T_n = 1 + R_n$

$\gamma_1\sin\theta_1 = \gamma_2\sin\theta_2$

$\cos\theta_2 = \sqrt{1-\sin^2\theta_2}$

$\cos\theta_2 = \sqrt{1-\frac{\gamma_1^2}{\gamma_2^2}\sin^2\theta_1}$

②



$$K_{0i}\cos\theta - K_{0r}\cos\theta_1 = K_{0t}\cos\theta_2$$

$$H_{0i} + H_{0r} = H_{0t}$$

$$H_{0i} = \frac{K_{0i}}{Z_1} \; ; \; H_{0r} = \frac{K_{0r}}{Z_1} \; ; \; H_{0t} = \frac{K_{0t}}{Z_2}$$

$$K_{0r}\cos\theta_1 = K_{0i}\cos\theta_1 - K_{0t}\cos\theta_2$$

$$K_{0r}\cos\theta_1 = K_{0i}\cos\theta_1 - H_{0t}\, Z_2\cos\theta_2$$

$$K_{0r}\cos\theta_1 = K_{0i}\cos\theta_1 - \left(\frac{K_{0i}}{Z_1} + \frac{K_{0r}}{Z_1}\right) Z_2\cos\theta_2$$

$$K_{0r}\left(\cos\theta_1 + \frac{Z_2}{Z_1}\cos\theta_2\right) = K_{0i}\left(\cos\theta_1 - \frac{Z_2}{Z_1}\cos\theta_2\right)$$

$$K_{0r} = \frac{Z_1\cos\theta_1 - Z_2\cos\theta_2}{Z_1\cos\theta_1 + Z_2\cos\theta_2}\, K_{0i} \quad (R_P)$$

$$K_{0t} = \frac{2Z_2\cos\theta_1}{Z_1\cos\theta_1 + Z_2\cos\theta_2}\, K_{0i} \quad (T_P)$$

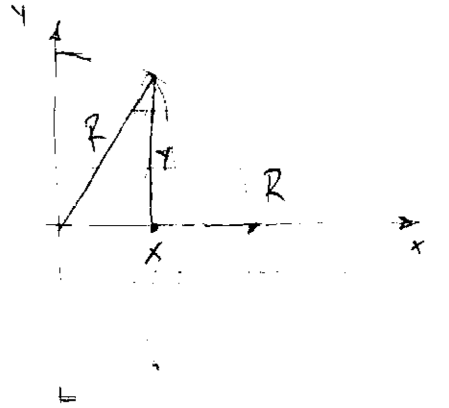
$$\boxed{K_{0r} = R_P \cdot K_{0i}}$$

$$\boxed{K_{0t} = T_P\, K_{0i}}$$

- $K_{0t}\cos\theta_2 = K_{0i}\cos\theta - Z_1(H_{0t} - H_{0i})\cos\theta_1 = K_{0i}\cos\theta_1 - Z_1\left(\frac{K_{0t}}{Z_2} - \frac{K_{0i}}{Z_1}\right)\cos\theta_1$

$$K_{0t}\cos\theta_2 + \frac{Z_1}{Z_2}K_{0t}\cos\theta_1 = 2\cdot K_{0i}\cos\theta_1$$

$$\boxed{K_{0t} = \frac{2Z_2\cos\theta_1}{Z_1\cos\theta_1 + Z_2\cos\theta_2}\, K_{0i}}$$



$$V = \int_{-R}^{R}\left(\sqrt{R^2 - x^2}\right)^2 \pi\, dx$$

$$= 2\int_0^R (R^2 - x^2)\pi\, dx = 2R^2\pi x\Big|_0^R - 2\pi\frac{x^3}{3}\Big|_0^R$$

$$V = 2R^2\pi R - 2\pi\frac{R^3}{3} = 2\pi\left(R^3 - \frac{R^3}{3}\right) = 2\pi\frac{2R^3}{3} = \frac{4\pi R^3}{3}$$

Ex. 34 ? DEMONSTRATE THAT $|\Gamma_\parallel|\; |\Gamma_\perp| \to 1$ IF $\theta_i \to 0$

$$\Gamma_\parallel = \frac{n_2\cos\theta_i - n_1\cos\theta_t}{n_2\cos\theta_i + n_1\cos\theta_t} \qquad T_\parallel = 1 + \Gamma = \frac{2n_2\cos\theta_i}{n_1\cos\theta_1 + n_2\cos\theta_2}$$

$$\lim_{\theta_i\to 0}\frac{n_2\cos\theta_i - n_1\cos\theta_t}{n_2\cos\theta_i + n_1\cos\theta_t} = \frac{n_2 - n_1\cos\theta_t}{n_2 + n_1\cos\theta_t} = \frac{\sqrt{\frac{\mu_2}{\varepsilon_2}} - \sqrt{\frac{\mu_1}{\varepsilon_1}}\cos\theta_t}{\sqrt{\frac{\mu_2}{\varepsilon_2}} + \sqrt{\frac{\mu_1}{\varepsilon_1}}\cos\theta_t}$$

$$\left|\varepsilon_1 = \varepsilon_r\,\varepsilon_0 = \varepsilon_0\right| = \frac{\sqrt{\frac{\mu}{\varepsilon_r\varepsilon_0}} - \sqrt{\frac{\mu}{\varepsilon_0}}\cos\theta_t}{\sqrt{\frac{\mu}{\varepsilon_r\varepsilon_0}} + \sqrt{\frac{\mu}{\varepsilon_0}}\cos\theta_t} = \frac{\frac{1}{\sqrt{\varepsilon_r}} - \cos\theta_t}{\frac{1}{\sqrt{\varepsilon_r}} + \cos\theta_t}$$

$$= \frac{1 - \sqrt{\varepsilon_r}\cos\theta_t}{1 + \sqrt{\varepsilon_r}\cos\theta_t}$$

$\mu_1 = \mu_2 = \mu \qquad \varepsilon = \varepsilon_0 \varepsilon_r \qquad \Gamma_\perp = \dfrac{n_2 \sin\theta_i - n_1 \sin\theta_t}{n_2 \sin\theta_i + n_1 \sin\theta_t}$

$n_2 = \sqrt{\dfrac{\mu}{\varepsilon_0 \varepsilon_r}} \qquad n_1 = \sqrt{\dfrac{\mu}{\varepsilon_0}} \qquad \Gamma_\perp = \dfrac{\sqrt{\frac{\mu}{\varepsilon_0\varepsilon_r}} \sin\theta_i - \sqrt{\frac{\mu}{\varepsilon_0}} \sin\theta_t}{\sqrt{\frac{\mu}{\varepsilon_0\varepsilon_r}} \sin\theta_i + \sqrt{\frac{\mu}{\varepsilon_0}} \sin\theta_t}$

$\Gamma_\perp = \dfrac{\frac{1}{\sqrt{\varepsilon_r}} \sin\theta_i - \sin\theta_t}{\frac{1}{\sqrt{\varepsilon_r}} \sin\theta_i + \sin\theta_t}$

$\sqrt{\mu \varepsilon_0}\, \sin(90 - \theta_i) = \sqrt{\mu \varepsilon_0 \varepsilon_r}\, \sin(90 - \theta_t)$

$\cos\theta_i = \sqrt{\varepsilon_r} \cdot \cos\theta_t$

$\cos\theta_i = \sqrt{\varepsilon_r} \sqrt{1 - \sin^2\theta_t} \qquad \varepsilon_r - \varepsilon_r \sin^2\theta_t = \cos^2\theta_i$

$\varepsilon_r \sin^2\theta_t = \varepsilon_r - \cos^2\theta_i \qquad \sqrt{\varepsilon_r} \sin\theta_t = \sqrt{\varepsilon_r - \cos^2\theta_i}$

$$\Gamma_\perp = \frac{\sin\theta_i - \sqrt{\varepsilon_r} \sin\theta_t}{\sin\theta_i + \sqrt{\varepsilon_r} \sin\theta_t} = \frac{\sin\theta_i - \sqrt{\varepsilon_r - \cos^2\theta_i}}{\sin\theta_i + \sqrt{\varepsilon_r - \cos^2\theta_i}}$$

$\Gamma_\parallel = \dfrac{n_2 \sin\theta_t - n_1 \sin\theta_i}{n_2 \sin\theta_t + n_1 \sin\theta_i} = \dfrac{\sqrt{\frac{\mu}{\varepsilon_0\varepsilon_r}} \sin\theta_t - \sqrt{\frac{\mu}{\varepsilon_0}} \sin\theta_i}{\sqrt{\frac{\mu}{\varepsilon_0\varepsilon_r}} \sin\theta_t + \sqrt{\frac{\mu}{\varepsilon_0}} \sin\theta_i}$

$\Gamma_\parallel = \dfrac{\sin\theta_t - \sqrt{\varepsilon_r} \sin\theta_i}{\sin\theta_t + \sqrt{\varepsilon_r} \sin\theta_i} \qquad \sin\theta_t = \dfrac{1}{\sqrt{\varepsilon_r}} \sqrt{\varepsilon_r - \cos^2\theta_i}$

$$\Gamma_\parallel = \frac{\sqrt{\varepsilon_r - \cos^2\theta_i} - \varepsilon_r \sin\theta_i}{\sqrt{\varepsilon_r - \cos^2\theta_i} + \varepsilon_r \sin\theta_i} \qquad \Gamma_\parallel = \frac{-\varepsilon_r \sin\theta_i + \sqrt{\varepsilon_r - \cos^2\theta_i}}{\varepsilon_r \sin\theta_i + \sqrt{\varepsilon_r - \cos^2\theta_i}}$$

Ex 3.4 $\quad |\Gamma_\perp|, |\Gamma_\parallel| \to 1$ if $\theta_i \to 0$

$$\lim_{\theta_i \to 0} \left| \frac{\sin\theta_i - \sqrt{\varepsilon_r - \cos^2\theta_i}}{\sin\theta_i + \sqrt{\varepsilon_r - \cos^2\theta_i}} \right| = \left| \frac{0 - \sqrt{\varepsilon_r - 1}}{0 + \sqrt{\varepsilon_r - 1}} \right| = |-1| = 1$$

$$\lim_{\theta_i \to 0} |\Gamma_\parallel| = \left| \frac{-0 + \sqrt{\varepsilon_r - 1}}{0 + \sqrt{\varepsilon_r - 1}} \right| = 1$$

- BREWSTER ANGLE

$\Gamma_\parallel = \dfrac{E_R}{E_i} = \dfrac{n_2 \cdot \sin\theta_t - n_1 \cdot \sin\theta_i}{n_2 \cdot \sin\theta_t + n_1 \cdot \sin\theta_i} = 0$

$n_2 \sin\theta_t - n_1 \cdot \sin\theta_i = 0 \qquad n_2 \sin\theta_t = n_1 \sin\theta_i$

$\sqrt{\dfrac{\mu_2}{\varepsilon_2}} \sin\theta_t = \sqrt{\dfrac{\mu_1}{\varepsilon_1}} \sin\theta_i \qquad \sqrt{\varepsilon_1 \mu_1} \sin(90 - \theta_i) = \sqrt{\varepsilon_2 \mu_2} \sin\left(\frac{\pi}{2} - \theta_t\right)$

$$\sqrt{\varepsilon_1\mu_1}\cdot\cos(\theta_i) = \sqrt{\varepsilon_2\mu_2}\cos\theta_t = \sqrt{\varepsilon_2\mu_2}\sqrt{1-\sin^2\theta_t}$$

$$\frac{\varepsilon_1\mu_1}{\varepsilon_2\mu_2}\cos^2(\theta_i) = 1-\sin^2\theta_t \qquad \sin\theta_t = \sqrt{1-\frac{\varepsilon_1\mu_1}{\varepsilon_2\mu_2}\cos^2(\theta_i)}$$

$$\sqrt{\frac{\mu_2}{\varepsilon_2}}\sqrt{1-\frac{\varepsilon_1\mu_1}{\varepsilon_2\mu_2}\cos^2\theta_i} = \sqrt{\frac{\mu_1}{\varepsilon_1}}\sqrt{1-\cos^2\theta_i}$$

$$\frac{\mu_2}{\varepsilon_2}\left(1-\frac{\varepsilon_1\mu_1}{\varepsilon_2\mu_2}\cos^2\theta_i\right) = \frac{\mu_1}{\varepsilon_1}\left(1-\cos^2\theta_i\right)$$

$$\frac{\mu_2}{\varepsilon_2}\,\frac{\varepsilon_2\mu_2-\varepsilon_1\mu_1\cos^2\theta_i}{\varepsilon_2\mu_2} = \frac{\mu_1}{\varepsilon_1}\left(1-\cos^2\theta_i\right)$$

$$\varepsilon_1\varepsilon_2\mu_2-\varepsilon_1^2\mu_1\cos^2\theta_i = \varepsilon_2^2\mu_1-\varepsilon_2^2\mu_1\cos^2\theta_i$$

$$\varepsilon_1\varepsilon_2\mu_2-\varepsilon_2^2\mu_1 = \left(\varepsilon_1^2\mu_1-\varepsilon_2^2\mu_1\right)\cos^2\theta_i$$

$$\cos^2\theta_i = \frac{\varepsilon_1\varepsilon_2\mu_2-\varepsilon_2^2\mu_1}{\varepsilon_1^2\mu_1-\varepsilon_2^2\mu_1}$$

$$1-\sin^2\theta_i = \frac{\varepsilon_1\varepsilon_2\mu_2-\varepsilon_2^2\mu_1}{\varepsilon_1^2\mu_1-\varepsilon_2^2\mu_1}$$

$$\frac{\varepsilon_1^2\mu_1-\cancel{\varepsilon_2^2\mu_1}-\varepsilon_1\varepsilon_2\mu_2+\cancel{\varepsilon_2^2\mu_1}}{\varepsilon_1^2\mu_1-\varepsilon_2^2\mu_1} = \sin^2\theta_i$$

$$\sin^2\theta_i = \frac{\varepsilon_1(\varepsilon_1\mu_1-\varepsilon_2\mu_2)}{\mu_1(\varepsilon_1^2-\varepsilon_2^2)} = \left|\mu_1=\mu_2=\mu\right| = \frac{\varepsilon_1\mu(\varepsilon_1-\varepsilon_2)}{\mu(\varepsilon_1^2-\varepsilon_2^2)}$$

$$\sin^2\theta_i = \frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2} \qquad \boxed{\sin\theta_B = \sqrt{\frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2}}}$$

- FREE SPACE VS. DIELECTRIC

$$\sin\theta_B = \sqrt{\frac{\varepsilon_0}{\varepsilon_0+\varepsilon_0\varepsilon_1}} = \sqrt{\frac{1}{1+\varepsilon_r}} = \sqrt{\frac{1-\varepsilon_r}{1-\varepsilon_r^2}}$$

$$\boxed{\sin\theta_B = \sqrt{\frac{\varepsilon_r-1}{\varepsilon_r^2-1}}}$$ ONLY FOR VERTICAL POLARIZATION

EX. 3.5 $\varepsilon_r = 4$ $\quad \sin\theta_B = \frac{1}{\sqrt{1+\varepsilon_r}}$ $\quad \theta_B = \arcsin\frac{1}{\sqrt{5}}$

$\theta_B = 0{,}46$ rad $\quad \frac{\pi}{2}$ rad $\;90°$ $\quad 1$ rad $= \frac{180°}{\pi}$

$\theta_B = 0{,}46\cdot\frac{180}{\pi} = 26{,}6°$

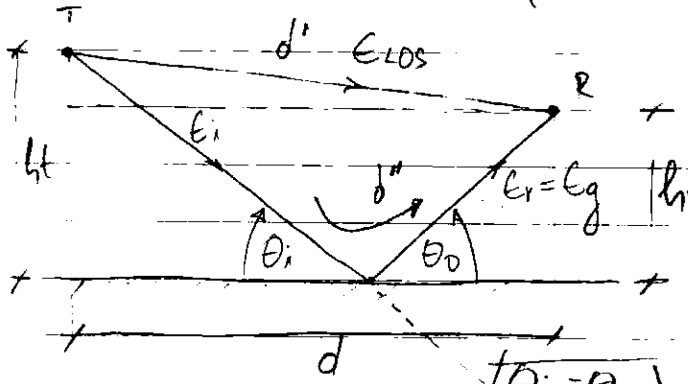
- NA POVRŠINA NA IDEALEN PROVODNIK

$K_{tang} = 0 \quad H_n = 0$ $\quad$ HORIZ. POLARIZACIJA: $E_i = -E_r,\ \theta_i = \theta_r$

VERTIKALNA -II-: $E_i = E_r,\ \theta_i = \theta_r$

$$\Gamma_\perp = \frac{n_2 \sin\theta_i - n_1 \sin\theta_t}{n_2 \sin\theta_i + n_1 \sin\theta_t} = 1 \qquad \Gamma_\parallel = \frac{n_2 \sin\theta_t - n_1 \sin\theta_i}{n_2 \sin\theta_t + n_1 \sin\theta_i} = -1$$

- **Ground reflection (2 ray model)**



$$E_{TOT} = E_{LOS} + E_g$$

$$E(d,t) = \frac{E_0 d_0}{d} \cos\left(w_c\left(t - \frac{d}{c}\right)\right) \qquad d > d_0$$

$$E_{LOS}(d',t) = \frac{E_0 d_0}{d'} \cos\left(w_c\left(t - \frac{d'}{c}\right)\right)$$

$$E_g(d'',t) = \Gamma \frac{E_0 d_0}{d''} \cos\left(w_c\left(t - \frac{d''}{c}\right)\right)$$

$$E_g = \Gamma E_i$$
$$E_t = (1+\Gamma) E_i \qquad \Gamma = -1 \qquad |E_{TOT}| = |E_{LOS} + E_g|$$

$$E_{TOT}(d,t) = \frac{E_0 d_0}{d'} \cos\left(w_c\left(t - \frac{d'}{c}\right)\right) + (-1) \frac{E_0 d_0}{d''} \cos\left(w_c\left(t - \frac{d''}{c}\right)\right)$$
(*) (**)

$$\Delta = d'' - d' = \sqrt{(h_t + h_r)^2 + d^2} - \sqrt{(h_t - h_r)^2 + d^2}$$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3 + \dots$$

$$\left(\sqrt{(h_t + h_r)^2 + d^2}\right)' = \frac{1}{2} \cdot \frac{2d}{\sqrt{(h_t + h_r)^2 + d^2}}$$

$$\left(\sqrt{(h_t - h_r)^2 + d^2}\right)' = \frac{1}{2} \frac{2d}{\sqrt{(h_t - h_r)^2 + d^2}}$$

$$\frac{d}{\sqrt{h_t^2 + 2h_t h_r + h_r^2 + d^2}} + \frac{d}{\sqrt{h_t^2 - 2h_t h_r + h_r^2 + d^2}}$$

$$\frac{d\sqrt{(h_t - h_r)^2 + d^2} - d\sqrt{(h_t + h_r)^2 + d^2}}{\sqrt{[(h_t + h_r)^2 + d^2][(h_t - h_r)^2 + d^2]}}$$
(▲)

$$▲ = (h_t + h_r)^2 (h_t - h_r)^2 + d^2 (h_t + h_r)^2 + d^2 (h_t - h_r)^2 + d^4$$

$$\boxed{\Delta = d'' - d' \approx \frac{2 h_t h_r}{d}}$$

$$f(x) = \sqrt{x}$$
$$f(x) = \sqrt{x_0} + \frac{1}{2}\frac{1}{\sqrt{x}}(x - x_0)$$
$$x_0 = 0 \Rightarrow f(x) = \frac{1}{2}\sqrt{x}$$

$f(x) = e^x \qquad f(x) \approx f(x_0) + f'(x_0)(x - x_0) = e^0 + e^{x_0}(x - 0) = 1 + x$

$f(x) = \frac{1}{x} \qquad f(x) \approx \frac{1}{x_0} - \frac{1}{x_0^2}(x - x_0) = 1 - \frac{1}{x_0^2}(x-1) = 1 - (x-1)$

$f(x) = \sqrt{1+x} \qquad \boxed{f(x) \approx 1 + \frac{1}{2}x} \qquad f'(x_0) = \frac{1}{2}\frac{1}{\sqrt{1+x_0}} = \frac{1}{2}$

$f'(x_0)(x-x_0) = \frac{1}{2}x$

$(*) = \sqrt{(l_t + l_r)^2 + d^2} = (l_t + l_r)\sqrt{1 + \left(\frac{d}{l_t + l_r}\right)^2} = (l_t + l_r)\left(1 + \frac{d^2}{(l_t+l_r)^2}\right)$

$(*) = (l_t + l_r) + \frac{d^2}{(l_t + l_r)} = \frac{(l_t + l_r)^2 + d^2}{(l_t + l_r)}$

$(**) = \frac{(l_t - l_r)^2 + d^2}{(l_t - l_r)} \qquad \cdot (l_t + l_r)$

$\Delta = d'' - d' = (*) - (**) = \frac{\left[(l_t + l_r)^2 + d^2\right](l_t - l_r) + \left[(l_t + l_r)^2 + d^2\right]}{l_t^2 - l_r^2}$

MMV

$d \gg l_t + l_r \qquad \frac{l_t + l_r}{d} \to 0$

$(*) = \sqrt{(l_t + l_r)^2 + d^2} = d\sqrt{1 + \frac{(l_t + l_r)^2}{d^2}} \approx d\left(1 + \frac{(l_t + l_r)^2}{2d^2}\right)$

$(**) = d\left(1 + \frac{(l_t - l_r)^2}{2d^2}\right)$

$(*) - (**) = d + \frac{(l_t + l_r)^2}{2d} - d - \frac{(l_t - l_r)^2}{2d} =$

$= \frac{l_t^2 + 2l_t l_r + l_r^2 - l_t^2 + 2l_t l_r - l_r^2}{2d} = \frac{2l_t l_r}{d}$

MMV

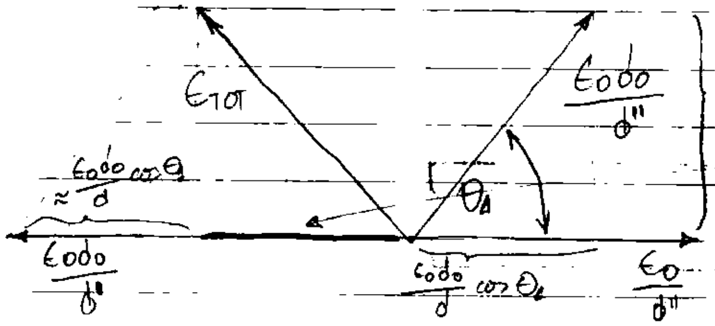
$\boxed{\Theta_\Delta = \frac{2\pi\Delta}{\lambda} = \frac{\Delta\,\omega_c}{c}} \qquad \tau_d \quad \frac{\Delta}{c} = \frac{\Theta_\Delta}{\omega_c}$

$\lambda = \frac{c}{f} = \frac{c}{\frac{\omega_c}{2\pi}} = \frac{2\pi c}{\omega_c} \qquad \frac{\Theta_\Delta}{2\pi f_c}$

$\left|\frac{E_0 d_0}{d}\right| \approx \left|\frac{E_0 d_0}{d'}\right| \approx \left|\frac{E_0 d_0}{d''}\right|$

$E_r(d, t - \frac{d''}{c}) = \frac{E_0 d_0}{d'}\cos\left[\omega_c\left(\frac{d'' - d'}{c}\right)\right] - \frac{E_0 d_0}{d''}\cos 0° ;$

$= \frac{E_0 d_0}{d'}\cos\left(\frac{\Delta \cdot \omega_c}{c}\right) - \frac{E_0 d_0}{d''} \approx \frac{E_0 d_0}{d}\left(\cos\Theta_\Delta - 1\right)$



$$\frac{E_0 d_0}{d}\sin\theta_d$$

$$\frac{E_0 d_0}{d'} - \frac{E_0 d_0}{d''}\cos\theta_d \approx \frac{E_0 d_0}{d}(1-\cos\theta_d)$$

$$|E_{TOT}(d)| = \sqrt{\left(\frac{E_0 d_0}{d}\right)^2(\cos\theta_d - 1)^2 + \left(\frac{E_0 d_0}{d}\right)^2\sin^2\theta_d} =$$

$$= \sqrt{\left(\frac{E_0 d_0}{d}\right)^2(\cos^2\theta_d - 2\cos\theta_d + 1 + \sin^2\theta_d)} =$$

$$= \sqrt{\left(\frac{E_0 d_0}{d}\right)^2 2(1-\cos\theta_d)} = \frac{\sqrt{2}E_0 d_0}{d}\sqrt{1-\cos\theta_d}$$

$$= \left|\cos\theta_d = \cos^2\frac{\theta_d}{2} - \sin^2\frac{\theta_d}{2}\right| = \frac{\sqrt{2}E_0 d_0}{d}\sqrt{\cos^2\frac{\theta_d}{2} + \sin^2\frac{\theta_d}{2} - \cos^2\frac{\theta_d}{2} + \sin^2\frac{\theta_d}{2}}$$

$$\boxed{|E_{TOT}(d)| = \frac{2E_0 d_0}{d}\cdot\sin\frac{\theta_d}{2}}$$

$$\sin\frac{\theta_d}{2} \approx \frac{\theta_d}{2} \qquad \frac{\theta_d}{2} < 0.3 \text{ rad}$$

$$\frac{\theta_d}{2} = \frac{\Delta \cdot w_c}{2cd} = \frac{2h_t h_r w_c}{2cd} = \frac{h_t h_r 2\pi f_c}{dc} = \frac{2\pi h_t h_r}{\lambda d}$$

$$\frac{\theta_d}{2} < 0.3 \qquad \frac{2\pi h_t h_r}{\lambda d} < \frac{3}{10}$$

$$\boxed{d > \frac{20\pi h_t h_r}{3\lambda} \approx \frac{20 h_t h_r}{\lambda}}$$

$$\text{If } \boxed{d > \frac{20 h_t h_r}{\lambda}} \Rightarrow$$

$$\boxed{E_{TOT} \approx \frac{2E_0 d_0}{d}\,\frac{2\pi h_t h_r}{\lambda d} \approx \frac{k}{d^2}} \qquad \boxed{E_{TOT} = \frac{4\pi E_0 d_0 h_t h_r}{\lambda d^2}}$$

$$P_r = \frac{G_T G_R \cdot \lambda^2 P_T}{(4\pi)^2 d^2 L}, \quad L = 1, \quad P_r = \frac{G_T G_R \cdot \lambda^2 \cdot P_T}{(4\pi)^2 d^2}$$

$$G = \frac{4\pi A_e}{\lambda^2} \qquad P_r = P_T G_T$$

$$\lambda \sim \frac{4\pi E_0 d_0 h_t h_r}{E_{TOT}\, d^2} \qquad P_r = \frac{G_T G_R P_T}{(4\pi)^2 d^2}\cdot\frac{(4\pi)^2 E_0 d_0 h_t^2 h_r^2}{E_{TOT}\, d^2}$$

$$P_r = P_T G_T G_R \frac{h_t^2 h_r^2}{d^4}\cdot\frac{E_0 d_0}{E_{TOT}} \qquad \boxed{P_r = P_T G_T G_R \frac{h_t^2 h_r^2}{d^4}}$$

$$P_L = 10\log\frac{P_T}{P_R} = 10\log d^4 - 10\log G_T G_R h_t^2 h_r^2$$

$$= 40\log d - (10\log G_T + 10\log G_R + 20\log h_t + 20\log h_r)$$

$$\theta_\Delta = \frac{2\pi\Delta}{\lambda} = \frac{2\pi\, 2 h_t h_r}{\lambda \cdot d} = \frac{4\pi h_t h_r}{\lambda d}$$

IF: $\theta_\Delta = \pi$ $\quad \pi = \frac{4\pi h_t h_r}{\lambda d}$ $\quad \boxed{\lambda = \frac{4 h_t h_r}{d}}$ NO!! GROUND IS IN FIRST FRESNEL ZONE

Ex. 36 $d = 5\,km$ ANTENNA MONOPOLE $\frac{\lambda}{4}$ $G_R|_{dB} = 2{,}55\,dB$

$2{,}55 = 10\log(G_R)$ $\quad 10^{0{,}255}\ G_R \Rightarrow \boxed{G_R = 1{,}799}$

$E_0 = 10^{-3}\,V/m$ AT $d_0 = 1\,km$; $f_c = 900\,MHz$

(a) $G_R = ?$ $\quad l_R = ?$

(b) 2-RAY MODEL $P_R = ?$ $\quad h_T = 50\,m \quad h_r = 1{,}5\,m$

(a) $P_r(d_0) = P_d \cdot A_e = \frac{|E|^2}{120\pi} A_e$; $\quad A_e = \frac{G_R \lambda^2}{4\pi}$

$\lambda = \frac{c}{f[MHz]} = \frac{300}{900} = \frac{1}{3}$

$P_r(d) = 10\log\frac{P_r(d_0)}{1\,mW} - 20\log\frac{d}{d_0}$

$\frac{P_r}{P_t} = \frac{G_T G_R \cdot \lambda^2}{(4\pi)^2 d^2}$ $\quad \frac{P_r(d)}{P_r(d_0)} = \frac{\frac{1}{d^2}}{\frac{1}{d_0^2}} = \frac{d_0^2}{d^2}$

$P_R(d) = \frac{d_0}{d} P_R(d_0) = \frac{d_0}{d} \cdot \frac{|E|^2}{120\pi} \cdot \frac{G_R \lambda^2}{4\pi}$

$\boxed{l_R = \frac{\lambda}{4} = \frac{1}{12} = 0{,}083\,m = 8{,}3\,cm}$ $\quad \boxed{G_R = 1{,}8}$

(b) $E_{TOT} = \frac{4\pi E_0 d_0 h_t h_r}{\lambda \cdot d^2} = \frac{4\pi\, 10^{-3} \cdot 10^3 \cdot 50 \cdot 1.5}{\frac{1}{3} \cdot 25 \cdot 10^6}$

$E_{TOT} = 12\pi \cdot 3 \cdot 10^{-6} = 36\pi \cdot 10^{-6}\,\frac{V}{m} = 113{,}1 \cdot 10^{-6}\,\frac{V}{m}$

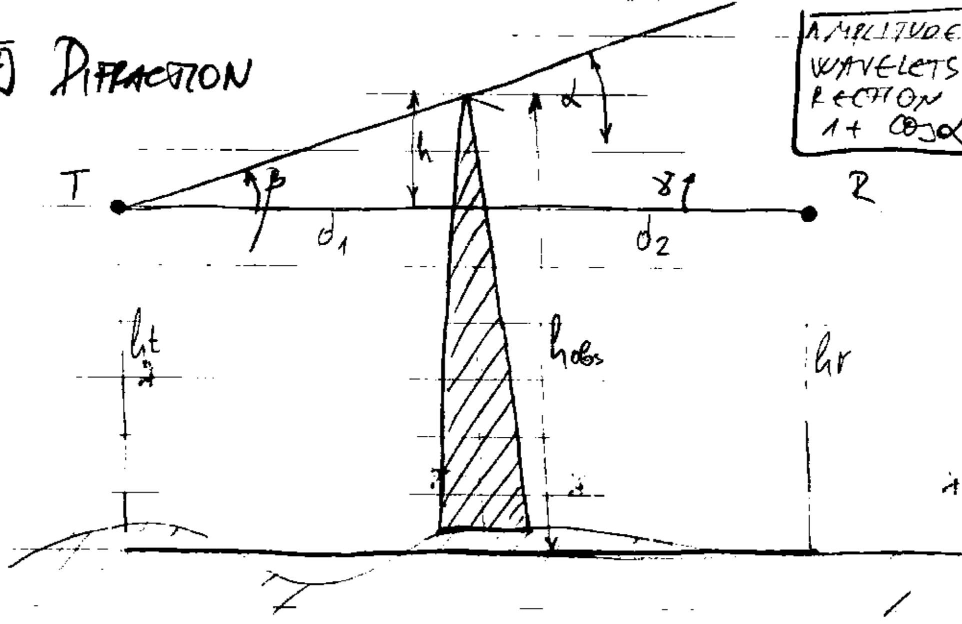
$P_r(d) = \frac{|E|^2}{120\pi} \cdot A_e = \frac{|E|^2}{120\pi} \frac{G_R \lambda^2}{4\pi} = \frac{(18\pi \cdot 10^{-6})^2}{120\pi} \cdot \frac{1.8 \cdot \frac{1}{9}}{4\pi}$

$P_r(d) = \frac{36^2 \cdot \pi^2\, 10^{-12} \cdot 1.8}{120 \cdot 36 \cdot \pi^2} = 0{,}54 \cdot 10^{-12} = 0{,}54\,pW$

$P_r|_{dBm}\ 10\log\frac{54{,}0\, 10^{-14}}{10^{-3}} = 10\log 54{,}0 \cdot 10^{-11} = -92{,}7\,dBm$

$P_r|_{dBW} = 10\log 5{,}4 \cdot 10^{-13} = -122\,dBW$

3.7 DIFFRACTION



AMPLITUDE OF THE SECONDARY WAVELETS IN A GIVEN DIRECTION IS PROPORTIONAL TO $1+\cos\alpha$.

$$\Delta = \frac{h^2(d_1+d_2)}{2d_1 d_2}$$

$$\Delta = d'' - d' = \sqrt{d_1^2+h^2} + \sqrt{d_2^2+h^2} - d_1 - d_2$$

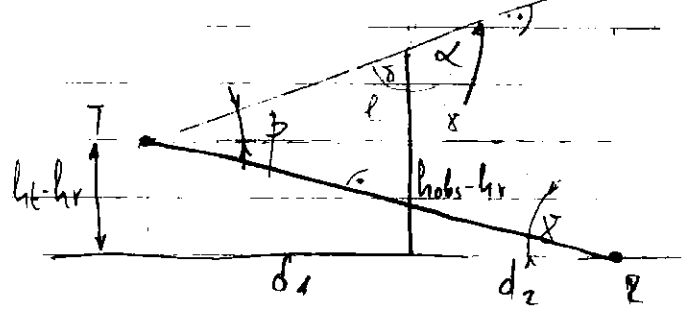
$$d_1, d_2 \gg h \Rightarrow \quad \sqrt{1+x^2} \doteq 1 + \frac{x^2}{2} \quad x \to 0$$

$$d_1\sqrt{1+\left(\frac{h}{d_1}\right)^2} \doteq d_1\left(1+\frac{h^2}{2d_1^2}\right) \qquad d_2\sqrt{1+\left(\frac{h}{d_2}\right)} \doteq d_2\left(1+\frac{h^2}{2d_2^2}\right)$$

$$\Delta = d_1 + \frac{h^2}{2d_1} + d_2 + \frac{h^2}{2d_2} - d_1 - d_2 = \frac{h^2 d_2 + h^2 d_1}{2d_1 d_2} = \frac{h^2(d_1+d_2)}{2d_1 d_2}$$

$$\boxed{\phi = \frac{2\pi\Delta}{\lambda} \doteq \frac{2\pi}{\lambda}\frac{h^2}{2}\frac{d_1+d_2}{d_1 d_2}} \qquad \boxed{\Delta = \frac{h^2(d_1+d_2)}{2d_1 d_2}}$$

- when $tg(x) \doteq x$, then $\alpha = \beta + \gamma$ $\quad \boxed{\alpha \doteq h\,\frac{d_1+d_2}{d_1 d_2}}$ ⊕



δ $180 - \beta - \gamma$

$$x = 180 - \delta = 180 - 180 + \beta + \gamma$$

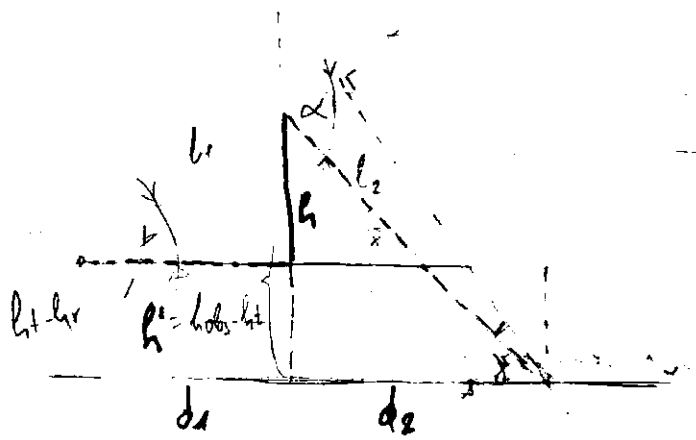
$$\boxed{\alpha = \beta + \gamma}$$

$$\nu = h\sqrt{\frac{2(d_1+d_2)}{\lambda d_1 d_2}\left(\frac{d_1 d_2}{(d_1+d_2)}\cdot\frac{d_1+d_2}{d_1 d_2}\right)} = h\cdot\frac{d_1+d_2}{d_1 d_2}\sqrt{\frac{2d_1 d_2}{\lambda(d_1+d_2)}}$$

$$\boxed{\nu = \alpha\sqrt{\frac{2d_1 d_2}{\lambda(d_1+d_2)}}}$$ Fresnel-Kirchoff diffraction parameter

$$\nu^2 = h^2\,\frac{2(d_1+d_2)}{\lambda\, d_1 d_2} = 2\cdot\frac{h^2}{2}\cdot 2\cdot\underbrace{\frac{2\pi}{\lambda}\cdot\frac{d_1+d_2}{d_1 d_2}}_{\phi}\cdot\frac{\lambda}{8\pi}$$

$$\boxed{\phi = \frac{\pi\nu^2}{2}} \qquad tg\,\beta = \frac{h}{d_1} \qquad \boxed{\nu = h\sqrt{\frac{2(d_1+d_2)}{\lambda d_1 d_2}}}$$



$$l_2 = \sqrt{l_1'^2 + d_2^2} \doteq l_1' \sqrt{1+\left(\frac{d_2}{l_1'}\right)^2}$$

$$l_2 \doteq l_1'\left(1+\frac{d_2^2}{2l_1'^2}\right) = l_1' + \frac{d_2^2}{2l_1'}$$

$$l_1 \doteq h_1 + \frac{d_1^2}{2l_1}$$

PROOF FOR (?)

$$tg(\beta) = \frac{h_1}{d_1} \qquad \alpha = \gamma + \beta$$

$$tg\beta = tg(\alpha-\gamma) = \frac{\sin(\alpha-\gamma)}{\cos(\alpha-\gamma)} = \frac{\sin\alpha\cdot\cos\gamma - \sin\gamma\cos\alpha}{\cos\alpha\cdot\cos\gamma + \sin\alpha\cdot\sin\gamma} =$$

$$= \frac{1}{\cos\alpha\cdot\cos\gamma}\;\frac{\sin\alpha\cdot\cos\gamma - \sin\gamma\cos\alpha}{1 + tg\alpha\cdot tg\gamma} = \frac{tg\alpha - tg\gamma}{1+tg\alpha\cdot tg\gamma}$$

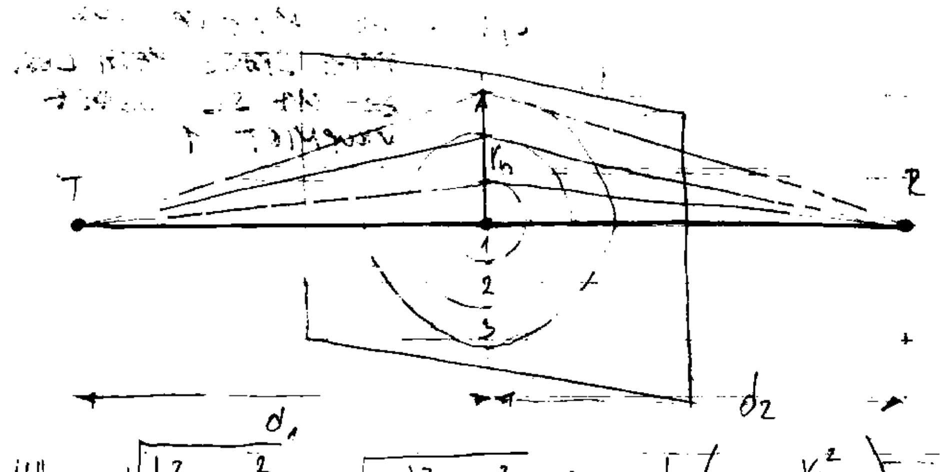
$$tg\beta = \frac{tg\alpha - \frac{h_1'}{d_2}}{1 + tg\alpha\,\frac{h_1'}{d_2}} \qquad \frac{d_2\, tg\alpha - h_1'}{d_2 + h_1'\, tg\alpha} = \frac{h_1}{d_1}$$

$$\boxed{h_1' \doteq h_1} \qquad (d_2\, tg\alpha - h_1)\, d_1 = h_1(d_2 + h_1\, tg\alpha)$$

$$d_1 d_2\, tg\alpha - h_1 d_1 = h_1 d_2 + h_1^2\, tg\alpha \qquad (d_1 d_2 - h_1^2)\, tg\alpha = h_1(d_1+d_2)$$

$$tg\alpha = \frac{h_1(d_1+d_2)}{d_1 d_2 - h_1^2} \qquad d_1 d_2 \gg h_1^2 \Rightarrow tg\alpha \doteq \frac{h_1(d_1+d_2)}{d_1 d_2}$$

$$\alpha \to 0 \quad tg\alpha \doteq \alpha \qquad \boxed{\alpha \doteq \frac{h_1(d_1+d_2)}{d_1 d_2}}$$



$$d'' = \sqrt{d_1^2 + r_n^2} + \sqrt{d_2^2 + r_n^2} \doteq d_1\left(1+\frac{r_n^2}{2d_1^2}\right) + d_2\left(1+\frac{r_n^2}{2d_2^2}\right)$$

$$n\cdot\frac{\lambda}{2} = \left(d_1 + \frac{r_n^2}{2d_1} + d_2 + \frac{r_n^2}{2d_2}\right) - (d_1+d_2)$$

$$2d_1 d_2\cdot n\frac{\lambda}{2} = 2d_1^2 d_2 + r_n^2 d_2 + 2d_1 d_2^2 + r_n^2 d_1 - 2d_1 d_2(d_1+d_2)$$

$$r_n^2(d_1+d_2) = d_1 d_2\, n\lambda - 2d_1 d_2(d_1+d_2) + 2d_1 d_2(d_1+d_2)$$

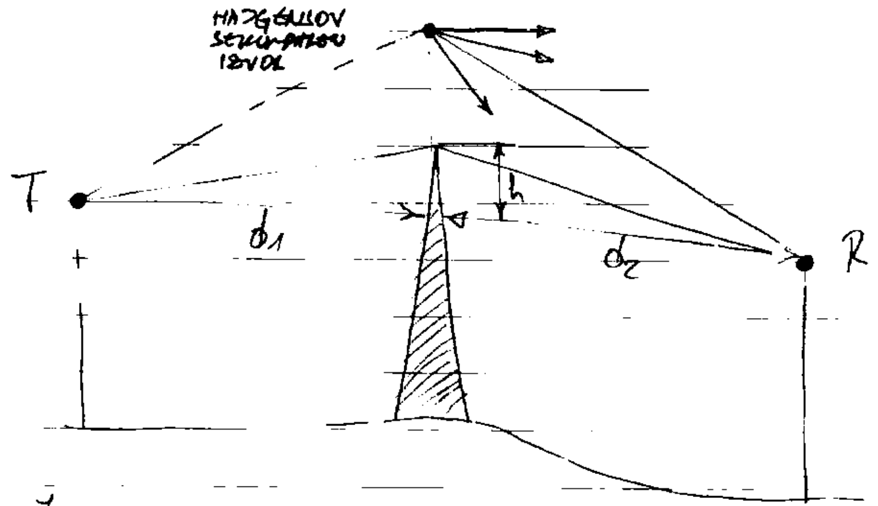
$$\boxed{r_n = \sqrt{\frac{n\lambda\, d_1 d_2}{d_1+d_2}}}$$ RADIUS NA FRENELOVE ZONY

Ex: $d_1 = 3km$ $d_2 = 3km$ $\lambda = \frac{1}{3} m$ ($f = 900 MHz$)

$$r_1 = \sqrt{\frac{\lambda \cdot d_1 d_2}{d_1 + d_2}} = \sqrt{\frac{1}{3} \cdot \frac{3 \cdot 10^3 \cdot 3 \cdot 10^3}{6 \cdot 10^3}} = \sqrt{\frac{3 \cdot 10^3}{6}} = \sqrt{5} \cdot 10^2$$

$$\boxed{r_1 = 223 m}$$

- KNIFE EDGE DIFFRACTION MODEL



- JAČINATA NA ELEKTRIČNOTO POLE ZA „KNIFE-EDGE DIFFRACTED WAVE" e:

$$\boxed{\frac{E_d}{E_0} = F(v) = \frac{1+j}{2} \int_v^{\infty} e^{-j\frac{\pi t^2}{2}} dt}$$

$F(v)$ - Fresnel integral

$$v = \alpha \sqrt{\frac{2 d_1 d_2}{\lambda (d_1 + d_2)}}$$

Fresnel - Kirchhoff parameter.

DIFRACTION GAIN:

$$\boxed{G_d(dB) = 20 \log |F(v)|}$$

$|G_d|$ - SE DODAVA NA FREE SPACE PATH LOSS ZA DA SE DOBIE VKUPNIOT PL.

- APPROXIMATE SOLUTION:

$$G_d = \begin{cases} 0 & v \leq -1 \\ 20 \log (0,5 - 0,62 v) & -1 \leq v \leq 0 \\ 20 \log (0,5 \exp(-0,95 v)) & 0 \leq v \leq 1 \\ 20 \log \left(0,4 - \sqrt{0,1184 - (0,38 - 0,1 v)^2}\right) & 1 \leq v \leq 2,4 \\ 20 \log \left(\frac{0,225}{v}\right) & v > 2,4 \end{cases}$$

Ex 3.7 diff. loss = ? FOR THE THREE CASES FROM Fig. 3.12

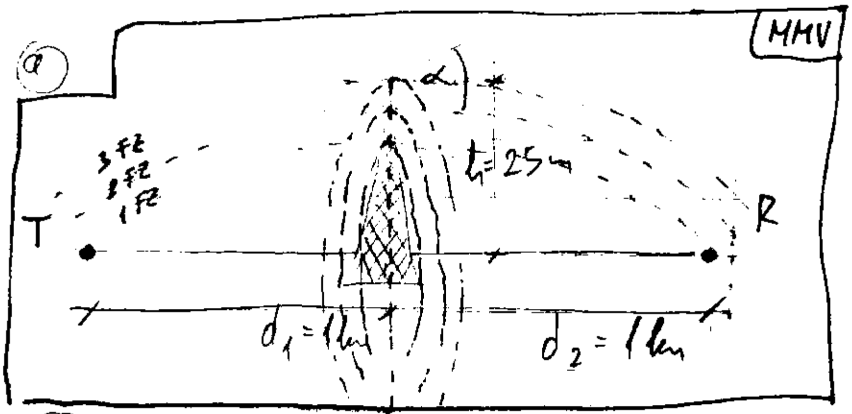
$\lambda = \frac{1}{3} m$ $d_1 = 1 km$ $d_2 = 1 km$

(a) $h = 25 m$ (b) $h = 0$ (c) $h = -25 m$

- COMPARE VALUES OBTAINED BY EXACT AND APPROX. SOLUTION

MMV

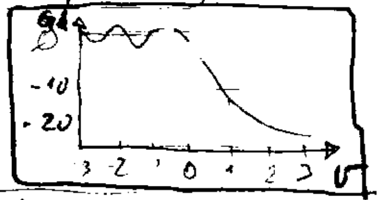
(a)



$$v = \alpha \sqrt{\frac{2 d_1 d_2}{\lambda (d_1 + d_2)}} = h_1 \frac{d_1 + d_2}{d_1 d_2} \sqrt{\frac{2 d_1 d_2}{d_1 + d_2}} = \frac{1}{2}\sqrt{30} \doteq 2{,}74$$

$G_d(v) = -21{,}7$ dB

(b) $h_1 = 0 \Rightarrow v = 0$ $G_d = -6$ dB



(c) $G_d(v) = -0{,}74$ dB

$$v = h_1 \frac{d_1 + d_2}{d_1 d_2} \sqrt{\frac{2 d_1 d_2}{\lambda (d_1 + d_2)}} = h_1 \sqrt{\frac{2(d_1 + d_2)}{\lambda d_1 d_2}}$$

(a) $$\Delta = \frac{h_1^2 (d_1 + d_2)}{2 d_1 d_2} = \frac{5}{8} = 0{,}625 \text{ m}$$

COMPLETELY BLOCKS FIRST 3 FZ HUGE LOSSES

$$\frac{\lambda}{2} = \frac{1}{6} \qquad \frac{1}{6} \cdot n = \frac{5}{8} \quad n = \frac{30}{8} = 3{,}75$$

(b) $\Delta = 0$ TIP LIES IN THE MIDDLE OF FIRST FZ

(c) $\Delta = 0{,}625$ $n = 3{,}75$ (COMPLETELY BLOCKS FIRST 3 FZ) BELOW LOS ⇒ NEGLEGIBLE LOSS

$$v = h_1 \sqrt{\frac{2(d_1 + d_2)}{\lambda d_1 d_2}} \qquad h_1 = r_n = \sqrt{\frac{n \lambda d_1 d_2}{d_1 + d_2}}$$

$$v = \sqrt{\frac{n \lambda d_1 d_2}{d_1 + d_2}} \sqrt{\frac{2(d_1 + d_2)}{\lambda d_1 d_2}} = \sqrt{2n}$$

$v = \sqrt{2n}$

IF $h_1 = 10$ m $\Rightarrow$ $$\Delta = \frac{h_1^2 (d_1 + d_2)}{2 d_1 d_2} = \frac{100 (2000)}{2 \cdot 10^6}$$

$$\Delta = \frac{2 \cdot 10^5}{2 \cdot 10^6} = \frac{1}{10} = 0{,}1 \text{ m} \qquad \Delta = n \frac{\lambda}{2}$$

$$n = \frac{2\Delta}{\lambda} = \frac{2 \cdot 1/10}{1/3} = \frac{2 \cdot 3}{10} = \frac{6}{10} = 0{,}6$$

$G_d(0{,}6) = -11$

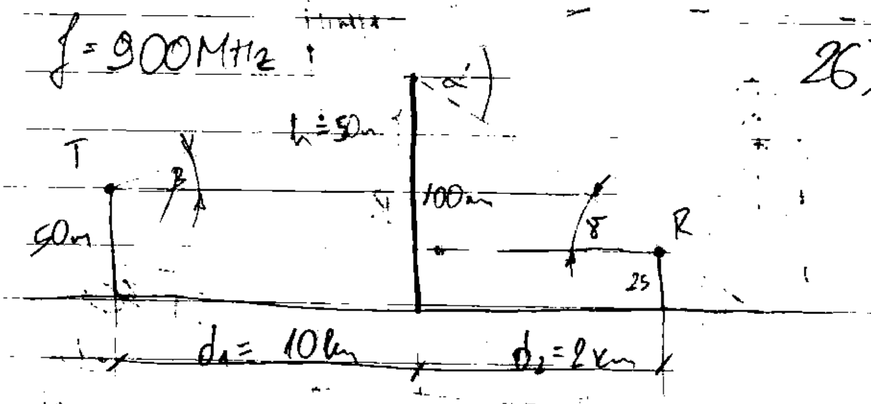
SEVISTE ZNAČITELNO

$h_1 = ?$ T.Š. 0,55 (55%) OD 1. FZ ... SLOBODNO

$$\frac{h_1^2 (d_1 + d_2)}{2 d_1 d_2} = 0{,}55 \cdot \frac{\lambda}{2} \qquad h_{max} = \sqrt{0{,}55 \frac{\lambda d_1 d_2}{d_1 + d_2}}$$

$$h_{max} = \sqrt{0{,}55 \frac{0{,}33 \cdot 10^6}{2 \cdot 10^3}} = \sqrt{\frac{0{,}55 \cdot 0{,}33 \cdot 10^3}{2}} = 9{,}57$$

Ex 3.8 $f = 900\,MHz$ 263 540



a) $G_d = ?$ b) $h = ?$ for $G_d = -6\,dB$

a) $v = h\sqrt{\frac{2(d_1+d_2)}{\lambda d_1 d_2}} = 50\sqrt{\frac{2 \cdot 12 \cdot 10^3}{\frac{1}{3} \cdot 20 \cdot 10^6}} = 100\sqrt{\frac{18}{5}\cdot 10^{-3}} = 3$

$G_d(v) = G_d(3) = -22{,}52\,dB \qquad PL = +22{,}52\,dB$

b) $G_d = -6\,dB \Rightarrow v = 0 \Rightarrow h = 0$

POTČEN PRISTUP

ALT: $\beta + \gamma + \pi - \alpha = \pi \qquad \alpha = \beta + \gamma$

$tg\,\beta = \frac{50}{d_1} = \frac{50}{10 \cdot 10^3} = 5 \cdot 10^{-3} \qquad \beta = arctg\, 5 \cdot 10^{-3} = 0{,}05\,rad = 0{,}286^\circ$

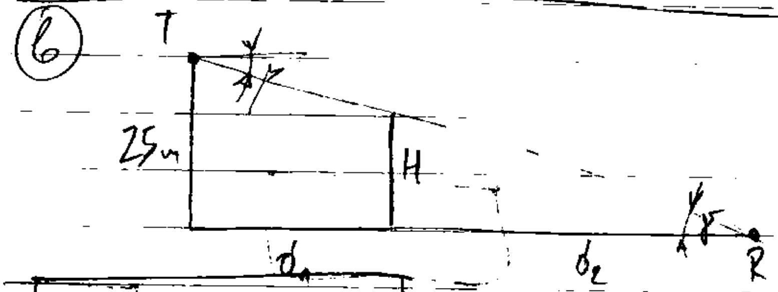
$tg\,\gamma = \frac{75}{d_2} = \frac{75}{2 \cdot 10^3} = 37{,}5 \cdot 10^{-3} \qquad \gamma = arctg\, 37{,}5 \cdot 10^{-3} = 0{,}037\,rad = 2{,}15^\circ$

$\alpha = \beta + \gamma = 2{,}434^\circ = 0{,}0425\,rad$

$v = \alpha\sqrt{\frac{2 d_1 d_2}{\lambda(d_1+d_2)}} = 0{,}0425\sqrt{\frac{2 \cdot 10^4 \cdot 2 \cdot 10^3}{\frac{1}{3}(12 \cdot 10^3)}} = 0{,}0425\sqrt{\frac{4 \cdot 10^7}{4 \cdot 10^3}}$

$v = 0{,}0425\sqrt{10^4} = 4{,}25 \qquad \boxed{G_d(v) = -25{,}5\,dB}$

b)



$\beta = \gamma$

$h = 0$

$\frac{50}{d_1 + d_2} = \frac{H}{d_2}$

$H = \frac{25\,d_2}{d_1+d_2} = \frac{25 \cdot 2 \cdot 10^3}{12 \cdot 10^3}$

$\boxed{H = 25/6 = 4{,}17\,m}$

· SCATTERING $h_c = \frac{\lambda}{8 \sin\theta_i}$ RAYLEIGH CRITERION

$h < h_c \rightarrow$ SMOOTH

$h > h_c$ - ROUGH

- if surface height is gaussian random variable

$\boxed{\rho_s = \exp\left[-8\left(\frac{\pi \sigma_h \sin\theta_i}{\lambda}\right)^2\right]}$ for rough surfaces reflection coefficients to be multiplied by ρ_s

$$\sigma_h^2 = \overline{(h-\bar{h})^2}$$

$$\rho_S = \exp\left[-8\left(\frac{\pi \sigma_h \sin\theta_i}{\lambda}\right)^2\right] I_0\left[8\left(\frac{\pi\sigma_h \sin\theta_i}{\lambda}\right)^2\right]$$

$$\Gamma_{rough} = \rho_S \Gamma$$

- RADAR CROSS SECTION MODEL

$$P_R(dBm) = P_T(dBm) + G_T(dBi) + 20\log(\lambda) + RCS[dB\,m^2] - 30\log(4\pi) - 20\log d_T - 20\log d_R$$

- PRACTICAL LINK BUDGET DESIGN USING PATH LOSS MODELS

$$\overline{PL}(d) \sim \left(\frac{d}{d_0}\right)^n \qquad \overline{PL}(dB) = \overline{PL}(d_0) + 10n\log\left(\frac{d}{d_0}\right)$$

$n=2$ FOR FREE SPACE — PATH LOSS EXPONENT

- LOG-NORMAL SHADOWING

$$PL(d)[dB] = \overline{PL}(d_0) + 10n\log\left(\frac{d}{d_0}\right) + X_\sigma$$ MMV

$$P_r(d) = P_t + G_t - PL(d) \quad [dB]$$

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{x} e^{-u^2}du = \frac{1}{\sqrt{\pi}}\left(\int_{-\infty}^{0} e^{-u^2}du + \int_0^x e^{-u^2}du\right) = \frac{1}{2} + \frac{1}{\sqrt{\pi}}\int_0^x e^{-u^2}du$$

(with $\int_{-\infty}^{0} e^{-u^2}du = \frac{\sqrt{\pi}}{2}$)

$$Q(z) = \frac{1}{\sqrt{2\pi}}\int e^{-\frac{x^2}{2}}dx = \frac{1}{2}\left[1-\operatorname{erf}\left(\frac{z}{\sqrt{2}}\right)\right]$$

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0 e^{-u^2}du$$

$$Q(z) = \quad \frac{x}{\sqrt{2}} = u \quad x = z \to u = \frac{z}{\sqrt{2}} \qquad \frac{1}{\sqrt{2\pi}}\int^{\infty} e^{-u^2}\sqrt{2}\,du$$

$$dx = \sqrt{2}\,du$$

$$= \frac{1}{\sqrt{\pi}}\int_{\frac{z}{\sqrt{2}}}^{\infty} e^{-u^2}du = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-u^2}du - \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\frac{z}{\sqrt{2}}} e^{-u^2}du = \circledast$$

$$Q(z) = 1 - \operatorname{erf}(z/\sqrt{2})$$

$$\operatorname{erf}\left(\frac{z}{\sqrt{2}}\right) = \frac{2}{\sqrt{\pi}}\int_0^{z/\sqrt{2}} e^{-u^2}du \quad \Big| \quad \circledast = 1 - \frac{1}{\sqrt{\pi}}\int_{-\infty}^{0} e^{-u^2}du - \frac{1}{\sqrt{\pi}}\int_0^{z/\sqrt{2}} e^{-u^2}du$$

$$= 1 - \frac{\sqrt{\pi}}{2} - \frac{2}{\sqrt{\pi}}\int_0 e^{-u^2}du$$

$$Q(z) = \frac{1}{2}\left[1 - \mathrm{erf}\left(\frac{z}{\sqrt{2}}\right)\right]$$

$$Q(z) = \frac{1}{\sqrt{2\pi}} \int_z^{\infty} e^{-\frac{x^2}{2}} dx \quad \mu = \frac{x}{\sqrt{2}} \quad d\mu = \frac{dx}{\sqrt{2}}$$

$$x = z \quad \mu = \frac{z}{\sqrt{2}}$$

$$Q(z) = \frac{1}{\sqrt{2\pi}} \sqrt{2} \int_{\frac{z}{\sqrt{2}}}^{\infty} e^{-\mu^2} d\mu = \frac{1}{2} \underbrace{\frac{2}{\sqrt{\pi}} \int_{z/\sqrt{2}}^{\infty} e^{-\mu^2} d\mu}_{\mathrm{erfc}\left(\frac{z}{\sqrt{2}}\right)}$$

$$Q(z) = \frac{1}{2} \mathrm{erfc}\left(\frac{z}{\sqrt{2}}\right) = \frac{1}{2}\left[1 - \mathrm{erf}\left(\frac{z}{\sqrt{2}}\right)\right]$$

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\mu^2} d\mu \qquad \mathrm{erfc} = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-\mu^2} d\mu$$

$$\underline{\mathrm{erfc}(z)} = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu^2} d\mu - \frac{2}{\sqrt{\pi}} \int_{-\infty}^{z} e^{-\mu^2} d\mu =$$

$$= \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu^2} d\mu - \frac{2}{\sqrt{\pi}} \int_{-\infty}^{0} e^{-\mu^2} d\mu - \underbrace{\frac{2}{\sqrt{\pi}} \int_0^z e^{-\mu^2} d\mu}_{\mathrm{erf}(z)}$$

$$= 2 - 1 - \mathrm{erf}(z) = \underline{1 - \mathrm{erf}(z)}$$

- PROBABILITY THAT THE RECEIVED SIGNAL LEVEL WILL EXCEED A CERTAIN VALUE γ

$$P_r\left[P_r(d) > \gamma\right] = Q\left(\frac{\gamma - \overline{P_r}(d)}{\sigma}\right) \Rightarrow \text{ABOVE } \gamma$$

$$P_r\left[P_r(d) < \gamma\right] = Q\left(\frac{\overline{P_r}(d) - \gamma}{\sigma}\right) \Rightarrow \text{BELOW } \gamma$$

CDF

- DETERMINATION OF PERCENTAGE OF COVERAGE AREA

$U(\gamma)$ – PERCENTAGE OF USEFUL SERVICE AREA

$$U(\gamma) = \frac{1}{\pi R^2} \int P_r\left[P_r(r) > \gamma\right] dA = \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R P_r\left[P_r(r) > \gamma\right] r\, dr\, d\theta$$

$$P_r\left[P_r(d) > \gamma\right] = \frac{1}{2}\left\{1 - \mathrm{erf}\left[\frac{\gamma - \left(P_t - \left(\overline{PL}(d_0) + 10n\log\frac{r}{d_0}\right)\right)}{\sqrt{2}\,\sigma}\right]\right\}$$

$$\overline{PL}(r) = \overline{PL}(d_0) + 10n\log\frac{R}{d_0} + 10n\log\left(\frac{r}{R}\right)$$

$$P_r[P_r(r)>\gamma] = \frac{1}{2}-\frac{1}{2}\,\mathrm{erf}\left[\frac{\gamma-(P_t-\overline{PL}(d_0)+10n\log\frac{R}{d_0}+10n\log\frac{r}{R}))}{\sigma\sqrt{2}}\right]$$

$$\boxed{a = \frac{\gamma - P_t + \overline{PL}(d_0) + 10n\log\frac{R}{d_0}}{\sigma\sqrt{2}}} \qquad \boxed{b = \frac{10n\log e}{\sigma\sqrt{2}}}$$

$$P_r[P_r(r)>\gamma] = \frac{1}{2}-\frac{1}{2}\,\mathrm{erf}\left[a+\frac{10n}{\sigma\sqrt{2}}\log\left(\frac{r}{R}\right)\right] = \frac{1}{2}-\frac{1}{2}\mathrm{erf}\left(a+b\ln\frac{r}{R}\right)$$

$$\log(x) = \frac{\ln x}{\ln 10} \qquad \log\frac{r}{R} = \frac{\ln\frac{r}{R}}{\log 10}$$

$$\mathrm{ld}(x) = \frac{\ln x}{\ln 2} \qquad \log(x) = \log e\,\ln x$$

$$x = e^{\ln x} \qquad \log(x) = \log e^{\ln x} = \ln x \log e$$

$$U(\gamma) = \frac{1}{\pi R^2}\int_0^{2\pi}\int_0^R\left[\frac{1}{2}-\frac{1}{2}\mathrm{erf}\left(a+b\ln\frac{r}{R}\right)\right] r\,dr\,d\Theta =$$

$$= \underbrace{\frac{2}{R^2}\int_0^R\frac{1}{2}r\,dr}_{(*)} - \frac{2}{R^2}\cdot\frac{1}{2}\int_0^R r\cdot\mathrm{erf}\left(a+b\ln\frac{r}{R}\right)dr =$$

$$(*) = \frac{2}{2R^2}\,\frac{R^2}{2} = \frac{1}{2} \qquad \boxed{U(\gamma) = \frac{1}{2}-\frac{1}{R^2}\int_0^R r\cdot\mathrm{erf}\left(a+b\ln\frac{r}{R}\right)dr}$$

$$t = \mathrm{erf}\left(a+b\ln\frac{r}{R}\right) \qquad \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du$$

$$g(x) = \int_0^x f(u)\,du \qquad \frac{dg(x)}{dx} = f(x)$$

$$dt = \frac{2}{\sqrt{\pi}}e^{-\left(a+b\ln\frac{r}{R}\right)^2}dr$$

$$U(\gamma) = \frac{1}{2}-\frac{1}{R^2}\int_0^R\frac{2r}{\sqrt{\pi}}e^{-\left(a+b\ln\frac{r}{R}\right)^2}dr = \frac{1}{2}-\frac{2}{\sqrt{\pi}R^2}\underbrace{\int_0^R r\,e^{-\left(a+b\ln\frac{r}{R}\right)^2}dr}_{(*)}$$

$$(*):\ \left|\begin{array}{l} u = a+b\ln\frac{r}{R} \quad \ln\frac{r}{R} = \frac{u-a}{b} \quad \left[r = R\,e^{\frac{u-a}{b}}\right] \\ du = b\,\frac{1}{\left(\frac{r}{R}\right)}\cdot\frac{1}{R}dr = b\,\frac{R}{r}\,\frac{1}{R}dr = \frac{b}{r}dr \end{array}\right. \qquad \begin{array}{l} r=0 \quad u=-\infty \\ r=R \quad u=a+b\ln\frac{R}{R}=a \end{array}$$

$$\int_{-\infty}^a r\,e^{-u^2}\frac{r\,du}{b} = \frac{1}{b}\int_{-\infty}^a r^2 e^{-u^2}du = \frac{1}{b}\int_{-\infty}^a R^2 e^{\frac{2(u-a)}{b}}\cdot e^{-u^2}du$$

$$= \frac{R^2}{b}\int_{-\infty}^a e^{\frac{2u}{b}}e^{-\frac{2a}{b}}e^{-u^2}du = \ldots \qquad +\frac{2u}{b}-u^2-\frac{1}{b^2} \qquad -\left(u-\frac{1}{b}\right)^2 = -\left(u^2-\frac{2u}{b}+\frac{1}{b^2}\right) = -u^2+\frac{2u}{b}-\frac{1}{b^2}$$

$$= \frac{R^2}{b}e^{-\frac{2a}{b}+\frac{1}{b^2}}\int_{-\infty}^{a-\frac{1}{b}} e^{-\left(u-\frac{1}{b}\right)^2}d\left(u-\frac{1}{b}\right) = \frac{R^2}{b}e^{\frac{-2ab+1}{b^2}}\left(\int_{-\infty}^0 e^{-v^2}dv+\int_0^{a-\frac{1}{b}}e^{-v^2}dv\right)$$

$$U(\delta) = \frac{1}{2} - \frac{e}{\sqrt{\pi} R^2} \frac{R^2}{b} e^{\frac{(1-2ab)}{b^2}} \left[\frac{\sqrt{\pi}}{2} + \frac{\sqrt{\pi}}{2} \operatorname{erf}\left(a - \frac{1}{b}\right) \right]$$

$$U(\delta) = \frac{1}{2} - \frac{1}{b} e^{\frac{1-2ab}{b^2}} \left[1 + \operatorname{erf}\left(a - \frac{1}{b}\right) \right] = \frac{1}{2} - \frac{1}{b} e^{\frac{1-2ab}{b^2}} \left[1 - \operatorname{erf}\left(\frac{1-a}{b}\right) \right]$$

$$U(\delta) = \frac{1}{2} - \frac{1}{b} e^{\frac{1-2ab}{b^2}} \left[1 - \operatorname{erf}\left(\frac{1-ab}{b}\right) \right]$$

if $a = 0 \Rightarrow$ $$U(\delta) = \frac{1}{2} - \frac{1}{b} e^{\frac{1}{b^2}} \left[1 - \operatorname{erf}\left(\frac{1}{b}\right) \right]$$

Book: $$U(\delta) = \frac{1}{2} \left[1 + \exp\left(\frac{1}{b^2}\right) \left(1 - \operatorname{erf}\left(\frac{1}{b}\right)\right) \right]$$ for $a = 0$

$b = \frac{10 n \log e}{\sigma \sqrt{2}}$, $a \neq 0$:

$$U(\delta) = \frac{1}{2} \left(1 - \operatorname{erf}(a) + \exp\left(\frac{1-2ab}{b^2}\right) \left[1 - \operatorname{erf}\left(\frac{1-ab}{b}\right) \right] \right)$$

$$(*) = \int_0^R r\, e^{-(a + b \ln \frac{r}{R})^2} dr = \left| \begin{matrix} u = r & du = dr \\ v = \int e^{-(a+b\ln\frac{r}{R})^2} dr \end{matrix} \right|$$

v: (1)

$$(1) = \left| \begin{matrix} u = a + b \ln \frac{r}{R} & r = R e^{\frac{u-a}{b}} \\ du = \frac{b}{\frac{r}{R}} \cdot \frac{1}{R} dr = \frac{b}{r} dr \end{matrix} \right| = \int e^{-u^2} \frac{r}{b} du = \frac{1}{b} \int e^{-u^2} \cdot R e^{\frac{u-a}{b}} du$$

$$= \frac{R}{b} \int e^{-u^2} \cdot e^{\frac{u}{b}} \cdot e^{-\frac{a}{b}} du =$$

Ex. 3.9 $d_0 = 100\,m$, $d_1 = 200\,m$, $d_2 = 1\,km$, $d_3 = 3\,km$

$$\overline{PL}(d) = \overline{PL}(d_0) + 10 n \log\left(\frac{d}{d_0}\right) + X_\sigma$$

(a) MMSE ~~(~~minimum mean square ~~estimate)~~ = ? for n

(b) $\sigma = ?$

(c) $P_r = ?$ AT $d = 2\,km$

(d) $P_r[P_r(2km) > -60\,dBm] = ?$

(e) $U[\delta = -60\,dBm] = ?$

	d	received PW	PL
0	100 m	0 dBm	~~0~~
1	200 m	-20 dBm	20
2	1000 m	-35 dBm	35
3	3000 m	-70 dBm	70

(a) $\overline{PL}(d_0) = 0\,dBm$

$$\overline{PL}(d_1) = 10 n \log\left(\frac{d_1}{10^2}\right) + X_\sigma \qquad n_1 = \frac{\overline{PL}(d_1) - X_\sigma}{10 \log\left(\frac{d_1}{100}\right)}$$

$$n_1 = \frac{20\,dBm - X_\sigma}{10 \log 2}; \quad n_2 = \frac{35 - X_\sigma}{10 \log 10}; \quad n_3 = \frac{70 - X_\sigma}{10 \log 30}$$

$|X_\sigma| = \emptyset$ $n_1 = 6{,}64$ $n_2 = 3{,}5$ $n_3 = 4{,}73$

$$n_{sq}^2 = \frac{1}{3}\left(n_1^2 + n_2^2 + n_3^2\right) \qquad \bar{n} = 4{,}96 \qquad n_{sq} = 5{,}127$$

(b) $$\sigma^2 = \left[(\bar{n}-n_1)^2 + (\bar{n}-n_2)^2 + (\bar{n}-n_3)^2\right]\frac{1}{3} = (1{,}2930)^2$$

$\boxed{\sigma = 1{,}2930}$

$$P_R = \frac{P_T \cdot G_1 \cdot G_2 \cdot \lambda^2}{(4\pi)^2 \cdot d^2 \cdot L}$$

$d_i = [100, 200, 1000, 3000]$

$p_i = [0, -20, -35, -70]$

$$20 = 10\, n \log 2 + X\sigma$$
$$35 = 10\, n \log 10 + X\sigma$$
$$70 = 10\, n \log 30 + X\sigma$$

(a) p_i – power at distance d_i

$\hat{p}_i$ – estimate of p_i using $\left(\frac{d}{d_0}\right)^n$ path loss model

$$J(n) = \sum_{i=1}^{k} (p_i - \hat{p}_i)^2 \qquad \frac{d}{dn}(J(n)) = \emptyset \Rightarrow n =$$

$$PL(dB) = PL(d_0) + 10 n \log\left(\frac{d}{d_0}\right)$$

$$\hat{p}_i = p_i(d_0) - 10 n \log \frac{d}{d_0} \qquad \hat{p}_1 = -10 n \log\left(\frac{d_1}{d_0}\right) =$$

$$\hat{p}_1 = -10 n \log\left(\frac{100}{100}\right) = \emptyset \qquad \hat{p}_2 = -10 n \log \frac{200}{100} = -3n$$

$$\hat{p}_3 = -10 n \log 10 = -10n \qquad \hat{p}_4 = -14{,}77\, n$$

$$J(n) = (0-0)^2 + (-20+3n)^2 + (-35+10n)^2 + (-70+14{,}77n)^2$$

$$J(n) = 327{,}2\, n^2 - 2887{,}8\, n + 6525$$

$$\frac{d}{dn}[J(n)] = 654{,}3\, n - 2887{,}8 = \emptyset \Rightarrow \boxed{n = 4{,}41}$$

(b) $X\sigma(0) = \emptyset$ $X\sigma(1) = 20 - 10 n \log 2 = 6{,}7246$

$X\sigma(2) = 35 - 10 n \log 10 = -9{,}1$

$X\sigma(3) = 70 - 10 n \log 30 = -4{,}85$ — $X\sigma = -1{,}97\, dB$ $\sigma_{X\sigma} = 5{,}123\, dB$

$$\sigma^2 = \frac{J(n)}{4} = \frac{1}{4}\left(327{,}2n^2 - 2887{,}8n + 6525\right) = 38{,}075$$

$$\sigma = \sqrt{38{,}075} = \underline{6{,}17\, dB}$$

(c) $P_r = ?$ $d = 2km$

$$P_r(d) = P_r(0) - 10n\log\left(\frac{d}{d_0}\right) = 0 - 10n\log\frac{2\cdot10^4}{10^3} = -10\cdot4{,}41\cdot\log(20)$$

$$P_r(d) = -57{,}4\,dB$$

$$P_r(d) = -57{,}4\,dB + X_\sigma = -57{,}4\,dB + 6{,}17\cdot\underbrace{randn(1,1)}_{=-0{,}89} = -62{,}9\,dBm$$

(d) $P_r\left[P_r(2km) > -60\,dBm\right] = ?$

$$\frac{P_r(d)-\gamma}{\sqrt2\,\sigma}$$

$$P_r\left[P_r(d) > \gamma\right] = \frac12\left[1 - erf\left(\frac{\gamma - P_r(d)}{\sqrt2\,\sigma}\right)\right] = \frac12\left(1 - \frac{2}{\sqrt\pi}\int_0 e^{-u^2}du\right) =$$

$$= \left|\frac{P_r(d)-\gamma}{\sqrt2\,\sigma} = \frac{-60+57{,}4}{\sqrt2\,6{,}17} = -0{,}3\right| = \frac12\left(1 - \frac{2}{\sqrt\pi}\int_0^{0.3} e^{-u^2}du\right)$$

$$\boxed{P_r\left[P_r(d) > \gamma\right] = 0{,}66}$$

$$\frac{1}{\sqrt{2\pi}\,\sigma}\int_0^{-60} e^{-\frac{P^2}{2\sigma^2}}\,dP = \frac{1}{\sqrt{2\pi}\,6{,}17}\int_0^{-60} e^{-\frac{P^2}{2(6{,}17)^2}}\,dP$$

(e) $U\left[\gamma = -60\,dBm\right] = 2$

$$a = \frac{\gamma - P_r(2km)}{\sigma\sqrt2} = \frac{-60+57{,}4}{6{,}17\sqrt2} = -0{,}2979 \doteq -0{,}3$$

$$b = \frac{10n}{\sigma\sqrt2}\log e = 2{,}19$$

$$U(\gamma) = \frac12\left(1 - erf(a) + exp\left(\frac{1-2ab}{b^2}\right)\left[1 - erf\left(\frac{1-ab}{b}\right)\right]\right)$$

$$\frac{\sigma}{n} = \frac{6{,}17}{4{,}41} = 1{,}4;\quad P_R\left[P_r > -60\right] = 0{,}66;\quad \boxed{U(\gamma) = 0{,}9}$$

- **OKUMURA MODEL**

$$\boxed{L_{50}(dB) = L_F + A_{mu}(f,d) - G(h_{te}) - G(h_{re}) - G_{AREA}}$$

$$G(h_{te}) = 20\log\left(\frac{h_{te}}{200}\right) \quad 1000m > h_{te} > 30m$$

$$G(h_{re}) = 10\log\left(\frac{h_{re}}{3}\right) \quad h_{re} \le 3m$$

$$G(h_{re}) = 20\log\left(\frac{h_{re}}{3}\right) \quad 10m > h_{re} > 3m$$

Ex 3.10 Find $L_{50} = ?$ $d = 50$ km $h_{te} = 100$ $h_{re} = 10$

EIRP = 1 kW 900 MHz $G_r = 1$

$$P_r = \frac{P_t G_t G_r \cdot \lambda^2}{(4\pi)^2 \cdot d^2} = \frac{EIRP \cdot \frac{1}{9}}{(4\pi)^2 d^2} = \frac{10^3}{16\pi^2 \cdot 9 \cdot 5^2 10^{8}}$$

$$P_r = \frac{1}{16 \cdot 9 \cdot 25\pi^2} \cdot 10^{-5} = 2{,}8 \cdot 10^{-5} \cdot 10^{-5} = 2{,}8 \cdot 10^{-10} = 0{,}28 \text{ nW}$$

$$L_{50}(dB) = L_F + A_{mu}(f,d) - G(h_{te}) - G(h_{re}) - G_{AREA} \doteq 9\,dB$$

$$G(h_{te}) = 20 \log (0.5) = -6\,dB$$

$$G(h_{re}) = 20 \log \left(\frac{h_{re}}{3}\right) = 10\,dB$$

$$L_F = 20 \log \frac{P_T}{P_R} = 10 \log \frac{10^3}{0{,}28 \cdot 10^{-9}} = 10 \log \frac{10^{12}}{0{,}28} = 125\,dB$$

$$L_{50} = 125 + A_{mu}(f,d) + 6 - 10 - 9 = 121\,dB + A_{mu} - 9$$

$$L_{50} = 112 + A_{mu}(900\,MHz, 50\,km) \qquad A_{mu} \doteq 35\,dB$$

$$L_{50} = 112 + 35 = \underline{147\,dB}$$

$$P_r(dB) = EIRP(dBm) - L_{50} + G_r(dB) = 10 \log \frac{10^3}{10^{-3}} - 150 + 0$$

$$P_r(dB) \doteq -90\,dB$$

- **HATA MODEL**

$$L_{50}(urban)(dB) = 69.55 + 26.16 \log f_c - 13.82 \log h_{te} - a(h_{re}) + (44{,}9 - 6{,}55 \log h_{te}) \log d$$

$$a(h_{re}) = (1.1 \log(f_c) - 0.7) h_{re} - (1.56 \log f_c - 0.8)\,dB$$

LARGE CITY

$$a(h_{re}) = 8.29 (\log 1.54 h_{re})^2 - 1.1\,dB \qquad f_c \leq 300\,MHz$$

$$a(h_{re}) = 3.2 (\log 11.75 h_{re})^2 - 4.97\,dB \qquad f_c > 300\,MHz$$

SUBURBAN $$L_{50}(dB) = L_{50}(urban) - 2[\log(f_c/28)]^2 - 5.4$$

OPEN RURAL $$L_{50}(dB) = L_{50}(urban) - 4.78 (\log f_c)^2 - 18.33 \log f_c - 40{,}98$$

- **WALFISH & BERTONI MODEL**

$$S = P_0 Q^2 P_1 \qquad P_0 = \left(\frac{\lambda}{4\pi R}\right)^2$$

$$d = 1\,m \quad \frac{P_{r0}}{P_T} = \frac{\lambda^2}{(4\pi)^2 d^2} = \frac{\left(\frac{300}{1900}\right)^2}{(4\pi)^2} = \frac{1}{16\pi^2 \cdot 19^2} = 1{,}6 \cdot 10^{-4}$$

$$P_1 = PL(d_0) = 10 \log \frac{1}{1{,}6 \cdot 10^{-4}} = \underline{38\,dB}$$

- **WIDEBAND PCS MICROCELL MODEL**

$$d_f = \frac{1}{\lambda} \sqrt{(\Sigma^2 - \Delta^2)^2 - 2(\Sigma^2 + \Delta^2)\left(\frac{\lambda}{2}\right)^2 + \left(\frac{\lambda}{2}\right)^4}$$ } FRESNEL I ZONE CLEARANCE DISTANCE

$$\overline{PL}(d) = \begin{cases} 10 n_1 \log(d) + p_1 & 1 < d < d_f \\ 10 n_2 \log(d/d_f) + 10 n_1 \log d_f + p_1 & d > d_f \end{cases}$$

$$p_1 = PL(d_0) = \frac{P_t \cdot G_t G_r \lambda^2}{(4\pi)^2 d^2} \qquad 10\log \frac{P_1}{P_t} = 10\log \frac{(0{,}158)^2}{(4\pi)^2}$$

$$\lambda = \frac{300}{1900} = \frac{3}{19} = 0{,}158 \quad ; \quad PL = 10\log \frac{P_t}{P_1} = 38\ dB$$

• OBS ~~CASE~~ : $\overline{PL}(d) = 10 n \log(d) + p_1$

3.11 Indoor Propagation Models

$$PL(dB) = PL(d_0) + 10 n \log\left(\frac{d}{d_0}\right) + X_\sigma$$

• Attenuation factor model

$$\overline{PL}(dB) = PL(d_0) + 10\, n_{SF} \log\left(\frac{d}{d_0}\right) + FAF(dB)$$

$$\overline{PL}(d) = \overline{PL}(d_0) + 10\log\left(\frac{d}{d_0}\right) + \alpha d + FAF(dB)$$

Example 3.11 $n = 3{,}27$ 1 floor $n = 5.22$ 3 floor
$FAF = 24{,}4$ 3 floors btw T & R

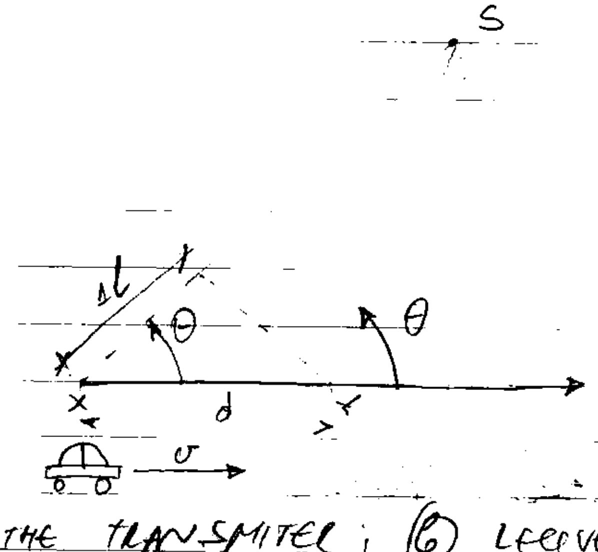
• $PL(30) = PL(d_0) + 10 \cdot 3{,}27 \log\left(\frac{30}{1}\right) + 24{,}4$ [3.94]

$$PL(d_0) = -10\log \frac{\frac{1}{9}}{16\pi^2} = -10\log \frac{1}{9 \cdot 16 \cdot \pi^2} = 31\ dB$$

$$PL(30) = 31 + 48{,}3 + 24{,}4 = 103{,}7\ dB$$

• [3.95] $PL(30) = PL(d_0) + 10 \cdot 5{,}22 \log 30 = 108{,}105\ dB$

CH4: Small-Scale Fading and Multipath



$$\Delta l = d \cdot \cos\theta = v \cdot \Delta t \cos\theta$$

$$\Delta\phi = \frac{2\pi \Delta l}{\lambda} = \frac{2\pi v \Delta t}{\lambda} \cos\theta$$

$$f_d = \frac{\omega_d}{2\pi} = \frac{\Delta\phi}{2\pi \Delta t} = \frac{v}{\lambda}\cos\theta$$

Example 4.1 $f = 1850\ MHz$
$v = 60$ mph

(a) Receiver is moving towards the transmiter ; (b) receiver away from transmiter
(c) direction perpendicular to the direction of arrival

(a.) $f = 1850\,MHz$ $\quad v = 60\,mph = 60 \cdot 1{,}61\,km/h = 26{,}8\,m/sec$

$f_d = \frac{v}{\lambda} \cdot \cos(\theta)$; $\quad \lambda = \frac{300}{1850} = 0{,}1622$

$f_d = \frac{26{,}8\,m/s}{0{,}1622\,m} \cos(\theta) = 165 \cos(\theta)$ $\quad f_{d\,max} = 165\,Hz$

$f_{c\,max} = f + f_{d\,max}$

$f_{c\,min} = f - f_{d\,min}$

(b) $f_{d\,min} = 165 \cos(\pi) = -165\,Hz$

$f_{c\,MAX} = 1850 \cdot 10^6 + 0.000165 \cdot 10^6 = 1850{,}000165\,MHz$

$f_{c\,MIN} = 1850 \cdot 10^6 - 0{,}000165 \cdot 10^6 = 1849{,}999835\,MHz$

(c) $f_d = \emptyset$

4.2. IMPULSE RESPONSE MODEL OF A MULTIPATH CHANNEL

$x(t)$ - TRANSMITED SIGNAL

$y(d,t)$ - RECEIVED SIGNAL AT DISTANCE „d"

$$y(d,t) = x(t) \otimes h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(d, t-\tau)\, d\tau$$

FOR CAUSAL SYSTEM $h(d,t) = \emptyset \quad t < \emptyset$

$$y(d,t) = \int_{-\infty}^{t} x(\tau)\, h(d, t-\tau)\, d\tau \qquad d = v\,t$$

$$y(vt,t) = \int_{-\infty}^{t} x(\tau) \cdot h(vt, t-\tau)\, d\tau \qquad v = const$$

$$y(t) = \int_{-\infty}^{t} x(\tau)\, h(vt, t-\tau)\, d\tau = x(t) \otimes h(vt,t) = x(t) \otimes h(d,t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t,\tau)\, d\tau = h(t) \otimes h(t,\tau)$$

$$r(t) = c(t) \otimes \frac{1}{2} h_b(t,\tau) \quad - h_b(t,\tau) \Rightarrow \text{BASEBAND IMPULSE RESPONSE}$$

$x(t)$ ▶ $h(t,\tau) = Re\{ h(t,\tau)\, e^{j\omega_c t} \}$ ▶ $y(t)$

$y(t) = Re\{ r(t)\, e^{j\omega_c t} \}$

$y(t) = x(t) \otimes h(t)$

$c(t)$ ▶ $\frac{1}{2} h_b(t,\tau)$ ▶ $r(t)$

$\frac{1}{2} r(t) = \frac{1}{2} c(t) \otimes \frac{1}{2} h_b(t)$

$x(t) = Re\{ c(t)\, e^{+j\omega_c t} \}$ $\qquad y(t) = Re\{ r(t)\, e^{j\omega_c t} \}$

$u(t) = u_m(t) \cdot \cos(\omega_0 t) = u_m(t) \left[\frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t} \right]$

$U(j\omega) = \frac{1}{2} U_m[j(\omega-\omega_0)] + \frac{1}{2} U_m[j(\omega+\omega_0)]$

$u(t) = k\, U_m \cos(\omega_m t) \cdot \cos(\omega_0 t) = \frac{k\,U_m}{2} \cos[(\omega_0-\omega_m)t] + \frac{k\,U_m}{2} \cos\ldots$

DFT $X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}$ $k = 0..N-1$

$W_N = e^{-j\frac{2\pi}{N}}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} [1 \; 1 \; 1] = \begin{bmatrix} x_1 & x_1 & x_1 \\ x_2 & x_2 & x_2 \\ x_3 & x_3 & x_3 \end{bmatrix}$ $e^{jx} = \cos x + j \sin x$

$Dt = 10 \cdot 10^{-6} = 10 \mu sec$ $\tau = 10 * Dt$ $T = 100 Dt$

$T = 1 msec$ $f = \frac{1}{T} = 100 Hz$

$N = 100$

$\cos(\pi [0:99])$ $\cos(\omega t)$ $t = n \cdot \Delta t$

$\cos(\frac{2\pi k}{N} n \Delta t)$

$\cos(2\pi f n \Delta t) = \cos(2\pi f t)$

$f = 50 kHz \Rightarrow \cos(2\pi \cdot 50 \cdot 10^3 \cdot n \cdot 10 \cdot 10^{-6}) =$

$= \cos(100 \pi \cdot n \cdot 10 \cdot 10^{-3}) = \cos(10^3 \pi \cdot n \cdot 10^{-3})$

$= \cos(\pi n) = / n = [0:99] / = \cos(\pi [0:99])$

$N = 100$ $[-50 : 50] \frac{1}{100 Dt} = [-50000 : 1000 : 50000]$

$r(t) = c(t) \otimes \frac{1}{2} h_b(t, \tau)$

BANDPASS
$x(t) \triangleright h(t,\tau) = Re\{h_b(t,\tau) e^{j\omega_c t}\} \triangleright y(t) = Re\{r(t) e^{j\omega_c t}\}$

$y(t) = x(t) \otimes h(t)$

BASEBAND
$c(t) \triangleright \frac{1}{2} h_b(t,\tau) \triangleright \frac{1}{2} r(t) = \frac{1}{2} c(t) \otimes \frac{1}{2} h_b(t)$

$x(t) = Re\{c(t) e^{-j2\pi f_c t}\}$
$y(t) = Re\{r(t) e^{-j2\pi f_c t}\}$

- AVERAGE POWER OF BANDPASS SIGNAL $\overline{x^2(t)}$ IS $\frac{1}{2} \overline{|c(t)|^2}$

EXCESS DELAY BINS:

TIME DELAY BINS : $\tau_{i+1} - \tau_i$ $\tau_0 = 0$

τ_0 - FIRST ARRIVAL SIGNAL AT RECEIVER

$\Delta\tau = \tau_1 - \tau_0$ $\tau_1 = \Delta\tau$ $\tau_i = i \Delta\tau$ $i = 0..N-1$

N - TOTAL NUMBER OF EQUALLY-SPACED MULTIPATH COMPONENTS

- FREQUENCY SPAN OF THE MODEL : $1/2\Delta\tau$ (BANDWIDTH OF THE TRANSMITTED SIGNAL SHOULD BE < $1/2\Delta\tau$)
- EXCESS DELAYS: $\tau_i \Rightarrow$ DELAY OF THE i-TH COMPONENT COMPARED TO THE FIRST ARRIVING COMPONENT

- BASEBAND IMPULSE RESPONSE OF MULTIPATH CHANNEL:

$$h_b(t,\tau) = \sum_{i=0}^{N-1} a_i(t,\tau) \exp[j2\pi f_c \tau_i(t) + \phi_i(t,\tau)] \delta(\tau - \tau_i(t))$$

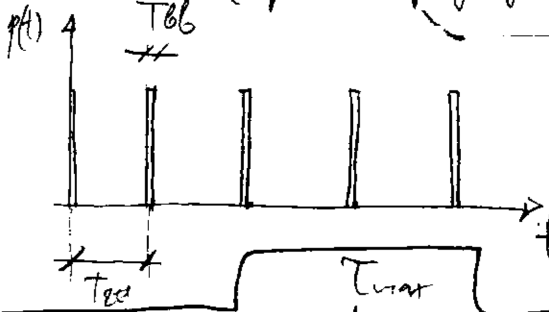
$a_i(t,\tau)$ - REAL AMPLITUDE $\tau_i(t)$ - EXCESS DELAY

- IF CHANNEL IMPULSE RESPONSE IS TIME INVARIANT. →

$$h_B(\tau) = \sum_{i=0}^{N-1} a_i \exp(-j\theta_i)\,\delta(\tau-\tau_i)$$

RELATIONSHIP BETWEEN BANDWIDTH & RX POWER

$$x(t) = \mathrm{Re}\{ p(t) \exp(j2\pi f_c t)\}$$



LET: $p(t) = 2\sqrt{\frac{\tau_{max}}{T_{bb}}}$ $0 < t < T_{bb}$

$$r(t) = \frac{1}{2}\sum_{i=0}^{N-1} a_i e^{-j\theta_i} p(t-\tau_i) =$$

$$= \sum_{i=0}^{N-1} a_i e^{-j\theta_i} \sqrt{\frac{\tau_{max}}{T_{bb}}}\, \mathrm{rect}\left[t - \frac{T_{bb}}{2} - \tau_i\right]$$

$$|r(t_0)|^2 = \frac{1}{\tau_{max}}\int_0^{\tau_{max}} r(t)\cdot r^*(t)\,dt$$

$= \Big|\, |r(t_0)|^2$ – POWER DELAY PROFILE $\Big| =$

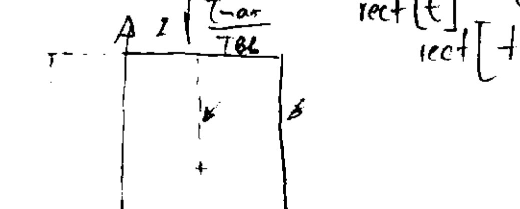
$$= \frac{1}{\tau_{max}}\int_0^{\tau_{max}} \frac{1}{4}\left[\sum_{i=0}^{N-1} a_i e^{-j\theta_i} p(t-\tau_i)\right]\left[\sum_{j=0}^{N-1} a_j e^{+j\theta_j} p(t-\tau_j)\right] =$$

$$= \frac{1}{4\tau_{max}}\int_0^{\tau_{max}}\left(\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} a_i(t_0)a_j(t_0)\,p(t-\tau_i)\,p(t-\tau_j)\,e^{-j(\theta_i-\theta_j)}\right)dt$$

$|\tau_j - \tau_i| > T_{bb}$ for $i \neq j$ →

$$|r(t_0)|^2 = \frac{1}{4\tau_{max}}\int_0^{\tau_{max}}\left[\sum_{k=0}^{N-1} a_k^2(t_0)\cdot p^2(t-\tau_k)\right]dt =$$

$$= \frac{1}{4\tau_{max}}\sum_{k=0}^{N-1} a_k^2(t_0)\int_0^{\tau_{max}}\left\{2\sqrt{\frac{\tau_{max}}{T_{bb}}}\,\mathrm{rect}\left[t-\frac{T_{bb}}{2}-\tau_k\right]\right\}^2 \quad \tau_k = 0$$



$$\int_0^{T_{bb}} \frac{\tau_{max}}{4 T_{bb}} \left(2\sqrt{\frac{\tau_{max}}{T_{bb}}}\right)^2 dt \quad \text{(I)}$$

$$\frac{\tau_{max}}{4T_{bb}} \quad \frac{4\tau_{max}}{T_{bb}} \cdot T_{bb} = \frac{\tau_{max}}{T_{bb}}$$

$$= \frac{1}{4\tau_{max}}\sum_{k=0}^{N-1} a_k^2(t_0)\,\frac{4\cdot\tau_{max}}{T_{bb}}\int_0^{T_{bb}} dt = \sum_{k=0}^{N-1} a_k^2(t_0)$$

$$E_{a,\theta}\left[P_{WB}\right] = E_{a,\theta}\left[\sum_{i=0}^{N-1}|a_i e^{j\theta_i}|^2\right] \doteq \sum_{i=0}^{N-1}\overline{a_i^2}$$

WIDE BAND

↳ ensemble average

• CW NARROWBAND SIGNAL AMPLITUDE $c(t) = 2$

$$r(t) = \sum_{i=0}^{N-1} a_i \exp(j\theta_i(t,\tau)) = \sum_{i=0}^{N-1} a_i e^{j\theta_i(t,\tau)}$$ PHASOR SUM

$$|r(t)|^2 = \left| \sum_{i=0}^{N-1} a_i \exp(j\theta_i(t,\tau)) \right|^2$$

AVERAGE RECEIVED POWER OVER LOCAL AREA IS:

$$E_{a,\theta}[P_{CW}] = E_{a,\theta}\left[\left| \sum_{i=0}^{N-1} a_i e^{j\theta} \right|^2 \right]$$ $\overline{\xi + \eta} = \overline{\xi} + \overline{\eta}$

$$E_{a,\theta}[P_{CW}] \doteq \overline{(a_0 e^{j\theta_0} + a_1 e^{j\theta_1} + \dots + a_{N-1} e^{j\theta_{N-1}}) \cdot (a_0 e^{-j\theta_0} + a_1 e^{-j\theta_1} + \dots a_{N-1} e^{-j\theta_{N-1}})}$$

$(*) = a_0 e^{j\theta_0} \cdot a_1 e^{-j\theta_1} = a_0 a_1 [\cos(\theta_0 - \theta_1) + j\sin(\theta_0 - \theta_1)]$

$(1) = a_0 e^{-j\theta_0} \cdot a_1 e^{j\theta_1} = a_0 a_1 [\cos(\theta_0 - \theta_1) - j\sin(\theta_0 - \theta_1)]$

$(*) + (1) = 2 a_0 a_1 \cos(\theta_0 - \theta_1)$

$$E_{a,\theta}[P_{CW}] \doteq \sum_{i=0}^{N-1} \overline{a_i^2} + 2 \sum_{i=0}^{N-1} \sum_{\substack{j=0 \\ i \neq j}}^{N-1} \overline{a_i a_j \cos(\theta_i - \theta_j)}$$

$$E_{a,\theta}[P_{CW}] = \sum_{i=0}^{N-1} \overline{a_i^2} + 2 \sum_{i=0}^{N-1} \sum_{\substack{j=0 \\ i \neq j}}^{N-1} V_{ij} \overline{\cos(\theta_i - \theta_j)}$$ $V_{ij} = E_a[a_i a_j]$

Example 4.2 (U) $\tau_{max} = 100\,\mu s$ – URBAN RADIO CHANNELS

(M) $\tau_{max} = 4\,\mu s$ – MICROCELLULAR –//–

$N = 64$ NUMBER OF MULTIPATH BINS

(a) $\Delta\tau = ?$ (b) MAXIMUM BANDWIDTH

(I) REPEAT THE EXERCISE WITH INDOOR CHANNEL MODEL $\tau_{max} = 500\,ns$

(a) $\tau_i = i \cdot \Delta\tau$, $i = 0 \dots N-1$ $\tau_{max} = \tau_{N-1} = (N-1)\Delta\tau$ $\Delta\tau = \frac{\tau_{max}}{N-1}$

(NAPAKA V KNJIGI) $\Delta\tau = \frac{\tau_{max}}{N}$

U: $\Delta\tau = \frac{100\,\mu s}{63} = 1{,}59\,\mu s$ M: $63{,}5\,ns$ I: $7{,}9\,ns$

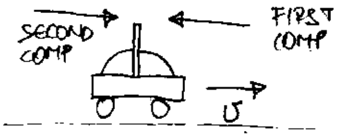
(b) $B = \frac{1}{2\Delta\tau}$ $B_U = \frac{1}{2 \cdot 1{,}59} 10^6 = 314\,kHz$; $B_M = 7{,}9\,MHz$; $B_I = 63{,}3\,MHz$

Example 4.3

$v = 10\,m/s$ RECEIVES TWO MULTIPATH COMPONENTS
$f_c = 1\,GHz$ $\tau_0 = 0$ WITH INITIAL PHASE 0° & -70 dBm

SECOND COMPONENT $\tau_1 = 1\,\mu s$ WITH INITIAL PHASE 0° & -73 dBm POWER

- COMPUTE NARROWBAND INSTANTANEOUS POWER AT TIME INTERVALS 0:0.1:0.5 s
- COMPUTE AVERAGE NARROWBAND & WIDEBAND POWERS OVER THE INTERVAL



- $v = 10\,m/s$ 0.1 TIME INTERVAL CORRESPONDS TO SPATIAL INTERVAL OF 1 m

$\lambda = \frac{300}{1000} = 0.3\,m$ $|r(t)|^2 = \left| \sum_{i=0}^{N-1} a_i e^{j\theta_i(t,\tau)} \right|^2$

$-70\,dBm = 10 \log \frac{P_1}{1\,mW} = 10 \log P_1 10^{-3} \Rightarrow 10^{-7} = P_1 10^{+3}$ $P_1 = 10^{-10}$

$P_1 = 100 \cdot 10^{-12} = 100\,pW$ $-73\,dBm = 10 \log \frac{P_2}{10^{-3}}$ $P_2 = 10^{-3} \cdot 10^{-7.3} = 50\,pW$

$|r(t)|^2 = \left| \sqrt{100\,pW}\, e^{j\theta} + \sqrt{50\,pW}\, e^{j\theta} \right|^2 = 2{,}9 \cdot 10^{-10} = 290\,pW$

$\theta_i = \frac{2\pi d}{\lambda} = \frac{2\pi v t}{\lambda}$, $t = 0.1\,s$ $\theta_1 = \frac{2\pi \cdot 10 \cdot 0.1}{0.3} = \frac{20\pi}{3}$ [rad]

$= \frac{20\pi}{3} \cdot \frac{1}{2\pi} = \frac{10}{3}$

$\frac{10}{3} 2\pi = k \cdot \frac{1}{3} 2\pi = \frac{2\pi}{3}$

$= 2{,}094$ rad

$\theta_1 = \frac{20\pi}{3} \cdot \frac{180}{\pi} = \text{REM}\left(\frac{1200°}{360}\right) \cdot 360 = \frac{360°}{3} = 120°$

$t = 0.1\,s$: $\theta_1 = 2{,}09$ $\theta_2 = -2{,}01$

$t = 0:0.1:0.5$ $R2 = 10^{-9} [0{,}2914;\ 0{,}0793;\ 0{,}0793;\ 0{,}2914;\ 0{,}0793;\ 0{,}0793]$

$\theta = [0;\ 120;\ 240;\ 0;\ 120;\ 240]$

AVERAGE NARROWBAND:

$\text{mean}(R2) = \frac{(2 \cdot 291{,}4 + 4 \cdot 79{,}3)\, 10^{-12}}{6} = 150\,pW$

WIDEBAND POWER: $E_{a,\theta}[P_{WB}] = E_{a,\theta}\left[\sum_{i=0}^{N-1} |a_i e^{j\theta_i}|^2 \right] = \sum_{i=0}^{N-1} \overline{a_i^2}$

$E_{a,\theta}[P_{WB}] = 100\,pW + 50\,pW = 150\,pW$ (P_1, P_2)

$\overline{\xi\eta} = \iint xy\, p_{\xi\eta}(x,y)\, dx\, dy$ $F_\xi(x) = \int_{-\infty}^{x} p(x)\, dt$

$F_{\xi\eta}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} p(x,y)\, dx\, dy$ $F_\xi(x) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} p(x,y)\, dx\, dy$

$\frac{d F_\xi(x)}{dx} = \int_{-\infty}^{\infty} p(x,y)\, dy = p_\xi(x)$

- **TIME DISPERSION PARAMETERS**

$\sigma_\xi^2 = \overline{(\xi - \bar{\xi})^2} = \overline{\xi^2} - \bar{\xi}^2$

- MEAN EXCESS DELAY ($\bar{\tau}$);
- RMS DELAY SPREAD (σ_τ);

$$\bar{\tau} = \frac{\sum_k a_k^2 \tau_k}{\sum_k a_k^2} = \frac{\sum_k P(\tau_k)\, \tau_k}{\sum_k P(\tau_k)}$$

$\bar{\tau}$ IS FIRST MOMENT OF THE POWER DELAY PROFILE

$$\sigma_\tau = \sqrt{\overline{\tau^2} - \bar{\tau}^2}$$ RMS DELAY SPREAD

$$\overline{\tau^2} = \frac{\sum_k a_k^2 \tau_k^2}{\sum_k a_k^2} = \frac{\sum_k P(\tau_k)\tau_k^2}{\sum_k P(\tau_k)}$$

- MAXIMUM EXCESS DELAY (X dB)

$\tau_x - \tau_0$; τ_0 - FIRST ARIVING SIGNAL

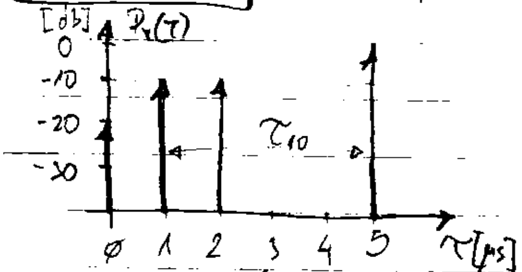
τ_x - MAXIMUM DELAY AT WHICH MULTIPATH COMPONENT IS WITHIN X dB OF THE STRONGEST ARIVING MULTIPATH COMPON.

- ~~COHERENCE~~ BANDWIDTH - B_C:

$$B_C \doteq \frac{1}{50\sigma_\tau}$$ FOR THE FREQ. CORELATION FUNCTION ABOVE 0.9

$$B_C \doteq \frac{1}{5\sigma_T}$$ FOR THE FREQ. CORRELATION FUNCTION ABOVE 0.5

Exam. 4.4 $\bar{\tau}, \sigma_\tau, \tau_x = ?$ $x = 10$ dB. Estimate 50% COHERENT BANDW. OF THE CHANNEL



$$\bar{\tau} = \frac{\sum_k a_k^2 \tau_k}{\sum_k a_k^2} = \frac{\sum_k P(\tau_k)\tau_k}{\sum_k P(\tau_k)}$$

$$\bar{\tau} = \frac{-20\cdot 0 - 10\cdot 1 - 10\cdot 2 - 0\cdot 5}{-20 - 10 - 10 + 0 + 0 - 0}$$

$$\bar{\tau} = \frac{-30}{-40} = 0{,}75\,\mu s$$

$$\overline{\tau^2} = \frac{\sum_k P(\tau_k)\tau_k^2}{\sum_k P(\tau_k)} = \frac{-20\cdot 0^2 - 10\cdot 1^2 - 10\cdot 4 - 0\cdot 5^2}{-40} = \frac{-50}{-40} = 1{,}25\,\mu s^2$$

$$\sigma_\tau = \sqrt{(1{,}25)^2 - (0{,}75)^2} = 1\,\mu s \qquad \tau_{10} = 4\,\mu s$$

$$B_C = \frac{1}{5\sigma_\tau} = 0{,}2\cdot 10^{+6} = 200\,kHz$$

OK

$10\log P(0) = -20$ $\quad 10^{-2} = P(0)$ $\quad P(0) = 0{,}01$ $\quad P(1) = P(2) = 0{,}1$ $\quad P(5) = 1$

$$\bar{\tau} = \frac{0{,}01\cdot 0 + 0{,}1\cdot 1 + 0{,}1\cdot 2 + 1\cdot 5}{0{,}01 + 0{,}1 + 0{,}1 + 1} = \frac{0{,}3 + 5}{1{,}21} = \frac{5{,}3}{1{,}21} = 4{,}38\,\mu s$$

$$\overline{\tau^2} = \frac{0{,}1\cdot 1^2 + 0{,}1\cdot 4 + 25}{1{,}21} = \frac{0{,}5 + 25}{1{,}21} = \frac{25{,}5}{1{,}21} = 21{,}074\,\mu s^2$$

$$\sigma_\tau = \sqrt{21{,}074 - (4{,}38)^2} = 1{,}375\,\mu s \qquad \tau_{10} = 4\,\mu s$$

$$B_C = \frac{1}{5\sigma_\tau} = 0.145\,10^{6} = 145\,kHz$$

AMPS : 30kHz < 145kHz "OK"

GSM : 200kHz > 145kHz NOT OK EQULIZER NEEDED

$$\Delta l = d\cdot\cos\theta \qquad \Delta\phi = \frac{2\pi\Delta l}{\lambda} = \frac{2\pi}{\lambda}\cdot d\cos\theta = \frac{2\pi}{\lambda}\,v\,\Delta t\cos\theta$$

$$f_d = \frac{\omega_d}{2\pi} = \frac{\Delta\phi}{\Delta t\cdot 2\pi} = \frac{1}{\Delta t\, 2\pi}\cdot\frac{2\pi}{\lambda}\, v\,\Delta t\cos\theta = \frac{v}{\lambda}\cdot\cos\theta$$

⊙ FREQUENCY DISPERSION

• COHERENCE TIME: (*) $T_C = \frac{1}{f_m}$ $f_m = f_{d,max}$ (MAXIMUM DOPPLER SHIFT)

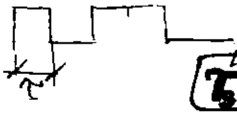
IF TIME CORRELATION FUNCTION > 0.5 THEN

(**) $T_C = \frac{9}{16\pi f_m}$ $f_m = \frac{v}{\lambda}$

287,5 : 209,8
144 : 105
SYMBOL PERIOD

GEOMETRICAL MEAN OF: (*) & (**)

(1) $T_C = \sqrt{\frac{9}{16\pi f_m^2}}$



$T_S < T_C$ NO DISTORTION

EXAMPLE: $v = 60$ mph $f_c = 900$ MHz $\lambda = 0{,}333$ m

$f_m = \frac{v}{\lambda} = \frac{60 \cdot 1{,}61 \cdot 10^3}{0{,}33 \cdot 3600} = 81{,}3$ Hz

(**) ⇒ $T_C = 2{,}2$ msec   (1) $T_C = 5{,}2$ msec

$1/T_C = 454$ bps $1/T_C = 192$ bps

EXAMPLE 4.5 PROPER SPATIAL SAMPLING INTERVAL = ?

HIGHLY CORRELATED SAMPLES IN TIME; $f_c = 1900$ MHz

$v = 50$ m/s, HOW LONG IT WILL TAKE TO MAKE THE MEASUR?

DOPPLER SPREAD $B_D = ?$

$\lambda = \frac{300}{1900} = \frac{3}{19} = 0{,}16$ m ; $B_D = f_m = \frac{v}{\lambda} = 316$ Hz   DOPPLER SPREAD

$T_C = \sqrt{\frac{9}{16\pi f_m^2}} = 1{,}34$ msec $d = v \cdot T_C = 50 \cdot 1{,}34 \cdot 10^{-3} = 6{,}68$ cm

SPATIAL SAMPLING < 6,68 cm

• CONSERVATIVE APPROACH (SMALLEST T_C)

$T_C = \frac{9}{16\pi f_m^2} = 565{,}42$ μs $d \leq v \cdot \frac{T_C}{2} = 1{,}41$ cm

10m TRAVEL DISTANCE: $N = \frac{10\text{m}}{0{,}0141} = 709$ SAMPLES

TIME FOR MEASUREMENT = $N \cdot T_C/2 = 0{,}2$ sec

TYPES OF SMALL SCALE FADING

SMALL SCALE FADING (BASED ON MULTIPATH TIME DELAY SPREAD) TIME DISPERSION

FLAT FADING
1. BW of signal < BW of channel ($B_S \ll B_C$)
2. Delay spread < Symbol period ($T_S \gg \sigma_\tau$)

FREQUENCY SELECTIVE FAD.
1.) BW signal > BW of channel
2.) Delay spread > Symbol period

SMALL SCALE FADING (BASED ON DOPPLER SPREAD) FRQ. DISPERSION

FAST FADING
1. High Doppler spread
2. Coherence time < Symbol period
3. Channel variations faster than the band signal variat.

SLOW FADING
1. Low Doppler spread
2. Coherence time > Symbol time
3. Channel variation slower than baseband signal variation

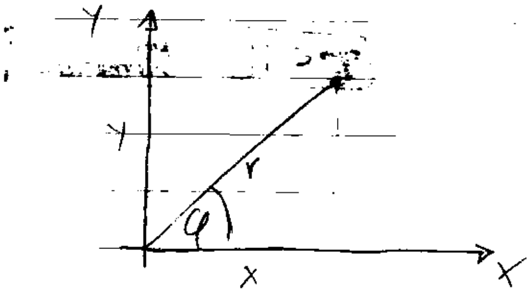
$u(t) = x(t)\cos(\omega_0 t) - y(t)\sin(\omega_0 t)$

$u(t) = r(t)\cos(\omega_0 t + \varphi(t))$

$r(t) = \sqrt{x^2(t) + y^2(t)}$

$\varphi(t) = \operatorname{arctg}\frac{y(t)}{x(t)}$

~~$u(t) = \frac{x(t)}{2}\left(e^{j\theta} + e^{-j\theta}\right) - \frac{y(t)}{2}\left(e^{j\theta} - e^{-j\theta}\right)$~~



$x = r \cdot \cos(\varphi)$

$y = r \cdot \sin(\varphi)$

$\varphi = \operatorname{arctg}\frac{y}{x}$

$u(t) = r\cos\varphi \cdot \cos(\omega_0 t) - r\sin\varphi \sin(\omega_0 t) = r\cos(\omega_0 t + \varphi)$

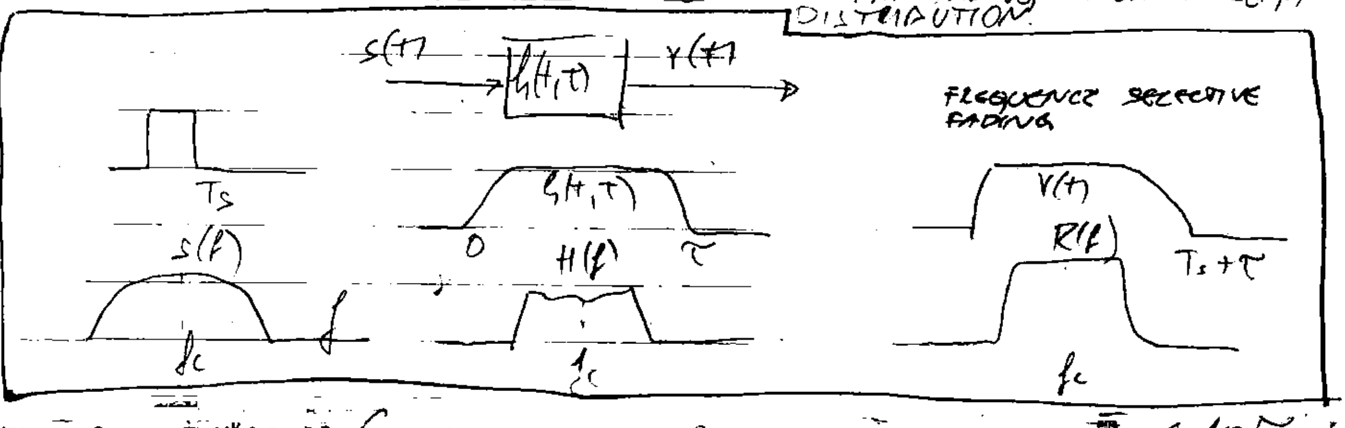
2.) FREQUENCY SELECTIVE FADING — FREQUENCY SELECTIVE CHANNELS = WIDEBAND CHANNELS

$B_S > B_C$ AND $T_S < \sigma_\tau$

RAILEIGH DISTRIBUTION

$$p(r) = \int_0^{2\pi} p_{r,\varphi}(r,\varphi)\, d\varphi = \int_0^{2\pi} \frac{r}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} d\varphi = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}$$

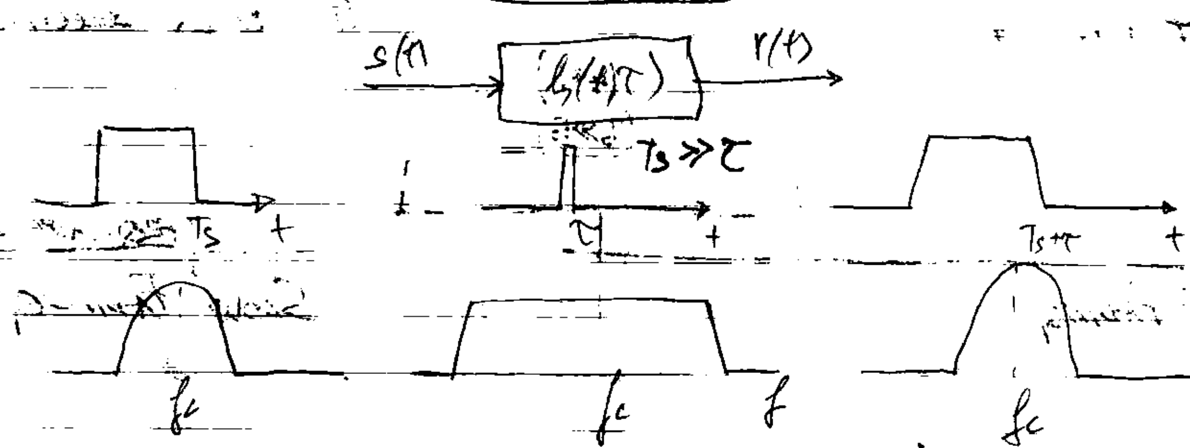
AMPLITUDE DISTRIBUTION IN FLAT FADING FOLLOWS RAILEIGH DISTRIBUTION



RULE OF THUMB: CHANNEL IS FREQ SELECTIVE IF: $T_S \leq 10\sigma_\tau$

1.) FLAT FADING

$B_S \ll B_C$ AND $T_S \gg \sigma_\tau$



FSF, FF PODOBNO E DA GI PAMTIŠ OD ASPEKT NA BANDWIDTH !!!

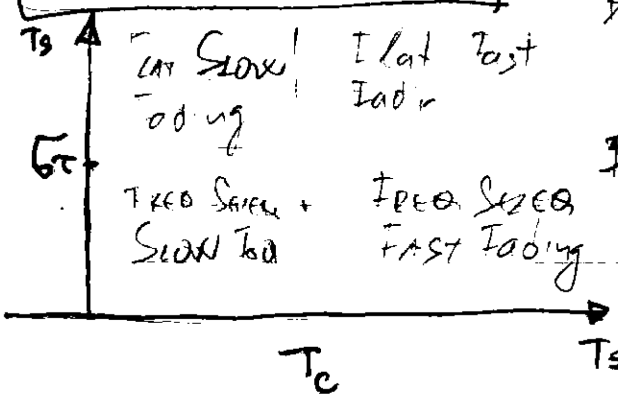
FADING EFFECTS DUE TO DOPPLER SPREAD

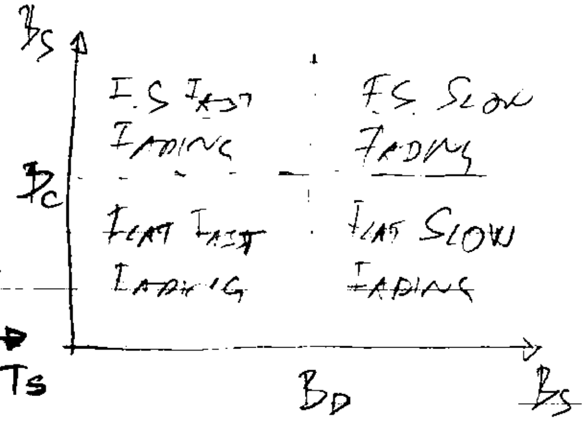
① • FAST FADING THE CHANNEL IMPULSE RESPONSE CHANGES RAPIDLY WITHIN SYMBOL DURATION

$T_s > T_c \qquad B_s < B_D$ OCCURS FOR VERY LOW DATA RATES

② SLOW FADING

$T_s < T_c \qquad B_s > B_D$





DESCRIBE STATISTICAL TIME VARYING NATURE OF RECEIVED ENVELOPE OF RECEIVED FLAT FADING

RAYLEIGH FADING DISTRIBUTION

• $$p(r) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \qquad 0 \le r \le \infty$$

σ – rms VALUE OF THE RECEIVED VOLTAGE SIGNAL BEFORE ENVELOPE DETECTION

$$P(R) = P_r(r \le R) = \int_0^R p(r)\,dr = -e^{-\frac{r^2}{2\sigma^2}}\Big|_0^R = -e^{-\frac{R^2}{2\sigma^2}} + 1$$

$$\int \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr = \frac{1}{2}\int e^{-\frac{r^2}{2\sigma^2}} d\left(\frac{r^2}{\sigma^2}\right) = \int e^{-\frac{r^2}{2\sigma^2}} d\frac{r^2}{2\sigma^2} = \int e^{-x} dx$$

$$= -e^{-x} = -e^{-\frac{r^2}{2\sigma^2}}$$

• $$P(R) = P_r(r \le R) = 1 - e^{-\frac{R^2}{2\sigma^2}}$$

THE PROBABILITY THAT THE ENVELOPE OF THE RECEIVED SIGNAL DOESN'T EXCEED SPECIFIED VALUE R IS GIVEN BY CORRESPONDING CDF.

• MEAN VALUE

$$\bar{r} = E(r) = \int_0^\infty r\,p(r)\,dr = \int_0^\infty r \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr = \int_0^\infty \frac{r^2}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr$$

$$v = \int \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr = -e^{-\frac{r^2}{2\sigma^2}} \quad = -r \cdot e^{-\frac{r^2}{2\sigma^2}} - \int -e^{-\frac{r^2}{2\sigma^2}} dr$$

$u = r$

$$\bar{r} = -r e^{-\frac{r^2}{2\sigma^2}} + \int e^{-\frac{r^2}{2\sigma^2}} dr \Big|_0^\infty \quad (*)$$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

$$\int e^{-x^2} dx = \quad x^2 = u \quad 2x\,dx = du \quad dx = \frac{du}{2x} = \int e^{-u} \frac{du}{\sqrt{u}} \quad ?$$

$$\bar r = \int_0^\infty \frac{r^2}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr = 2\int_0^\infty \frac{r^2}{2\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr$$

$$u = e^{-\frac{r^2}{2\sigma^2}} \qquad du = e^{-\frac{r^2}{2\sigma^2}}\left(-\frac{2r}{2\sigma^2}\right) dr = r e^{-\frac{r^2}{2\sigma^2}} dr$$

$$v = \int \frac{r^2}{2\sigma^2} dr = \frac{1}{2\sigma^2}\frac{r^3}{3}$$

$$u = \frac{r^2}{2\sigma^2} \qquad du = \frac{2r}{2\sigma^2} dr = \frac{r}{\sigma^2} dr \qquad dr = \frac{\sigma^2}{r} du$$

$$r = \sigma\sqrt{2}\sqrt{u} \; ; \quad dr = \frac{\sigma^2}{\sigma\sqrt{2}\sqrt{u}} du = \frac{du}{\sqrt{2u}}$$

(*) $$\bar r = -r e^{-\frac{r^2}{2\sigma^2}}\Big|_0^\infty + \int_0^\infty e^{-\frac{r^2}{2\sigma^2}} dr = -\lim_{r\to\infty} \frac{r}{e^{\frac{r^2}{2\sigma^2}}} + 0 + \sigma\sqrt{2}\int_0^\infty e^{-u^2} du$$

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$\bar r = \sigma\sqrt{2}\int_0^\infty e^{-u^2} du = \sigma\sqrt{2}\,\frac{\sqrt{\pi}}{2} = \sigma\frac{\sqrt{\pi}}{\sqrt{2}}$$

$$\boxed{\bar r_{mean} = \sigma\sqrt{\frac{\pi}{2}} = 1.25\sigma}$$

- **VARIANCE:**

$$\sigma_r^2 = E(r^2) - E^2(r) = \underbrace{\int_0^\infty \frac{r^3}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr}_{E(r^2)} - \left(\sigma\sqrt{\frac{\pi}{2}}\right)^2$$

$$E(r^2) = \int_0^\infty r^2 \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr, \qquad v = \int \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr = -e^{-\frac{r^2}{2\sigma^2}}$$

$$E(r^2) = -r^2 e^{-\frac{r^2}{2\sigma^2}}\Big|_0^\infty + \int_0^\infty e^{-\frac{r^2}{2\sigma^2}} d(r^2) = 0 + \int_0^\infty 2r e^{-\frac{r^2}{2\sigma^2}} dr = \ldots$$

$$= 2\sigma^2 \int_0^\infty \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr = 2\sigma^2 \qquad \boxed{\sigma_r = 2\sigma^2 - \sigma^2\frac{\pi}{2}}$$

$$\boxed{\sigma_r = \sigma^2\left(2 - \frac{\pi}{2}\right)}$$

- **MEDIAN** $\quad \frac{1}{2} = \int_0^{r_{median}} p(r)\,dr \qquad \boxed{r_m = 1.177\sigma}$

$$\int_0^{r_m} \frac{r}{\sigma} e^{-\frac{r^2}{2\sigma^2}} dr = \left| \begin{array}{l} u = \frac{r}{\sqrt{2}\sigma} \\ du = \frac{dr}{\sqrt{2}\sigma} \end{array} \quad \begin{array}{l} r = r_m \\ u = \frac{r_m}{\sqrt{2}\sigma} \end{array} \right|$$

$$= 2\sigma \int_0^{\frac{r_m}{\sqrt{2}\sigma}} \frac{r}{\sqrt{2}\sigma} e^{-u^2} du = 2\sigma \int_0^{\frac{r_m}{\sqrt{2}\sigma}} u e^{-u^2} du$$

$\frac{r_m}{\sqrt{2}\sigma} = 1.81157$

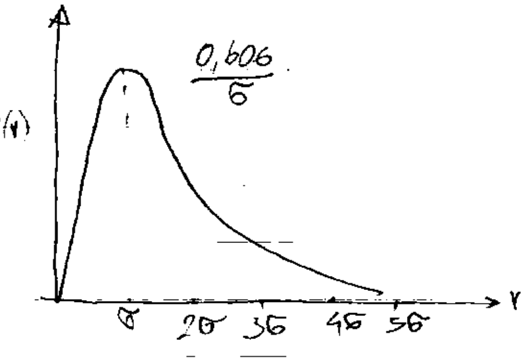
2K

$\bar{r} = 1.25\,\sigma$ $r_{med} = 1.177\,\sigma$

$20\log\frac{\bar{r}}{r_m} = 0.52$ dB

$p(r) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}$

$p(\sigma) = \frac{1}{\sigma} e^{-\frac{1}{2}} = \frac{1}{\sigma\sqrt{e}} = \frac{0{,}606}{\sigma}$



• Ricean fading distribution

If there is a dominant (non-fading) signal component present, such as line-of-sight propagation path, small-scale fading envelope distribution is **Rician**.

$$p(r) = \begin{cases} \frac{r}{\sigma^2} e^{-\frac{(r^2 + A^2)}{2\sigma^2}} I_0\left(\frac{A \cdot r}{\sigma^2}\right) & A \geq 0,\ r \geq 0 \\ \emptyset & r < \emptyset \end{cases}$$

A – peak amplitude of dominant signal

I_0 – Bessel function of first kind and zero order

$$I_0\left(\frac{Ar}{\sigma^2}\right) = \frac{1}{2\pi}\int_0^{2\pi} e^{\frac{Ar}{\sigma^2}\cos\varphi}\, d\varphi$$

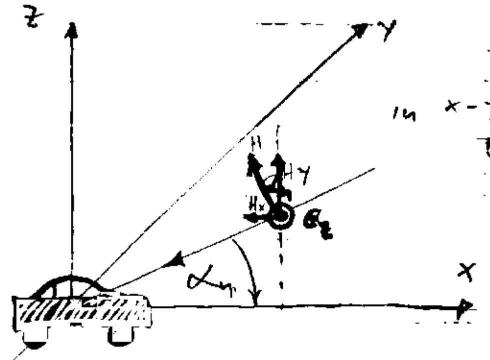
$$K = \frac{A^2}{2\sigma^2}$$

ratio between deterministic signal power and "variance" of the multipath.

$K(dB) = 10\log\frac{A^2}{2\sigma^2}$ [dB] Rician factor

Statistical models for multipath fading channels

• Clarke model for ~~flat fading~~:



$d = v \cdot \Delta t \cos(\alpha_n)$

$\Delta\phi = \frac{2\pi d}{\lambda} = \frac{2\pi v \cdot \Delta t \cos(\alpha_n)}{\lambda}$

$\omega_n = \frac{\Delta\phi}{\Delta t} = \frac{2\pi v}{\lambda} \cdot \cos(\alpha_n)$

$f_n = \frac{\omega_n}{2\pi} = \frac{v}{\lambda} \cdot \cos(\alpha_n)$

$H_y = H \cdot \cos(\alpha_n)$ $H_x = H \cdot \sin(\alpha_n)$

$$E_z = E_0 \sum_{n=1}^{N} C_n \cos(2\pi f_c t + \Theta_n)$$

$$H_x = -\frac{E_0}{\eta} \sum_{n=1}^{N} C_n \sin(\alpha_n) \cos(2\pi f_c t + \Theta_n)$$

$$H_y = -\frac{E_0}{\eta} \sum_{n=1}^{N} C_n \cos(\alpha_n) \cos(2\pi f_c t + \Theta_n)$$

C_n – real random variable representing amplitudes of individual waves

η – intrinsic impedance of free space ($377\,\Omega$)

Θ_n – random phase of n-th arriving component

$$\Theta_n = 2\pi f_n t + \phi_n$$

- AMPLITUDES OF E&H ARE NORMALIZED SUCH THAT ENSAMBLE AVERAGE OF C_n'S IS GIVEN BY

$$\sum_{n=1}^{N} \overline{C_n^2} = 1$$

- THREE COMPONENT E_z, H_x & H_y CAN BE APROXIMATED AS GAUSSIAN RANDOM VARIABLES IF N IS SUFFICIENTLY LARGE.
- PHASE ANGLES ARE WITH UNIFORM PDF $[0, 2\pi]$

$$E_z = T_c(t)\cos(2\pi f_c t) - T_s(t)\sin(2\pi f_c t)$$

$$E_z = E_0 \sum_{n=1}^{N} C_n \cos(2\pi f_c t + \Theta_n) = E_0 \sum_{n=1}^{N} C_n \cos(2\pi f_c t)\cos\Theta_n -$$

$$- E_0 \sum_{n=1}^{N} C_n \sin(2\pi f_c t)\sin\Theta_n = \Big/ \Theta_n = 2\pi f_n t + \phi_n \Big/ =$$

$$= \underbrace{E_0\left[\sum_{n=1}^{N} C_n \cos(2\pi f_n t + \phi_n)\right]}_{T_c(t)} \cos(2\pi f_c t) - \underbrace{E_0\left[\sum_{n=1}^{N} C_n \sin(2\pi f_n t + \phi_n)\right]}_{T_s(t)} \sin(2\pi f_c t)$$

$$E_z = T_c(t)\cos(2\pi f_c t) - T_s(t)\sin(2\pi f_c t)$$

$$T_c(t) = E_0 \sum_{n=1}^{N} C_n \cos(2\pi f_n t + \phi_n) \qquad T_s(t) = E_0 \sum_{n=1}^{N} C_n \sin(2\pi f_n t + \phi_n)$$

~~$T_c(t)$, $T_s(t)$~~ ARE GAUSSIAN RANDOM PROCESSES DENOTED T_c & T_s. AT ANY TIME "t", T_c & T_s ARE UNCORELATED ZERO MEAN GAUSSIAN RANDOM VARIABLES WITH VARIANCE:

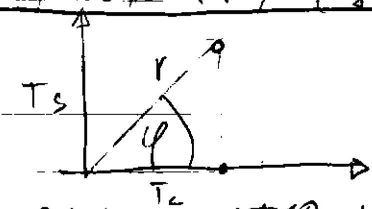
$$\overline{T_c^2} = \overline{T_s^2} = \overline{|E_z|^2} = \frac{E_0^2}{2} \quad ?$$

$$E_z = r(t)\cdot\cos\varphi(t)\cdot\cos(2\pi f_c t) - r(t)\sin(\varphi(t))\sin(2\pi f_c t) = r(t)\cdot\cos(2\pi f_c t + \varphi(t))$$

$$T_c(t) = r(t)\cos\varphi(t)$$

$$T_s(t) = r(t)\sin\varphi(t)$$

$$\varphi(t) = \mathrm{arctg}\,\frac{T_s(t)}{T_c(t)}$$



$$r(t) = \sqrt{T_c^2(t) + T_s^2(t)}$$

- E_0 KOMPONENTE NA ZAKODIAN KAKO VO DIGITALNI TK (RAYSON MORE NA DETEKTIRAN ŠUM) ⇒ SLUČAJNIOT PROMENLIV SIGNAL IMA AMPLITUDA "r" KOJA IMA RELEIEVA RASPREDELBA NA GUSTINA NA VEROJATNOST:

$$p(r) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \qquad 0 < r < \infty \qquad \text{KADE } \sigma^2 = \frac{E_0^2}{2}$$

$$p_{T_c T_s}(T_c, T_s) = p(T_c)\cdot p(T_s) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{T_c^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{T_s^2}{2\sigma^2}}$$

$$p(T_c, T_s) = \frac{1}{2\pi\sigma^2} e^{-\frac{T_c^2 + T_s^2}{2\sigma^2}}$$

$$T_c = r\cdot\cos\varphi$$

$$T_s = r\cdot\sin\varphi$$

$$J = \begin{vmatrix} \frac{\partial T_c}{\partial r} & \frac{\partial T_c}{\partial \varphi} \\ \frac{\partial T_s}{\partial r} & \frac{\partial T_s}{\partial \varphi} \end{vmatrix}$$

$$J = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r\cos^2\varphi + r\sin^2\varphi = r$$

$$\boxed{d\tau_c d\tau_s = |J|\, d\varphi\, dr}$$

$$\boxed{p(r,\varphi) = |J|\, p(\tau_c,\tau_s)}$$

$$p(r,\varphi) = \frac{1}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \qquad p(r) = \int_0^{2\pi} p(r,\varphi)\cdot|J|\, d\varphi$$

$$p(r) = \int_0^{2\pi} \frac{1}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \cdot r\, d\varphi = \frac{r}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \cdot 2\pi \qquad \boxed{p(r) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}}$$

$$p(\tau_c,\tau_s)\, d\tau_c\, d\tau_s = p(r,\varphi)\cdot dr\, d\varphi \qquad p(r,\varphi) = \frac{p(\tau_c,\tau_s)}{\frac{d\tau_c d\tau_s}{dr\, d\varphi}} = \frac{p(\tau_c,\tau_s)}{\frac{1}{|J|}}$$

$$\boxed{p(r,\varphi) = |J|\, p(\tau_c,\tau_s)} \qquad p(r,\varphi) = r\cdot\frac{1}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \qquad p(r) = \int_0^{2\pi} p(r,\varphi)\, d\varphi$$

- $$\overline{T_c^2} = \int_0^{\infty} T_c\, p(T_c)\, dT_c \quad !!$$

$$\overline{|E_z|^2} = \overline{\left(r\cos\varphi\cdot\cos(2\pi f_c t) - r\sin\varphi\sin(2\pi f_c t)\right)^2} =$$

$$= \underbrace{\overline{r^2\cos^2\varphi\cos^2(2\pi f_c t)}}_{(*)} - \underbrace{\overline{2r\cos\varphi\sin\varphi\cdot\cos(\omega_c t)\cdot\sin(\omega_c t)}}_{(**)} + \underbrace{\overline{r^2\sin^2\varphi\cdot\sin^2(\omega_c t)}}_{(2)}$$

$$(*) = \cos^2(\omega_c t)\int_0^{\infty} r^2\cos^2\varphi\, \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}\, dr = 2\sigma^2\cos^2(\omega_c t)\cos^2\varphi \qquad (2): 2\sigma^2\sin^2\varphi\cdot\sin^2\omega_c t$$

$$(*) + (2) = 2\sigma^2 \qquad (**) = \sigma\sqrt{\frac{\pi}{2}}\cos\varphi\sin\varphi\cos\omega_c t\sin\omega_c t$$

SPECTRAL ~~STATE~~ DUE TO DOPPLER SPREAD IN CLARKE'S MODEL

- $p(\alpha)\,d\alpha$ – FRACTION OF TOTAL INCOMING POWER WITHIN $d\alpha$ OF THE ANGLE α
- A AVERAGE RECEIVED POWER WITH RESPECT TO ISOTROPIC ANTENNA
- $G(\alpha)$ AZIMUTAL GAIN PATTERN OF MOBILE ANTENNA

$$P_r = \int_0^{2\pi} A\, G(\alpha)\cdot p(\alpha)\, d\alpha \quad \rightarrow \text{TOTAL RECEIVED POWER WITH ANGLE}$$

(1) $f(\alpha) = f = \frac{v}{\lambda}\cdot\cos(\alpha) + f_c = f_m\cos\alpha + f_c$ — f_m – MAXIMUM POWER SHIFT

$$\boxed{f(\alpha) = f(-\alpha)}$$

$S(f)$ – POWER SPECTRUM

$S(f)\,df$ → DIFFERENTIAL VARIATION OF RECEIVED POWER WITH FREQUENCY

(2) $S(f)\,df = A\left[p(\alpha)\,G(\alpha) + p(-\alpha)\,G(-\alpha)\right]|d\alpha|$

(1) → $df = f_m\,|-\sin\alpha|\cdot d\alpha$

(1) → $\cos\alpha = \frac{f-f_c}{f_m}$ $\qquad \alpha = \cos^{-1}\frac{f-f_c}{f_m} = \arccos\frac{f-f_c}{f_m}$

$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-\left(\frac{f-f_c}{f_m}\right)^2}$$

(2) → $S(f)\cdot f_m\,\sin\alpha\cdot d\alpha = A\left[p(\alpha)\,G(\alpha) + p(-\alpha)\cdot G(-\alpha)\right] d\alpha$

$$\boxed{S(f) = \frac{A\left[p(\alpha)\cdot G(\alpha) + p(-\alpha)\,G(-\alpha)\right]}{f_m\cdot\sqrt{1-\left(\frac{f-f_c}{f_m}\right)^2}}} \qquad S(f) = 0 \text{ if } |f-f_c| > f_m$$

- In case of GP antenna ($\lambda/4$) and uniform distribution: $G(\alpha) = 1.5$, $\psi(\alpha) = \frac{1}{2\pi}$ $[0 .. 2\pi]$

4.78 $$S_{\varepsilon z}(f) = \frac{1 \cdot \left[\frac{1}{2\pi} \cdot 1.5 + \frac{1}{2\pi} 1.5\right]}{f_m \sqrt{1 - \left(\frac{f - f_c}{f_m}\right)^2}} = \frac{1.5}{\pi f_m \sqrt{1 - \left(\frac{f - f_c}{f_m}\right)^2}}$$

4.79 $$S_{bb}(f) = \frac{1}{8\pi f_m} K\left[\sqrt{1 - \left(\frac{f}{2f_m}\right)^2}\right]$$

$K[\cdot]$ – elliptical integral of first kind

$10\,dB = 10 \log \frac{x}{y}$ $\quad \log \frac{x}{y} = 1$ $\quad \frac{x}{y} = 10$ $\quad x = 10y$

$X[dB] = 10 \log \frac{X(100)}{1W} = 10 \log \frac{0.153}{1W} = -8{,}2\,dB$

$Y[dB] = 10 \log \frac{Y(100)}{1W} = 10 \log \frac{0.2644}{1W} = -5{,}77\,dB$

• Complex Gaussian Distribution

$z = x + jy$ $\quad m_x, m_y$ $\quad m_z = m_x + j m_y = E(z)$

$\sigma^2 = E\left((x - m_x)^2\right) = E\left[(y - m_y)^2\right] = \frac{1}{2} E\left(|z - m_z|^2\right)$

– Probability density function of 2D random variable (x, y)

$$p(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{(x - m_x)^2 + (y - m_y)^2}{2\sigma^2}}$$

$$p(z) = \frac{1}{2\pi\sigma^2} e^{-\frac{|z - m_z|^2}{2\sigma^2}}$$

$x^2 - 2x m_x + m_x^2 + y^2 - 2y m_y + m_y^2$

$|z - m_z|^2 = z^2 - 2z m_z + m_z^2 = (x + jy)^2 - 2(x + jy)(m_x + j m_y) + (m_x + j m_y)^2$

$= x^2 + 2jxy - y^2 - 2(x m_x + jx m_y + jy m_x - y m_y) + m_x^2 + 2 m_x m_y$??

$$|z - m_z|^2 = |x + jy - m_x - j m_y|^2 = |(x - m_x) + j(y - m_y)|^2 = (x - m_x)^2 + (y - m_y)^2$$

$E\left(|z - m_z|^2\right) = E\left[(x - m_x)^2 + (y - m_y)^2\right] = E\left[(x - m_x)^2\right] + E\left[(y - m_y)^2\right]$ (each $= \sigma^2$)

$2\sigma^2 = E\left(|z - m_z|^2\right)$ $\quad \sigma^2 = \frac{1}{2} E\left(|z - m_z|^2\right)$

$\cos(2\pi f \cdot n \Delta t)$ $\quad f = 50\,kHz = 5 \cdot 10^4$ $\quad \Delta t = 10^{-5}$

$\cos(2\pi \cdot 5 \cdot 10^4 \cdot n \cdot 10^{-5})$ $\quad$ / $n = [1:1000]$ / $= \cos(10\pi \cdot 10^{-1} n) = \cos(\pi n)$

$$\begin{bmatrix}1\\2\\3\\4\end{bmatrix} [1, 1, 1, 1] = \begin{bmatrix}1 & 1 & 1 & 1\\ 2 & 2 & 2 & 2\\ 3 & 3 & 3 & 3\\ 4 & 4 & 4 & 4\end{bmatrix}$$

50k

$T = \frac{1}{5 \cdot 10^4} = 0.2 \cdot 10^{-4} = 2 \cdot 10^{-5}$

5k

$T = \frac{1}{5} \cdot 10^{-3} = 0.2 \cdot 10^{-3} = 2 \cdot 10^{-4}$

$N \cdot Dt = 10^3 \cdot 10^{-5} = 10^{-2}$

$N = 1000 \quad Dt = 10^{-4}$

$f = 100 Hz \quad c(t) = \cos(2\pi f \cdot n Dt) = \cos(2\pi \cdot 10^2 \cdot n \cdot 10^{-4}) = \cos(2 \cdot 10^{-2} \pi n)$

$c(t) = \cos(0.02 \pi n)$

$T = \frac{1}{10^{-2}} = 0{,}01$ sec

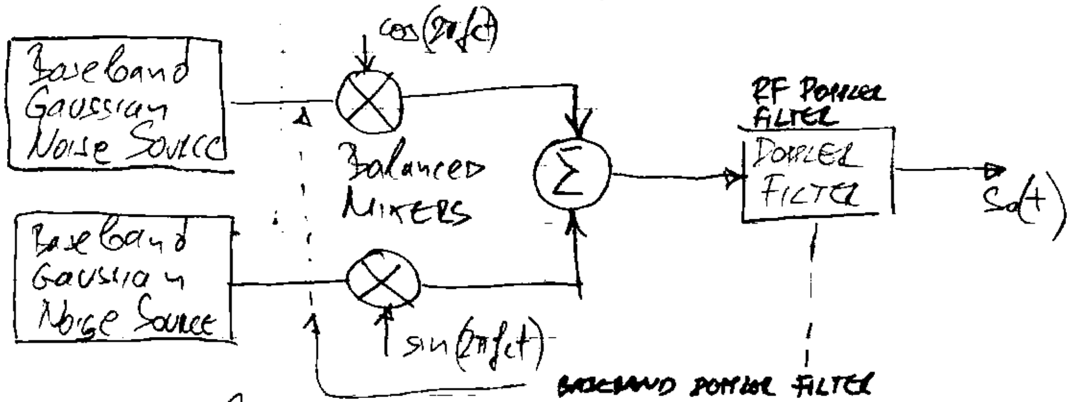
$t = 0 : Dt : 0.1$

$t = (0 : N-1) \cdot Dt$

$f = [0 : N-1] / (N \cdot Dt)$

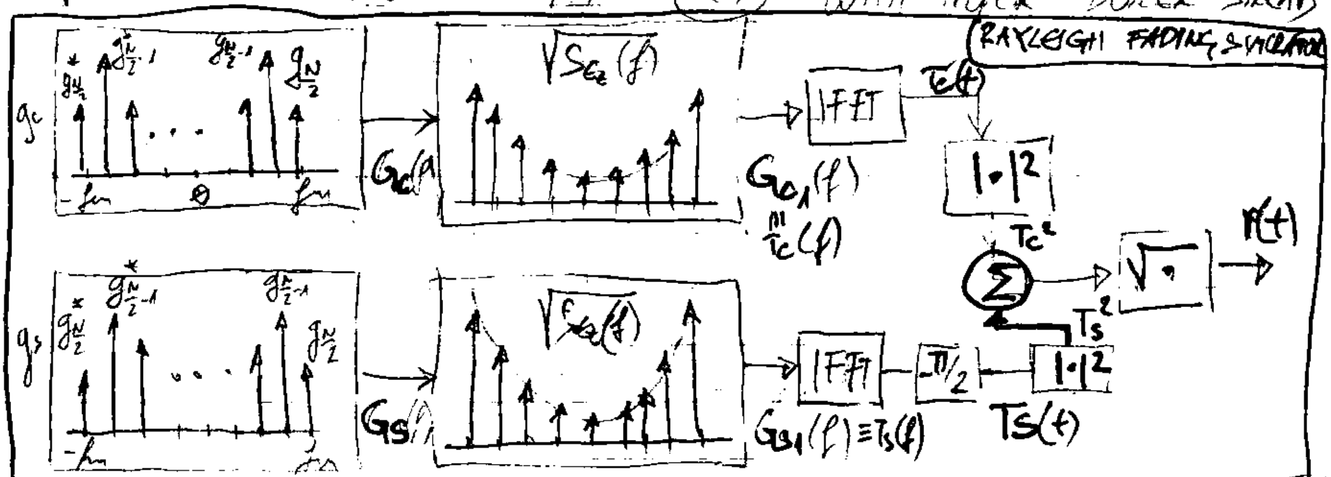
$\frac{1}{Dt} = 10$ kHz

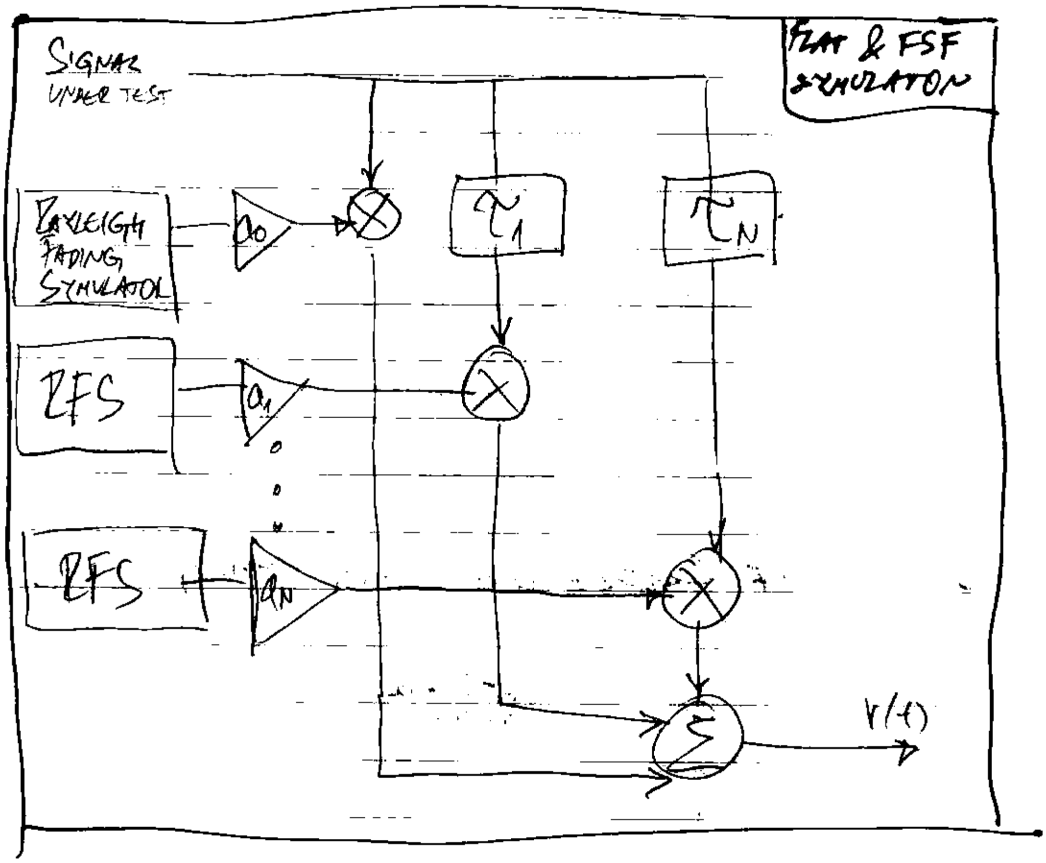
SIMULATION OF CLARKE and GANS FADING MODELS



- IMPLEMENT SMITH SIMULATOR.

1) (a) SPECIFY N FREQ. DOMAIN POINTS.
   (b) f_m - DEFINE MAX DOPPLER FREQ
2) $\Delta f = \frac{2 f_m}{N-1}$ → CHECK NEXT PAGE PROOF
   $T = \frac{1}{\Delta f}$ } TIME DURATION OF FADING WAVEFORM
3) GENERATE GAUSSIAN RANDOM VARIABLES FOR EACH OF THE N/2 POSITIVE FREQ COMPONENTS
4) CONSTRUCT THE NEGATIVE FREQ COMPONENTS OF NOISE SOURCE BY CONJUGATING POSITIVE VALUES (VIDI OTK AMASO)
5) MULTIPLY IN-PHASE AND QUADRATURE NOISE BY $\sqrt{S_{Ez}(f)}$
6) PERFORM IFFT + $|T_S|^2 + |T_C|^2$
7) $\sqrt{\cdot}$ IN ORDER TO GET $r(t)$ → RAYLEIGH FADING SIGNAL WITH PROPER DOPPLER SPREAD





- $2\pi f_c Dt = 0.02$ $\quad f_c = \frac{0.02}{2\pi Dt}$ $\quad f_c = 100 Hz$
- $f = 900 MHz$ $\quad \lambda = \frac{300}{900} = 0,33$ $\quad v = 100 km/h = 27,7 m/s$

$f_m = \frac{v}{\lambda} = \frac{27,7}{0,33} = 83,33 Hz$

$f_m = 100 Hz \Rightarrow v = f_m \cdot \lambda = 100 \cdot \frac{1}{3} = 33,3 \frac{m}{s} = 120 \frac{km}{h}$

$f_c = 900 MHz$ $\quad v = 120 km/h = 33,33 m/s$ $\quad f_m = 100 Hz$

$T_c = \frac{1}{9 \cdot 10^8} = 0,111 \cdot 10^{-8} = 1,11 \cdot 10^{-9} = 1,11$ nsec

N=1000

$N \cdot Dt = T_c \Rightarrow N = 1000 \Rightarrow Dt = \frac{T_c}{1000} = 1,11$ psec

$k = 2 f_c \cdot Dt$ $\quad k = 2 \cdot 9 \cdot 10^8 \cdot 1,11 \cdot 10^{-12} = 20 \cdot 10^{-4} = 2 \cdot 10^{-3}$

$\cos(0.002 * \pi * t)$

N=10.000 $\quad N \cdot Dt = T_c$ $\quad Dt = \frac{T_c}{10^4} = \frac{1.111 \cdot 10^{-9}}{10^4} = 1.111 \cdot 10^{-13} = 0,11 \cdot 10^{-12}$ sec

$f_c \cdot Dt$ $\quad k = 2 \cdot 9 \cdot 10^8 \cdot 0.11 \cdot 10^{-12} = 2 \cdot 10^{-4} = 0.2 \cdot 10^{-3}$

- DOPPLER FILTER

$$S_{E_z}(f) = \frac{1.5}{\pi f_m \sqrt{1 - \left(\frac{f - f_c}{f_m}\right)^2}}$$

$f_c = 0$ BASEBAND

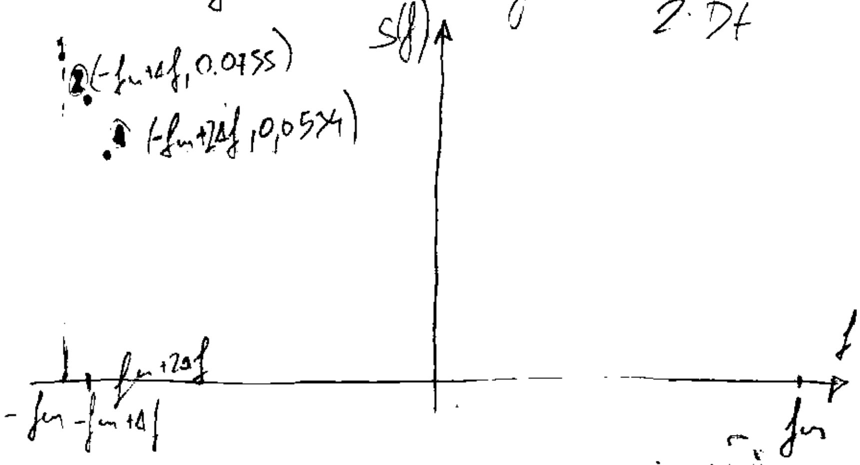
$f_m = 100 Hz$

- $f_{max} = 100 Hz$ $\quad \cos(2\pi f_{max} \cdot Dt \cdot n)$ $\quad T = \frac{1}{f_m} = 10^{-2} = 0,01$ sec

$Dt = 10^{-4}$ $\quad k = 2\pi f_{max} \cdot Dt = 2 \cdot 10^2 \cdot 10^{-4} = 2 \cdot 10^{-2} = 0,02$

$\Delta t = 10^{4}$; $N = 5000$; $\cos(0.02\pi n)$

$0.02 = 2 \cdot f_c \cdot \Delta t$ $\quad f_c = \frac{0.02}{2 \cdot \Delta t} = 0.01 \cdot 10^{4} = \underline{10^{2}}$



$B = 2 f_m$

$\Delta f = \frac{2 f_m}{N-1}$

$K = \frac{0{,}0755 - 0{,}0534}{\Delta f}$

$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

$$S(f) - S(-f_m + 2\Delta f) = \frac{0{,}0755 - 0{,}0534}{-f_m + \Delta f - (-f_m + 2\Delta f)} (f + f_m - 2\Delta f)$$

$$S(f) - 0{,}0534 = \frac{0.0221}{-f_m + \Delta f + f_m - 2\Delta f} (f + f_m - 2\Delta f)$$

$$S(f) - 0.0534 = -\frac{0.0221}{\Delta f} (f + f_m - 2\Delta f)$$

$$S(f) = -\frac{0{,}0221}{\Delta f}(f + f_m - 2\Delta f) + 0.0534$$

$$S(-f_m) = -\frac{0{,}0221}{\Delta f}(-f_m + f_m - 2\Delta f) + 0.0534 = +2 \cdot 0{,}0221 + 0.0534 = 0{,}0976$$

$$e^{+j\frac{\pi}{2}} = \cos\frac{\pi}{2} + j \sin\frac{\pi}{2} = 0 + j$$

$$g_x + j g_y = \sqrt{g_x^2 + g_y^2}\, e^{j \arctan \frac{g_y}{g_x}} \cdot e^{j\frac{\pi}{2}}$$

$$z = 1 + j1 \qquad \sqrt{z} = \left(\sqrt{2^2 + 1^2} \cdot e^{j \arctan \frac{1}{2}}\right)^{1/2} = \sqrt[4]{5}\, e^{j\frac{0{,}4636}{2}} = 1{,}4953 \cdot e^{+j0{,}2318}$$

$\sqrt{z} = 1.4553 + 0.3406 j$

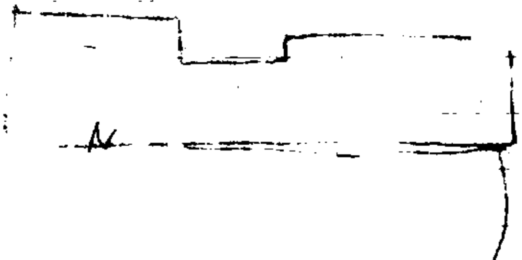
$c = \cos(2\pi f_c t)$

$c_7 = r \cdot \cos(2\pi f_c t + \varphi(t))$

$\cos\left(\alpha - \frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2} - \alpha\right) = \sin\alpha$

$W_N = e^{-j\frac{2\pi}{N}}$

$$\mathcal{F}\{f(t-\tau)\} = F(j\omega) \cdot e^{-j\omega\tau}$$



$$F(k) = \sum_{n=1}^{N} f(n)\, e^{-j\frac{2\pi nk}{N}}, \quad k = 1 \ldots N$$

$$\mathcal{F}[f(n - n_1)] = F(k)\, e^{-j\frac{2\pi k n_1}{N}}$$

sawtooth (20*pi*t) = sawtooth (20*pi*n*Dt) = sawtooth (0,002 π n)

$Dt = 10^{-4}$ sec $2 f_c Dt = 0{,}002$ $f_c = \frac{0{,}002}{2} 10^4 = 0{,}001 \cdot 10^4 = 10$ Hz

cos (0,2·π·n) $f_c = 900$ MHz

$2 \cdot f_c \cdot Dt = 0{,}2$ $f_c = \frac{0{,}1}{Dt}$ $Dt = \frac{0{,}1}{f_c} = 0{,}11$ nsec

$Dt = 1{,}11 \cdot 10^{-12} \Rightarrow f_c = \frac{0{,}1}{1{,}11 \cdot 10^{-12}} = 90$ GHz

$0{,}1 \cdot 10^{-8} = 10^{-1} \cdot 10^{-8} = 10^{-9} = 1$ ns $T_c = 1$ ns

$f_c = 900$ MHz $= 9 \cdot 10^{8} \Rightarrow T_c = \frac{1}{f_c} = 1{,}11$ ns

⊙ Level Crossing and Fading Statistics

- **LCR** – expected rate at which the Rayleigh fading envelope, normalized to a local rms signal level, crosses ~~a specified level~~ in positive-going

the number of level crossings per second are:

$$N_R = \int_0^{\infty} \dot{r}\, p(R, \dot{r})\, d\dot{r} = \sqrt{2\pi} f_m \rho e^{-\rho^2}$$

$\dot{r}$ – time derivate of $r(t)$ (i.e. the slope)

$p(R, \dot{r})$ – joint density function of „r" and „$\dot{r}$" at $r = R$

f_m – maximum Doppler frequency

$\rho = R/R_{rms}$ value of the specified level R (normalized).

Ex. 26 Rayleigh fading signal LCR = ? for $\rho = 1$ $f_m = 20$ Hz

$v_{max} = ?$ if $f_c = 900$ MHz

$LCR = N_R = \sqrt{2\pi} f_m \rho e^{-\rho^2} = \sqrt{2\pi} \cdot 20 \cdot 1 \cdot e^{-1} = 18{,}443$

$f_m = \frac{v_{max}}{\lambda}$ $v_{max} = f_m \lambda = 20 \cdot 0{,}33 = 6{,}6 \frac{m}{s} = 24 \frac{km}{h}$

- **Average fade duration** – average duration of time for which received signal is below specified level R for Rayleigh fad.:

$\tau = \frac{1}{N_R} P_r[r \leq R]$ $P_r[r \leq R] = \frac{1}{T} \sum \tau_i$

$$P_r[r \leq R] = \int_0^R \frac{r}{\sigma^2} e^{-\frac{r^2}{\sigma^2}} dr = -e^{-\frac{r^2}{2\sigma^2}} \Big|_0^R = -e^{-\frac{R^2}{2\sigma^2}} + 1 = 1 - e^{-\frac{R^2}{2\sigma^2}}$$

$$P_r[r \leq R] = 1 - e^{-\rho^2}$$

$\rho = \frac{\frac{R}{\sqrt{2}}}{\sigma} = \frac{R}{\sqrt{2}\sigma}$

$$\tau = \frac{1}{\sqrt{2\pi} f_m \rho e^{-\rho^2}} \cdot \left(1 - e^{-\rho^2}\right) = \frac{e^{\rho^2} - 1}{\sqrt{2\pi} f_m \rho}$$

Ex. 4.7. $\rho = 0.01, \rho = 0.1$ & $\rho = 1$ ... $f_m = 200\,Hz$

$$\tau = \frac{e^{\rho^2} - 1}{\sqrt{2\pi} f_m \rho}$$

$\rho = 0.01$ $\bar{\tau} = 199 \cdot 10^{-6} = 199\,\mu s$

$\rho = 0.1$ $\bar{\tau} = 2005\,\mu s = 2005 \cdot 10^{-6}$

$\rho = 1$ $\bar{\tau} = 3{,}4\,ms$

Ex. 4.8. $\bar{\tau} = ?$ $\rho = 0.707$ $f_m = 20\,Hz$ binary digital word with bit duration 50 bps FAST/SLOW = ?

- AVERAGE NUMBER OF BIT ERRORS / SEC = ?

BIT ERROR = FADE $\rho < 0.1$

- $$\tau = \frac{\rho^2 - 1}{\sqrt{2\pi} f_m \rho} = 0{,}0183 = 18\,msec$$

$$T_s = \frac{1}{50} = 0{,}02 = 20\,msec$$

FAST FADING $T_C < T_S$ $T_C = \frac{9}{16\pi f_m} = 8{,}95\,msec \approx 9\,msec$

$T_s = 20\,ms > T_C = 9\,msec \Rightarrow$ FAST FADING

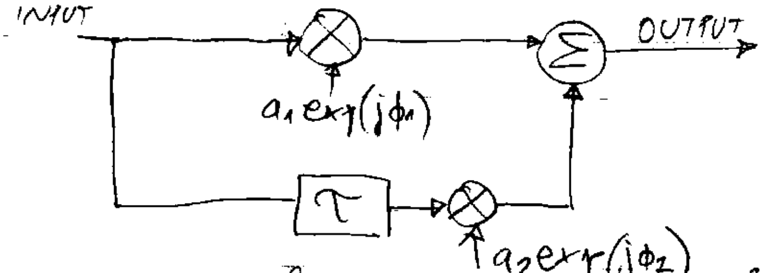
- $\rho = 0{,}1 \Rightarrow \bar{\tau} = 2\,msec$

$N_L = \sqrt{2\pi} f_m \rho e^{-\rho^2} = 5{,}06$ level cross/sec

$$BER = N_L / v_B = \frac{5\,\frac{1}{sec}}{50\,bps} = \frac{1}{10} = 0{,}1 = 10\%$$

TWO-RAY RAYLEIGH FADING MODEL

- USED TO MODEL THE MULTIPATH DELAY SPREAD



$$h_b(t) = a_1 \exp(j\phi_1)\,\delta(t) + a_2 \exp(j\phi_2)\,\delta(t - \tau)$$

SPECIAL CASE: $a_2 = 0$

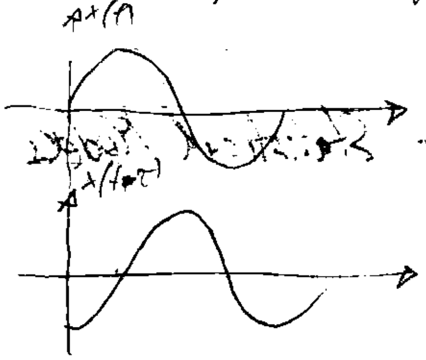
$h_b(t) = a_1 \exp(j\phi_1)\,\delta(t)$ } FLAT RAYLEIGH FADING

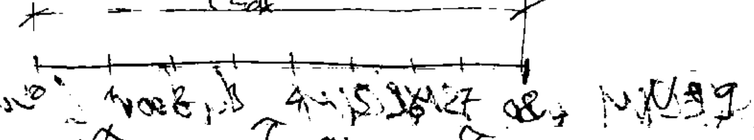
$\sigma_\tau > T_s$ frequency selective fading

$\sigma_\tau \leq T_s$ flat fading

$-20\,dBW = 10 \log \frac{WGN}{1W}$ $WGN = 10^{-2} = 0{,}01\,W = 10\,mW$

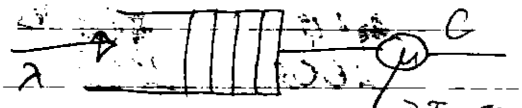
$-60\,dBW \Rightarrow WGN = 10^{-6} = 1\,\mu W$





$$\Delta\tau = \frac{\tau_{max}}{N-1} = \frac{\tau_{max}}{8}$$

• exponential distribution of inter-arrival times

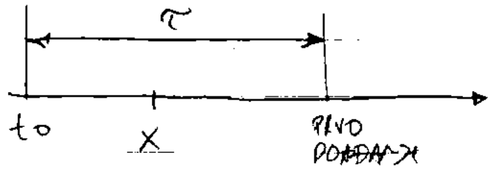


λ - SREDEN BROJ NA PAKETI VO 1 sec
μ - " - OPSLUZENI VO 1 sec
C - KAPACITET NA KANALOT

$\rho = \frac{\lambda}{\mu}$ INTENZITET NA SOOBRAKAJ

• VEROJATNOSTA DEKA VO INTERVALOT „T" ĆE SE SLUČAT „K" NASTANI E SOGLASNO POASONOVATA RASPREDELBA

$$P(k) = \frac{(\lambda T)^k}{k!} e^{-\lambda T} \qquad E(k) = \sum_{k=0}^{\infty} k P(k) = \lambda T$$



$$P(\tau > x) = \frac{(\lambda \tau)^0}{0!} e^{-\lambda x} = e^{-\lambda x}$$

$$P(\tau \le x) = 1 - e^{-\lambda x}$$

• FUNKCIJA NA GUSTINA NA VEROJATNOST

$$f(x) = \frac{dP(\tau \le x)}{dx} = \lambda \cdot e^{-\lambda x}$$

PRIMER: $f(x|\mu) = \frac{1}{\mu} e^{-\frac{x}{\mu}}$ $\qquad f(2|2) = \frac{1}{2} e^{-\frac{2}{2}} = 0.184$

$$f(3|2) = \frac{1}{2} e^{-\frac{3}{2}} = 0.112$$

$$E(x) = \sum_{x=0}^{\infty} x \cdot f(x) = \sum_{x=0}^{\infty} \frac{x}{\mu} e^{-\frac{x}{\mu}}$$

} NE VAKA!! SE KORISTI ZA KONTINUALNA DISTRIBUCIJA

$$I = \frac{1}{\mu} \int_0^{\infty} x e^{-\frac{x}{\mu}} dx = \qquad u = x, \quad v = \int e^{-\frac{x}{\mu}} dx = -\mu \int e^{-\frac{x}{\mu}} d\left(\frac{x}{\mu}\right) = -\mu e^{-\frac{x}{\mu}}$$

$$I = \frac{1}{\mu}\left(-x \cdot \mu e^{-\frac{x}{\mu}} \Big|_0^{\infty} + \int_0^{\infty} \mu e^{-\frac{x}{\mu}} dx \right) = \frac{1}{\mu}\left(-\lim_{x \to \infty} \frac{\mu x}{e^{\frac{x}{\mu}}} + \mu e^{-\frac{x}{\mu}}(-\mu) \Big|_0^{\infty} \right)$$

$$I = \frac{1}{\mu}\left(-\mu^2 e^{-\frac{x}{\mu}} \Big|_0^{\infty} \right) = -\mu e^{-\frac{x}{\mu}} \Big|_0^{\infty} = -(0 - \mu e^0) = +\mu$$

• SIRCIM AND SMRCIM Indoor & Outdoor Statistical Models

$$h_b(t, X_l, S_n, P_n, P_n) = \sum_i A_i(\tau_i, X_l, S_n, P_n, P_n) e^{j\theta_i(\tau_i, X_l, S_n, P_n, P_n)} \delta(t - \tau_i(X_l, S_n, P_n, P_n))$$

X_l - receiver spacing
S_n - topography (LOS or OLOS)
P_n - PARTICULAR MEASUREMENT LOCATION

$$y(n) = \sum_{k=0}^{\infty} x(k)\, h(n-k) \qquad n = 0,1 \ldots \infty$$

$$h_b(t,\tau) = \sum_{i=0}^{N-1} a_i(t,\tau) \exp\left[j\left(2\pi f_c \tau_i(t) + \phi_i(t,\tau)\right)\right] \delta(\tau - \tau_i(t))$$

$$h_b(\tau) = \sum_{i=0}^{N-1} a_i \exp(-j\theta_i) \cdot \delta(\tau - \tau_i) \qquad \text{TIME INVARIANT}$$

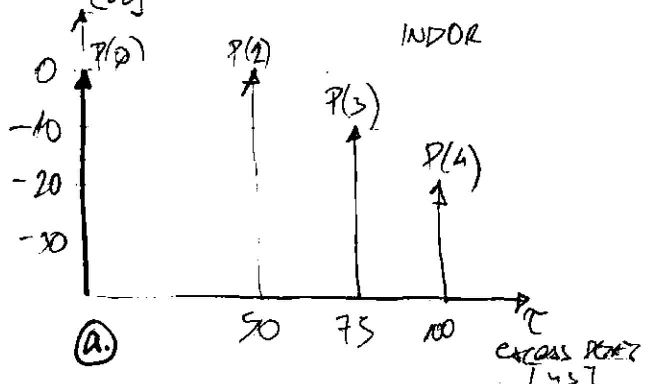
PROBABILITY FOR MULTIPATH ARRIVING AT A PARTICULAR EXCESS DELAY

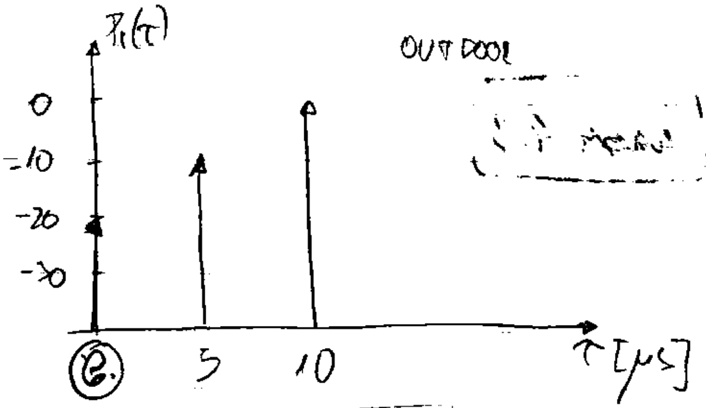
$$P_r(T_i, S_1) = \begin{cases} 1 - \frac{T_i}{367} & T_i < 110\,\mu s \\ 0{,}65 - \frac{T_i - 110}{360} & 110\,\mu s < T_i < 200\,\mu s \\ 0{,}22 - \frac{T_i - 200}{1360} & 200\,\mu s < T_i < 500\,\mu s \end{cases}$$

LOS

$$P_r(T_i, S_2) = \begin{cases} 0.55 + \frac{T_i}{667} & T_i < 100\,\mu s \\ 0.08 + 0.62 \exp\left(\frac{T_i - 100}{75}\right) & 100\,\mu s < T_i < 500\,\mu s \end{cases}$$

PROBLEM 4.2 SUITABLE BER PERFORMANCE WHENEVER $\frac{\sigma}{T_s} \leq 0.1$ $\min\{T_s\} = ?$ THAT CAN BE SENT OVER:





$$T_s \geq 10\,\sigma_\tau \qquad \boxed{\sigma_\tau = \overline{\tau^2} - \bar{\tau}^2 = \overline{(\tau - \bar{\tau})^2}}$$

(a) $$\bar{\tau} = \frac{\sum P(k)\tau_k}{\sum P(k)}$$

$$0\,dB = 10 \log \frac{P(k)}{1W} \qquad P(0) = 10^{\frac{0}{10}}\, W = 1W \qquad P(2) = 1W$$

$$P(3) = 10^{-1} = 0{,}1\,W \qquad P(4) = 10^{-2} = 0{,}01\,W$$

$$\bar{\tau} = \frac{0 \cdot 0 + 1 \cdot 50 \cdot 10^{-9} + 0{,}1 \cdot 75 \cdot 10^{-9} + 0{,}01 \cdot 10^{-9}}{1 + 1 + 0{,}1 + 0{,}01} = \frac{50 \cdot 10^{-9} + 7{,}5 \cdot 10^{-9} + 0{,}01 \cdot 10^{-9}}{2{,}11}$$

$$\bar{\tau} = \frac{57{,}51 \cdot 10^{-9}}{2{,}11} = 27{,}256\,ns$$

$$\sigma_\tau^2 = \frac{(0 - 51{,}81 \cdot 10^{-9})^2 + \left[(50 - 51{,}81) 10^{-9}\right]^2 + \left[(75 - 51{,}81) 10^{-9}\right]^2 + \left[(100 - 51{,}81) 10^{-9}\right]^2}{4}$$

$$= 0{,}25\left(27 \cdot 10^3 + 3{,}28 + 537{,}8 + 2{,}3 \cdot 10^3\right) \cdot 10^{-18} = 1{,}385 \cdot 10^{-15}$$

$\sigma_\tau = \sqrt{1,385\cdot 10^{-15}} = 37,22\,\mu s$

ATTENTION: $\overline{\tau^2} = \dfrac{\sum P(\tau)\tau_k^2}{\sum P(\tau)} = \dfrac{0\cdot 0 + 1\cdot 50^2 + 0.1\cdot 75^2 + 0,01\cdot 100^2}{2,11}\cdot 10^{-18}$

$\overline{\tau^2} = 0,47\,(2500 + 562,5 + 100)\cdot 10^{-18} = 1,4986\cdot 10^{-15}$

$\sigma_\tau = \sqrt{\overline{\tau^2} - \bar{\tau}^2} = \sqrt{1,4986\cdot 10^{-15} - (27,256\cdot 10^{-9})^2} = 27,49\,\mu s$

$\boxed{\sigma_\tau = 27,44\,\mu s}$

$T_s \geq 10\,\sigma_\tau \qquad T_s \geq 275\cdot 10^{-9} \qquad f_s \leq 3,63\ \text{MHz} \qquad f_s \leq 3,63\ \text{Mbps}$

(b) $\bar{\tau} = \dfrac{0,1\cdot 5 + 1\cdot 10}{0,01 + 0,1 + 1} = \dfrac{10,5}{1,11} = 9,46\,\mu s$

$\overline{\tau^2} = \dfrac{0,1\cdot 25 + 1\cdot 100}{1,11} = \dfrac{102,5}{1,11} = 92,34\,\mu s^2$

$\sigma_\tau = \sqrt{\overline{\tau^2} - \bar{\tau}^2} = \sqrt{92,34 - 9,46^2} = 1,688\,\mu s$!

$T_s \geq 10\,\sigma_\tau = 16,88\,\mu s \qquad f_s \leq \dfrac{1}{T_s} = 59,24\ \text{kbps}$

Problem 4.3 For the power delay profiles from 4.2 estimate 90% correlation and 50% correlation coherence bandwidths

Correlation > 0,5 $\qquad T_c = \dfrac{9}{16\pi f_m}$

(a) $|r(t_0)|^2 = \sum_{k=0}^{N-1} a_k(t_0) = 1 + 1 + 0,1 + 0,01 = 2,11\ \text{W}$

$B_c = \dfrac{1}{5\sigma_\tau} = \dfrac{1}{5\cdot 27.49\cdot 10^{-9}} = 7.28\ \text{MHz}$ correlation > 0,5

$B_c = \dfrac{1}{50\sigma_\tau} = 0,728\ \text{MHz}$ correlation > 0.9

(b) $B_c = \dfrac{1}{5\sigma_\tau} = \dfrac{1}{5\cdot 1,688\cdot 10^{-6}} = 118\ \text{kHz}$ correlation > 0.5

$B_c = \dfrac{1}{50\sigma_\tau} = 11,8\ \text{kHz}$ correlation > 0.9

Problem 4.4 (a) $R_b = 25$ kbps binary modulated signal

RMS = ?

(b) 8-PSK $R_b = 75$ kbps

(a) BPSK $B_{QPSK} = R_b = 25\,kHz$ $T_s = \frac{1}{25kHz} = 40\mu s$

$\sigma_\tau \le 0{,}1\, T_s$ $\sigma_\tau \le 0.1 \cdot 40\mu s = 4\mu s$

(б) 8PSK $B_{8PSK} = \frac{1}{3} R_b = \frac{1 \cdot 75}{3} = 25\,kHz$

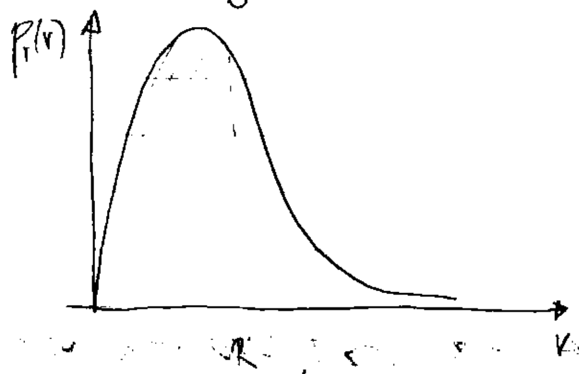
$\sigma_\tau \le 0.1\, T_s = 0.1 \frac{1}{25kHz} = 0.1 \cdot 40\mu s = 4\mu s$

PROBLEM 4.5

$p_r(r) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}$ σ^2 – VARIANCE; $P(r<R)$

$P(r \le R) = \int_0^R p_r(r)\, dr$

$P(r \le R) = \int_0^R \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr$ $d\mu$



$\mu = \frac{r^2}{2\sigma^2}$ $d\mu = \frac{2r}{2\sigma^2} dr = \frac{r\, dr}{\sigma^2}$ $r=0$ $\mu = 0$; $r = R$ $\mu = \frac{R^2}{2\sigma^2}$

$\int_0^{R^2/2\sigma^2} e^{-\mu} d\mu = -e^{-\mu}\Big|_0^{\frac{R^2}{2\sigma^2}} = -\left(e^{-\frac{R^2}{2\sigma^2}} - 1\right) = 1 - e^{-\frac{R^2}{2\sigma^2}}$

~~$10dB = 20 \log R$ $R = 10^{1/2} = \sqrt{10} = 3{,}16$~~

~~$P(r < 3{,}16) = 1 - e^{-\frac{(3{,}16)^2}{2\sigma^2}} = 1 - e^{-\frac{10}{2\sigma^2}}$~~

$-10\, dB = 10 \log \frac{R^2}{2\sigma^2}$ $\frac{R}{\sqrt{2}\sigma} = 10^{-\frac{1}{2}} = 0{,}316$

$P\left(\frac{r}{\sqrt{2}\sigma} < 0{,}316\right) = 1 - e^{-(0{,}316)^2} = 1 - e^{-0{,}1} = 1 - 0{,}904 = 0{,}095 = 9{,}5\%$

ПРОЦЕНТОТ НА ВРЕМЕТО ВО КОЕ СИГНАЛОТ ЌЕ ВИДЕ −10 dB ПОМАЛ ОД σ Е 9,5%

PROBLEM 4.6 $LCR = \sqrt{2\pi}\, f_m\, \rho\, e^{-\rho^2}$ $f_m = \frac{v}{\lambda} = \frac{13{,}8}{0{,}333} = 41{,}67$

(a) $\frac{dLCR}{d\rho} = 0$ $\frac{d}{d\rho}\left(\sqrt{2\pi} f_m \rho e^{-\rho^2}\right) = \sqrt{2\pi} \frac{d}{d\rho}\left(\rho e^{-\rho^2}\right) =$

$= \rho' e^{-\rho^2} + \rho \cdot \left(e^{-\rho^2}\right)' = e^{-\rho^2} + \rho\, e^{-\rho^2}(-2\rho) = e^{-\rho^2} - 2\rho^2 e^{-\rho^2} = 0$

$e^{-\rho^2} = 2\rho^2 e^{-\rho^2}$ $\rho^2 = \frac{1}{2}$ $\rho = \frac{1}{\sqrt{2}} = 0{,}707 = -1{,}5\,dB$

(б) $v = 50\, km/h = 13{,}8\, m/s$ $f_c = 900\,MHz$ $T_c = \frac{1}{f_c} = 1{,}11\,ns$

$LCR = \sqrt{2\pi} f_m \rho e^{-\rho^2} = \sqrt{2\pi} \cdot 41{,}67 \cdot 0{,}707\, e^{-0{,}707^2} = 44{,}8$ CROSSING/s = [illegible] CROSSING

ⓒ $\rho = 0{,}707 \qquad \tau = \frac{e^{\rho^2} - 1}{\sqrt{2\pi} f_m \rho} = \frac{e^{\frac{1}{2}} - 1}{\sqrt{2\pi}\, 41{,}67 \cdot 0{,}707} = 8{,}8 \text{ ms}$

Problem 4.7 $\quad f_c = 900 \text{ MHz}, \; v = 10s\,?, \; \tau|_{10dB} = 1 \text{ ms}$

- $d = ?$ for 10s interval?
- LCR = ? at RMS threshold level during 10s

• $10 \log \rho = -10 \text{ dB} \qquad \rho = 10^{-1} = 0{,}1$

$10^{-3} = \frac{e^{0{,}01} - 1}{\sqrt{2\pi} f_m \, 0{,}1} \qquad f_m = \frac{e^{0{,}01} - 1}{\sqrt{2\pi} \cdot 10^{-4}} = 4 \cdot 10^{-3} \cdot 10^{4} = 4 \cdot 10^{1} = 40 \text{ Hz}$

$f_m = \frac{v}{\lambda} \qquad v = f_m \lambda = 40 \cdot \frac{1}{3} = 13{,}3 \frac{m}{s} = 48 \frac{km}{h} \qquad d = v \cdot s = 133 \text{ m}$

• $LCR = \sqrt{2\pi} f_m \rho e^{-\rho^2} = \sqrt{2\pi} \cdot 40 \cdot 0{,}1\, e^{-0{,}01} = 9{,}93 \frac{\text{cross}}{\text{sec}}$

$T = 10s \qquad LCR|_{10s} = 99{,}3 \text{ cross} / 10\text{sec}$ $\Big\}\; \rho = 0{,}1$

$\rho = 1 \qquad LCR = \sqrt{2\pi} f_m \rho e^{-1} = \sqrt{2\pi} \cdot 40 \cdot 0{,}1 \cdot e^{-1} = 3{,}7 \frac{\text{cross}}{\text{sec}}$

SEKAK NE TREBA DA SE TRETIRA KAKO SNAGA TUKU KAKO NIVO:
$\rho = R / R_{RMS}$ — root mean square !!!

• $20 \log \rho = -10 \text{ dB} \qquad \rho = 10^{-\frac{1}{2}} = \frac{1}{\sqrt{10}} = 0{,}316$

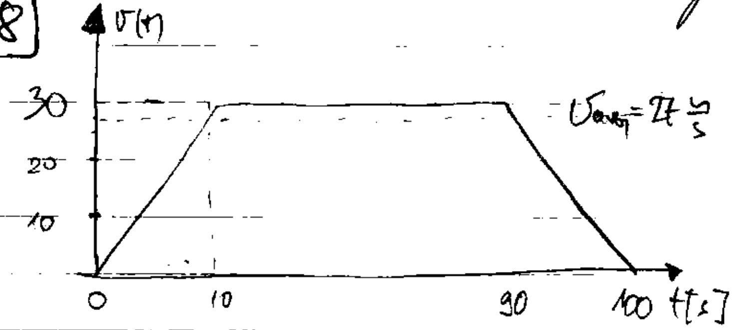
$f_m = \frac{e^{+0{,}1} - 1}{\sqrt{2\pi}\, 0{,}316 \cdot 10^{-3}} = 0{,}133 \cdot 10^{3} = 133 \text{ Hz} \qquad f_m = \frac{v}{\lambda}$

$v = f_m \cdot \lambda = \frac{133}{3} = 44{,}3 \frac{m}{sec} = 160 \frac{km}{sec} \qquad d = v \cdot 10s = 443 \text{ m}$

• $\rho = 1 \qquad LCR = \sqrt{2\pi} \cdot 133 \cdot 1 \cdot e^{-1} = 122{,}6 \frac{\text{cross}}{\text{sec}}$

10s: $\quad LCR \cdot 10 \text{sec} = 1226$ crossings

Problem 4.8



$f_c = 900 \text{ MHz}, \; \rho = 0{,}1$
LCR = ?
Fade duration $\tau = ?$ $\Big\}$ 100s period

$LCR = \sqrt{2\pi} f_m \rho e^{-\rho^2} \qquad f_m = \frac{v}{\lambda}$

$t = 0 \ldots 10s \quad v = 3t \qquad t = 10 \ldots 90s \quad v = 30 \frac{m}{s}$

$t = 90 \ldots 100 \qquad v = -3(t - 100) = -3t + 300$

$v_{avg} = \frac{1}{b-a} \int_a^b v(t)\, dt = \frac{1}{100} \left[\int_0^{10} 3t\, dt + \int_{10}^{90} 30\, dt + \int_{90}^{100} (-3t + 300)\, dt \right.$

$$v_{avg} = \frac{1}{100}\left[3\,\frac{t^2}{2}\Big|_0^{10} + 30(90-10) + \left(-\frac{3t^2}{2} + 300t\right)\Big|_{90}^{100}\right] =$$

$$= \frac{1}{100}\left[3\left(\frac{100}{2}-0\right) + 2400 - \frac{3}{2}(10^4 - 8100) + 300 \cdot 10\right] =$$

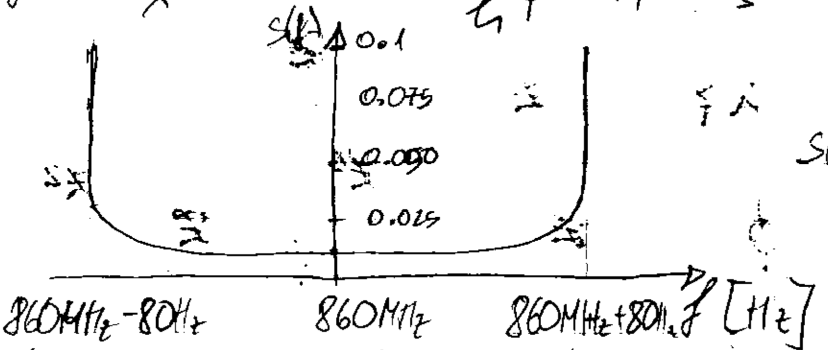
$$= \frac{1}{100}\left(150 + 2400 - 2850 + 3000\right) = \frac{2700}{100} = 27\ \frac{m}{s}$$

$N_R = 20{,}101\ \frac{cross}{sec} \doteq 2010\ \frac{crossings}{1000\ sec}$

$\bar{\tau} = \frac{e^{\rho^2}-1}{\sqrt{2\pi} f_m \rho} = 494{,}99\ \mu s \doteq 495\ \mu s$

Problem 4.9: $f_c = 860\ MHz$ $\quad \lambda = \frac{300}{860} = 0{,}35\ m$

$f_m = \frac{v}{\lambda}$ $\quad v = 100\ \frac{km}{h} = 27{,}78\ \frac{m}{s}$ $\quad f_m = 79{,}36 \doteq 80\ Hz$



$$S(f) = \frac{1.5}{\pi f_m \sqrt{1 - \left(\frac{f-f_c}{f_m}\right)^2}}$$

$\rho_{dB} = -20\ dB = 20 \log \rho \quad \rho = 10^{-1} = 0.1$

$N_R = \sqrt{2\pi} f_m \rho e^{-\rho^2} = 19.85\ \frac{cross}{sec}$ $\quad \tau = \frac{1 - e^{-\rho^2}}{N_R} = 501 \cdot 10^{-6}\ sec$

Problem 4.10 Maximum RMS delay spread = ?

	System	RF Data Rate	Modulation	Spectral Efficiency
1	USDC	48.6 kbps	π/4 DQPSK	2 bps/Hz
2	GSM	270.833 kbps	0.3 GMSK	3,33 bps/Hz
3	DECT	1152 kbps	0.3 GMSK	3,33 bps/Hz

$T_S \geq 10\sigma_\tau$ $\quad T_{S1} = \frac{2}{48.6 \cdot 10^{+3}} = 41.15 \cdot 10^{-6} = 41{,}15\ \mu sec$

$\sigma_{\tau 1} \leq 4{,}115\ \mu sec$

$T_{S2} = \frac{1}{0.3 \cdot 270.833 \cdot 10^{+3}} = 12{,}31\ \mu sec$ $\quad \sigma_{\tau 2} \leq 1{,}23\ \mu sec$

$T_{S3} = \frac{1}{0.3 \cdot 1152\ kbps} = 2{,}89\ \mu sec$ $\quad \sigma_{\tau 3} \leq 0{,}289\ \mu sec$

Prob 4.11 RF Doppler spectrum = ? $\quad \frac{5}{8}\lambda$ vertical monopole

$f_m = \frac{v}{\lambda}$

$$S(f) = \frac{1.75A}{\pi f_m \sqrt{1 - \left(\frac{f-f_c}{f_m}\right)^2}} \qquad S_{BB}(f) = \frac{1}{8\pi f_m} K\left[\sqrt{1 - \left(\frac{f}{2f_m}\right)^2}\right]$$

↓ elliptical integral of first kind

ZA $\frac{5}{8}\lambda$ monopole $G(\lambda) = 1.75$ [illegible]

[illegible]

!!! [illegible]

4.12 Show that the magnitude (envelope) of the sum of two independent identically distributed complex (quadrature) Gaussian sources is Rayleigh distributed

$$f_{X,Y}(X,Y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{X^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{Y^2}{2\sigma^2}}$$

X, Y – two independent, zero mean, Gaussian random variables with equal variance σ^2

$$f_{X,Y}(X,Y) = \frac{1}{2\pi\sigma^2} e^{-\frac{X^2+Y^2}{2\sigma^2}}$$

$X = Z\cos\phi \quad X^2+Y^2=Z^2$

$Y = Z\sin\phi \quad \phi = \arctg\frac{Y}{X}$

$$|J| = \begin{vmatrix} \frac{\partial X}{\partial Z} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial Z} & \frac{\partial Y}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \cos\phi & -Z\sin\phi \\ \sin\phi & Z\cos\phi \end{vmatrix} = Z\cos^2\phi + Z\sin^2\phi = Z$$

$$f_{Z,\varphi}(Z,\phi) = |J| \cdot f_{X,Y}(X,Y) = \frac{Z}{2\pi\sigma^2} e^{-\frac{Z^2}{2\sigma^2}}$$

$$f_Z(Z) = \int_0^{2\pi} f_{Z,\varphi}(Z,\phi)\, d\phi = \int_0^{2\pi} \frac{Z}{2\pi\sigma^2} e^{-\frac{Z^2}{2\sigma^2}} d\phi = \frac{Z}{\sigma^2} e^{-\frac{Z^2}{2\sigma^2}}$$

Z – represents magnitude (envelope) of the sum of two independent identically distributed complex (quadrature) Gaussian sources

$$F_\xi(\xi \le x) = \int_{-\infty}^{x} p_\xi(x)\, dx \qquad \boxed{p_\xi(x) = \frac{dF_\xi(x)}{dx}}$$

ALTERNATIVO: $F_Z(Z) = \iint_{\sqrt{X^2+Y^2}<Z \text{ (krugnica)}} f_{X,Y}(X,Y)\, dX\, dY =$

$$= \int_0^{2\pi}\int_0^{Z} \frac{r}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr\, d\phi = 2\pi \frac{1}{2\pi\sigma^2}\int_0^{Z} r e^{-\frac{r^2}{2\sigma^2}} dr$$

$$= \int_0^{Z} e^{-\frac{r^2}{2\sigma^2}} d\frac{r^2}{2\sigma^2} = \int_0^{\frac{Z^2}{2\sigma^2}} e^{-u}\, du = -e^{-u}\Big|_0^{\frac{Z^2}{2\sigma^2}} =$$

$$= 1 - e^{-\frac{Z^2}{2\sigma^2}}$$

$$f_Z(Z) = \frac{dF_Z(Z)}{dZ} = +e^{-\frac{Z^2}{2\sigma^2}} \frac{2Z}{2\sigma^2} = \frac{Z}{\sigma^2} e^{-\frac{Z^2}{2\sigma^2}} \quad Z \ge 0$$

COMPLEX GO SAKA ZA SIGNALI SO QUADRATURE ZOŠTO MOŽE DA SE RAZGLEDUVA KOMPLEKSOT VAKO: $z = x + jy = r\cdot e^{j\varphi}$

ŠTO SMETAM DEKA E NAJPOPULAR PRISTAP !!!

PR.4.13 GENERATE TIME SEQUENCE of 8192 SAMPLE VALUES OF RAYLEIGH FADING SIGNAL FOR:

(a) $f_m = 20Hz$ (b) $f_m = 200Hz$

$\Delta f = \frac{2 f_m}{N-1} = 4 mHz$ $\quad t = \frac{1}{\Delta f} = 204 s$

PR.4.14 Generate 100 sample functions BASED ON PR 4.13 AND COMPARE TEORETICAL & SIMULATED VALUES FOR R_{RMS}, N_L, and $\bar{\tau}$ for $\rho = 1; 0,1; 0,01$

$N_R = \sqrt{2\pi} f_m \rho e^{-\rho^2}$ $\quad \bar{\tau} = \frac{e^{+\rho^2} - 1}{\sqrt{2\pi} f_m \rho}$

(a) $f_m = 20Hz$

$\rho = 1$ $\quad N_L = 18,44$ $\quad \bar{\tau} = 34 msec$

$\rho = 0.1$ $\quad N_L = 4,96$ $\quad \bar{\tau} = 2 msec$

$\rho = 0.01$ $\quad N_L = 0.5$ $\quad \bar{\tau} = 0,2 msec$

$\rho = \frac{R}{R_{MS}}$

(b) $f_m = 200Hz$

$\rho = 1$ $\quad N_\rho = 184,4$ $\quad \bar{\tau} = 3,4 msec$

$\rho = 0.1$ $\quad N_L = 49,6$ $\quad \bar{\tau} = 0.2 msec$

$\rho = 0.01$ $\quad N_L = 5,01$ $\quad \tau = 0.02 msec$

---

rfs_example3.m

$Df = 0.111 \cdot 10^{-3}$ $\quad N = 1000$ $\quad \Delta t > \tau$

$\Delta f = \frac{2 f_m}{N-1} = \frac{2 \cdot 100}{1000-1} = 0,2 Hz$

$\Delta t_{1000} = \frac{1}{f_s} = \frac{\Delta t}{N} = \frac{1}{N \Delta f} = \underline{5 \cdot 10^{-3}}$

$\rho = \frac{1}{\sqrt{2}} = 0,707$ $\quad \frac{R}{R_{RMS}} = 0,707$ $\quad 20 \log \frac{R}{R_{RMS}} = -3dB$

SIMULATION OF FLAT FADING USING MATLAB FOR CLASSROOM INSTRUCTIONS

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mfs_example4 $\quad f_m = 200Hz$ $\quad T_C = \frac{9}{16\pi f_m} = 1,8 ms$

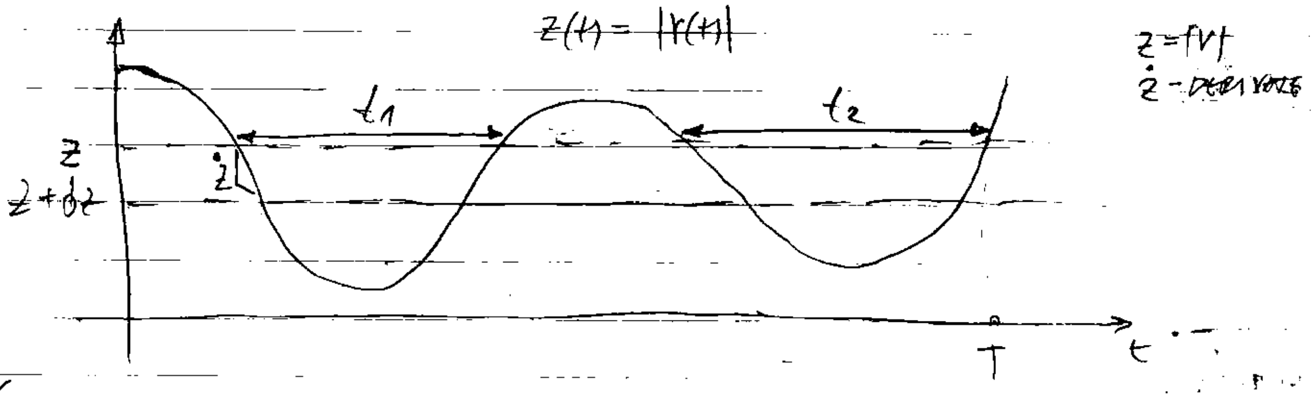
$T_S = 11,1 e^{-3} = 11,1 ms$ $\quad T_S \ll T_C \Rightarrow$ **SLOW** FADING

mfs_fastfading $\quad T_C = 1,8 ms$ $\quad T_S = 11.1 ms$ $\quad T_S > T_C$ FAST FAD

LCR FROM GOLDSMITH $\quad L_Z$

$z = |r|$ $\quad \dot{z}, p(z, \dot{z})$ $\quad \dot{z} \rightarrow$ DERIVATE OF SIGNAL ENVELOPE

↳ SIGNAL ENVELOPE $\quad p(z, \dot{z})$ JOINT DISTRIBUTION



EXPECTED AMOUNT OF TIME THAT SIGNAL ENVELOPE SPENDS IN INTERVAL $z, z+dz$ WITH ENVELOPE SLOPE WITHIN RANGE $[\dot z, \dot z + d\dot z]$ OVER TIME DURATION dt IS:

$$A = p(z, \dot z)\, dz\, d\dot z\, dt$$

TIME TO CROSS FROM z TO $z+dz$ once for given ENVELOPE SLOPE $\dot z$ IS:

$$B = \frac{dz}{\dot z}$$

EXPECTED CROSSINGS OF THE ENVELOPE z WITHIN INTERVAL $(z, z+dz)$ FOR A GIVEN ENVELOPE SLOPE $\dot z$ OVER dt IS:

$$\frac{A}{B} = \frac{p(z,\dot z)\, dz\, d\dot z\, dt}{\frac{dz}{\dot z}} = \dot z\, p(z, \dot z)\, d\dot z\, dt$$

- THE EXPECTED NUMBER OF CROSSINGS OF ENVELOPE LEVEL z for SLOPES BETWEEN $\dot z \div \dot z + d\dot z$ in interval $[0, T]$ IN DOWNWARD DIRECTION IS:

$$\int_0^T \dot z\, p(z, \dot z)\, d\dot z\, dt = \dot z\, p(z, \dot z)\, d\dot z\, T$$

$$N_z = T \int_0^{\infty} \dot z\, p(z, \dot z)\, d\dot z$$ } EXPECTED CROSSINGS OF ENVELOPE z WITH ~~NEGATIVE~~ SLOPE OVER INTERVAL $[0, T]$

$$L_z = \frac{N_z}{T} = \int_0^{\infty} \dot z\, p(z, \dot z)\, d\dot z$$ } LEVEL CROSSING RATE

$$L_z = \sqrt{2\pi(K+1)}\, f_D\, \rho\, e^{-K-(K+1)\rho^2}\, I_0\left(2\rho\sqrt{K(K+1)}\right) \qquad \rho = \frac{Z}{\sqrt{P_r}}$$

↑ FOR RICIAN FADING — $\sqrt{P_r}$: AVERAGE ENVELOPE LEVEL

FOR RAYLEIGH FADING $K=0$

$$L_z = \sqrt{2\pi}\, f_D\, \rho\, e^{-\rho^2}$$

- AVERAGE FADE DURATION

$$p(z(t) < Z) = \frac{1}{T} \sum_i t_i \qquad \bar t_z = \frac{1}{T L_z} \sum_i t_i = \frac{p(z(t) < Z)}{L_z}$$

$$\bar t_z = \frac{e^{\rho^2} - 1}{\rho f_D \sqrt{2\pi}}$$

$$s(t) = \sum_{i=1}^{N} a_i \cos(\omega_c t + \varphi_i) \qquad v \cdot \Delta t \qquad p_\phi(\phi) = \frac{1}{2\pi}$$

$$\Delta\phi_D = \frac{2\pi \Delta S}{\lambda} = \frac{2\pi d \cdot \cos\alpha}{\lambda} = \frac{2\pi d}{\lambda} \cdot \cos\alpha \qquad \omega_D = \frac{\phi_D}{\Delta t} = \frac{2\pi}{\Delta t} \cdot \frac{v \Delta t}{\lambda} \cdot \cos\alpha$$

$$f_D = \frac{\omega_D}{2\pi} = \frac{1}{2\pi} \frac{2\pi v}{\lambda} \cos\alpha = \frac{v}{\lambda} \cdot \cos\alpha$$

$$P(\lambda, k) = \frac{(c\tau)^k}{k!} e^{-c\tau} = \frac{\lambda^k}{k!} e^{-\lambda}$$

$$\omega_{di} = \frac{\omega_c v}{c} \cos\psi_i \rightarrow \text{ANGLE OF ARRIVAL}$$

$$\omega_c = 2\pi f_c \qquad \omega_{di} = 2\pi f_d \qquad f_{di} = \frac{f_c v}{c} \cos\psi_i$$

$$v = d \cdot \Delta t \qquad \phi_d = \frac{2\pi \Delta S}{\lambda} = \frac{2\pi d \cdot \cos\psi_i}{\lambda}$$

$$\omega_{di} = \frac{\phi_{di}}{\Delta t} = \frac{2\pi d \cos\psi_i}{\Delta t\, \lambda} = \frac{2\pi \cdot v \Delta t \cos\psi_i}{\Delta t \cdot \lambda}$$

$$f_{di} = \frac{\omega_{di}}{2\pi} = \frac{v}{\lambda} \cdot \cos\psi_i \qquad f_{di} = \frac{\frac{k}{\lambda} \cdot v}{k} \cos\psi_i = \frac{v}{\lambda} \cdot \cos\psi_i$$

$$s(t) = \sum_{i=1}^{N} a_i \cos(\omega_c t + \omega_{di} t + \phi_i) \quad \text{DOPLER} \quad \text{MULTIPATH}$$

$$s(t) = \sum_{i=1}^{N} a_i \cos\omega_c t \cdot \cos(\omega_{di} t + \phi_i) - \sum_{i=1}^{N} a_i \sin\omega_c t \cdot \sin(\omega_{di} + \phi_i)$$

$$s(t) = \underbrace{\left(\sum_{i=1}^{N} a_i \cos(\omega_{di} t + \phi_i)\right)}_{I(t) \text{ - in-phase}} \cdot \cos(\omega_c t) - \underbrace{\left(\sum_{i=1}^{N} a_i \sin(\omega_{di} t + \phi_i)\right)}_{Q(t) \text{ - quadrature}} \sin(\omega_c t)$$

$$I(t) = \sum_{i=1}^{N} a_i \cos(\omega_{di} t + \phi_i) \qquad Q(t) = \sum_{i=1}^{N} a_i \sin(\omega_{di} t + \phi_i)$$

$$R = \sqrt{I^2(t) + Q^2(t)} \qquad f(r) = \frac{r}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right) \quad \text{PDF FOR } r(t)$$

$I(t)$ & $Q(t)$ ARE DISTRIBUTED NORMALLY WHEN N IS LARGE

$$f_c = 900\,\text{MHz} \qquad \lambda = \frac{c}{f} = \frac{300}{900} = 0.33 \qquad c = \lambda f_c = \frac{\lambda}{T_c}$$

$$T_c = \frac{1}{f_c} = \frac{1}{9 \cdot 10^8} = \frac{1}{9} \cdot 10^{-8} = 1{,}11 \cdot 10^{-9} = 1{,}11\,\text{ns}$$

1250 WAVELENGTHS : $1250 \cdot T_c = 1{,}4\,\mu\text{sec}$

- a_i – Weibull DISTRIBUTED

$$f(x, k, \lambda) = \frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k} \qquad x > 0$$

$k > 0$ – shape PARAMETER $\qquad \lambda > 0$ – scale parameter

COMPLEMENTARY CUMULATIVE DISTRIBUTION IS STRETCHED EXPONENT.

weibrud ; unifrud ; demod

$e^{x} = \cos x + j \sin x$

$e^{-x} = \cos x - j \sin x$

OTK MODULACII

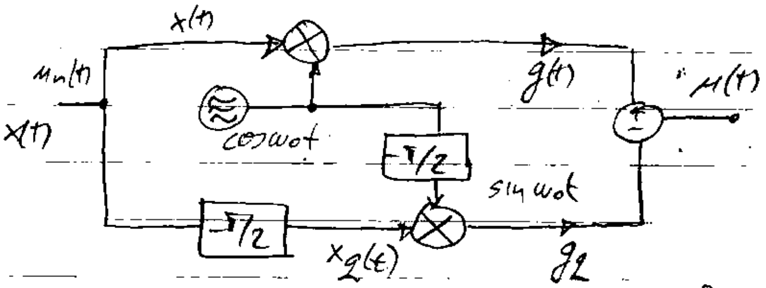
· **KAM SIGNAL**

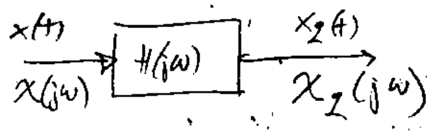
$$u(t) = U_0 \cos\omega_0 t + k\, u_m(t) \cos(\omega_0 t) = [U_0 + k\,u_m(t)] \cos\omega_0 t$$

$$u(t) = U_0 \left[1 + \frac{k\,u_m(t)}{U_0}\right] \cos\omega_0 t \qquad m(t) = \frac{u_m(t)}{U_0}$$

AMPLITUDA NA MODULIRANIOT SIGNAL SE MENUVA $U_0 - k u_m(t) \div U_0 + k u_m(t)$

· **AMISO**

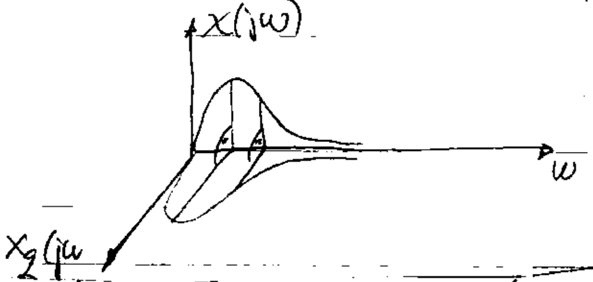




$$g(t) = x(t) \cdot \cos(\omega_0 t) \qquad G(j\omega) = \frac{1}{2} X[j(\omega - \omega_0)] + \frac{1}{2} X[j(\omega + \omega_0)]$$

$$x_2(t) \to X_2(j\omega)$$

$$H(j\omega) = \begin{cases} 1 \cdot e^{-j\frac{\pi}{2}} = -j & \omega > 0 \\ 1 \cdot e^{j\frac{\pi}{2}} = j & \omega < 0 \end{cases}$$



$$f(t) = \sin(\omega_0 t) = \cos\left(\omega_0 t - \frac{\pi}{2}\right) \qquad F(j\omega) = \mathcal{F}\left[\cos\left(\omega_0 t - \frac{\pi}{2}\right)\right]$$

$$= \mathcal{F}\left[f\left(\omega_0 t - \frac{\pi}{2}\right)\right] = G(j\omega)\, e^{-j\frac{\pi}{2}} \qquad G(j\omega) = \mathcal{F}\{\cos(\omega_0 t)\}$$

$$G(j\omega) = \mathcal{F}\left[\frac{1}{2} e^{+j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t}\right] = \frac{2\pi}{2} \delta(\omega - \omega_0) + \frac{2\pi}{2} \delta(\omega + \omega_0)$$

$$\mathcal{F}\{f(t) \cdot e^{\pm j\omega_0 t}\} = F[j(\omega \mp \omega_0)] \qquad \mathcal{F}\{e^{j\omega_0 t}\} = 2\pi\,\delta(\omega - \omega_0)$$

$$\mathcal{F}\{\sin\omega_0 t\} = [\pi\,\delta(\omega - \omega_0) + \pi\,\delta(\omega + \omega_0)]\, e^{-j\frac{\pi}{2}} = -j\pi\delta(\omega - \omega_0) - j\pi\,\delta(\omega + \omega_0) \quad (?)$$

$$\mathcal{F}\{k\} = \int_{-\infty}^{\infty} k \cdot e^{-j\omega t}\, dt = \qquad \delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t}\, d\omega$$

$\omega \to t$, $t \to -\omega$, $d\omega = -dt$

$$\delta(\omega) = -\frac{1}{2\pi} \int e^{-j\omega t}\, dt$$

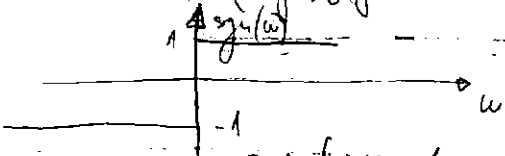
$$\int_{-\infty}^{\infty} e^{-j\omega t}\, d\omega = 2\pi\,\delta(-\omega), \quad \delta(-\omega) = \delta(\omega)$$

$$\mathcal{F}\{k\} = k \int_{-\infty}^{\infty} e^{-j\omega t}\, dt = 2k\pi\,\delta(\omega)$$

$$\mathcal{F}\{\sin\omega_0 t\} = \frac{1}{2j} \mathcal{F}\{e^{+j\omega_0 t} - e^{-j\omega_0 t}\} = \frac{1}{2j}[2\pi\,\delta(\omega - \omega_0) - 2\pi\,\delta(\omega + \omega_0)]$$

$$= -\pi j\,\delta(\omega - \omega_0) + \pi j\,\delta(\omega + \omega_0)$$

$X_q(j\omega) = \begin{cases} -jX(j\omega) & \omega>0 \\ jX(j\omega) & \omega<0 \end{cases}$ $X_q(j\omega) = -j\,\mathrm{sgn}(\omega)\cdot X(j\omega)$



$g_q(t) = x_q(t)\cdot \sin(\omega_0 t)$

$G_q(j\omega) = \mathcal{F}\{x_q(t)\cdot \frac{1}{2j}(e^{+j\omega_0 t} - e^{-j\omega_0 t})\} = \frac{1}{2j}X_q[j(\omega-\omega_0)] - \frac{1}{2j}X_q[j(\omega+\omega_0)]$

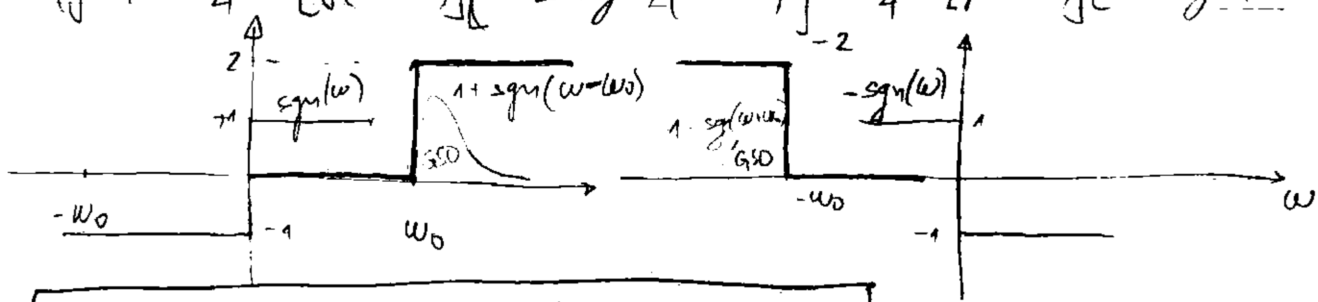
$G_q(j\omega) = -\frac{1}{2}X(j(\omega-\omega_0))\,\mathrm{sgn}(\omega-\omega_0) + \frac{1}{2}X(\omega+\omega_0)\,\mathrm{sgn}(\omega+\omega_0)$

$u(t) = \frac{1}{2}g(t) \mp \frac{1}{2}g_q(t) = \frac{1}{2}x(t)\cdot\cos(\omega_0 t) \mp \frac{1}{2}x_q(t)\sin(\omega_0 t)$

$U(j\omega) = \frac{1}{2}G(j\omega) \mp \frac{1}{2}G_q(j\omega) = \frac{1}{4}X[j(\omega-\omega_0)] + \frac{1}{4}X[j(\omega+\omega_0)] \mp$
$\{\frac{1}{4}X(\omega+\omega_0)\,\mathrm{sgn}(\omega+\omega_0) - \frac{1}{4}X[j(\omega-\omega_0)]\,\mathrm{sgn}(\omega-\omega_0)\}$

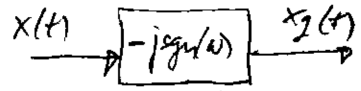
GSO

$U(j\omega) = \frac{1}{4}X[j(\omega-\omega_0)][1 \pm \mathrm{sgn}(\omega-\omega_0)] + \frac{1}{4}X[j(\omega+\omega_0)][1 \mp \mathrm{sgn}(\omega+\omega_0)]$



$u(t) = \frac{1}{2}x(t)\cos(\omega_0 t) \mp \frac{1}{2}x_q(t)\sin(\omega_0 t)$

• $x_q(t) = ?$



$x_q(t) = \int_{-\infty}^{\infty} x(\tau)\cdot h(t-\tau)\,d\tau$ $h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} -j\,\mathrm{sgn}(\omega)\,e^{+j\omega t}\,d\omega =$ (*)

$= \frac{1}{2\pi}\left[-j\int_{-\infty}^{0}(-1)e^{j\omega t}d\omega - j\int_{0}^{\infty}e^{j\omega t}d\omega\right] = \frac{1}{2\pi}\left[\int_{-\infty}^{0}e^{j\omega t}d\omega - \int_{0}^{\infty}e^{j\omega t}d\omega\right]$

$\mathcal{F}\{\mathrm{sgn}(t)\} = \int_{-\infty}^{\infty}\mathrm{sgn}(t)\,e^{-j\omega t}dt = -\int_{-\infty}^{0}e^{-j\omega t}dt + \int_{0}^{\infty}e^{-j\omega t}dt = \frac{2}{j\omega}$

$\left|\begin{matrix}\omega\to t\\ t=-\omega\end{matrix}\right|$ $\mathcal{F}\{\mathrm{sgn}(\omega)\} = +\int_{+\infty}^{0}e^{j\omega t}d\omega + \int_{0}^{-\infty}e^{j\omega t}(-1)d\omega = \int_{-\infty}^{0}e^{j\omega t}d\omega - \int_{0}^{\infty}e^{j\omega t}d\omega$

$= -\int_{0}^{\infty}e^{j\omega t}d\omega + \int_{-\infty}^{0}e^{j\omega t}d\omega = \int_{-\infty}^{0}e^{j\omega t}d\omega - \int_{0}^{\infty}e^{j\omega t}d\omega = (*) = \frac{2}{jt}$

$h(t) = \frac{j}{2\pi}\cdot(*) = \frac{j}{2\pi}\cdot\frac{2}{jt} = \frac{1}{\pi t}$ $h(t-\tau) = \frac{1}{\pi(t-\tau)}$

$x_q(t) = \int_{-\infty}^{\infty}\frac{x(\tau)}{\pi(t-\tau)}\,d\tau = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(\tau)}{t-\tau}\,d\tau = \hat{x}(t)$ hilbertowa transformacja

HILBERT TRANSFORM:

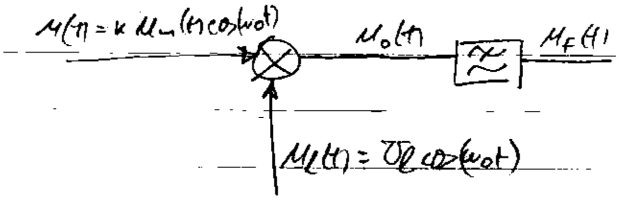
$$F(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{t-s}\,dt$$

AMJSO

$$u(t) = \frac{1}{2}x(t)\cos(\omega_0 t) \mp \frac{1}{2}\hat{x}(t)\sin(\omega_0 t) = k\,u_m(t)\cos(\omega_0 t) \mp k\,\hat{u}_m(t)\sin(\omega_0 t)$$

3. DEMODULACIJA NA AM SIGNALI

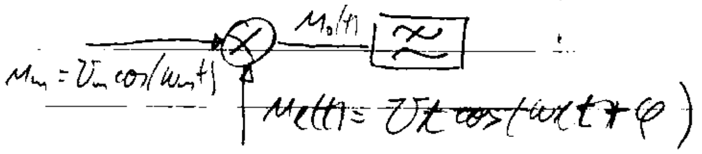
- SINHRONA DEMODULACIJA NA AMDSO



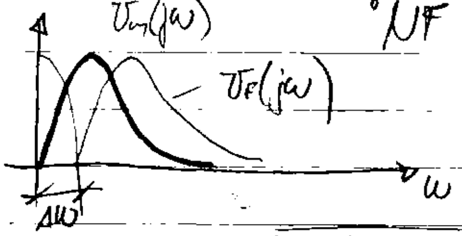
$$u_0(t) = k\,u_m(t)\cos(\omega_0 t)\cdot U_l\cos(\omega_0 t) = k\cdot u_m(t)\cdot U_l\left\{\frac{1}{2}\cos[(\omega_0-\omega_0)t] + \frac{1}{2}\cos[(\omega_0+\omega_0)t]\right\}$$

$$= \frac{k}{2}u_m(t)\cdot U_l + \frac{k}{2}u_m(t)\cdot U_l\cdot\cos[2\omega_0 t].$$

- AKO NOSITELOT NE SE SINHRONI SINFAZNI



$$u_0(t) = \underbrace{\frac{k}{2}u_m(t)\cdot U_l\cos[(\omega_0-\omega_l)t-\varphi]}_{NF} + \underbrace{\frac{k}{2}u_m(t)U_l\cos[(\omega_0+\omega_l)t+\varphi]}_{VF}$$



$$\Delta\omega = \omega_0 - \omega_l$$

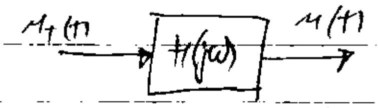
AMJSO $\quad u(t) = \frac{1}{2}u_m(t)\cos(\omega_0 t) \mp \frac{1}{2}\hat{u}_m\sin(\omega_0 t)$

SINHRONA DEMOD: $\quad u_d(t) = u(t)\cdot U_l\cos(\omega_l t)$

$\Delta\omega = 0 \Rightarrow \quad u_F(t) = \frac{1}{4}U_l u_m(t)\cos\varphi \pm \frac{1}{4}U_l\hat{u}_m(t)\cdot\sin\varphi$

$\varphi = 0 \Rightarrow \quad u_F(t) = \frac{1}{4}U_l u_m(t)\cos\varphi = k\,u_m(t)$

- SINHRONA DEMODULACIJA NA AMNSO

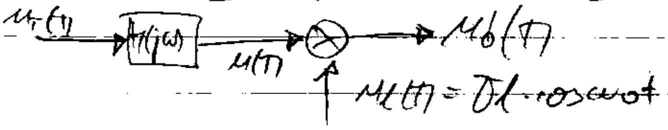


$$u_1(t) = k\,u_m(t)\cos(\omega_0 t) + U_0\cos(\omega_0 t)$$

$$u(t) \rightarrow U(j\omega) = U_1(j\omega)\cdot H(j\omega)$$

$$U_1(j\omega) = \frac{k}{2}U_m[j(\omega-\omega_0)] + \frac{k}{2}U_m[j(\omega+\omega_0)]$$

→ NE GO ODFRLAMUVAME NOSITELOT !!



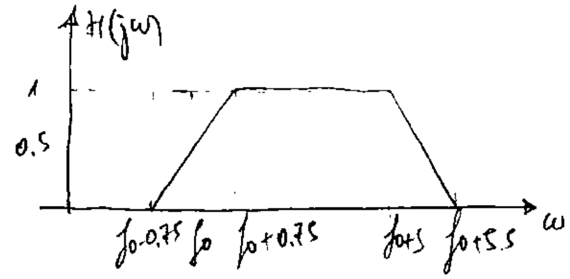
$$U_d(j\omega) = \frac{U_l}{2}U[j(\omega-\omega_0)] + \frac{U_l}{2}U[j(\omega+\omega_0)]$$

$$U_d(j\omega) = \frac{U_L}{2} U_T(j\omega)_{\omega_0} H(j\omega)_{\omega_0} + \frac{U_L}{2} U_T[j(\omega+\omega_0)] H[j(\omega+\omega_0)] =$$

$$= \frac{U_L}{2} H[j(\omega-\omega_0)] \left\{ \frac{k}{2} U_m[j(\omega-\omega_0)] + \frac{k}{2} U_m[j(\omega+\omega_0)] \right\} +$$

$$+ \frac{U_L}{2} H[j(\omega+\omega_0)] \left\{ \frac{k}{2} U_m[j(\omega-\omega_0)] + \frac{k}{2} U_m[j(\omega+\omega_0)] \right\}$$

$$U[j(\omega-\omega_0)] = U_T(j\omega)_{\omega_0} H(j\omega)_{\omega_0} \quad \frac{k}{2}\left\{ U_m[j(\omega-\omega_0)] + U_m[j(\omega+\omega_0)] \right\} H[j(\omega-\omega_0)]$$



$$\omega-\omega_0+\omega-\omega_0 = \omega-2\omega_0$$
$$\omega-\omega_0+\omega_0 = \omega$$

$$U[j(\omega-\omega_0)] = \frac{k}{2} U_m[j(\omega-2\omega_0)] H[j(\omega-\omega_0)] + \frac{1}{2} k U_m(j\omega) H[j(\omega-\omega_0)]$$

$$U[j(\omega+\omega_0)] = \frac{k}{2} U_m(j\omega) H[j(\omega+\omega_0)] + \frac{k}{2} U_m[j(\omega+2\omega_0)] H[j(\omega+\omega_0)]$$

$$U_d(j\omega) = \frac{k}{4} U_L U_m(j\omega) \left\{ H[j(\omega-\omega_0)] + H[j(\omega+\omega_0)] \right\} +$$ (= 1)

$$\frac{k}{4} U_L \left\{ U_m[j(\omega-2\omega_0)] H[j(\omega-\omega_0)] + \frac{k}{2} U_m[j(\omega+2\omega_0)] H[j(\omega+\omega_0)] \right\}$$

OUT FILTER

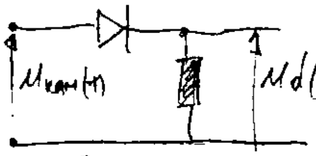
$$u_F(t) = \frac{k U_L}{4} u_m(t)$$

TR. ①

$$\cos(\omega_0 t)\cdot\cos(\omega_0 t) = \frac{1}{2}\cos(2\omega_0 t) + \frac{1}{2}\cos(\emptyset) = \frac{1}{2}\cos(2\omega_0 t) + \frac{1}{2}$$

(3.6) Детекција на КАМ сигнали

- Детекција на КАМ со нелинеарен детектор



$$u_d(t) = a_1 u_{kam}(t) + a_2 u_{kam}^2(t)$$

$$u_{kam}(t) = U_0[1 + m_0 m(t)] \cos\omega_0 t \qquad m(t) = \frac{u_m(t)}{U_m} \quad m_0 = \frac{U_m}{U_0}$$

$$a_2 u_{kam}^2(t) = a_2 U_0^2 [1 + m_0 m(t)]^2 \cos^2\omega_0 t = \frac{a_2 U_0^2}{2}[1+m_0 m(t)]^2(1+\cos 2\omega_0 t)$$

$$\cos^2\omega_0 t = \cos(\omega_0 t)\cos(\omega_0 t) = \frac{1}{2}\cos[(\omega_0-\omega_0)t] + \frac{1}{2}\cos 2\omega_0 t = \frac{1}{2}(1+\cos 2\omega_0 t)$$

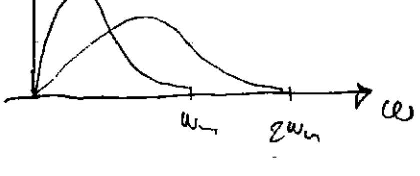
$$a_2 u_{kam}^2(t) = \frac{a_2 U_0^2}{2}[1 + 2m_0 m(t) + m_0^2 m^2(t)] + \frac{a_2 U_0^2}{2}[\ldots]\cos 2\omega_0 t$$

ФИЛТЕРОД

$$u_F(t) = \underbrace{\frac{a_2 U_0^2}{2}}_{(1)} + \underbrace{a_2 U_0^2 m_0 m(t)}_{(2)} + \underbrace{\frac{a_2 U_0^2}{2} m_0^2 m^2(t)}_{(3)}$$

(2) - корисен сигнал

(3) -



$$m(t) = \cos(\omega_m t)$$

$$\frac{a_2 U_0^2}{2} m_0^2 m^2(t) = \frac{a_2 U_0^2}{2} m_0^2 (1 + \cos 2\omega_m t)\frac{1}{2}$$

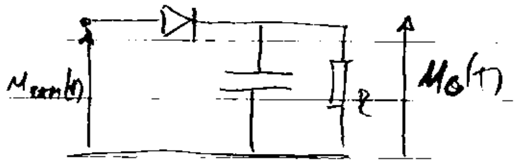
$$= \underbrace{\frac{a_2 U_0^2}{4} m_0^2}_{\text{отпаѓа}} + \frac{a_2 U_0^2}{4}\cos(2\omega_m t)$$

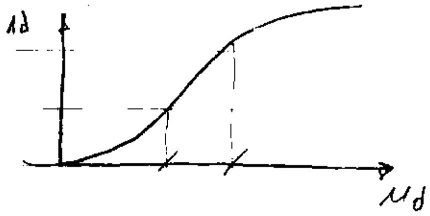
$$d_2 = \left(\frac{\frac{a_2 U_0^2 m_0^2}{4}}{a_2 U_0^2 m_0}\right)^2 = \left(\frac{m_0}{4}\right)^2$$

$a_{H2} = 20 \log \frac{4}{m_0}$ - Y SLABEENJE NA HARMONICNOTO IZOBLICUVANJE OD II RED

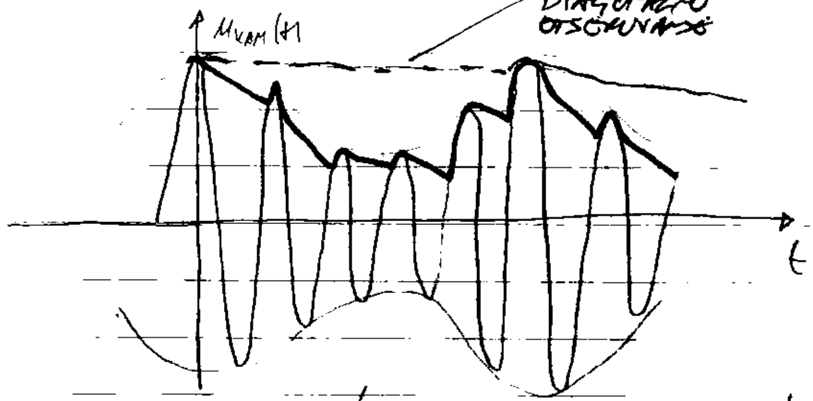
$m_0 \downarrow$ POGOLEMO SLABEENJE NA HARM. IZOBLIC.

· LINEAREN DETEKTOR NA ANVELOPA





$\tau_d = R_d \cdot C \qquad \tau = R \cdot C$



ZA DA NE DOJDE DO DIAG. OTSEKUVANJE
$S_a(t_0) = S_{ce}(t_0)$

$u_{AM}(t) = U_0(1 + m_0 m(t)) \cos(\omega_0 t)$

$u_a(t) = U_0(1 + m_0 m(t))$

$u_c(t) = U e^{-\frac{t - t_0}{RC}}$

$$S_a(t) = \left.\frac{d u_a(t)}{dt}\right|_{t=t_0} \qquad S_{ce}(t_0) = \left.\frac{d u_c(t)}{dt}\right|_{t=t_0}$$

NEKA: $m(t) = \cos(\omega_m t)$ - TEST TON

$S_a(t_0) = [U_0(1 + m_0 \cos\omega_m t)]' = -U_0 m_0 \cdot \sin(\omega_m t_0) \cdot \omega_m = -U_0 m_0 \omega_m \sin(\omega_m t_0)$

$S_{ce}(t_0) = \frac{U}{RC} e^{-\frac{t - t_0}{RC}} \cdot (-1) = -\frac{U}{RC} \cdot e^0 = -\frac{U}{RC}$

(*) $U = u_a(t_0) = U_0(1 + m_0 m(t_0)) = U_0[1 + m_0 \cos(\omega_m t_0)]$

$-\frac{U_0}{RC}(1 + m_0 \cos(\omega_m t_0)) = -U_0 m_0 \omega_m \sin(\omega_m t_0)$

$$\tau = RC = \frac{1 + m_0 \cos(\omega_m t_0)}{m_0 \omega_m \sin(\omega_m t_0)} \qquad \frac{d\tau}{dt_0} = 0$$

$$\frac{-m_0 \omega_m \sin\omega_m t_0 \cdot m_0 \omega_m \sin(\omega_m t_0) - (1 + m_0 \cos(\omega_m t_0)) m_0 \omega_m^2 \cos(\omega_m t_0)}{m_0^2 \omega_m^2 \sin^2(\omega_m t_0)} = 0$$

$-m_0^2 \omega_m^2 \sin^2 \omega_m t_0 - m_0 \omega_m^2 \cos(\omega_m t_0) - m_0^2 \omega_m^2 \cos^2(\omega_m t_0) = (*) = 0$

~~$-m_0^2 \omega_m^2 - m_0 \omega_m^2 \cos(\omega_m t_0) = 0 \qquad -m_0^2 \omega_m^2 - m_0 \omega_m^2 \cos(\omega_m t_0)$~~

$-m_0 \omega_m [m_0 \omega_m \sin^2 \omega_m t_0 + \omega_m \cos(\omega_m t_0) + m_0 \omega_m \cos^2(\omega_m t_0)] = 0$ (A)

$$\frac{(*)}{m_0^2 \omega_m^2 \sin^2(\omega_m t_0)} = -1 - \frac{m_0 \omega_m \cos(\omega_m t_0) + m_0^2 \omega_m \cos^2(\omega_m t_0)}{m_0^2 \omega_m^2 \sin^2(\omega_m t_0)} =$$

$$= -1 - \frac{\cos(\omega_m t_0)[1 + m_0 \cos(\omega_m t_0)]}{m_0 \omega_m \sin^2(\omega_m t_0)} = 0$$

~~$-\cos(\omega_m t_0)[1 + m_0 \cos(\omega_m t_0)] = m_0 \omega_m \sin^2(\omega_m t_0)$~~

~~$-\cos(\omega_m t_0) + m_0 \cos^2(\omega_m t_0) + m_0 \omega_m \sin^2(\omega_m t_0) = 0$~~

$$m_0\left[\omega_m \sin^2(\omega_m t_0) + \cos^2(\omega_m t_0)\right] = -\cos(\omega_m t_0)$$

$$\boxed{m_0\Big|_{max} = \frac{-\cos\omega_m t_0}{\omega_m \sin^2(\omega_m t_0) + \cos^2(\omega_m t_0)}}$$

ISTO

~~(1) ⇒ $m_0[\omega_m \sin^2(\omega_m t_0) + \cos^2(\omega_m t_0)] = -\cos(\omega_m t_0)$~~

(1) ⇒ $m_0\omega_m \sin^2\omega_m t_0 + \omega_m\cos(\omega_m t_0) + m_0\omega_m\cos^2(\omega_m t_0) = 0$

$m_0\omega_m(\sin^2\omega_m t_0 + \cos^2\omega_m t_0) = -\omega_m\cos(\omega_m t_0)$

$$\boxed{m_0 = -\cos(\omega_m t_0)}$$

$$\tau_{max} = RC_{max} = \frac{1-\cos^2(\omega_m t_0)}{-\cos(\omega_m t_0)\cdot\omega_m\cdot\sin(\omega_m t_0)} = \frac{\sin^2(\omega_m t_0)}{\cos(\omega_m t_0)\cdot\omega_m\cdot\sin(\omega_m t_0)}$$

$$\tau_{max} = -\frac{\sin(\omega_m t_0)}{\omega_m\cos(\omega_m t_0)} = -\frac{1}{\omega_m}\,tg(\omega_m t_0)$$

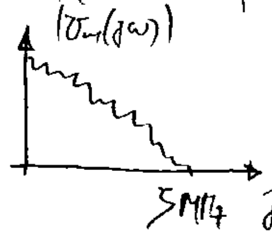
$$\tau_{max} = -\frac{\sqrt{1-\cos^2\omega_m t_0}}{\omega_m\cos(\omega_m t_0)} = -\frac{1}{\omega_m}\sqrt{\frac{1}{\cos^2(\omega_m t_0)}-1} = -\frac{1}{\omega_m}\sqrt{\frac{1}{m_0^2}-1}$$

$$\boxed{\tau_{max} = -\frac{1}{\omega_m}\sqrt{\frac{1}{m_0^2}-1}}$$

- DETEKCIJA NA AMNSO SO D.A

– AMNSO NE MOŽE DA SE DETEKTIRA SO DA

DOKAZ:
$$u(t) = u_m(t)\cos(\omega_0 t) \mp \hat{u}_m(t)\sin(\omega_0 t)$$
$$u(t) = u_a(t)\cos(\omega_0 t + \varphi(t))$$
$$u_a(t) = \sqrt{u_m^2(t) + \hat{u}_m^2(t)} \neq u_m(t)$$



$$u_m(t) = U_m\cos(\omega_m t)$$
$$m(t) = \cos(\omega_m t)$$
$$u(t) = U_0(1 + m_0 m(t))\cos(\omega_0 t) =$$
$$= U_0(1 + m_0\cdot\cos(\omega_m t))\cos(\omega_0 t) =$$
$$= U_0\cos\omega_0 t + \frac{U_0 m_0}{2}\cos[(\omega_0-\omega_m)t] + \frac{U_0 m_0}{2}\cos[(\omega_0+\omega_m)t] \qquad m_0 = \frac{kU_m}{U_0}$$

$$u_F(t) = U_0 H_0\cos(\omega_0 t) + \frac{U_0 m_0}{2}H^+\cos[(\omega_0-\omega_m)t] + \frac{U_0 m_0}{2}H^-\cos[(\omega_0+\omega_m)t]$$
$$u_F(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) = u_a(t)\cos(\omega_0 t + \varphi)$$
$$u_a(t) = \sqrt{A^2+B^2} = ?$$
$$u_F(t) = U_0 H_0\cos(\omega_0 t) + \frac{U_0 m_0 H^+}{2}\cos(\omega_0 t)\cdot\cos\omega_m t + \frac{U_0 m_0 H^+}{2}\sin(\omega_0 t)\sin(\omega_m t) +$$
$$+ \frac{U_0 m_0}{2}H^-\cos\omega_0 t\cdot\cos\omega_m t - \frac{U_0 m_0 H^-}{2}\sin\omega_0 t\cdot\sin\omega_m t$$

$$u_F(t) = U_0\left(H_0 + \frac{m_0 H^+}{2}\cos\omega_m t + \frac{m_0 H^-}{2}\cos\omega_m t\right)\cdot\cos(\omega_0 t) +$$
$$+ U_0\left(\frac{m_0 H^+}{2}\sin\omega_m t - \frac{m_0 H^-}{2}\sin\omega_m t\right)\cdot\sin(\omega_0 t)$$

$$u_F(t) = U_0\left[H_0 + \frac{m_0}{2}(H^+ + H^-)\cos\omega_m t\right]\cos\omega_0 t + \frac{U_0 m_0}{2}(H^+ - H^-)\sin\omega_m t \sin(\omega_0 t)$$

$$u_a(t) = U_0\sqrt{\left(H_0 + \frac{m_0}{2}(H^+ + H^-)\cos\omega_m t\right)^2 + \left(\frac{1}{2}m_0(H^+ - H^-)\sin\omega_m t\right)^2}$$

$$u_a(t) = U_{a0} + U_{a1}\cos(\omega_m t) + U_{a2}\cos(2\omega_m t)$$

$$d_2 = \frac{U_{a2}^2}{U_{a1}^2} = \left[\frac{m_0(H^+ - H^-)^2}{4}\right]^2 \qquad \Delta H = H^+ - H^- = 1 \text{ NAJGORE} \qquad d_2 = \left(\frac{m_0}{4}\right)^2$$

$$a_{H2} = 10\log\left(\frac{4}{m_0}\right)^2 = 20\log\frac{4}{m_0}$$

• SNAGA NA SLOŽEN SIGNAL

① $x(t) = U_1\cos(\omega_1 t) + U_2\cos(\omega_2 t)$ $\qquad P_+ = \frac{U_1^2}{2R} + \frac{U_2^2}{2R}$

② $x(t) = U_1\cos(\omega_1 t) + U_2\cos(\omega_1 t)$ $\qquad P_+ = \frac{(U_1 + U_2)^2}{2R}$

③ $x(t) = U_1\cos(\omega_1 t + \varphi_1) + U_2\cos(\omega_1 t + \varphi_2) =$

$$U_1\cos(\omega_1 t)\cdot\cos\varphi_1 - U_1\sin(\omega_1 t)\sin\varphi_1 + U_2\cos(\omega_1 t)\cdot\cos\varphi_2 - U_2\sin(\omega_1 t)\sin\varphi_2$$

$$= \underbrace{(U_1\cos\varphi_1 + U_2\cos\varphi_2)}_{A}\cos(\omega_1 t) - \underbrace{(U_1\sin\varphi_1 + U_2\sin\varphi_2)}_{B}\sin(\omega_1 t)$$

$$x(t) = \sqrt{A^2 + B^2}\cdot\cos(\omega_1 t + \varphi) \qquad \varphi = \operatorname{arctg}\frac{B}{A}$$

$$A = R\cos\varphi \quad B = R\sin\varphi;\ R = \sqrt{A^2 + B^2},\ \frac{B}{A} = \operatorname{tg}\varphi \Rightarrow \varphi = \operatorname{arctg}\frac{B}{A}$$

$U_1 = U_2$ $\qquad \underline{\underline{x(t)}} = \sqrt{\underbrace{U_1^2(\cos\varphi_1 + \cos\varphi_2)^2 + U_1^2(\sin\varphi_1 + \sin\varphi_2)^2}_{*}}\cdot\cos(\omega_1 t + \varphi)$

$$* = U_1^2(\cos^2\varphi_1 + 2\cos\varphi_1\cos\varphi_2 + \cos^2\varphi_2 + \sin^2\varphi_1 + 2\sin\varphi_1\sin\varphi_2 + \sin^2\varphi_2) =$$

(where $\cos^2\varphi_1 + \sin^2\varphi_1 \to 1$, $\cos^2\varphi_2 + \sin^2\varphi_2 \to 1$)

$$= 2U_1^2(1 + \underbrace{\cos\varphi_1\cos\varphi_2 + \sin\varphi_1\sin\varphi_2}_{\cos(\varphi_1 - \varphi_2)}) =$$

$$= \left| \begin{array}{l} 1 + \cos\alpha \qquad \cos\left(\frac{\alpha}{2} + \frac{\alpha}{2}\right) = \cos\frac{\alpha}{2}\cdot\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\cdot\sin\frac{\alpha}{2} = \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} \\ 1 + \cos\alpha = 1 + \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} = 2\cos^2\frac{\alpha}{2} \end{array} \right| =$$

$$= 2U_1^2\cdot 2\cos^2\left(\frac{\varphi_1 - \varphi_2}{2}\right) = \boxed{4U_1^2\cdot\cos^2\frac{\varphi_1 - \varphi_2}{2}}$$

$$x(t) = 2U_1\cdot\cos\frac{\varphi_1 - \varphi_2}{2}\cdot\cos(\omega_1 t + \varphi)$$

• AKO: $\varphi_1 = \varphi_2$ $\qquad P = \frac{(2U_1)^2}{2R} = 4\frac{U_1^2}{2R}$

AKO: $\varphi_1 - \varphi_2 = \pi \Rightarrow \pi = 0$ $\quad$ AKO: $\varphi_1 - \varphi_2 = \frac{\pi}{2}$ $\quad \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$$P = \frac{\left(2U_1\cdot\frac{\sqrt{2}}{2}\right)^2}{2R} = \frac{2U_1^2}{2R} = 2\frac{U_1^2}{2R}$$

④ Šum

- Spektralna gustina na sredna snaga na termički šum

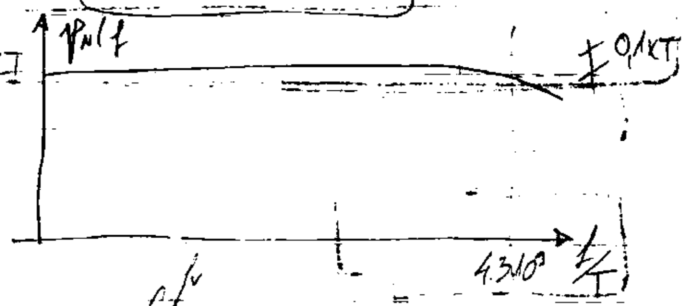
$$\varphi_N(f) = \frac{hf}{e^{\frac{h\cdot f}{KT}} - 1}$$

$h = 6.62 \cdot 10^{-34}\ J\cdot s$ $\qquad \frac{hf}{KT} \ll 1$

$K = 1.38 \cdot 10^{-23}\ J/K$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots \qquad e^{\frac{hf}{KT}} \doteq 1 + \frac{h\cdot f}{KT}$$

$$\varphi_N(f) = \frac{h\cdot f}{1 + \frac{h\cdot f}{KT} - 1} = K\cdot T$$



- Raspoloživa sredna snaga

$$\varphi_N(f) = \frac{dP_{RN}}{df} \qquad P_{RN} = \int_{f_N}^{f_V} \varphi_N(f)\,df = K\cdot T\cdot B \quad [W/Hz]$$

Pr. $B = 1 Hz \quad P_{RN} = K\cdot T\cdot B = 4\cdot 10^{-21}\ [W/Hz]$

$$T = \frac{4\cdot 10^{-21}}{1.38\cdot 10^{-23}} \doteq \underline{290\ K}$$

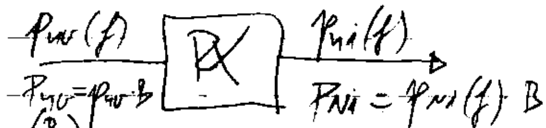
$$10\log\frac{P_{RN}}{1mW} = -174\ dB \qquad B = 4\ kHz \quad P_{RN} = 1{,}656\cdot 10^{-17}\ \frac{W}{Hz} = -138\ dB$$

$$E_{Neff} = \sqrt{4P_{RN}\cdot R} = \sqrt{4RKTB}$$

- Aproksimacija na term. šum so PP signali

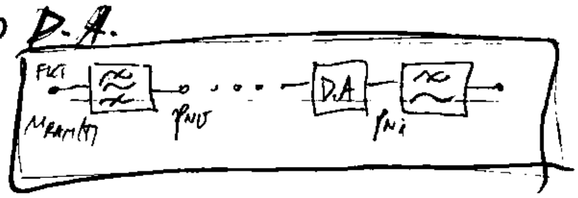
$$e(t) = \sum_{i=1}^{n} E_i \cos(\omega_i t + \varphi_i)$$

- Odnos signal šum

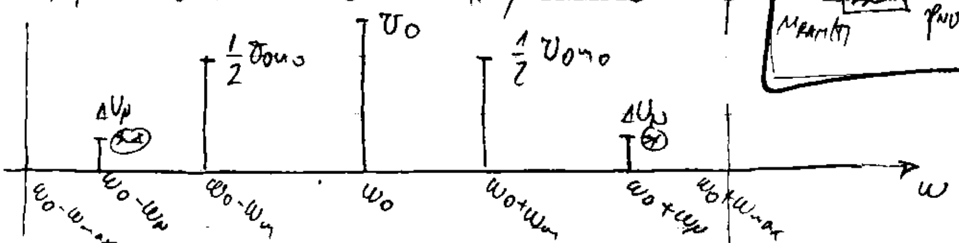


$$F = \frac{\varphi_{NI}}{\varphi_{NU}} \qquad \left(\frac{P_S}{P_N}\right)_i \qquad 10\log\left(\frac{P_S}{P_N}\right)_i$$

- **S/N kaj priemnik na KAM so D.A.**



$$u_{KAM} = U_0(1 + m_0\cos\omega_m t)\cos\omega_0 t$$



$$u(t) = U_0(1 + m_0\cos\omega_m t)\cos\omega_0 t + \Delta U_N\cdot\cos((\omega_0 + \omega_N)t + \varphi_N) =$$

$$= U_0(1 + m_0\cos\omega_m t)\cos\omega_0 t + \Delta U_N\cos(\omega_0 t)\cos(\omega_N t + \varphi_N) - \Delta U_N\sin(\omega_0 t)\sin(\omega_N t + \varphi_N)$$

$$= \underbrace{[U_0(1 + m_0\cos\omega_m t) + \Delta U_N\cos(\omega_N t + \varphi_N)]}_{A}\cos\omega_0 t - \underbrace{\Delta U_N\sin(\omega_N t + \varphi_N)}_{B}\sin(\omega_0 t)$$

$$u_0(t) = \sqrt{A^2 + B^2} = \Big\{U_0^2(1 + m_0\cos\omega_m t)^2 + 2U_0(1 + m_0\cos\omega_m t)\cdot\Delta U_N\cos(\omega_N t + \varphi_N) +$$

$$\Delta U_N^2\underbrace{(\cos^2(\omega_N t + \varphi_N) + \sin^2(\omega_N t + \varphi_N))}_{1}\Big\}^{1/2} \qquad \Delta U_N \ll U_0 \qquad \Delta U_N^2 \to \emptyset$$

$$u_a(t) = \left[U_0^2(1+m_0\cos\omega_m t)^2 + 2U_0\Delta U_N(1+m_0\cos\omega_m t)\cdot\cos(\omega_N t+\varphi_N)\right]^{1/2}$$

~~$u_a(t) = U_0(1+m_0\cos\omega_m t)\sqrt{1 + \frac{2\Delta U_N\cos(\omega_N t+\varphi_N)}{U_0(1+m_0\cos\omega_m t)}}$~~

$$\sqrt{1+x} \doteq 1 + \frac{x}{2}$$

$$u_a(t) = U_0(1+m_0\cos\omega_m t)\left(1 + \frac{\Delta U_N\cos(\omega_N t+\varphi_N)}{U_0(1+m_0\cos\omega_m t)}\right) \doteq$$

$$u_d(t) = U_0(1+m_0\cos\omega_m t) + \Delta U_N\cos(\omega_N t+\varphi_N) \quad (\Delta)$$

$$u_d(t) = U_0 + U_0 m_0\cos\omega_m t + \Delta U_N\cos(\omega_N t+\varphi_N)$$

MMV

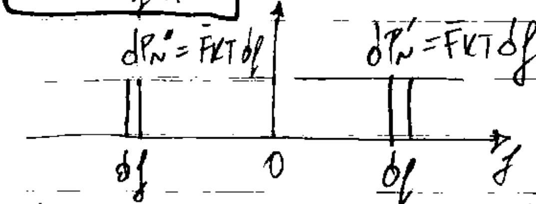
$$u_{Nd} = \Delta U_N\cos(\omega_N t+\varphi_{N\ominus}) + \Delta U_N\cos(\omega_N t+\varphi_{N\oplus})$$

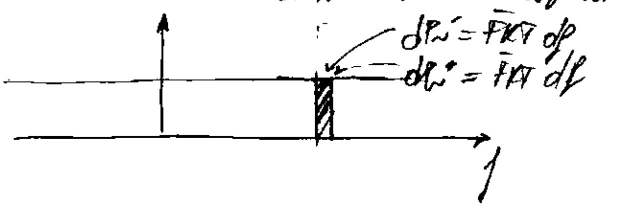
$$P_N = \sum_{i=1} \Delta P_{N_i} \qquad \Delta_N = \Delta P_N' + \Delta P_D''$$

$P_{ND} = P_{ND1} + P_{ND2}$

$\Delta P_N' \sim U_N^2 \quad \omega_0+\omega_N$

$\Delta P_N'' \sim U_N^2 \quad \omega_0-\omega_N$





$$dP_{Ni} = 2FKT\,df = 2p_{NV}\,df$$

$$P_{SV} = \left(\frac{1}{2}U_0m_0\right)^2\frac{1}{2R} + \left(\frac{1}{2}U_0m_0\right)^2\frac{1}{2R} = 2P_{m1}$$

enak stranskima comp. $\left(\frac{1}{2}U_0m_0\right)^2\frac{1}{2R}$ (Δ)

$$P_{NV} = \int_{f_0-f_{max}}^{f_0+f_{max}} p_{NV}(f)\,df = FKT\,B_{VF}$$

$$P_{Si} = (m_0U_0)^2\frac{1}{2R} = 4P_{m1}$$

$$P_{Ni} = \int_0^{f_{max}} 2p_{NV}\,df = 2FKT\cdot B_{NF}$$

$$B_{VF} = 2B_{NF}$$

$$\left(\frac{P_S}{P_N}\right)_V = \frac{2P_{m1}}{2FKT\cdot B_{NF}} = \frac{2P_{m1}}{2FKT\,B_{NF}} = \frac{P_{m1}}{FKT\,B_{NF}}$$

$$\left(\frac{P_S}{P_N}\right)_i = \frac{4P_{m1}}{2FKT\,B_{NF}} = \frac{2P_{m1}}{FKT\,B_{NF}} = 2\left(\frac{P_S}{P_N}\right)_V$$

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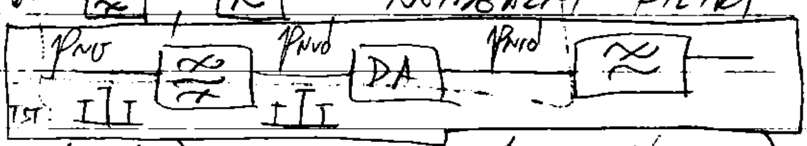


$p_{NV} \neq p_{NVD}$

$U_0A(\omega_0)$ – NOSILEC, $\quad \frac{1}{2}m_0U_0A(\omega_0+\omega_m)$ GSO; $\quad \frac{1}{2}m_0U_0A(\omega_0-\omega_m)$ DSO

$$P_{m1} = \left(\frac{1}{2}m_0U_0A(\omega_0+\omega_m)\right)^2\frac{1}{2R} \qquad P_{m2} = \left[\frac{1}{2}m_0U_0A(\omega_0-\omega_m)\right]^2\frac{1}{2R}$$

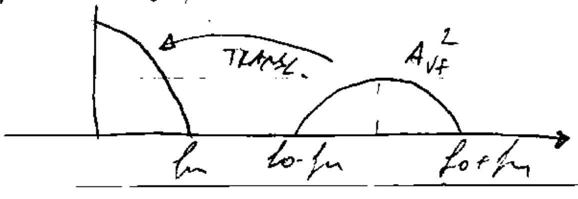
~~$P_{SVD} = P_{m1} + P_{m2}$~~

$$P_{m1} = P_{m1V}\,A^2(\omega_0+\omega_m) \qquad P_{m2} = P_{m2V}\,A^2(\omega_0-\omega_m)$$

~~$p_{NVD} = p_{NV}\cdot A_{VF}^2(\omega)$~~

$$P_{NVD} = \int_{f_1}^{f_2} p_{NV}\,A_{VF}^2(f)\,df$$

$$P_{NVD} = FKT\int_{f_1=f_0+f_{max}}^{f_2=f_0+f_{max}} A_{VF}(f)\,df$$



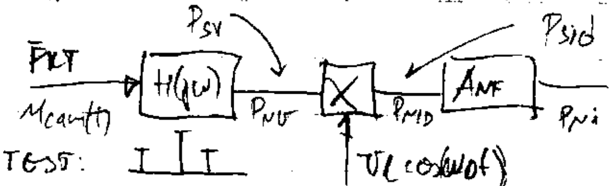
$$p_{Nid} = 2FKT\,A_{id}^2(f) \qquad p_{Ni} = p_{Nid}\,A_{NF}^2(f)$$

$$P_{Ni} = \int_0^{f_{max}} p_{Nid}\,A_{NF}^2(f)\,df = \int_0^{f_{max}} 2\,p_{Nv}\,A_{VF}^2(f)\cdot A_{NF}^2(f)\,df$$

$$P_{sid} = \left\{ m\,U_0 \left[A_{VF}(f_0 - f_m) + A_{VF}(f_0 + f_m) \right] \right\}^2 \frac{1}{2R} \qquad P_{si} = P_{sid}\cdot A_{NF}^2(f)\cdot\frac{1}{2R}$$

$$\left(\frac{S}{N}\right)_v = \frac{P_{svd}}{P_{Nvd}} \qquad \left(\frac{S}{N}\right)_i = \frac{P_{si}}{P_{Ni}}$$

- S/N KOD SINHRONE DEMODULACIJE



$$m(t) = U_0\cos\omega_0 t + \frac{1}{2} m_0 U_0 \cos(\omega_0 \pm \omega_m)t$$

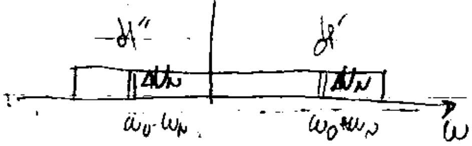
$$m_{di}(t) = m(t)\cdot m_L(t) = (VF) + \frac{1}{2} U_L U_0 m_0 \cos(\omega_m t) + \frac{U_0 U_L}{2}$$

$$P_{m_1} = P_{m_2} = \left(\frac{1}{2} m U_0\right)^2 \frac{1}{2R} \qquad P_{sv} = 2P_{m_1}$$

$$m(t)\cdot U_L\cos\omega_0 t = \frac{U_0 U_L}{2}\left[\cos(\omega_0 - \omega_0)t + \cos(\omega_0 + \omega_0)t\right] + \frac{U_L}{2} m_0 U_0 \cos(\omega_0 + \omega_m)t\cos\omega_0 t + \frac{U_L}{2} m_0 U_0 \cos(\omega_0 - \omega_m)t\cos\omega_0 t$$

$$= \frac{U_0 U_L}{2} + \frac{U_0 U_L}{2}\cos 2\omega_0 t + \frac{U_L}{4} m_0 U_0\left[\cos(2\omega_0 + \omega_m)t + \cos(\omega_m t)\right] +$$
$$+ \frac{U_L}{4} m_0 U_0 \left[\cos(\omega_m t) + \cos(2\omega_0 + \omega_m)t\right] = (VF) + \frac{U_0 U_L}{2} + \frac{U_L}{2} m_0 U_0 \cos\omega_m t$$

$$P_{sid} = \left(\frac{1}{2} m_0 U_0 U_L\right)^2 \frac{1}{2R} = \left(\frac{1}{2} U_L\right)^2 \left(m_0 U_0\right)^2 \frac{1}{2R} = 4 D_P P_{m_1} = 2 D_P P_v$$



$$\Delta U_N \cos(\omega_0 - \omega_N)t \cdot U_0 \cos\omega_0 t = \frac{\Delta U_N U_0}{2}\cos(\omega_N t + \varphi_N) + (VF)$$

$$\Delta P' = \frac{\Delta U_N^2}{2R} \xrightarrow{\text{TRANSLIRANO NA NF}} \qquad \Delta P_{Ni}' = \left(\frac{\Delta U_N U_0}{2}\right)^2 \cdot \frac{1}{2R} = \left(\frac{U_0}{2}\right)^2 \cdot \Delta U_N^2 \cdot \frac{1}{2R}$$

$$\Delta P_{Ni}' = D_P \cdot \Delta U_N^2 \cdot \frac{1}{2R}$$

$$\Delta P'' = \frac{\Delta U_N^2}{2R} \xrightarrow{NF} \qquad \Delta P_{Ni}'' = D_P\,\Delta U_N^2 \cdot \frac{1}{2R}$$

$$\Delta P_{Ni} = \Delta P_{Ni}' + \Delta P_{Ni}'' = 2 D_P\,\Delta U_N^2\,\frac{1}{2R} = 2\cdot\Delta P'$$

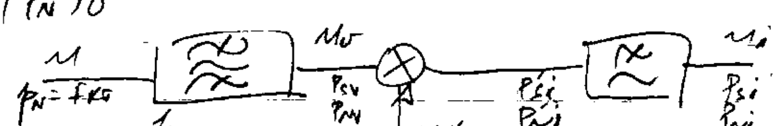
$$dP_{Ni} = D_P \cdot dP_{Nv} \qquad dP_{Ni} = D_P \cdot p_{Nv}\cdot df$$

$$P_{Nv} = FKT\cdot B_{VF} \qquad P_{Ni} = 2FKT\cdot D_P\cdot B_{NF} \qquad P_{sv} = 2P_{m_1} \qquad P_{si} = 4D_P P_{m_1} = 2D_P P_{sv}$$

$$\left(\frac{P_s}{P_n}\right)_i = \frac{2D_P\cdot P_{sv}}{2FKT\cdot D_P\cdot B_{NF}} = \frac{2P_{sv}}{FKT\cdot B_{VF}} = 2\cdot\frac{P_{sv}}{P_{Nv}} = 2\left(\frac{P_s}{P_N}\right)_v$$

AM 1SO $\quad \left(\frac{P_s}{P_N}\right)_i = \left(\frac{P_s}{P_N}\right)_v$

- Realna veza:



$$P_{Nv} = \int_{-f_1}^{f_2} p_{Nv}\,df = \int_{-f_1}^{f_2} p_N \cdot A_{VF}^2(f)\,df$$

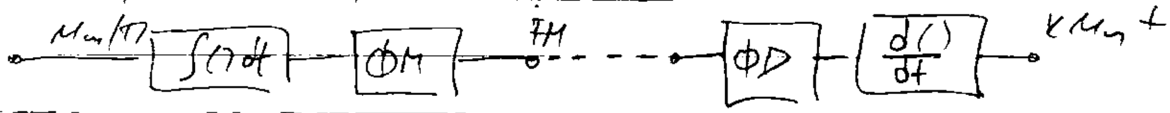
$$P_{Si} = 2D_p \cdot P_{SV} \qquad \gamma_{Ni}' = 2D_p \cdot \gamma_N \cdot A_{VF}^2(f)$$

$$\gamma_{Ni} = \gamma_{Ni}' \cdot A_{NF}^2(f) = 2D_p\, \gamma_N \cdot A_{VF}^2 \cdot A_{NF}^2 \qquad dP_{Ni} = \gamma_{Ni}\, df$$

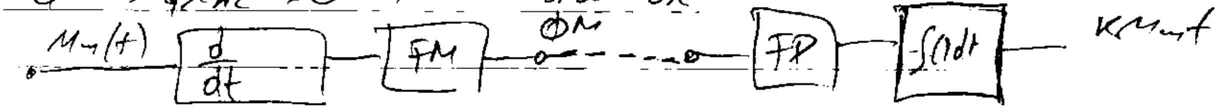
$$P_{Ni} = \int_0^{f_m} 2D_p\, \gamma_N \cdot A_{VF}^2 \cdot A_{NF}^2\, df = 2D_p\, \gamma_N \int_0^{f_m} A_{VF}^2(f+f_0)\, A_{NF}^2(f)\, df$$

ОТК Аглова модулација

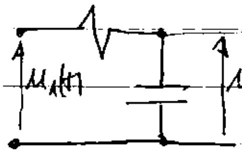
- FM сигнал со ΦM модулатор



- ΦM сигнал со FM модулатор

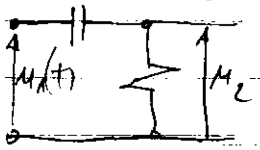


$$u_{\Phi M}(t) = U_0 \cos(\omega_0 t + k_\varphi u_m(t)) \qquad u_{FM}(t) = U_0 \cos\left(\omega_0 t + k_\omega \int u_m(t)dt\right)$$



$$u_2(t) \doteq \frac{1}{RC}\int u_1(t)\,dt \qquad \left[\frac{1}{\omega_m C} \ll R\right]$$

$$\left(i = RC\ \frac{du_c}{dt}\ (=)\ \Omega F\ \frac{V}{s}\right)$$



$$u_2(t) = RC\ \frac{du_1(t)}{dt} \qquad \left[\frac{1}{\omega C} \gg R\right]$$

- Нека $u_m(t) = U_m \cos(\omega_m t)$

$$u_{\Phi M}(t) = U_0 \cos(\omega_0 t + k_\varphi U_m \cos(\omega_m t)) = U_0 \cos(\omega_0 t + \Delta\Phi_0 \cos(\omega_m t))$$

$$\delta\Phi_i = \Delta\Phi_0 \cos(\omega_m t) \qquad \delta\omega_i = -\Delta\Phi_0 \sin(\omega_m t) \qquad \left[\omega_i = \frac{d\Phi_i}{dt}\right]$$

$$u_{FM}(t) = U_0 \cos\left(\omega_0 t + k_\omega U_m \int \cos\omega_m t\, dt\right) = U_0 \cos\left(\omega_0 t + \frac{k_\omega U_m}{\omega_m} \sin(\omega_m t)\right)$$

$$u_{FM}(t) = U_0 \cos\left(\omega_0 t + \frac{\Delta\omega_0}{\omega_m} \sin(\omega_m t)\right)$$

$$u_{FM}(t) = U_0 \cos\left(\omega_0 t + \frac{\Delta\omega_0}{\omega_m} \cdot \cos\left(\omega_m t - \frac{\pi}{2}\right)\right)$$

$$\delta\omega_i = \frac{d}{dt}(\delta\Phi_i) = \frac{\Delta\omega_0}{\omega_m}(-)\sin\left(\omega_m t - \frac{\pi}{2}\right) \cdot \omega_m = \Delta\omega_0(-)(-)\sin\left(\frac{\pi}{2} - \omega_m t\right)$$

$$\boxed{\delta\omega_i = \Delta\omega_0 \cos\omega_m t}$$

- Спектар на ΦM и FM сигнали

$$\cos(0.2\pi n) = \cos(\omega_m t) \qquad t = nDt \qquad \omega = 2\pi f$$

$$\cos(\omega t) = \cos(2\pi f \cdot nDt) = \cos(2\pi \cdot 10^5 \cdot n \cdot 0.1 \cdot 10^{-5}) = \cos(0.2 \cdot \pi \cdot n)$$

$$u_{\Phi M}(t) = U_0 \cos\{\omega_0 t + k_\varphi U_m \cos\omega_m t\} = U_0 \cos\{\omega_0 t + m\cos(\omega_m t)\}$$

$$m = \Delta\Phi_0 = k_\varphi U_m \qquad e^{jm\sin\omega_m t} = \sum_{n=-\infty}^{\infty} J_n(m)\, e^{jn\omega_m t}$$

$$u_{FM}(t) = U_0 \cos\left\{\omega_0 t + \frac{\Delta\omega_0}{\omega_m}\cdot\cos\left(\omega_m t - \frac{\pi}{2}\right)\right\} = U_0 \cos\left\{\omega_0 t + m\cdot\cos\left(\omega_m t - \frac{\pi}{2}\right)\right\}$$

$$m = \frac{\Delta\omega_0}{\omega_m} = \frac{K_\omega \cdot U_m}{\omega_m} = \frac{\Delta f_0}{f_m}$$

- u_{AM} & $u_{FM}(t)$ СЕ ПЕРИОДИЧНИ ФУНКЦИИ ТА МОЖЕ ДА СЕ РАЗВИЈАТ ВО Ф.РЕД

$$\cos(\alpha + m\cos\varphi) = \sum_{n=-\infty}^{\infty} J_n(m)\cdot\cos\left(\alpha + n\varphi + \frac{n\pi}{2}\right)$$

$$\cos(\alpha + m\sin\varphi) = \sum_{n=-\infty}^{\infty} J_n(m)\cos(\alpha + n\varphi)$$

$$\sin(\alpha + m\cos\varphi) = \sum_{n=-\infty}^{\infty} J_n(m)\cdot\sin\left(\alpha + n\varphi + \frac{n\pi}{2}\right)$$

$$\sin(\alpha + m\sin\varphi) = \sum_{n=-\infty}^{\infty} J_n(m)\sin(\alpha + n\varphi)$$

$$F_n = \frac{1}{T}\int_{-T/2}^{T/2} \cos(\alpha + m\cos\varphi)\cdot e^{-jn\omega_0 t}\,dt = ? \Rightarrow$$ НЕ СЕ РЕШАВА ВО ЗАТВОРЕНА ФОРМА ⇒ БЕСЕЛОВИ ФУНКЦИИ

ОПШТ ИЗРАЗ ЗА αFM (ОМ/ФМ)

- $$u(t) = U_0\cos(\omega_0 t + m\cos\omega_m t) = \sum_{n=-\infty}^{\infty} J_n(m)\cdot\cos\left(\omega_0 t + n\cdot\omega_m t + \frac{n\pi}{2}\right) = (*)$$

$$e^{jm\sin\omega_m t} = \sum_{n=-\infty}^{\infty} J_n(m)\cdot e^{jn\omega_m t}$$

$$J_n(m) = \frac{1}{T}\int_{-T/2}^{T/2} e^{jm\sin(\omega_0 t)}\, e^{-jn\omega_0 t}\,dt$$

$$m \ll 1 \Rightarrow J_n(m) \doteq \frac{m^n}{2^n n!}$$

$$d(\omega t) \quad 2\pi f\,dt = \frac{2\pi}{T}\,dt$$

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j(n\tau - x\sin\tau)}\,d\tau = \Bigg| \begin{array}{l}\tau = \omega_0 t\\ d\tau = \omega_0\,dt = \frac{2\pi}{T}dt\\ \omega_0 = 2\pi f = \frac{2\pi}{T}\end{array} \Bigg| \begin{array}{l}\cdot\ \tau = \pi \quad t = \frac{\tau}{\omega_0} = \frac{\pi}{2\pi}T,\ t = T/2\\ \cdot\ \tau = -\pi \quad t = \frac{-\pi}{2\pi}\cdot T = -\frac{T}{2}\end{array}$$

$$= \frac{1}{2\pi}\cdot\frac{2\pi}{T}\int_{-T/2}^{T/2} e^{-jn\omega_0 t}\, e^{+jx\cdot\sin(\omega_0 t)}\,dt = \frac{1}{T}\int_{-T/2}^{T/2} e^{jx\sin(\omega_0 t)}\, e^{-jn\omega_0 t}\,dt$$

$$(*) = U_0 J_0(m)\cos\omega_0 t + \sum_{n=-\infty}^{-1} J_n(m)\cos\left(\omega_0 t + n\omega_m t + \frac{n\pi}{2}\right) + \sum_{n=1}^{\infty} J_n(m)\cos\left(\omega_0 t + n\omega_m t + \frac{n\pi}{2}\right)$$

$$J_{-n}(m) = (-1)^n J_n(m)$$

$$(**) \sum_{n=1}^{\infty} J_{-n}(m)\cos\left(\omega_0 t - n\omega_m t - \frac{n\pi}{2}\right) = \Bigg| \cos\left(\omega_0 t - n\omega_m t - \frac{n\pi}{2}\right) = (-1)^n\cdot\cos\left(\omega_0 t - n\omega_m t + \frac{n\pi}{2}\right)$$

$$\cos(\alpha - \pi) = \cos\alpha\cdot\cos\pi + \sin\alpha\,\sin\pi = -\cos\alpha$$

$$\cos\left(\alpha - \frac{\pi}{2}\right) = \cos\alpha\cos\frac{\pi}{2} + \sin\alpha\,\sin\frac{\pi}{2}$$

$$\cos\left(\alpha - \frac{n\pi}{2}\right) = \cos\alpha\cdot\cos\left(\frac{n\pi}{2}\right) - \sin\alpha\cdot\sin\left(\frac{n\pi}{2}\right)$$

$n = 0, 2, 4, 6$ $\quad \cos\left(\alpha - \frac{n\pi}{2}\right) = \cos(\alpha)$

$n = 1, 3, 5, 7$ $\quad \cos\left(\alpha - \frac{n\pi}{2}\right) = -\sin\alpha\cdot(-1)^{2n-1}$

$$\cos\left(\alpha - \frac{n\pi}{2}\right) = \sin\left(\frac{\pi}{2} - \alpha\right)\cos\left(\frac{n\pi}{2}\right) - \sin\alpha\cdot\sin\left(\frac{n\pi}{2}\right) =$$

$n = 0, 1, 2, 3, \ldots$

$\cos(n \cdot \pi/2) = [1, 0, -1, 0, 1, 0, -1, \ldots]$

$\sin(n\pi/2) = [0, 1, 0, -1, 0, 1, 0, \ldots]$

$\cos\left(\alpha - \frac{n\pi}{2}\right) = \cos(\alpha)\cos\frac{n\pi}{2} + \sin\alpha \sin\frac{n\pi}{2} =$

$= [\cos(\alpha), \sin(\alpha), -\cos(\alpha), -\sin(\alpha), \cos(\alpha), \sin(\alpha), -\cos(\alpha), -\sin(\alpha), \cos(\alpha), \sin(\alpha), -\cos(\alpha)]$

$\quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10$

$\cos\left(\alpha - n\frac{\pi}{2}\right) = (-1)^n \sin(\alpha)$?? $\quad n = 1 \quad \cos\left(\alpha - \frac{\pi}{2}\right) = -\sin(\alpha)$ #

$\cos\left(\alpha + \frac{n\pi}{2}\right) = [\cos\alpha, -\sin\alpha, -\cos\alpha, \sin\alpha, \cos\alpha, -\sin\alpha, -\cos\alpha, \sin\alpha, \cos\alpha, -\sin\alpha, -\cos\alpha]$

$= (-1)^n \cos\left(\alpha - \frac{n\pi}{2}\right) = \cos\alpha, -\sin\alpha, -\cos\alpha, \sin\alpha, \cos\alpha, -\sin\alpha$

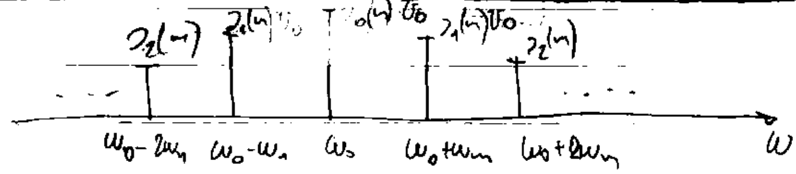
$\quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5$

$(*) = \sum_{n=1}^{\infty} J_n \cos\left(\omega_0 t - n\omega_m t - \frac{n\pi}{2}\right) = \sum_{n=1}^{\infty} (-1)^n J_n (m)^n \cos\left(\omega_0 t - n\omega_m t + \frac{n\pi}{2}\right) =$

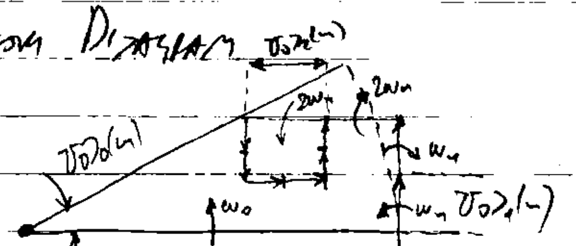
$J_{-n}(m) = (-1)^n J_n(m) \qquad \cos\left(\alpha - \frac{n\pi}{2}\right) = (-1)^n \cos\left(\alpha + \frac{n\pi}{2}\right)$

$= \sum_{n=1}^{\infty} J_n(m) \cdot \cos\left(\omega_0 t - n\omega_m t + \frac{n\pi}{2}\right)$

$$u(t) = U_0 \cdot J_0(m) \cos\omega_0 t + U_0 \sum_{n=1}^{\infty} J_n(m)\left[\cos\left((\omega_0 - n\omega_m)t + \frac{n\pi}{2}\right) + \cos\left((\omega_0 + n\omega_m)t + \frac{n\pi}{2}\right)\right]$$



· Fazorski dijagram



· Novo: $u(t) = U_0 \cos[\omega_0 t + m_1 \cos\omega_1 t + m_2 \cos\omega_2 t] =$

$= U_0 \cos\left[\frac{\omega_0 t}{2} + m_1 \cos\omega_1 t + \frac{\omega_0 t}{2} + m_2 \cos\omega_2 t\right] =$

$= U_0 \cos\left(\frac{\omega_0 t}{2} + m_1 \cos\omega_1 t\right) \cos\left(\frac{\omega_0 t}{2} + m_2 \cos\omega_2 t\right) - U_0 \sin\left(\frac{\omega_0 t}{2} + m_1 \cos\omega_1 t\right) \sin\left(\frac{\omega_0 t}{2} + m_2 \cos\omega_2 t\right)$

$= U_0 \sum_{p=-\infty}^{\infty} J_p(m) \cos\left(\frac{\omega_0 t}{2} + p\omega_1 t + \frac{p\pi}{2}\right) \sum_{q=-\infty}^{\infty} J_q(m) \cos\left(\frac{\omega_0 t}{2} + q\omega_2 t + \frac{q\pi}{2}\right) - U_0 \sum_{p=-\infty}^{\infty} J_p(m) \sin\left(\frac{\omega_0 t}{2} + p\omega_1 t + \frac{p\pi}{2}\right) \sum_{q=-\infty}^{\infty} J_q(m) \sin\left(\frac{\omega_0 t}{2} + q\omega_2 t + \frac{q\pi}{2}\right)$

$= U_0 \sum_{p=-\infty}^{\infty} J_p(m) J_q(m) \cos\left[\omega_0 t + (p\omega_1 + q\omega_2)t + \frac{p+q}{2}\pi\right]$

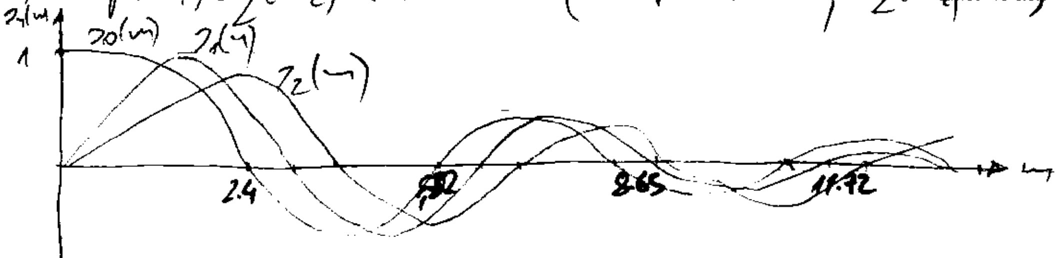
$p = 0$	$q = 0$	$\rightarrow$	$U_0 J_0(m) J_0(m)$	ω_0
$p = 0$	$q = 1$	$\rightarrow$	$U_0 J_0(m) J_1(m)$	$\omega_0 + \omega_2$
$p = 1$	$q = 0$	$\rightarrow$	$U_0 J_1(m) J_0(m)$	$\omega_0 + \omega_1$

$\ldots\ldots$
$\omega_0 + \omega_1 + \omega_2$
$\omega_0 + \omega_1 - \omega_2$
$\omega_0 - \omega_1 + \omega_2$
$\omega_0 - \omega_1 - \omega_2$

• ЗНАЧАЈНИ КОМПОНЕНТИ ВО СПЕКТАРОТ НА ФАЗНО МОДУЛИРАНИТЕ СИГНАЛИ

1) $J_n(m) \leq 1$ (само $J_0(0) = 1$)

2) $J_p(m_1) J_q(m_2) \leq 1$ $\left(J_1(m_1) \leq 1 , J_2(m_2) \leq 1 \right)$



$$u(t) = U_0 \cos(\omega_0 t) + U_0 \sum_{n=1}^{\infty} J_n(m) \left\{ \cos\left(\omega_0 t + n\omega_m t + \frac{n\pi}{2}\right) + \cos\left(\omega_0 t - n\omega_m t + \frac{n\pi}{2}\right) \right\}$$

$$J_n(m) = \frac{m^n}{2^n n!} \left[1 - \frac{m^2}{2(n+1)} + \frac{m^4}{2 \cdot 2^4 (n+1)(n+2)} - \dots + \frac{(-1)^p m^{2p}}{p!\, 2^{2p} (n+1) \cdots (n+p)} \right]$$

⊙ $m < 1$ $\quad J_n(m) \doteq \dfrac{m^n}{2^n n!}$ $\quad J_0(m) \doteq 1$; $J_1(m) \doteq \dfrac{m}{2}$; $J_2 \doteq \dfrac{m^2}{8}$; $J_3 \doteq \dfrac{m^3}{48}$

• КАРЛСОНОВ КРИТЕРИУМ ЗА ЗНАЧАЈНИ КОМПОНЕНТИ

Треба да се опфатат сите компоненти со снага ≥ 1% од немодулираниот носител

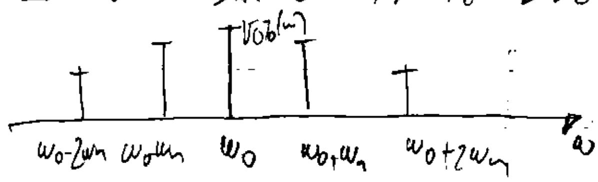
$$100 \frac{U_0^2 J_n^2(m)}{2R} < \frac{U_0^2}{2R}$$ → компонентата не е значајна

$U_0 J_0(m) \doteq U_0 \quad U_0 J_1(m) \doteq \dfrac{U_0 m}{2} \quad U_0 J_2(m) = \dfrac{U_0 m^2}{8}$

$$\frac{U_0^2}{2R} > 100 \frac{U_0^2}{2R} \cdot \frac{m^4}{64} \qquad \left(\frac{m^2}{8}\right)^2 < \frac{1}{100} = \left(\frac{1}{10}\right)^2$$

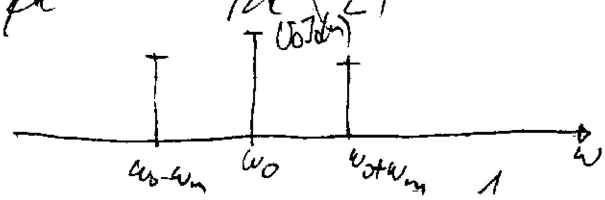
$$\frac{m^2}{8} < \frac{1}{10} \qquad m^2 < \frac{8}{10} \qquad m \leq \sqrt{0.8}$$ → компонентата не е значајна

За: $m \leq 0.9$ само првите две компоненти се значајни



$B_{FM} = 4 f_m$

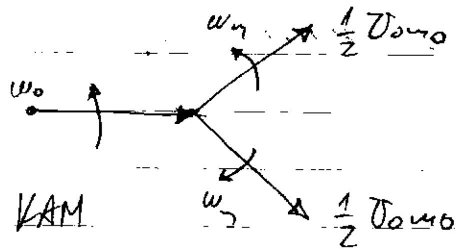
• $$\frac{U_0^2}{2R} > 100 \frac{U_0^2}{2R} \left(\frac{m}{2}\right)^2 \qquad m \leq \frac{2}{10} = \frac{1}{5} = 0.2$$



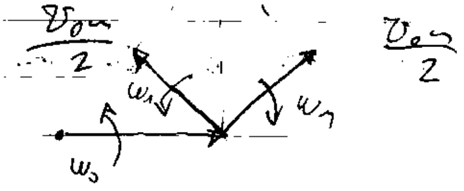
$B_{FM} = 2 f_m$

$$u(t) = U_0 \cos \omega_0 t + U_0 \sum_{n=1}^{1} J_n(m) \left\{ \cos\left[(\omega_0 + \omega_m)t + \frac{n\pi}{2}\right] + \cos\left[(\omega_0 - \omega_m)t + \frac{n\pi}{2}\right] \right\}$$

$$= U_0 \cos \omega_0 t + U_0 \underbrace{J_1(m)}_{\frac{m}{2}} \cos\left[(\omega_0 + \omega_m)t + \frac{\pi}{2}\right] + U_0 \underbrace{J_1(m)}_{\frac{m}{2}} \cos\left[(\omega_0 - \omega_m)t + \frac{\pi}{2}\right]$$



KAM



$J_1(m)$ $J_1(m)$

$$u(t) = U_0 \cos \omega_0 t + \frac{U_{0m}}{2} \cos\left[(\omega_0 + \omega_m)t + \frac{\pi}{2}\right] + \frac{U_{0m}}{2} \cos\left[(\omega_0 - \omega_m)t + \frac{\pi}{2}\right]$$

· Ekvivalentiranje oblici za $m = 0.3 \div 0.4$

KAM: $\left[U_0 + \frac{U_{0m}}{2} e^{j\omega_m t} + \frac{U_{0m}}{2} e^{-j\omega_m t}\right] \cdot e^{+j\omega_0 t}$

FM: $\left[U_0 + \frac{U_{0m}}{2} e^{j(\omega_m t + \frac{\pi}{2})} + \frac{U_{0m}}{2} e^{-j(\omega_m t - \frac{\pi}{2})}\right] e^{j\omega_0 t}$

$$J_0(m) = \frac{1}{2\pi}\int_0^{2\pi} e^{jm\cos\varphi}\, d\varphi \qquad J_n(m) = \frac{1}{T}\int_{-T/2}^{T/2} e^{jm\cos(\omega_0 t)} e^{-jn\omega_0 t}\, dt$$

$$J_n(m) = \frac{1}{2\pi}\int_0^{2\pi} e^{jm\cos\varphi} e^{-jn\varphi}\, d\varphi$$

$$J_n(m) = \frac{1}{T}\int_{-T/2}^{T/2} e^{jm\cos(\omega_0 t)} e^{-jn\omega_0 t}\, dt = \frac{1}{T}\frac{1}{\omega_0}\int_{-T/2}^{T/2} e^{j(m\cos(\omega_0 t) - n\omega_0 t)}\, d(\omega_0 t),$$

$\varphi = \omega_0 t \qquad d\varphi = \omega_0 dt = 2\pi f dt = \frac{2\pi}{T} dt \qquad dt = \frac{T}{2\pi} d\varphi$

$t = \frac{T}{2} \qquad \varphi = \frac{2\pi}{T} \cdot \frac{T}{2} = \pi \qquad t = -\frac{T}{2} \qquad \varphi = \frac{2\pi}{T} \cdot \left(-\frac{T}{2}\right) = -\pi$

$$J_n(m) = \frac{1}{T}\frac{1}{2\pi f}\int_{-\pi}^{\pi} e^{j(m\cos\varphi - n\varphi)}\, d\varphi = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{jm\cos\varphi} e^{-jn\varphi}\, d\varphi$$

$$\boxed{J_n(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{jm\cos\varphi} e^{-jn\varphi}\, d\varphi}$$

• $\boxed{m > 1}$ $\quad J_n(m) = \sqrt{\frac{2}{\pi m}} \cos\left(m - \frac{n\pi}{2} - \frac{\pi}{4}\right)$

$\boxed{J_{n \geq m+2}(m) < 0.1}$ $\qquad \boxed{\frac{\left[U_0 J_{n \geq m+2}(m)\right]^2}{2R} < 0{,}01 P_0}$

$\Rightarrow$ $\boxed{m+1}$ komponenti levo i desno od ω_0

$\boxed{B_{FM} = 2(m+1) f_m}$

za $m \gg 1$ $\qquad B_{FM} \cong 2m f_m$

• Потребен опсег за ΦМ

$u_{\Phi M} = U_0 \cos(\omega_0 t + k_\varphi U_m \cdot \cos(\omega_m t))$

$m = k_\varphi U_m$ $U_m \uparrow \Rightarrow m \uparrow \Rightarrow B_{VF} = 2(m+1) f_m \uparrow$

$B_{VF}^{\Phi M} = 2(m+1)\cdot f_{max}$

• Потребен конден опсег за FM

$\Delta f_0 = k_f \cdot U_m$ $\Delta\omega_0 = k_\omega U_m$

$u_{FM} = U_0 \cos(\omega_0 t + \frac{\Delta\omega_0}{\omega_m}\sin\omega_m t)$ $m = \frac{\Delta\omega_0}{\omega_m} = \frac{k_\omega U_m}{\omega_m}$

$B_{VF} = 2(m+1) f_m = 2\left(\frac{\Delta f_0}{f_m} f_m + f_m\right) = 2(\Delta f_0 + f_m) = 2(k_f U_m + f_m)$

$B_{VF}^{FM} = 2(\Delta f_0 + f_{max})$ во квалитетни ST: $\Delta f_0 = 140 kHz$

• Следна снага на FM & ΦМ ∝

$$u(t) = U_0\cos[\omega_0 t + m\cdot\cos(\omega_m t)] = U_0\sum_{n=-\infty}^{\infty} J_n(m)\cdot\cos\left[(\omega_0 + n\omega_m)t + \frac{n\pi}{2}\right]$$

↳ Опсег исто за ∠M

Парсевалова теорема: $P = \frac{1}{R}\sum_{-\infty}^{\infty}|F_n|^2$

$$[\text{eff } f(t)]^2 = \frac{1}{T}\int_{-T/2}^{T/2} f^2(t)\,dt = \sum_{-\infty}^{\infty}|F_n|^2$$

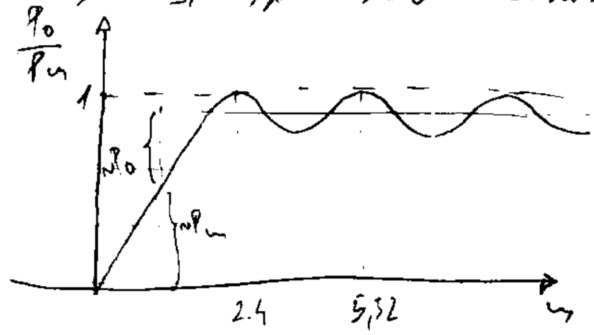
$$P = \frac{U_0^2}{2R}\sum_{-\infty}^{\infty} J_n^2(m) = P_0 \sum_{-\infty}^{\infty} J_n^2(m)$$

$\sum_{n=-\infty}^{\infty} J_n^2(m) = 1 \Rightarrow$ вкупна снага на ∠M $P = P_0$

! $P_0 = P_m + \frac{U_0^2 J_0^2(m)}{2R} = P_m + P_0 \cdot J_0^2(m)$

$P_m = P_0(1 - J_0^2(m))$ $m = 2,4$ $P_m = P_0$; $m = 5,52$ $P_m = P_0$; $m = \dots$ } целата снага оди во страничните компоненти

→ снага што отпаѓа на страничните компоненти



$\frac{P_m}{P_0} = 1 - J_0^2(m)$

$m\uparrow \Rightarrow P_m \uparrow$

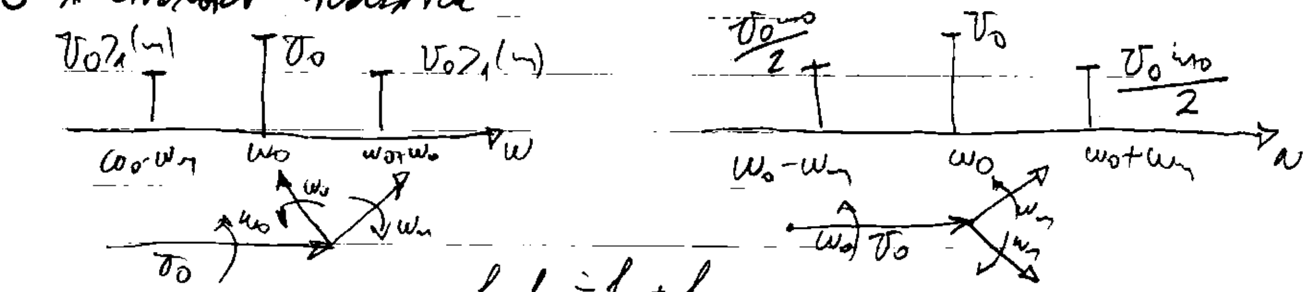
KAM $u(t) = U_0(1 + m_0 m(t))\cos\omega_0 t = U_0\left(1+\frac{U_m}{U_0}\cos\omega_m t\right)\cos(\omega_0 t)$

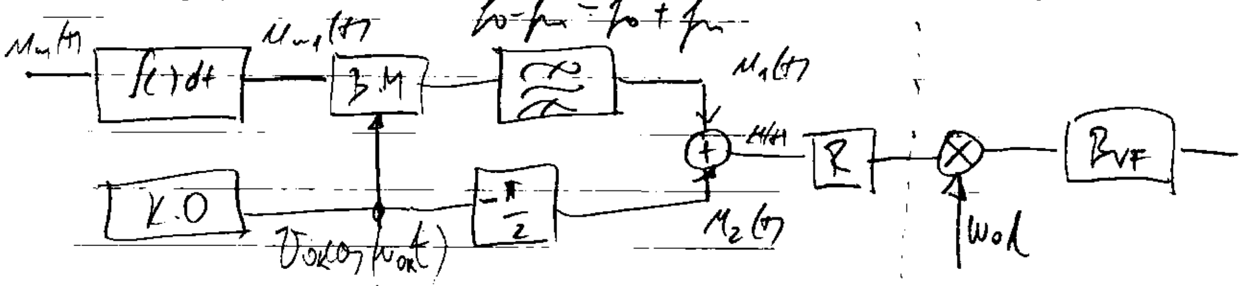
$= U_0\cos\omega_0 t + \frac{U_0 U_m}{2}\cos(\omega_0-\omega_m)t + \frac{U_0U_m}{2}\cos(\omega_0+\omega_m)t$

$P = \frac{U_0^2}{2R} + \frac{U_0^2U_m^2}{2\cdot 4R} + \frac{U_0^2U_m^2}{2\cdot4R} = \frac{U_0^2}{2R} + \frac{U_0^2U_m^2}{4R} = \frac{U_0^2}{2R}\left(1 + \frac{U_0^2}{2}\right)$

$m_0 = 1$ $\max(u(t)) = U_0(1+1) = 2U_0$ $P = \frac{4U_0^2}{2R} = 4P_0$

① ARMSTRONGOV MODULATOR





$$u_{FM}(t) = V_0 \cos\omega_0 t + \frac{m V_0}{2}\left\{\cos\left[(\omega_0-\omega_m)t + \frac{\pi}{2}\right] + \cos\left[(\omega_0+\omega_m)t + \frac{\pi}{2}\right]\right\}$$

$$u_{AM}(t) = V_0 \cos\omega_0 t + \frac{m_0 V_0}{2}\cos(\omega_0-\omega_m)t + \frac{m_0 V_0}{2}\cos(\omega_0+\omega_m)t$$

$u_1(t) = K_0\, u_{m1}(t)\cos(\omega_{0K}t)$ | $u_m(t) = V_m \cdot \cos(\omega_m t)$

$u_2(t) = V_{0K}\sin(\omega_{0K}t)$

$$u_{m1}(t) = \frac{1}{RC}\int u_m(t)dt = \frac{V_m}{RC\,\omega_m}\sin(\omega_m t) = \frac{V_{m1}}{\omega_m}\sin(\omega_m t)$$

$V_{m1} = V_m/RC = K_2 V_m$

$$u_1(t) = \frac{K_1 V_{m1}}{\omega_m}\sin(\omega_m t)\cos(\omega_{0K}t) = \frac{K V_m}{\omega_m}\sin(\omega_m t)\cdot\cos(\omega_{0K}t)$$

$$u(t) = V_{0K}\sin(\omega_{0K}t) + \underbrace{\frac{K V_m}{\omega_m}\sin(\omega_m t)}_{B}\cos(\omega_{0K}t) = V(t)\sin(\omega_{0K}t + \varphi(t))$$

① $A\cos\alpha + B\sin\alpha = \sqrt{A^2+B^2}\cdot\cos\left(\alpha + \operatorname{arctg}\frac{-B}{A}\right)$

$A = C\cos\varphi$ $\qquad \varphi = \operatorname{arctg}\frac{-B}{A}$

$B = -C\sin\varphi$ $\qquad A^2+B^2 = C^2(\cos^2\varphi + \sin^2\varphi)$

$C\cos\varphi\cos\alpha - C\sin\varphi\sin\alpha = C\cos(\alpha+\varphi)$

② $A\sin\alpha + B\cos\alpha = \sqrt{A^2+B^2}\,\sin\left(\alpha + \operatorname{arctg}\frac{B}{A}\right)$

$A = C\cos\varphi$ $\qquad \varphi = \operatorname{arctg}\frac{B}{A}$

$B = C\sin\varphi$ $\qquad A^2+B^2 = C^2 \qquad C = \sqrt{A^2+B^2}$

$C\cos\varphi\sin\alpha + C\sin\varphi\cos\alpha = C\sin(\alpha+\varphi)$

$$V(t) = \sqrt{V_{0K}^2 + \left(\frac{K V_m}{\omega_m}\right)^2 \sin^2(\omega_m t)}$$

$$\varphi(t) = \operatorname{arctg}\frac{B}{A} = \operatorname{arctg}\left[\frac{K V_m}{V_{0K}\,\omega_m}\sin(\omega_m t)\right] \doteq \frac{K V_m}{V_{0K}\,\omega_m}\sin(\omega_m t)$$

(K_ω)

$$\varphi(t) \doteq K_\omega \frac{V_m}{\omega_m}\sin(\omega_m t)$$

$$V(t) = V_{0K}\sqrt{1 + \left(\frac{K V_m}{V_0\,\omega_m}\right)^2 \sin^2(\omega_m t)} \doteq \left| \frac{K_\omega V_m}{\omega_m} \ll 1 \right| \doteq V_{0K}$$

$$u(t) = U_{0x} \sin\left[\omega_{0x} t + k_\omega \frac{U_m}{\omega_m} \sin(\omega_m t)\right] \Rightarrow \text{FM signal}$$

- $m \uparrow$ множење на фреквенција

$$r\omega_i = r\omega_0 + r\delta\omega_i \qquad \omega_i = \frac{d\Phi_i}{dt} = \omega_0 + \underbrace{k_\omega U_m \cos(\omega_m t)}_{\delta\omega_i}$$

$$r\omega_i = r\omega_0 + r \cdot k_\omega \cdot U_m \cdot \cos(\omega_m t)$$

$$\Delta\omega_{02} = r \cdot k_\omega U_m = r \cdot \Delta\omega_{01} \qquad \omega_{02} = r\,\omega_{01}$$

$$m_1 = \frac{\Delta\omega_{01}}{\omega_m} \qquad m_2 = \frac{\Delta\omega_{02}}{\omega_m} = r \cdot m_1$$

$$\left(\frac{k_\omega U_m}{\omega_{min}} \ll 1\right) \quad \text{максималниот коеф. на модулација} \ll 1$$

$$\omega_{02} = r \cdot \omega_{01} \qquad r\omega_{0x} \pm \omega_{0l} = \omega_0 \qquad \boxed{\omega_{0l} = \omega_0 - r\omega_{0x}}$$

$$\boxed{B_{vf} = 2(m+1) f_m}$$

- Доколку не е исполнето $m \ll 1$

$$\varphi(t) = \operatorname{arctg}\left[\frac{kU_m}{U_{0x}\omega_m} \cdot \sin(\omega_m t)\right] = \operatorname{arctg}(\Delta\Phi_{0x} \sin\omega_m t)$$

$$\delta\omega_i = \frac{d\varphi(t)}{dt} = \frac{\Delta\Phi_{0x} \cdot \omega_m \cos(\omega_m t)}{1 + \Delta\Phi_{0x}^2 \sin^2(\omega_m t)} = \sum_{n=-\infty}^{\infty} a_n \cos(n\omega_m t)$$

$$y = \operatorname{arctg} x \qquad x = \operatorname{tg} y \qquad \frac{dx}{dy} = \frac{\cos^2(y) + \sin^2(y)}{\cos^2(y)} = \frac{1}{\cos^2(y)}$$

$$= 1 + \operatorname{tg}^2(y) = 1 + x^2 \qquad \frac{dx}{dy} = 1 + x^2 \qquad \boxed{\frac{dy}{dx} = \frac{1}{1+x^2}}$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t) \cdot \cos(n\omega_m t)\,dt \qquad n = 0, 1, 2, \ldots \qquad (f(t) = \delta\omega_i)$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} \delta\omega_i \cos(n\omega_m t)\,dt = \frac{2}{T}\int_{-T/2}^{T/2} \frac{\Delta\Phi_{0x}\omega_m \cos(\omega_m t)\cos(n\omega_m t)}{1 + \Delta\Phi_{0x}^2 \sin^2(\omega_m t)}\,dt$$

$$n=1 \qquad a_1 = \frac{2\Delta\Phi_{0x}\omega_m}{T}\int_{-T/2}^{T/2} \frac{\cos^2(\omega_m t)}{1 + \Delta\Phi_{0x}^2 \sin^2(\omega_m t)}\,dt$$

$$I = \omega_m \int_{-T/2}^{T/2} \frac{\cos^2(\omega_m t)}{1 + \Delta\Phi_{0x}^2 \sin^2(\omega_m t)}\,dt = \left| \begin{array}{l} u = \omega_m t \\ du = \omega_m\,dt \\ t = \frac{T}{2} \quad u = \frac{2\pi}{T} \cdot \frac{T}{2} = \pi \\ t = -\frac{T}{2} \quad u = -\pi \end{array} \right|$$

$$I = \int_{-\pi}^{\pi} \frac{\cos^2(u)}{1 + \Delta\Phi_{0x}^2 \sin^2(u)}\,du = \frac{2\pi}{\Delta\Phi_{0x}^2}\left(\sqrt{1 + \Delta\Phi_{0x}^2} - 1\right)$$

$$a_1 = \frac{2\Delta\Phi_{0x}}{T} \cdot \frac{2\pi}{\Delta\Phi_{0x}^2}\left(\sqrt{1 + \Delta\Phi_{0x}^2} - 1\right) = \frac{4\pi}{T\,\Delta\Phi_{0x}}\left(\sqrt{1 + \Delta\Phi_{0x}^2} - 1\right)$$

$$T = \frac{1}{f_m} = \frac{1}{\omega_m / 2\pi} = \frac{2\pi}{\omega_m} \qquad \boxed{a_1 = \frac{2\omega_m}{\Delta\Phi_{0x}}\left(\sqrt{1 + \Delta\Phi_{0x}^2} - 1\right)}$$

$$\delta\omega_i = \frac{2\omega_m}{\Delta\phi_{ox}}\left[\sqrt{1+\Delta\phi_{ox}^2}-1\right]\cos(\omega_m t) + \frac{2\omega_m}{\Delta\phi_{ox}^3}\left[\sqrt{1+\Delta\phi_{ox}^2}-1\right]^3\cos(3\omega_m t)+\dots$$

$$d_3 = \left(\frac{\Delta\omega_{i3}}{\Delta\omega_{i1}}\right)^2 = \left(\frac{\left[\sqrt{1+\Delta\phi_{ox}^2}-1\right]^2}{\Delta\phi_{ox}^2}\right)^2$$

$\Delta\phi_{ox} = m < 1$

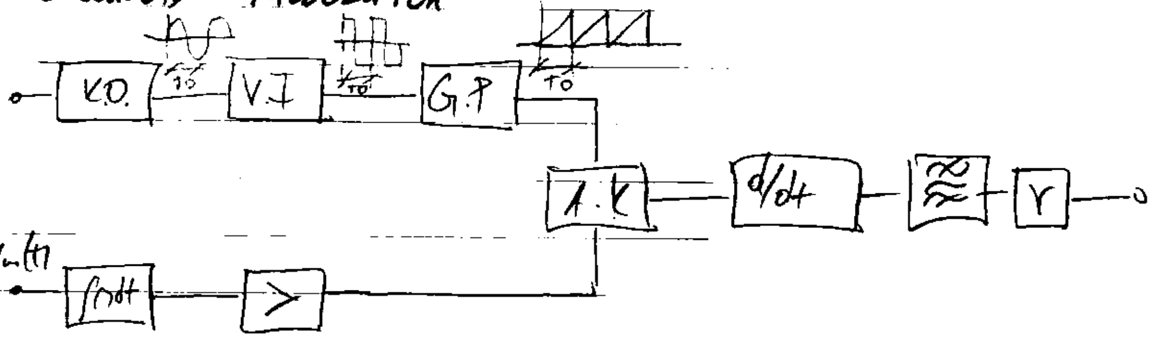
$\sqrt{1+\Delta\phi_{ox}^2} = 1 + \frac{\Delta\phi_{ox}^2}{2}$

$$d_3 = \left(\frac{\left(1+\frac{\Delta\phi_{ox}^2}{2}-1\right)^2}{\Delta\phi_{ox}^2}\right)^2 = \left(\frac{\Delta\phi_{ox}^4}{4\cdot\Delta\phi_{ox}^2}\right)^2 = \left(\frac{\Delta\phi_{ox}^2}{4}\right)^2$$

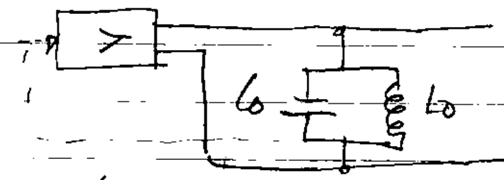
$$a_{H_3} = 10\log\frac{1}{d_3} = 20\log\frac{4}{\Delta\phi_{ox}^2} = 40\log\frac{2}{\Delta\phi_{ox}}$$

$\Delta\phi_{ox} = m \nearrow \Rightarrow$ поголеми изобличувања

- Serrasoid модулатор



- Директна постапка за добивање FM сигнали



$u_0(t) = U_0\cos(\omega_0 t)$

$$\omega_0 = \frac{1}{\sqrt{L_0C_0}}\ ; \quad \omega_i = \frac{1}{\sqrt{L_0C(t)}} = \frac{1}{\sqrt{L_0(C_0+\delta C(t))}} = \frac{1}{\sqrt{L_0C_0}}\cdot\frac{1}{\sqrt{1+\frac{\delta C(t)}{C_0}}}$$

$\frac{\delta C(t)}{C_0} \ll 1$; $\quad f(x) = f(0) + \frac{f'(0)}{1!}\cdot x + \frac{f''(0)}{2!}x^2 + \dots$

$f(x) = \frac{1}{\sqrt{1+x}}$; $f(0) = 1$; $f'(x) = -\frac{1}{2}(1+x)^{-\frac{1}{2}-1} = -\frac{1}{2\sqrt[3]{1+x}}$

$f'(0) = -\frac{1}{2}$; $\boxed{f(x) = 1 - \frac{1}{2}\cdot x}$

$$\omega_i = \omega_0\left(1 - \frac{1}{2}\frac{\delta C(t)}{C_0}\right) = \omega_0 - \frac{1}{2}\omega_0\frac{\delta C(t)}{C_0} = \omega_0 + \delta\omega_i$$

$\delta\omega_i = -\frac{1}{2}\omega_0\frac{\delta C(t)}{C_0}$; "−" $C\nearrow \Rightarrow \omega\downarrow$

$\delta\omega_i = -\frac{1}{2}\frac{\omega_0}{C_0}k\,m_m(t) = -\frac{1}{2}\frac{\omega_0}{C_0}k\,U_m\,m(t) \qquad \Delta\omega_0 = -\frac{1}{2}\frac{\omega_0}{C_0}k\,U_m$

- Директен модулатор со реактивен спој

$I = I_1 + I_a = \frac{U}{R+\frac{1}{j\omega C}} + S\,U_g$; $U_g = \frac{UR}{R+\frac{1}{j\omega C}}$

$$Y = \frac{I}{U} = \frac{1}{R + \frac{1}{j\omega C}} + \frac{SR}{R + \frac{1}{j\omega C}} = \frac{1 + SR}{R + \frac{1}{j\omega C}} \qquad Z = Y^{-1}$$

$$Z = \frac{R}{1 + SR} + \frac{1}{j\omega C(1 + SR)} = R_e + \frac{1}{j\omega C_e}$$

- $SR \gg 1 \qquad R_e = \frac{1}{S} \qquad C_e \doteq SRC \qquad S(t) = S_0 + \delta S(t)$

$$\delta S(t) = K\mu_m(t) \qquad C_e = RCS_0 + RC\delta S(t) = RCS_0 + RCK\mu_m(t)$$

AA'
$$C = \underbrace{C_0 + RCS_0}_{\text{fiksna}} + \underbrace{RCK\mu_m(t)}_{\delta C(t)}$$

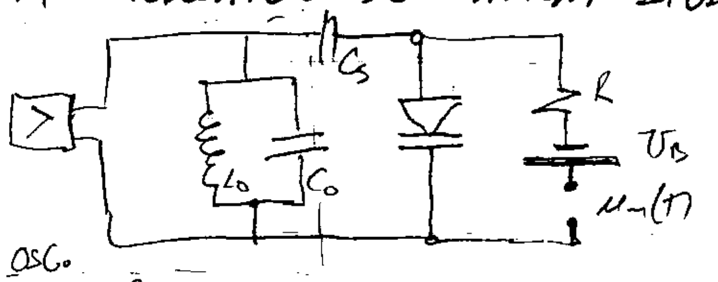
$$\omega_0 = \frac{1}{\sqrt{L_0(C_0 + RCS_0)}} \qquad \omega_i = \omega_0 + \delta\omega_i = \omega_0 - \frac{1}{2}\omega_0 \frac{CRK\mu_m(t)}{C_0 + RCS_0}$$

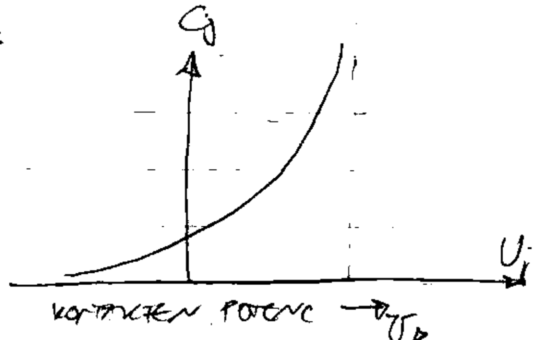
$$\omega_i = \frac{1}{\sqrt{L_0(C_0 + RCS_0 + RCK\mu_m(t))}} = \left(\frac{1}{\sqrt{L_0(C_0 + RCS_0)}}\right) \frac{1}{\sqrt{1 + \frac{RCK\mu_m(t)}{C_0 + RCS_0}}}$$

$$\omega_i = \omega_0\left(1 - \frac{1}{2}\frac{RCK\mu_m(t)}{C_0 + RCS_0}\right) = \omega_0 - \underbrace{\frac{\omega_0}{2}\frac{RCK\mu_m(t)}{C_0 + RCS_0}}_{\delta\omega_i}$$

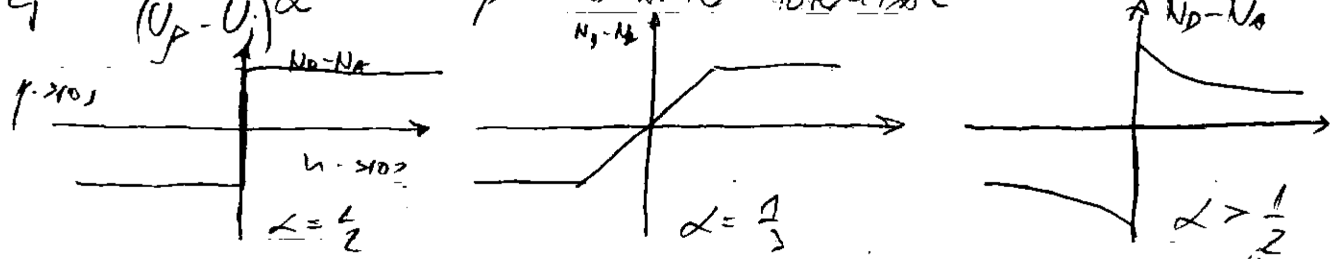
$$\boxed{\delta\omega_i = -\underbrace{\frac{\omega_0}{2}\frac{RCKU_m}{C_0 + RCS_0}}_{\Delta\omega_0} \cdot m(t)} \qquad \left(\mu_{FM}(t) = U_0 \cos\left(\omega_0 t + \int K\mu_m(t)\,dt\right)\right)$$

- FM MODULATOR SO VARICAP DIODA





$$C_j = \frac{C_K}{(U_p - U_j)^{\alpha}} \qquad U_p \text{ – kontakten potencijal}$$



- $\mu_m(t) = 0$

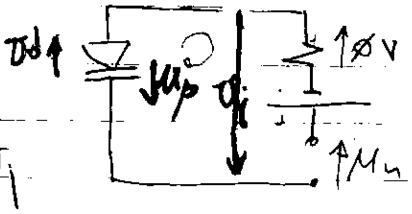
$$C_{j0} = \frac{C_K}{(U_p - U_B)^{\alpha}} = C_K U_Q^{-\alpha} \qquad \boxed{U_Q = U_p - U_B} \qquad \mu_j = U_B - \mu_m(t)$$

$$U = U_p - \mu_j = U_Q + \mu_m(t) \qquad C_j = \frac{C_K}{(U_p - U_j)^{\alpha}} = \frac{C_K}{(U_Q + \mu_m(t))^{\alpha}}$$

$$C_j = C_K\left[U_Q + \mu_m(t)\right]^{-\alpha} \ ; \ C_{vk} = C_0 + C_j \ ; \ C_j \gg C_0 \ ; \ C_{vk} \doteq C_j$$

$$\omega_i = \frac{1}{(L_0 C_j)^{1/2}} = (L_0 C_j)^{-1/2} = \underbrace{L_0^{-\frac{1}{2}} C_K^{-\frac{1}{2}}}_{K} \left[U_Q + u_m(t)\right]^{\frac{\alpha}{2}} = K\left[U_Q + u_m(t)\right]^{\alpha/2}$$

$$C_{j0} = C_K \cdot U_Q^{-\alpha} \qquad \omega_0 = (L_0 C_{j0})^{-\frac{1}{2}} = L_0^{-\frac{1}{2}} C_K^{-\frac{1}{2}} \cdot U_Q^{\alpha/2} = K\, U_Q^{\alpha/2}$$



$U_d = U_p - U_j$

$$u_p + 0V - U_B + u_m = U_p - U_B + u_m$$

$$C_j = \frac{C_K}{U_p - u_j} \qquad u_j = +U_B - u_m$$

$$C_j = \frac{C_K}{u_p - U_B + u_m(t)} = \frac{C_K}{U_Q + u_m}$$

$$\delta\omega_i = \omega_i - \omega_0 = K\left[U_Q + u_m(t)\right]^{\alpha/2} - K\, U_Q^{\alpha/2} = K\left\{\left[U_Q + u_m(t)\right]^{\alpha/2} - U_Q^{\alpha/2}\right\}$$

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b^{1} + \binom{n}{2} a^{n-2} b^{2} + \ldots + \binom{n}{n-1} a^{1} b^{n-1} + \binom{n}{n} b^n$$

$$\delta\omega_i = K\left\{\binom{\frac{\alpha}{2}}{0} U_Q^{\alpha/2} - U_Q^{\alpha/2} + \binom{\frac{\alpha}{2}}{1} U_Q^{\frac{\alpha}{2}-1} u_m^{1} + \binom{\frac{\alpha}{2}}{2} U_Q^{\frac{\alpha}{2}-2} u_m^{2} + \ldots\right\}$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!}$$

$$\binom{\frac{\alpha}{2}}{0} = \frac{\frac{\alpha}{2}!}{0!\,\frac{\alpha}{2}!} = 1 \qquad \binom{\frac{\alpha}{2}}{1} = \frac{\alpha}{2} \qquad \binom{\frac{\alpha}{2}}{2} = \frac{1}{2}\frac{\alpha}{2}\left(\frac{\alpha}{2} - 1\right) = \frac{\alpha}{4}\left(\frac{\alpha}{2} - 1\right)$$

$$\delta\omega_i = K\left\{\cancel{U_Q^{\alpha/2}} - \cancel{U_Q^{\alpha/2}} + \frac{\alpha}{2} U_Q^{\frac{\alpha}{2}-1} u_m^{1} + \frac{\alpha}{4}\left(\frac{\alpha}{2} - 1\right) U_Q^{\frac{\alpha}{2}-2} u_m^{2} + \ldots\right\}$$

$$\binom{\frac{\alpha}{2}}{1} = \frac{\frac{\alpha}{2}!}{\left(\frac{\alpha}{2} - 1\right)!} = \frac{\alpha}{2} \qquad \binom{n}{1} = \frac{n!}{(n-1)!} = \frac{1 \cdot (n-1) \cdot n}{1 \cdots (n-1)} = n$$

$$\binom{\frac{\alpha}{2}}{2} = \frac{\frac{\alpha}{2}!}{2!\left(\frac{\alpha}{2} - 2\right)!} = \frac{1}{2}\left(\frac{\alpha}{2} - 1\right) \cdot \frac{\alpha}{2}$$

$$\delta\omega_i = K\left\{\frac{\alpha}{2} U_Q^{\frac{\alpha}{2}-1} u_m(t) + \frac{\alpha}{4}\left(\frac{\alpha}{2} - 1\right) U_Q^{\frac{\alpha}{2}-2} u_m^{2} + \ldots\right\}$$

$\alpha = 2$

$$\delta\omega_i = K\left\{\frac{2}{2} U_Q^{0} \cdot u_m(t) + 0\right\} = K\, u_m(t) = K_\omega u_m(t)$$

ZANEMAREN ČLAN=0
DRUGI SE ZANEMARUJU

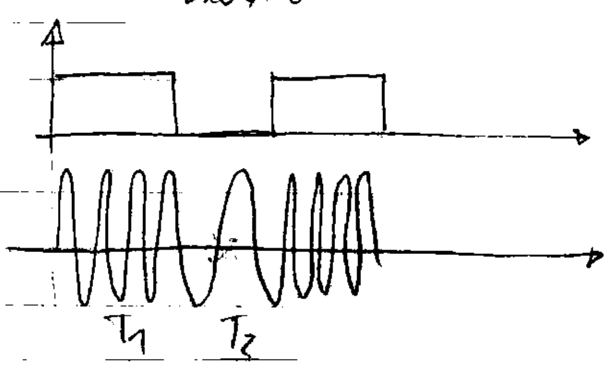
- DETEKCIJA FM SIGNALA

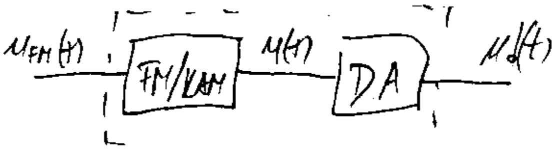
$u_D(t) = K_D\, \delta\omega_i$

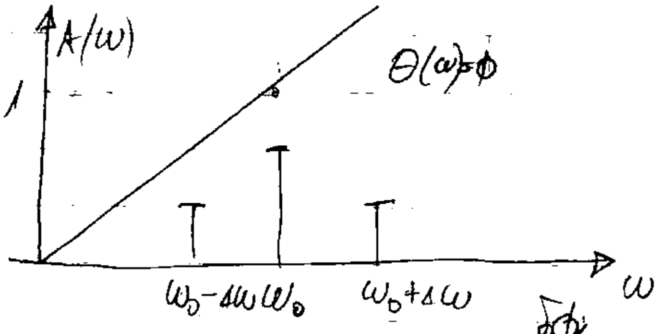
(a) MJERENJE ZA PERIODE

$$T_1 = \frac{1}{f_0 + \Delta f} \qquad T_2 = \frac{1}{f_0 - \Delta f}$$

(DIGITALNA MODULACIJA SIGNALI)







$\omega_0 \quad U_0 \to U_0$

$\omega_0 - \Delta\omega \quad U_0 \to U_0 A(\omega_0 - \Delta\omega)$

$\omega_0 + \Delta\omega \quad U_0 \to U_0 A(\omega_0 + \Delta\omega)$

- ИДЕАЛЕН СЛУЧАЈ

$$\boxed{U = U_0 + D\delta\omega_1}$$

$$u(t) = (U_0 + D\delta\omega_i)\cos\Big[\omega_0 t + \underbrace{k_\omega \int u_m(t)\,dt}_{\delta\phi_i}\Big]$$

$$u_a(t) = U_0\left(1 + \frac{D\delta\omega_i}{U_0}\right); \qquad \boxed{\delta\omega_i = \frac{d\,\delta\phi_i}{dt} = k_\omega u_m(t)}$$

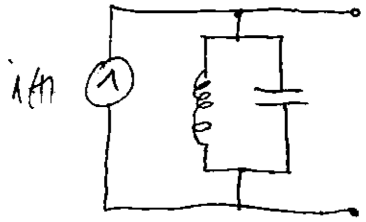
$$u_a(t) = U_0\left(1 + \frac{D k_\omega u_m(t)}{U_0}\right) = U_0\Big(1 + \underbrace{\frac{D k_\omega U_m}{U_0}}_{m_0} \cdot m(t)\Big)$$

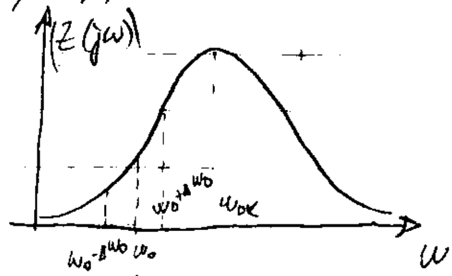
$$m_0 = \frac{D k_\omega U_m}{U_0} \qquad u_a = U_0(1 + m_0 m(t))$$

$$u_d(t) = k_d \cdot \delta\omega_i = k_d k_\omega \cdot u_m(t) = k_D u_m(t)$$

• РЕАЛИЗАЦИЈА НА FM/KAM КОНВЕРТОРОТ

$$\boxed{f_0 \gg 2\Delta f_0}$$





$$i(t) = I(t)\cos\left(\omega_0 t + k_\omega \int u_m(t)\,dt\right) = I_0\cos(\omega_0 t + \varphi)$$

$$i(t) = I_0\cos(\omega_0 t + \varphi) \xrightarrow{\text{complex phasors}} I(t) = I_0 e^{+j(\omega_0 t + \varphi)} \qquad I(j\omega) = \mathcal{F}\{I(t)\}$$

$$\omega_i = \omega_0 + \frac{d\varphi}{dt} = \omega_0 + \varphi'(t) \qquad \delta\omega_i = \frac{d\varphi(t)}{dt} = \varphi'(t)$$

$$Z(j\omega) = Z(j\omega_0) + \frac{Z'(j\omega_0)}{1!}\cdot j(\omega-\omega_0) + \frac{Z''(j\omega_0)}{2!} j^2(\omega-\omega_0)^2 + \dots$$

$$\mathcal{F}\{f'(t)\} = j\omega F(j\omega) \qquad F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\,dt \qquad \mathcal{F}\{f(t)\} = F(j\omega)$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega) e^{j\omega t}\,d\omega$$

$$f'(t) = \frac{1}{2\pi} j\omega \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t}\,d\omega = j\omega \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega) e^{j\omega t}\,d\omega$$

~~$$\int_{-\infty}^{\infty} f'(t) e^{-j\omega t}\,dt = \int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega F(j\omega) e^{j\omega t}\,d\omega\right) e^{-j\omega t}\,dt$$~~

$$\mathcal{F}\{f'(t)\} = j\omega \cdot F(j\omega)$$

$$\mathcal{F}\left\{\int_{-\infty}^{t} f(x)\,dx\right\} = \circledast \qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega) e^{+j\omega t} d\omega$$

$$\frac{1}{2\pi}\int_{-\infty}^{t}\left(\int_{-\infty}^{\infty} F(j\omega) e^{j\omega x} d\omega\right) dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega) d\omega \int_{-\infty}^{t} e^{j\omega x} dx$$

$$I = \int_{-\infty}^{t} e^{j\omega x} dx = \frac{1}{j\omega}\int_{-\infty}^{t} e^{j\omega x} d(j\omega x) = \frac{1}{j\omega} e^{j\omega x}\Big|_{-\infty}^{t} = \frac{1}{j\omega}\left(e^{j\omega t} - e^{-\infty}\right)$$

$$I = \frac{e^{j\omega t}}{j\omega}; \quad \circledast = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\,\frac{e^{j\omega t}}{j\omega}\, d\omega = \frac{1}{j\omega} F(j\omega)$$

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MARINA VELIČKOVA

ODLUKA OD 17 ULI ZA VODENJE TRANSAK SMETKA.

$$I(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} I(j\omega) e^{j\omega t} d\omega; \quad I_1(t) = I'(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega I(j\omega) e^{j\omega t} d\omega$$

$$I_1(t) = \frac{dI(t)}{dt} \qquad I(t) = I_0 e^{+j\omega_0 t + \varphi(t)}$$

$$U(t) = \left[Z(j\omega_0) + \frac{Z'(j\omega_0)}{1!}\left(\frac{d}{dt} - j\omega_0\right) + \frac{Z''(j\omega_0)}{2!}\left(\frac{d}{dt} - j\omega_0\right)^2 + \dots\right] I_0 e^{j(\omega_0 t + \varphi(t))}$$

$$U(t) = \mathcal{F}^{-1}\left\{Z(j\omega)\cdot I(j\omega)\right\}$$

$$\bullet \left(\frac{d}{dt} - j\omega_0\right) e^{j\omega_0 t + \varphi(t)} = \left(j\omega_0 + j\varphi'(t) - j\omega_0\right) e^{j\omega_0 t + \varphi(t)} = j\varphi'(t) e^{j(\omega_0 t + \varphi(t))}$$

$$\bullet \left(\frac{d}{dt} - j\omega_0\right)^2 e^{j\omega_0 t + \varphi(t)} = \left(\frac{d}{dt} - j\omega_0\right) j\varphi'(t) e^{j[\omega_0 t + \varphi(t)]} =$$

$$= j\left[\varphi''(t)\cdot e^{jx} + \varphi'(t)\left[j\omega_0 + j\varphi'(t)\right] e^{jx} - j\omega_0 e^{jx}\varphi'(t)\right] =$$

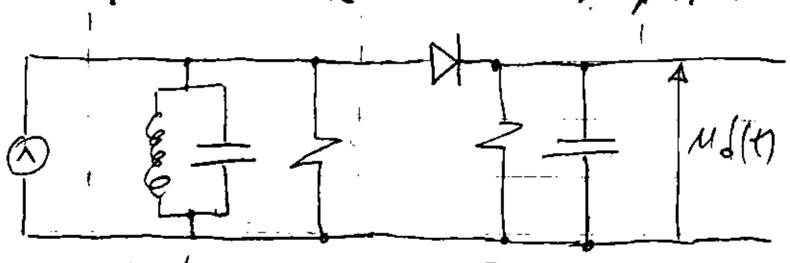
$$= j\left[\varphi'' + j\omega_0\varphi' + j\varphi'^2(t) - j\omega_0\varphi'\right] e^{j[\omega_0 t + \varphi(t)]}$$

$$= \left[j\varphi''(t) - \varphi'^2(t)\right] e^{j[\omega_0 t + \varphi(t)]}$$

$$U(t) = \left\{Z(j\omega_0) + Z'(j\omega_0)\, j\varphi' + \frac{Z''(j\omega_0)}{2}\left[j\varphi''(t) - \varphi'^2(t)\right]\right\} I_0 e^{j(\omega_0 t + \varphi(t))}$$

$$Z'', Z''' \to 0 \qquad u(t) = (a + b\cdot\delta\omega_i)\, I_0 e^{j[\omega_0 t + \varphi(t)]}$$

• Комплетен детектор на FM сигнали



FM/KAM конверзија D.A.

$$\frac{1}{Z(j\omega)} = \frac{1}{j\omega L} + \frac{1}{R} + j\omega C$$

→ импеданса на ОК

Стационарен режим: $L = 2C\cdot R^2$ $\quad |U| = |Z(j\omega)|\,|I|$

$$\omega_k = \frac{1}{\sqrt{LC}}\ ; \quad \frac{1}{Z(j\omega)} = \frac{R + j\omega L - \omega^2 CLR}{j\omega LR} = \frac{(R-\omega^2 LCR) + j\omega L}{j\omega LR}$$

$$Z(j\omega) = \frac{j\omega LR}{(R-\omega^2 LCR) + j\omega L} \qquad |Z| = \frac{\omega LR}{\sqrt{(R-\omega^2 LCR)^2 + \omega^2 L^2}}$$

$$|Z| = \frac{\omega LR}{\sqrt{R^2(1-\omega^2 LC)^2 + \omega^2 L^2}} = \frac{\omega LR}{\sqrt{R^2(1-2\omega^2 LC + \omega^4 L^2 C^2) + \omega^2 L^2}}$$

$$= \frac{\omega L}{\sqrt{1-2\omega^2 LC + \omega^4 L^2 C^2 + \frac{\omega^2 L^2}{R^2}}} = \left| R^2 = \frac{L}{2C} \right| = \frac{\omega L}{\sqrt{1+\omega^4 L^2 C^2}}$$

$$\boxed{|Z| = \frac{\omega L}{\sqrt{1 + \frac{\omega^4}{\omega_k^4}}}} \qquad \left(\frac{\omega}{\omega_k}\right) < 1 \qquad Z(j\omega) \doteq \omega L\left(1 - \frac{\omega^4}{2\omega_k^4}\right) = A(\omega)$$

$$A(\omega) = L\left(\omega - \frac{\omega^5}{2\omega_k^4}\right) \qquad \boxed{\delta\omega_i = k_\omega \cdot U_m \cos}$$

$$\omega_i = \omega(t) = \omega_0 + \delta\omega \qquad \boxed{\delta\omega_i = \omega_i - \omega_0} = \omega(t) - \omega_0$$

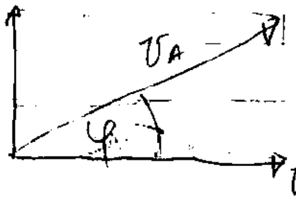
$$A(\omega) = A(\omega_0) + \frac{A'(\omega_0)}{1!}\,\delta\omega_i + \frac{A''(\omega_0)}{2!}\,\delta\omega_i^2 + \frac{A'''(\omega_0)}{3!}\,\delta\omega_i^3 + \ldots$$

$$|Z(j\omega)| = L\left(\omega_0 - \frac{\omega_0^5}{2\omega_k^4}\right) + L\left(1 - \frac{5}{2}\frac{\omega_0^4}{\omega_k^4}\right)\cdot\delta\omega_i - \frac{L}{2}\cdot 10\,\frac{\omega_0^3}{\omega_k^4}\cdot\delta\omega_i^2 - \frac{30L}{3!}\,\frac{\omega_0^2}{\omega_k^4}\,\delta\omega_i^3 - \ldots$$

$$|U| \doteq u_d(t) = I\cdot L\left\{\left(\omega_0 - \frac{\omega_0^5}{2\omega_k^4}\right) + \left(1 - \frac{5}{2}\frac{\omega_0^4}{\omega_k^4}\right)\cdot k_\omega U_m \cos(\omega_m t) - \right.$$

$$\left. - \frac{10}{2!}\,\frac{\omega_0^3}{\omega_k^4}\,(k_\omega U_m)^2 \cos^2(\omega_m t) - \frac{30}{3!}\,\frac{\omega_0^2}{\omega_k^2}\,(k_\omega U_m)^3\cdot\cos^3(\omega_m t) - \ldots\right\}$$

• Фостер-Сили дискриминатор



$$u_a = U_1 \cos\omega t + U_2 \sin\omega t$$

$$U_1 = U_A \cos\varphi \qquad \varphi = \mathrm{arctg}\,\frac{-U_2}{U_1}$$

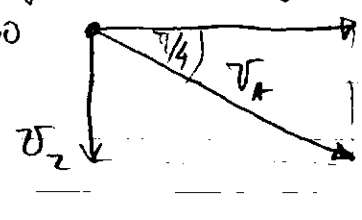
$$U_2 = -U_A \sin\varphi$$

$$u_a = U_A(\cos\omega t\cdot\cos\varphi - \sin\omega t\cdot\sin\varphi) = U_A\cos(\omega t + \varphi)$$

Ако: $U_1 = U_2 \Rightarrow \varphi = \mathrm{arctg}(-1) \quad \mathrm{tg}\,\varphi = \frac{-1}{U_1} \quad \left(\varphi = -\frac{\pi}{4}\right)$

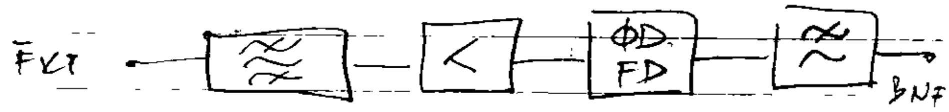
$$\boxed{u_a = U_A\cos\left(\omega t - \frac{\pi}{4}\right)}$$

Рамно



$$U_A = \sqrt{U_1^2 + U_2^2}$$

• Odnos S/N kaj FM, ΦM signali · OTS 200777



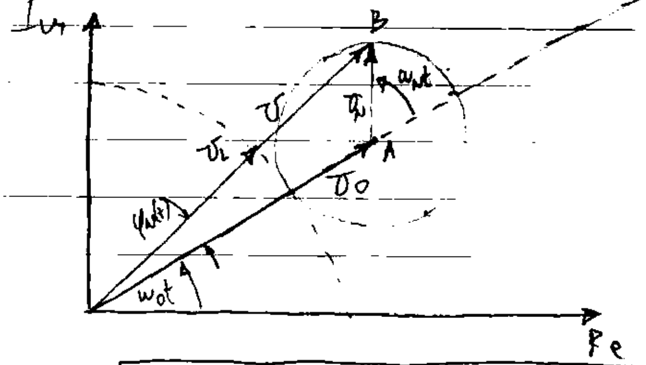
• NA VLEZ NEMODULIRAN NOSITEL + 1 FN PEČKA

$$u(t) = U_0 \cos\omega_0 t + U_N \cos(\omega_0+\omega_N)t = U_0\cos\omega_0 t + U_N\cos\omega_0 t\cdot\cos\omega_N t - U_N \sin\omega_0 t\cdot\sin\omega_N t = \underbrace{(U_0 + U_N\cos\omega_N t)}_{A}\cos\omega_0 t - \underbrace{U_N\sin\omega_N t}_{B}\sin\omega_0 t$$

$A = C\cdot\cos\varphi_N \qquad C = \sqrt{A^2+B^2}$

$B = C\cdot\sin\varphi_N \qquad \varphi_N = \operatorname{arctg}\frac{B}{A} = \operatorname{arctg}\frac{U_N\sin\omega_N t}{U_0+U_N\cos\omega_N t}$

$$u(t) = \sqrt{(U_0+U_N\cos\omega_N t)^2 + U_N^2\sin^2\omega_N t}\cdot\cos(\omega_0 t+\varphi_N)$$



$$U(t) = \sqrt{U_0^2 + 2U_0U_N\cos\omega_N t + \underbrace{U_N^2\cos^2\omega_N t + U_N^2\sin^2\omega_N t}_{U_N^2}}$$

$$= U_0\sqrt{\left(1+\frac{2U_N}{U_0}\cos\omega_N t + \frac{U_N^2}{U_0^2}\right)}$$

LIMITER ⇒ $U(t) = U_{L0}$ $\quad u(t) = U_{L0}\cdot\cos(\omega_0 t+\varphi_n)$

$U_0 \gg U_N \quad \operatorname{arctg}\frac{U_N\sin\omega_N t}{U_0+U_N\cos\omega_N t} \doteq \frac{U_N}{U_0}\sin\omega_N t$

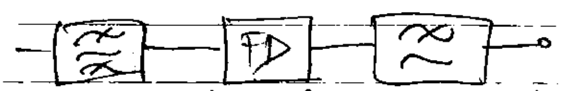
$$u(t) = U_{L0}\cos\Big(\omega_0 t + \underbrace{\frac{U_N}{U_0}\sin\omega_N t}_{\delta\Phi_i}\Big)$$

INDEX NA MODULACIJA $m_0 = \frac{U_N}{U_0}$

$$u_{d_{FM}}(t) = D_f\cdot\delta f_i \quad D_\omega\delta\omega_i = D_\omega\cdot\frac{U_N}{U_0}\omega_N\cos(\omega_N t) = D_f\cdot\frac{U_N}{U_0}f_N\cos(\omega_N t)$$

$$u_{d_{\Phi M}} = D_\Phi\cdot\delta\Phi_i = D_\Phi\cdot\frac{U_N}{U_0}\sin\omega_N t$$

• ~~Slabea~~ SNAGA NA S[...] NA IZLEZ OD FM ~~..~~



$u_d(t) = D_f\cdot\delta f_i$

$\delta f_{iN} = \frac{U_N}{U_0} f_N \cos(\omega_N t+\varphi_N) \qquad \Delta P_{Ni} = 2D_f^2\frac{U_N^2}{U_0^2}f_N^2$

$\Delta P_{Ni} = 2D_f^2\frac{\Delta P_{NV}}{P_0}f_N^2$ → REDUKCIJA NA SNAGATA NA ŠUM

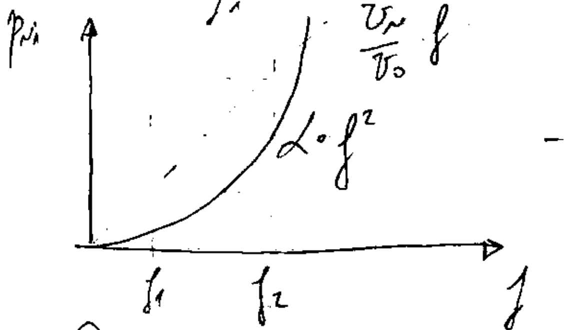
ΔP_{NV} → $\delta P_{NV} = FKT\cdot\delta f \qquad D_{FP} = D_F^2$

$$dP_{Ni} = 2D_{FP}\frac{FKT}{P_0} f^2\, df$$

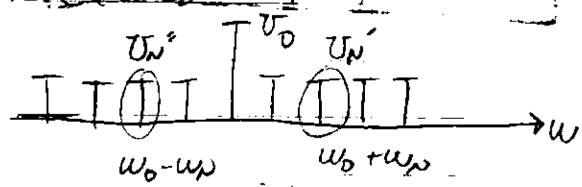
$$\boxed{p_{Ni} = 2D_{FP}\frac{FKT}{P_0} f^2 = \alpha f^2}$$

where $\alpha = 2D_{FP}\frac{FKT}{P_0}$.

$$P_{Ni} = \int_{f_1}^{f_2} p_{Ni}\, df = 2D_{FP}\frac{FKT}{P_0}\int_{f_1}^{f_2} f^2\, df = 2D_{FP}\frac{FKT}{P_0}\frac{f^3}{3}\Big|_{f_1}^{f_2}$$



- SLEDNA SLIKA NA ŠUM NA IZLEZ OD ΦM DA



$$\delta\phi_N = \frac{U_N}{U_0}\cos\omega_N t = \Delta\phi_{0N}\cos\omega_N t \qquad u_d(t) = D_\Phi \frac{U_N}{U_0}$$

$$\delta\phi_N' = \frac{U_N'}{U_0}\cos(\omega_N t + \varphi_N) \qquad \delta\phi_N'' = \frac{U_N''}{U_0}\cos(\omega_N t + \varphi_N)$$

$$\Delta P_{NU}' \sim (U_N')^2 \qquad \Delta P_{NU}'' \sim (U_N'')^2$$

$$\Delta P_{Ni}' \sim \left(D_\Phi \frac{U_N'}{U_0}\right)^2 \qquad \Delta P_{Ni}'' \sim \left(D_\Phi \frac{U_N''}{U_0}\right)^2 \qquad \boxed{\Delta P_{Ni} = \Delta P_{Ni}' + \Delta P_{Ni}''}$$

$$\Delta P_{Ni} = \frac{D_\Phi^2}{U_0^2}\left(U_N'^2 + U_N''^2\right) = D_{\Phi}^2\left(\frac{P_{NU}'}{P_0} + \frac{P_{NU}''}{P_0}\right)$$

$$dP_{NU} = FKT\, df \Rightarrow \Delta P_{Ni} = 2D_{\Phi}\frac{\Delta P_{NU}}{P_0}$$

$$dP_{Ni} = D_{\Phi}\frac{FKT}{P_0}\, df \qquad dP_{Ni} = D_{\Phi}\frac{FKT}{P_0}\, df \qquad p_{NU} = FKT$$

$$dP_{Ni} = 2D_{\Phi}\frac{FKT}{P_0}\, df = p_{Ni}\cdot df \qquad p_{Ni} = 2D_{\Phi}\frac{FKT}{P_0}$$

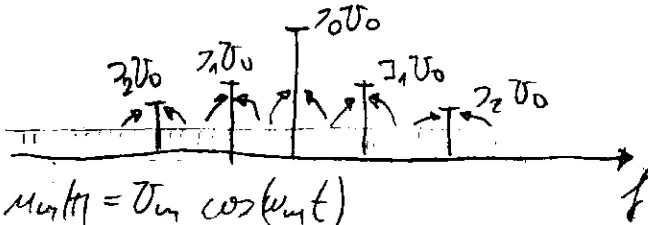
$$\boxed{P_{Ni} = p_{Ni}\cdot B_{NF}}$$

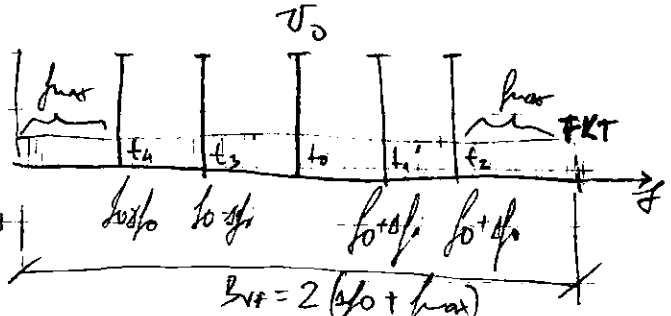
$$P_{Ni} = \int_{f_1}^{f_2} 2D_{\Phi}\frac{FKT}{P_0}\cdot df = 2D_{\Phi}\frac{FKT}{P_0}(f_2 - f_1)$$

$$P_{Ni} = 2D_{\Phi}\frac{FKT}{P_0}\cdot B_{NF} \qquad P_{NV} = FKT\cdot B_{VF}$$

- ŠUM NA IZLEZOT OD PRIEMNIKOT VO MESENJETO NA SIGNAL.

$$u_m(t) = U_m\cos\omega_m t$$





$$u_m(t) = U_m\cos(\omega_m t)$$

$$u_m(t) = -U_m \Rightarrow f_0 - \Delta f_0$$

$$u_m(t) = \theta \Rightarrow f_0 + \Delta f_0$$

– КАНОНСКА ... ЕЛАЗА –

$P_{sv} = P_0$ $\quad P_{NV} = FKT \cdot B_{VF} = P_{NV}'$ $\quad \left(\frac{S}{N}\right)_V = \frac{P_{sv}}{P_{NV}} = \frac{P_0}{FKT B_{VF}}$

- $\left(\frac{S}{N}\right)_i^{\Phi M} = ?$ $\quad \Delta\Phi_{os} = K_\varphi U_m \quad U_i = D_\Phi \cdot \Delta\Phi_{os} \; ; \; u_{di}(t) = D_\Phi \cdot \delta\Phi_i$

$u_{di}(t) = D_\Phi \cdot \Delta\Phi_{os} \cos\omega_m t$ $\quad P_{si} \approx D_\Phi^2 \Delta\Phi_{os}^2 = D_{\Phi\Phi} \Delta\Phi_{os}^2$

$P_{Ni} = D_{\Phi\Phi} \frac{FKT}{P_0}$ $\quad P_{Ni} = 2 D_{\Phi\Phi} \frac{FKT}{P_0} B_{NF}$

$$\frac{P_{si}}{P_{Ni}} = \frac{D_{\Phi\Phi} \Delta^2\Phi_{os}}{2 D_{\Phi\Phi} \frac{FKT}{P_0} B_{NF}} = \frac{\Delta\Phi_{os}^2}{2} \cdot \frac{P_0 \cdot B_{VF}}{(FKT \cdot B_{VF}) B_{NF}} = \frac{\Delta\Phi_{os}^2}{2} \frac{B_{VF}}{B_{NF}} \cdot \frac{P_0}{P_{NV}}$$

$$\frac{P_{si}}{P_{Ni}} = \frac{\Delta\Phi_{os}^2}{2} \frac{B_{VF}}{B_{NF}} \frac{P_0}{P_{NV}} \qquad \frac{(S/N)_i}{(S/N)_V} = \frac{\frac{\Delta\Phi_{os}^2}{2} \frac{B_{VF}}{B_{NF}} \frac{P_0}{P_{NV}}}{\frac{P_0}{FKT B_{VF}}} = \frac{\Delta\Phi_{os}^2}{2} \frac{B_{VF}}{B_{NF}}$$

- Ако се зема само P_{NL} која поминува низ NF филт:

$$\frac{P_{si}}{P_{Ni}} = \frac{\Delta\Phi_{os}^2}{2} \frac{P_0 \cdot B_{NF}}{FKT \cdot B_{NF} \cdot B_{NF}} = \Delta\Phi_{os}^2 \frac{P_0}{P_N} = \Delta\Phi_{os}^2 \frac{P_{sv}}{P_N}$$

$P_N = 2FKT \cdot B_{NF}$

- $\left(\frac{S}{N}\right)_i^{FM} = ?$ $\quad \delta f_i = \Delta f_{os} \cos\omega_m t = K_f U_m \quad u_{di}(t) = D_F \cdot \delta f_i$

$u_{di}(t) = D_F \cdot \Delta f_{os} \cos\omega_m t$ $\quad P_{si} = D_F^2 \cdot \Delta^2 f_{os} = D_{FF} \Delta^2 f_{os}$

$P_{Ni} = 2 D_{FF} \frac{FKT}{P_0} \int_{f_1}^{f_2} f^2 df$ $\quad \frac{P_{si}}{P_{Ni}} = \frac{D_{FF} \Delta^2 f_{os}}{2 D_{FF} \frac{FKT}{P_0} \int_{f_1}^{f_2} f^2 df}$

$$\frac{P_{si}}{P_{Ni}} = \frac{\Delta^2 f_{os} B_{VF}}{2 \int_{f_1}^{f_2} f^2 df} \cdot \frac{P_0}{P_{NR}} \qquad \frac{(S/N)_i}{(S/N)_V} = \frac{\Delta^2 f_{os} B_{VF}}{2 \int_{f_1}^{f_2} f^2 df}$$

- Праг на шум на системите за FM

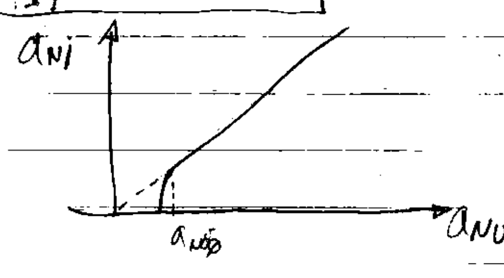
$B_{VF} = 2(\Delta f_0 + f_2)$ $\quad P_{NR} = FKT \cdot B_{VF}$

SUM: $20 \log \frac{U_m}{U_{eff}} = 13 dB$ $\quad$ ТТ: $20 \log \frac{U_m}{U_{eff}} = 3 dB$

$$10 \log \frac{P_{si}}{P_{Ni}} = 10 \log \frac{\Delta f_{os}^2 B_{VF}}{2 \int_{f_1}^{f_2} f^2 df} + 10 \log \frac{P_{sv}}{P_{NL}} = \gamma + a_{NV}$$

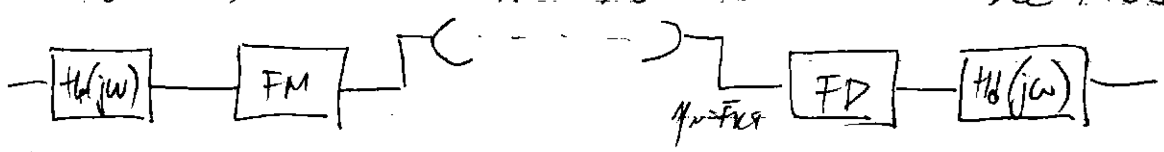
$a_{Ni} = \gamma + a_{NV}$ $\quad \gamma$ – фактор на подобрување на S/N

a_{NVO} – праг на шум

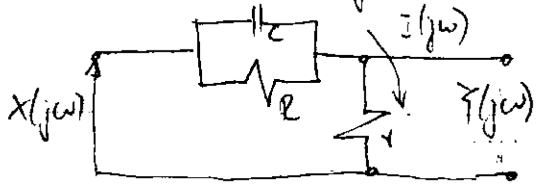


FM: $a_{NVO} = 10 dB$ т.е. $P_0 = 10 P_{NL}$

Подобрување на S/N со преемфазис и деемфазис



- Класично $H_p(j\omega)$ за музички сигнали



$$H_p(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

$$Y(j\omega) = r \cdot I(j\omega) \qquad I(j\omega) = \frac{X(j\omega)}{r + \frac{R\cdot\frac{1}{j\omega C}}{R+\frac{1}{j\omega C}}}$$

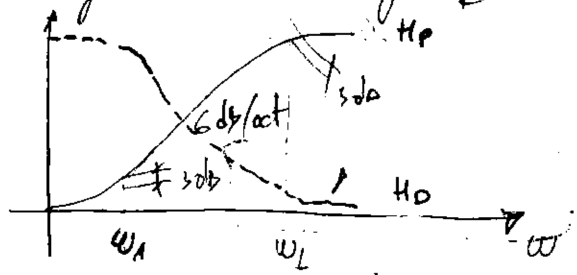
$$Y(j\omega) = \frac{r\,X(j\omega)}{r + \frac{\frac{R}{j\omega C}}{\frac{j\omega RC+1}{j\omega C}}} \; ; \quad H_p(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{r(j\omega RC+1)}{R+r+j\omega rRC}$$

$$R \gg r \qquad H_p(j\omega) = \frac{r(j\omega RC+1)}{R(1+j\omega rC)} = \frac{r}{R}(1+j\omega RC)$$

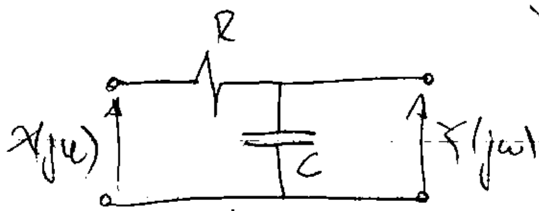
$$|H_p(j\omega)| = \frac{r}{R}\sqrt{1+\omega^2R^2C^2} = \Big/\omega_1 = \frac{1}{RC}\Big/ = \frac{r}{R}\sqrt{1+\frac{\omega^2}{\omega_1^2}}$$

$$|H_p(j\omega)| = H_p(\omega) = \frac{r}{R}\sqrt{1+\frac{\omega^2}{\omega_1^2}}$$

$$20\log H_p(\omega) = 20\log\frac{r}{R} + 10\log\left(1+\frac{\omega^2}{\omega_1^2}\right)$$



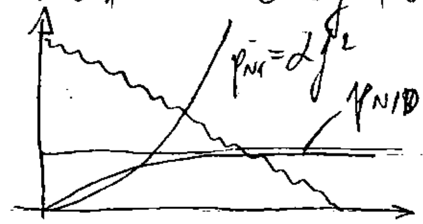
- Класично $H_d(j\omega)$ коло



$$H_d(j\omega) = \frac{\frac{1}{j\omega C}}{R+\frac{1}{j\omega C}} = \frac{1}{1+j\omega RC} = \frac{1}{1+j\frac{\omega}{\omega_1}} \qquad \omega_1 = \frac{1}{RC}$$

$$|H_d(j\omega)| = H_d(\omega) = \frac{1}{\sqrt{1+\frac{\omega^2}{\omega_1^2}}} \qquad 20\log H_d(\omega) = -10\log\left(1+\frac{\omega^2}{\omega_1^2}\right)$$

$$\omega \gg \omega_1 \qquad 20\log H_d(\omega) = -10\log\frac{\omega}{\omega_1}$$



$$p_{NI}(f) = 2\pi\frac{FKT}{T_0}f^2 = \alpha\cdot f^2 \quad \Big\}\ \text{SGSS без } H_d$$

$$p_{NID} = |H(j\omega)|^2\cdot p_{NI} = \frac{\alpha f^2}{1+\frac{f^2}{f_1^2}}$$

Ако: $\frac{f}{f_1} < 1 \Rightarrow \left(\frac{f}{f_1}\right)^2 \ll 1 \quad \varphi_{NID} = \alpha \cdot f^2$

Ако: $\frac{f}{f_1} > 1 \Rightarrow \left(\frac{f^2}{f_1}\right) \gg 1 \quad \varphi_{NID} = \frac{\alpha f^2}{\frac{f^2}{f_1^2}} = \alpha \cdot f_1^2$

$$P_{Ni} = \int_0^{f_{max}} \varphi_{Ni}(f)\,df = 2D_{FF}\frac{FKT}{P_0}\frac{f_{max}^3}{3}$$

$$P_{NID} = \int_0^{f_{max}} \frac{\alpha f^2}{1+\frac{f^2}{f_1^2}}\,df = \int_0^{f_{max}} \frac{\alpha \cdot f_1^2 f^2}{f_1^2+f^2}\,df = \alpha \cdot f_1^2 \underbrace{\int_0^{f_{max}} \frac{f^2 df}{f_1^2+f^2}}_{I}$$

$$I = \int_0^{f_{max}} \frac{(f_1^2+f^2)df}{f_1^2+f^2} - \int_0^{f_{max}} \frac{f_1^2 df}{f_1^2+f^2} = f_{max} - f_1 \int_0^{f_{max}} \frac{d\left(\frac{f}{f_1}\right)}{1+\left(\frac{f}{f_1}\right)^2}$$

$I = f_{max} - f_1 \operatorname{arctg}\frac{f_{max}}{f_1} \qquad P_{NID} = \alpha \cdot f_1^2\left[f_{max} - f_1 \operatorname{arctg}\frac{f_{max}}{f_1}\right]$

$$\boxed{P_{NID} = 2D_{FF}\frac{FKT}{P_0} f_1^3\left[\frac{f_{max}}{f_1} - \operatorname{arctg}\frac{f_{max}}{f_1}\right]}$$

$$\boxed{Q = \frac{P_{Ni}}{P_{NID}} = \frac{\frac{f_{max}^3}{3}}{f_1^3\left[\frac{f_{max}}{f_1} - \operatorname{arctg}\frac{f_{max}}{f_1}\right]} = \frac{(S/N)_{id}}{(S/N)_i}}$$

- МУЗИЧКИ СИГНАЛИ: $f_{max} = 15\,kHz \quad f_1 = 2{,}1\,kHz$

$Q = \frac{1}{3}\left(\frac{f_{max}}{f_1}\right)^2 \doteq 20 \qquad 10\log 20 = 10(\log 10 + \log 2\ldots) = 13\,db$

- $\Delta f_{os} = 75\,kHz \quad f_{max} = 15\,kHz$

$P_{Ni} = 2D_{FF}\frac{FKT}{P_0}\frac{f_{max}^3}{3} \qquad \boxed{P_{si} = D_{FF}(\Delta f_{os})^2}$

$$\frac{P_{si}}{P_{Ni}} = \frac{D_{FF}(\Delta f_{os})^2}{2D_{FF}\frac{FKT}{P_0}\frac{f_{max}^3}{3}} = \frac{3\,\Delta f_{os}^2 P_0}{2\,FKT f_{max}^3} = \left(\frac{S}{N}\right)_i$$

- So $H_y(kHz)$ $\qquad \frac{P_{SiD}}{P_{NiD}} = \frac{P_{si}}{P_{Ni}/Q} = Q\frac{P_{si}}{P_{Ni}}$

- S/N ВО РАДИОРЕЛЕЈНИТЕ СИСТЕМИ ЗА ПРЕНОС НА TV.

$\Delta f_{v-v} = 8\,MHz$

$\Delta f_{ov-v} = 0.7 \cdot 8 = 5{,}6\,MHz$

$\Delta f_{ov-v} > 2\Delta f_{ow}$

625 ЛИНИ $\quad f_{max} = 5\,MHz$

$$\left(S/N\right)_i = \frac{U_{sv-v}}{U_{Neff}} = \frac{U_{-OT} \text{ ЗА СКИКАЊА ВРВ-ВРВ}}{\text{eff. вр. НА } U_{OT} \text{ НА ШУМ}}$$

$a_{N1} = 20\log\frac{U_{SV-V}}{U_{Neff}} = 10\log A_{N1} = 10\log\left(\frac{U_{SV-V}}{U_{Neff}}\right)^2$

• $U_{Neff} = ?$ $P_{N1} = 2D_{FP}\frac{FKT}{P_0}\int_0^{f_{max}} f^2\,df$

$P_{N1} = \frac{U_{eff}^2}{R_d}$, $U_{eff}^2 = P_{N1}\cdot R_d = 2D_{FP}R_d\frac{FKT}{P_0}\int_0^{f_{max}} f^2\,df$

• ПОСЛЕ ДЕМОДУЛАТОРОТ $U_{SV-V} = (\Delta f_{0V-V})\cdot D_F$

$$S/N = \frac{D_F\,\Delta f_{0V-V}}{\left[2D_{FP}\cdot R_d\cdot\frac{FKT}{P_0}\int_0^{f_{max}} f^2\,df\right]^{1/2}} = \frac{U_{SV-V}}{U_{Neff}}$$

$U_S = D_F\cdot\Delta f_0$ $\left(u_S(t) = D_F\,\Delta f_0\cos(\omega_m t) = D_F\cdot\Delta f_i\right)$

$\boxed{U_S^2 = D_F^2\cdot\Delta f_0^2}$ $\frac{D_F^2\cdot\Delta f_0^2}{2\cdot R_d} = \frac{U_S^2}{2R_d} = D_{FP}\,\Delta f_0^2$ $\boxed{D_{FP} = \frac{D_F^2}{2R_d}}$

$$a_{N1} = 10\log\frac{D_F^2\,\Delta f_{0V-V}^2}{2D_{FP}R_d\,\frac{FKT}{P_0}\int_0^{f_{max}} f^2\,df} = 10\log\frac{\Delta f_{0V-V}^2}{\frac{FKT}{P_0}\int_0^{f_{max}} f^2\,df} =$$

$$= 10\log\frac{\Delta f_{0V-V}^2}{\frac{FKT}{P_0}\,\frac{f_{max}^3}{3}} = 10\log 3\left(\frac{\Delta f_{0V-V}}{f_{max}}\right)^2\frac{P_0}{FKT\cdot f_{max}}$$

$K\cdot T = 4\cdot10^{-21}$ W/Hz ; $\Delta f_{0V-V} = 5{,}6$ MHz ; $f_{max} = 5$ MHz

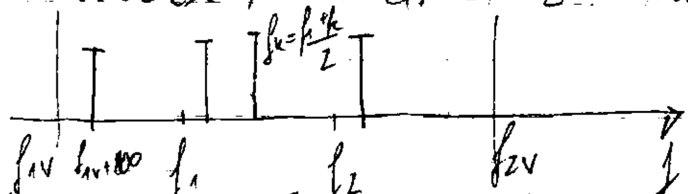
$$a_{N1} = 10\log 3\left(\frac{5{,}6}{5}\right)^2\cdot\frac{1}{4\cdot10^{-21}\cdot5\cdot10^{+6}} + 10\log F + 10\log\frac{P_0}{1mW} + 10\log10^{-3}$$ (p_{P0})

$$= 10\log 3.7632\cdot\frac{1}{20\cdot10^{-15}} + (-30) - 10\log F + p_{P0} =$$

$$= 142{,}75 - 30 - 10\log F + p_{P0} = \boxed{112{,}75 - 10\log F + p_{P0}}$$

• ПРЕНОС НА АУДИО (ГОВОР) И ТОНСКИ КАНАЛ ОД ТВ СИСТЕМОТ

• РАДИОРЕЛЕЈНИ СИГНАЛИ ЗА ПРЕНОС НА МУЛ СИГНАЛИ

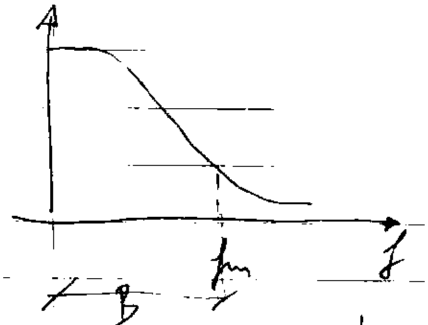


$$\left(\frac{P_S}{P_N}\right)_{10\,CH} = \frac{\Delta f_0^2\cdot P_0}{2FKT\int_{f_1}^{f_2} f^2\,df} = \left|\begin{array}{l} f = \frac{f_1+f_2}{2} = f_k \\ B_k = f_2 - f_1\end{array}\right| = \frac{\Delta f_0^2\cdot P_0}{2FKT\,f_k^2\,B_k} = \left(\frac{\Delta f_0}{f_k}\right)^2\frac{P_0}{2FKT\,B_k}$$

- Ако се стави H_q и H_n

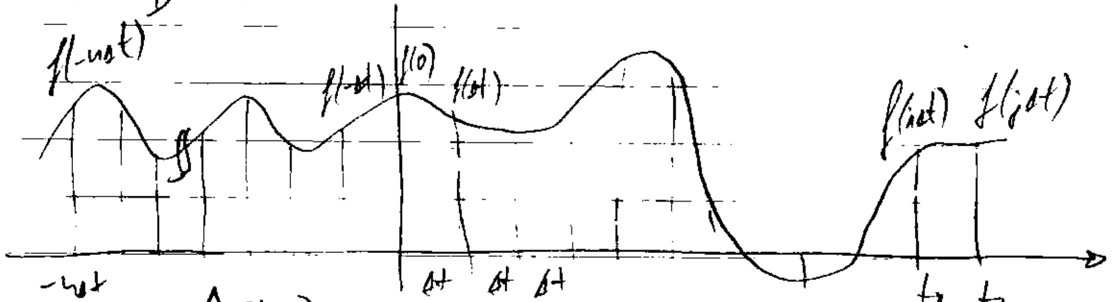
$$\frac{P_{SD}}{P_{ND}} = A_p^2(f_c)\,\frac{P_s}{P_n}$$

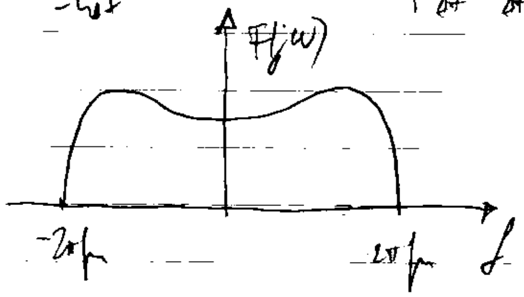
- Дискретизација на континуални сигнали



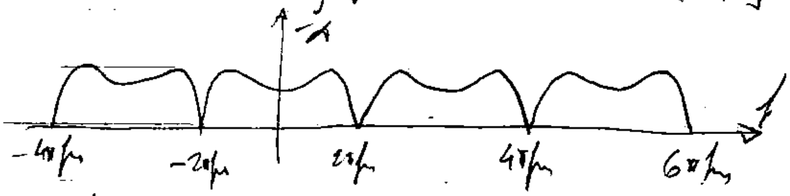
$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\,e^{j\omega t}\,d\omega = \frac{1}{2\pi}\int_{-\omega_N}^{\omega_N} F(j\omega)\,e^{j\omega t}\,d\omega$$

$$\boxed{\Delta t = t_j - t_i \le \frac{1}{2f_m}}$$





$$F(j\omega) = \int_{-\infty}^{\infty} f(t)\,e^{-j\omega t}\,dt = \mathcal{F}\{f(t)\}$$



$$F_p(j\omega) = \sum_{-\infty}^{\infty} F_n\,e^{jn\frac{2\pi}{"T"}\omega} = \sum_{n=-\infty}^{\infty} F_n\,e^{jn\frac{1}{2f_m}\omega} \qquad "T" = 4\pi f_m$$

$$F_n = \frac{1}{4\pi f_m}\int_{-2\pi f_m}^{2\pi f_m} F_p(j\omega)\,e^{-j\frac{n}{2f_m}\cdot\omega}\,d\omega = \frac{1}{4\pi f_m}\int_{-2\pi f_m}^{2\pi f_m} F_p(j\omega)\,e^{-j\frac{n}{2f_m}\omega}\,d\omega$$

(*)

$$f(t) = \frac{1}{2\pi}\int_{-2\pi f_m}^{2\pi f_m} F(j\omega)\,e^{+j\omega t}\,d\omega \qquad t = -n\Delta t = -\frac{n}{2f_m} \qquad f(t) = ?$$

$$f\left(-\frac{n}{2f_m}\right) = \frac{1}{2\pi}\int_{-2\pi f_m}^{2\pi f_m} F(j\omega)\,e^{-j\frac{n}{2f_m}\cdot\omega}\,d\omega \qquad (*) = 2\pi f\left(-\frac{n}{2f_m}\right)$$

$$F_n = \frac{1}{4\pi f_m}\cdot f\left(-\frac{n}{2f_m}\right)\cdot 2\pi \qquad \boxed{F_n = \frac{1}{2f_m}\,f\left(-\frac{n}{2f_m}\right)}$$

$$f\left(-\frac{n}{2f_m}\right) = 2f_m F_n\,; \quad F_p(j\omega) = \sum_{-\infty}^{\infty}\frac{1}{2f_m}\,f\left(-\frac{n}{2f_m}\right)e^{jn\frac{1}{2f_m}\cdot\omega}$$

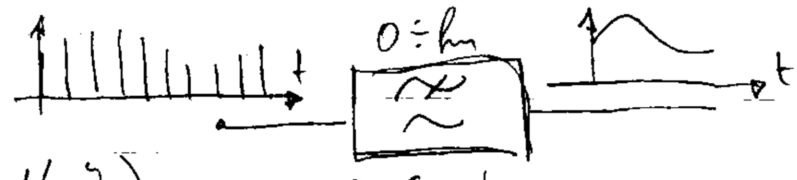
$$f(t) = \frac{1}{2\pi}\int_{-2\pi f_m}^{2\pi f_m}\sum_{-\infty}^{\infty}\frac{1}{2f_m}\,f\left(-\frac{n}{2f_m}\right)e^{jn\frac{1}{2f_m}\omega}\,e^{j\omega t}\,d\omega = \frac{1}{2\pi}\int_{-2\pi f_m}^{2\pi f_m} F(j\omega)\,e^{+j\omega t}\,d\omega$$

$$f(t)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{1}{2f_m}f\left(-\frac{n}{2f_m}\right)\underbrace{\int_{-2\pi f_m}^{2\pi f_m} e^{j\omega\left(t+\frac{n}{2f_m}\right)}d\omega}_{(*)}$$

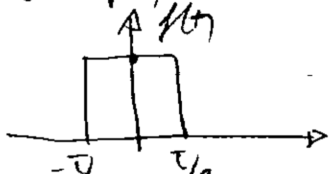
$$(*)=\frac{1}{j\left(t+\frac{n}{2f_m}\right)}e^{j\omega\left(t+\frac{n}{2f_m}\right)}\Bigg|_{-2\pi f_m}^{2\pi f_m}=\frac{1}{j\left(t+\frac{n}{2f_m}\right)}\left[e^{j2\pi f_m\left(t+\frac{n}{2f_m}\right)}-e^{-j2\pi f_m\left(t+\frac{n}{2f_m}\right)}\right]$$

$$f(t)=\sum_{n=-\infty}^{\infty}f\left(-\frac{n}{2f_m}\right)\frac{e^{j2\pi f_m\left(t+\frac{n}{2f_m}\right)}-e^{-j2\pi f_m\left(t+\frac{n}{2f_m}\right)}}{4\pi f_m j\left(t+\frac{n}{2f_m}\right)}=\sum_{n=-\infty}^{\infty}f\left(-\frac{n}{2f_m}\right)\frac{\sin\left(2\pi f_m\left(t+\frac{n}{2f_m}\right)\right)}{2\pi f_m\left(t+\frac{n}{2f_m}\right)}$$

$$\boxed{f(t)=\sum_{n=-\infty}^{\infty}f\left(\frac{n}{2f_m}\right)\frac{\sin\left(2\pi f_m\left(t-\frac{n}{2f_m}\right)\right)}{2\pi f_m\left(t-\frac{n}{2f_m}\right)}}$$



$f\left(\frac{n}{2f_m}\right)\cdot\tau$



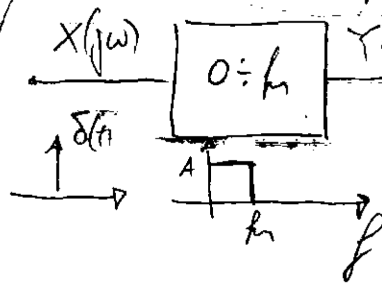
A.P. $\mathcal{F}\{f(t)\} = E\tau\,\frac{\sin\frac{\omega\tau}{2}}{\frac{\omega\tau}{2}} = F(j\omega)$

PERIOD

$$f(t)=\sum_{n=-\infty}^{\infty}E\alpha\cdot\frac{\sin(n\pi\alpha)}{(n\pi\alpha)}e^{+jn\omega_0 t}$$

$$\boxed{f(t)=E\alpha+2E\alpha\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi\alpha}{2}}{\frac{n\pi\alpha}{2}}\cdot\cos(n\omega_0 t)}$$

$$y_n(t)=2f_m\tau\, f\left(\frac{n}{2f_m}\right)A\cdot\frac{\sin 2\pi f_m\left(t-t_0-\frac{n}{2f_m}\right)}{2\pi f_m\left(t-t_0-\frac{n}{2f_m}\right)}$$



$$Y(j\omega)=\begin{cases}A\cdot X(j\omega) & 0\le\omega\le 2\pi f_m\\ 0 & \omega>2\pi f_m\end{cases}$$

$$y(t)=\frac{1}{2\pi}\int_{-2\pi f_m}^{2\pi f_m}A\cdot e^{j\omega t}d\omega=\frac{A}{2\pi}\frac{1}{jt}e^{j\omega t}\Big|_{-2\pi f_m}^{2\pi f_m}$$

$$y(t)=\frac{A}{2\pi}\frac{1}{jt}\frac{e^{j2\pi f_m t}-e^{-j2\pi f_m t}}{2}\cdot 2=\frac{A}{\pi t}\cdot\sin(2\pi f_m t)=2Af_m\frac{\sin(2\pi f_m t)}{2\pi f_m t}$$

$$\boxed{y(t)=2Af_m\frac{\sin(2\pi f_m t)}{2\pi f_m t}}$$

$$y(t)=\sum_{n=-\infty}^{\infty} 2f_m\tau\, f\left(\frac{n}{2f_m}\right) A\, \frac{\sin 2\pi f_m\left(t-t_0-\frac{n}{2f_m}\right)}{2\pi\left(t-t_0-\frac{n}{2f_m}\right)}$$

$$y(t)=2f_m\tau \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2f_m}\right) \frac{\sin 2\pi f_m\left(t-t_0-\frac{n}{2f_m}\right)}{2\pi\left(t-t_0-\frac{n}{2f_m}\right)} = 2f_m\tau\cdot f(t-t_0)$$

• OTSTRANUVANJE NA ISI

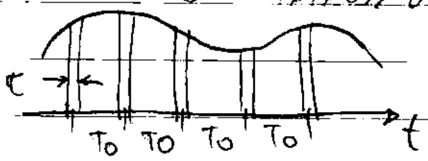
$$2\pi f_g\left(t-t_0+\frac{n}{2f_m}\right)=k\pi \qquad t-t_0+\frac{n}{2f_m}=\frac{k}{2f_g}$$

$$t=t_0-\frac{n}{2f_m}+\frac{k}{2f_g} \qquad \frac{1}{2f_g}=\frac{\Delta t}{N}=\frac{1}{2f_m N}$$

$$\boxed{f_g=N\cdot f_m}$$

• IMPULSNO AMPLITUDNO MODULIRANI SIGNALI

① IAM SO PRIRODNO ZEMANJE NA PRIMEROCI



$$m(t)=\begin{cases} m_n(t) & nT_0-\frac{\tau}{2}\le t\le nT_0+\frac{\tau}{2}\\ 0 & \text{elsewhere}\end{cases}$$

$$m_{IAM}(t)=k\,m_n(t)\,s(t) \qquad s(t)=U_0\sum s(t-nT_0)$$

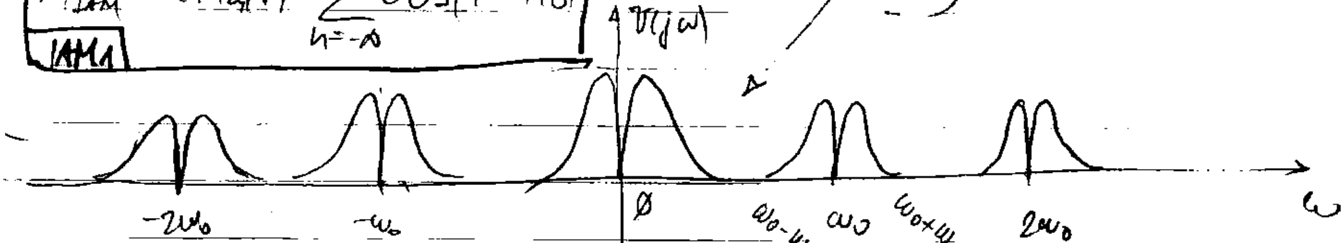
$m_n(t)\to U_m(j\omega)$ ograničen so f_m

$$m_0(t)=\left[\frac{\tau}{T_0}+2\frac{\tau}{T_0}\sum_{n=1}^{\infty}\frac{\sin\frac{n\omega\tau}{2}}{\frac{n\omega\tau}{2}}\cos(n\omega_0 t)\right]U_0$$

$$m_{IAM}(t)=k\,m_n(t)\frac{\tau U_0}{T_0}+2k\,m_n(t)\frac{\tau U_0}{T_0}\sum_{n=1}^{\infty}\frac{\sin\frac{n\omega\tau}{2}}{\frac{n\omega\tau}{2}}\cos(n\omega_0 t)$$

$$U(j\omega)=k\frac{\tau U_0}{T_0}U_m(j\omega)+2k\frac{\tau U_0}{T_0}\sum_{n=1}^{\infty}\frac{\sin\frac{n\omega\tau}{2}}{\frac{n\omega\tau}{2}}\left[U_m[j(\omega-n\omega_0)]+U_m[j(\omega+n\omega_0)]\right]$$

$$\boxed{m_{IAM}=k\,m_n(t)\sum_{n=-\infty}^{\infty}U_0\, s(t-nT_0)}$$ IAM1



② IAM SO REGULARNO ZEMANJE PRIMEROCI I NEGOV SPEKTAR

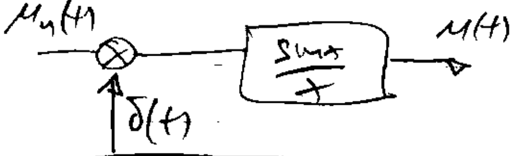
$$\delta_T(t)=\lim_{\tau\to 0}\frac{\tau}{T}\sum_{n=-\infty}^{\infty}\frac{\sin\frac{n\omega\tau}{2}}{\frac{n\omega\tau}{2}}e^{+jn\omega_0 t}=\frac{1}{T}\sum_{n=-\infty}^{\infty}e^{jn\omega_0 t}$$

$\tau\to 0$ $\quad s(t)\to\delta(t)$ $\quad m(t)=k\,m_n(t)\cdot\delta_T(t)=k\,m_n(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_0)$

$$\delta_T(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t} \qquad u(t) = k\, u_m(t)\cdot\frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}$$

$$U(j\omega) = \frac{k}{T} U_m\left[j(\omega-n\omega_0)t\right] + \frac{k}{T} U_m\left[j(\omega+n\omega_0)\right] \quad n = 0 \ldots \infty$$

$$U(j\omega) = k_1 \sum_{n=-\infty}^{\infty} U_m\left[j(\omega-n\omega_0)\right] \quad u(t) = k\sum_{n=-\infty}^{\infty} u_m(nT_0)\,\delta(t-nT_0)$$



$$u(t) = k\sum_{n=-\infty}^{\infty} u_m(nT_0)\, d(t-nT_0)$$

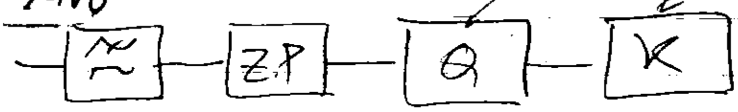
/AM₂

$$U(j\omega) = \frac{\sin\frac{\omega\tau}{2}}{\frac{\omega\tau}{2}}\, k_1 \sum_{n=-\infty}^{\infty} U_m\left[j(\omega-n\omega_0)\right]$$

spektarot dobien so $\delta_T(t)$

- Kvantizacija po MVO

$$q = \frac{U}{\Delta u}$$



$q = 2^n$ q – broj na kvant. nivoa $n = ld\, q$

$q = 8$ $n = ld\, 8 = 3$ $u_N(t) = u(t) - u_q(t)$

- Ramnomerna kvantizacija

$u_i < u < u_{i+1}$

$$\overline{u_{Ni}^2} = \int_{u_i}^{u_{i+1}} (u-u_{qi})^2\, p(u)\,du$$

$$\overline{u_{Ni}^2} = p_0 \int_{u_{qi}-\frac{\Delta u_i}{2}}^{u_{qi}+\frac{\Delta u_i}{2}} (u-u_{qi})^2\,du = p_0 \frac{(u-u_{qi})^3}{3}\Bigg|_{u_{qi}-\frac{\Delta u_i}{2}}^{u_{qi}+\frac{\Delta u_i}{2}} =$$

$$= \frac{p_0}{3}\left[\frac{\Delta u_i^3}{8} + \frac{\Delta u_i^3}{8}\right] = p_0\frac{\Delta u_i^3}{12}$$

$$\overline{u_N^2} = \sum_{i=1}^{q}\overline{u_{Ni}^2} = \frac{1}{12}\sum_{i=1}^{q} p_0\,\Delta u_i\,(\Delta u_i)^2 = \frac{\Delta u_i^2}{12}\sum_{i=1}^{q} p_0\Delta u_i = \frac{\Delta u^2}{12}$$

$$\overline{u_N^2} = \frac{\Delta u^2}{12} \qquad P_{Nq} = \frac{\Delta u^2}{12}$$

— snaga na šum na kvantizacija

$$P_s = \int_{-U/2}^{U/2} u^2 p(u)\,du = p_0\frac{u^3}{3}\Bigg|_{-U/2}^{U/2} = \frac{p_0}{3}\left(2\frac{U^3}{8}\right) = \frac{p_0 U^3}{12}$$

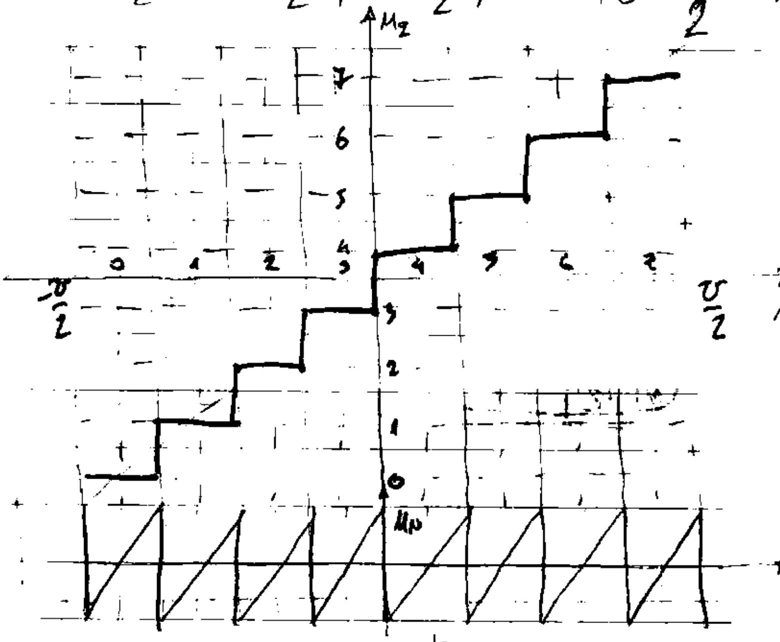
$p_0 U = 1$ $P_s = \frac{U^2}{12}$ $q\cdot\Delta u = U$ $P_s = \frac{q^2\Delta u^2}{12}$

- Snaga na kvantiziran signal

$$P = \frac{1}{\gamma}\left(f_1^2 + f_2^2 + \ldots + f_\gamma^2\right) = \frac{1}{\gamma}\sum_{i=1}^{\gamma} f_i^2 \qquad \Delta t = \frac{1}{2f_m}$$

$f_1, f_2, \ldots f_\gamma$ → primeroci vo int (t_1, t_2)

$\pm\frac{\Delta M}{2}, \pm\frac{3\Delta M}{2}, \pm\frac{5\Delta M}{2}, \dots, \pm\frac{q-1}{2}\Delta M$



$$P_q = 2\cdot\frac{1}{q}\frac{\Delta M^2}{4}\left[1^2+3^2+5^2+\dots(q-1)^2\right]$$

$$S=\sum_{x=1}^{N-1} x^2 = \frac{1}{6}N^3-\frac{1}{6}N$$

$$P_q = 2\cdot\frac{1}{q}\frac{\Delta M^2}{2^4}\left[\frac{1}{6}q^3-\frac{1}{6}q\right]$$

$$\boxed{P_q = \frac{\Delta M^2}{12}\left[q^2-1\right]}$$

$$\boxed{S=\frac{n(n+1)(2n+1)}{6}}$$

$$S = 1+q+q^2+\dots q^n$$
$$qS = q+q^2+q^3+\dots+q^{n+1}$$
$$S-qS = 1-q^{n+1} \qquad S=\frac{1-q^{n+1}}{1-q}$$

$$n\to\infty \quad \lim \frac{1-q^{n+1}}{1-q} = \frac{1}{1-q} \quad \boxed{q<1}$$

$$S = a_1+a_2+a_3+\dots a_n \qquad a_2=a_1+d \quad a_3=a_1+d=a_1+2d$$
$$a_n=a_1+(n-1)d$$

$$S = a_1+a_1+d+a_1+2d+\dots a_1+(n-1)d =$$
$$= a_1+a_1+d+a_2+d+\dots a_{n-1}+d$$
$$= \underbrace{a_1+a_2+\dots a_{n-1}+a_n}_{S}+(n-1)d-a_n+a_1 = S+(n-1)d-a_n+a_1$$
$$\cancel{S}=\cancel{S}+a_1-a_n+(n-1)d \qquad a_n=a_1+(n-1)d$$

$$S = a_1+a_2+a_3+\dots a_n$$
$$S = a_n+a_{n-1}+a_{n-2}+\dots a_1$$
$$2S = (a_1+a_n)+(a_2+a_{n-1})+(a_3+a_{n-2})+(a_{n-1}+a_2)\dots+(a_n+a_1)$$
$$[a_1+a_1+(n-1)d]+[a_1+d+a_1+(n-2)d]+\dots+[a_1+(n-1)d+a_1]$$
$$=[2a_1+(n-1)d]+[2a_1+(n-1)d]+\dots+[2a_1+(n-1)d]$$

$$2S = n\cdot(2a_1+(n-1)d) \qquad S=a_1n+\frac{n(n-1)d}{2}$$

SUMA SE DOKAŽE $\boxed{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}}$

$$\sum_{i=0}^{\frac{q}{2}-1}(2i+1)^2=\frac{q^3}{6}-\frac{q}{6}$$

$2i-2+1=2i-1$

$2\left(\frac{q}{2}-1\right)+1 = q-2+1 = (q-1)$

$$P_{Nq}=\frac{\Delta M^2}{12}$$
$$P_{Sq}=\frac{q^2\Delta M^2}{12}$$

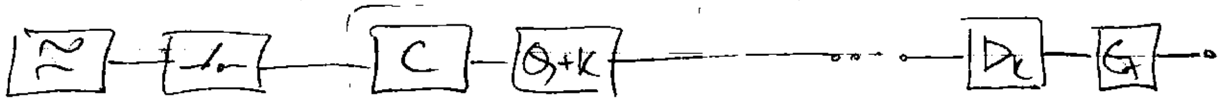
$$P_q=\frac{\Delta M^2}{12}\left[q^2-1\right] \qquad P_q=P_S-P_{Nq}$$

$$A_{Nq}=S/N=\frac{P_q}{P_{Nq}}=\frac{\frac{\Delta M^2}{12}[q^2-1]}{\frac{\Delta M^2}{12}}=q^2-1\doteq q^2$$
$$10\log A_{Nq}=20\log q$$

$q = 2^8 = 256 \qquad 20 \log q = 48{,}16 \text{ db}$

- NELINEARNA KVANTIZACIJA

$p(m) \neq const \qquad \Delta m \neq const \qquad$ KOMPRESIJA



$x_i + 1 \qquad Q(\Delta m \neq const)$

$$\overline{m_{Ni}^2} = \int_{x_i}^{x_i+1} (x - y_i)^2 p(x)\, dx \qquad x_i = \bar{x}_i - \frac{\Delta x_i}{2} \quad x_{i+1} = \bar{x}_i + \frac{\Delta x_i}{2}$$

$$q \uparrow\uparrow \Rightarrow \Delta x_i = \Delta x \downarrow\downarrow$$

$$p(x) \doteq p(\bar{x}_i) \qquad \bar{x}_i = m q_i = y_i$$

$$\overline{m_{Ni}^2} = p(\bar{x}_i) \int_{\bar{x}_i - \frac{\Delta x_i}{2}}^{\bar{x}_i + \frac{\Delta x_i}{2}} (x - y_i)^2\, dx = p(\bar{x}_i) \frac{1}{3}\left[\left(\bar{x}_i + \frac{\Delta x_i}{2} - y_i\right)^3 - \left(\bar{x}_i - \frac{\Delta x_i}{2} - y_i\right)^3\right]$$

$$\overline{m_{Ni}^2} = p(\bar{x}_i) \cdot \frac{1}{3}\left[2\frac{\Delta x_i^3}{8\,4}\right] = \frac{1}{12} p(\bar{x}_i) \frac{\Delta x_i^3}{12}$$

$$\Delta x_i = \frac{\Delta m_i}{F'(x)} \qquad \overline{m_{Ni}^2} = \frac{1}{12} p(\bar{x}_i) \frac{\Delta m_i^2}{[F'(x)]^2} \cdot \Delta x_i = \frac{1}{12} \frac{\Delta m_i^2}{[F'(x)]^2} p(\bar{x}_i) \Delta x_i$$

$$\frac{1}{2} U = \frac{1}{2} X \qquad q = \frac{U}{\Delta m} \qquad \Delta m = \frac{U}{q}$$

$$\overline{m_{Ni}^2} = \frac{1}{12} \frac{U^2}{q^2} p(\bar{x}_i) \cdot \Delta x_i \cdot \frac{1}{[F'(x)]^2}$$

$$\overline{m_N^2} = \sum_{i=0}^{q-1} \overline{m_{Ni}^2} = \frac{1}{12} \frac{U^2}{q^2} \sum_{i=1}^{q-1} \frac{p(\bar{x}_i)\Delta x_i}{[F'(x)]^2}$$

$m = F(x) \Rightarrow$ CONTINUIRANA FUNKCIJA

$$\overline{m_N^2} = \frac{1}{12} \frac{U^2}{q^2} \int_{-U/2}^{U/2} \frac{p(x)}{[F'(x)]^2}\, dx$$

RAVNOM. KVANT. $F'(x) = 1$

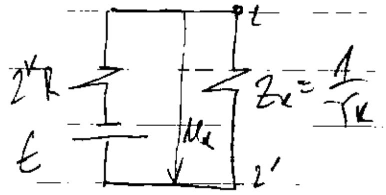
$$\overline{m_N^2} = \frac{1}{12} \frac{U^2}{q^2} = \frac{\Delta m^2}{12} \qquad \int_{-U/2}^{U/2} p(x) = 1$$

$$\overline{U_N^2} = \frac{1}{12} \frac{U^2}{q^2} \left[\int_{-U/2}^{U_7} \frac{p(x)\,dx}{(C_7)^2} + \int_{U_7}^{U_6} \frac{p(x)\,dx}{(C_6)^2} + \dots + \int_{U_6}^{U_7} \frac{p(x)\,dx}{(C_7)^2}\right]$$

- KODER I DEKODER

$$Y = \frac{1}{2^0 R} + \frac{1}{2^1 R} + \dots + \frac{1}{2^k R} + \dots + \frac{1}{2^n R} = \frac{1}{R}\left(2 - 2^{-n}\right)$$

$$S = \frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} = 2 - \frac{1}{2^n} = \left(2 - 2^{-n}\right) \qquad m' = \frac{1}{4} \cdot i$$



$$Y_k = \sum_{\substack{r=0 \\ r \neq k}}^{n-1} \frac{1}{2^r R} = Y - \frac{1}{2^k R}$$

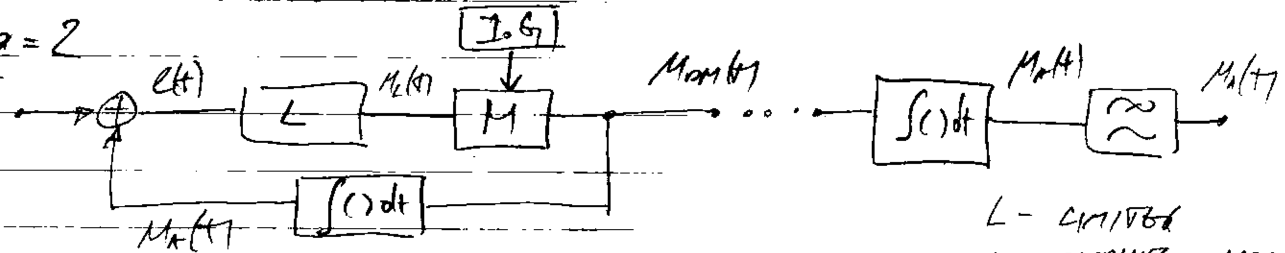
$$u_k = \frac{E}{2^k R + \frac{1}{Y_k}} \cdot \frac{1}{Y_k} = \frac{E}{2^k R\left(Y - \frac{1}{2^k R}\right) + 1} = \frac{E}{2^k R Y - 1 + 1} = \frac{E}{2^k R Y}$$

$$u = u' - u = \frac{i}{Y} - \frac{E}{2^k R Y} = \frac{1}{Y}\left(i - \frac{E}{2^k R}\right)$$

OPŠTI SLUČAJ: $u = \frac{1}{Y}\left(i - \frac{E}{2^k R} - \frac{E}{2^r R}\right)$

• DELTA MODULACIJA

$q = 2$



L – LIMITER
M – MODULATOR MOD

$$u_0(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s) \qquad T_s = \frac{1}{f_s}$$

$$u_L(t) = \Delta\, \mathrm{sgn}[u_1(t) - u_A(t)] = \begin{cases} +\Delta & e(t) > 0 \\ -\Delta & e(t) < 0 \end{cases}$$

$$e(t) = u_1(t) - u_A(t)$$

$$u_{DM}(t) = \sum_k \Delta\, \mathrm{sgn}[u_1(kT_s) - u_A(kT_s)]\, \delta(t - kT_s)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega\, e^{j\omega t} d\omega = \frac{j}{2\pi}\int_{-\infty}^{\infty} \omega\, e^{j\omega t} d\omega \qquad u = \omega \quad \sigma = \int e^{j\omega t} d\omega = \frac{e^{j\omega t}}{jt}$$

$$I = \left(\omega \cdot \frac{e^{j\omega t}}{jt} - \frac{1}{jt}\int e^{j\omega t} d\omega\right)\Big|_{-\infty}^{\infty} = \left(\omega \frac{e^{j\omega t}}{jt} - \frac{e^{j\omega t}}{(jt)^2}\right)\Big|_{-\infty}^{\infty}$$

$$= \left(\omega \frac{e^{j\omega t}}{jt} + \frac{e^{j\omega t}}{t^2}\right)\Big|_{-\infty}^{\infty} = \frac{e^{j\omega t}}{t}\left(\frac{1}{t} - j\omega\right)\Big|_{-x}^{x}$$

$$\lim_{x \to \infty}\left[\frac{e^{jxt}}{t}\left(\frac{1}{t} - jx\right) - \frac{e^{jxt}}{t}\left(\frac{1}{t} + jx\right)\right]$$

HEVISAJDOVA

$$u_A(t) = \sum_k \Delta\, \mathrm{sgn}[u_1(kT_s) - u_A(kT_s)]\, u_H(t - kT_s)$$

$$T_s << \Delta t \qquad \Delta t = \frac{1}{2 f_m}$$

GREŠKA KOJU PRAVI KVANTIZACIJA E: $e(t) = u_1(t) - u_A(t)$

$e(t) \downarrow$ AKO $T_s \downarrow$ I E $\Delta \downarrow$

• USLOV DA NE DOJDE DO STRMINSKO OPTOVARUVANJE

$$|m_m(t+T_s) - m_m(t)| \le \Delta \qquad \left|\frac{m_m(t+T_s)-m_m(t)}{T_s}\right| \le \frac{\Delta}{T_s} = f_s \cdot \Delta$$

$$\left|\frac{m_m(t+T_s)-m_m(t)}{T_s}\right| \le \left|\frac{dm_m(t)}{dt}\right|_{max} \qquad \boxed{\left|\frac{dm_m(t)}{dt}\right|_{max} \le \Delta \cdot f_s}$$

ZA DA SE IZBEGNE S.RM. OPTOV. TREBA ILI $\Delta \uparrow$ ILI $f_s \uparrow$

NEKA: $m_m(t) = U_m \cos(\omega_m t)$ $\qquad \left|\frac{dm_m(t)}{dt}\right|_{max} = U_m \omega_m$

$U_m \omega_m \le \Delta \cdot f_s$

• SLUČAEN SIGNAL $\qquad \boxed{2\pi f_m \cdot U_m \le \Delta \cdot f_s}$

• GRANULATEN ŠUM: $\quad e(t) = m_m(t) - m_a(t) \qquad |e(t)| \le \Delta$

$P_{NG} = C\Delta^2 \frac{f_c}{f_s}$ $\qquad f_c$ - GRANIČNA FRQ NA [Σ]

$p(e) = const \quad (-\Delta, \Delta) \qquad p(e) = \begin{cases} \frac{1}{2\Delta} & -\Delta < e(t) < \Delta \\ 0 & \text{NADVOR} \end{cases}$

$$\overline{e^2(t)} = \overline{e^2} = \int_{-\Delta}^{\Delta} e^2 p(e)\,de = \frac{1}{2\Delta}\int_{-\Delta}^{\Delta} e^2 de = \frac{1}{2\Delta}\frac{e^3}{3}\Big|_{-\Delta}^{\Delta} = \frac{1}{\Delta}\frac{\Delta^3}{3} = \frac{\Delta^2}{3}$$

$\boxed{P_{NG} = \frac{1}{3}\Delta^2 \frac{f_c}{f_s}}$ $\quad \Delta\downarrow \Rightarrow P_{NG}\downarrow$ NO MOŽE DA NASTANE STRMINSKO PREOPTOVARUVANJE

• **ADAPTIVNA Δ MODULACIJA (ADPCM)**

M - EN SIGNAL $\qquad f_b = f_d \cdot \log M$

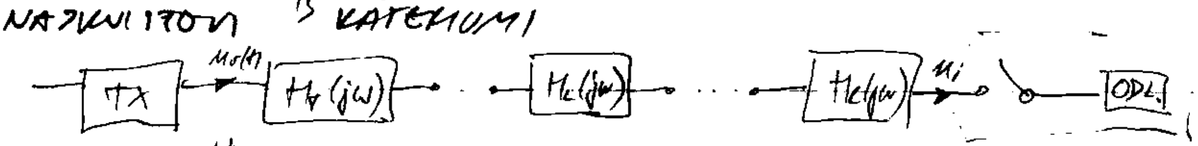
$M = 4 \qquad \boxed{f_b = 2 \cdot f_o} \qquad$ 2 bit / SIMBOL

• $U_x(f_n) = U_x \cdot \tau \qquad y(t) = 2f_c \cdot E\tau \cdot A \cdot \frac{\sin 2\pi f_c(t-t_0)}{2\pi f_c(t-t_0)}$

$2\pi f_c(t-t_0) = \pi \quad \boxed{t = t_0 + \frac{1}{2f_c}} \qquad 2\pi f_c \cdot T_s = \pi \quad \boxed{T_s = \frac{1}{2f_c}}$ (*)

(*) NAJKVISTOV KRITERIUM ZA IDEALEN SP PROTOKOL $\quad \frac{1}{T_s} = f_s \Rightarrow$ NAJKVISTOVA BRZINA

• **NAJKVISTOVI KRITERIUMI**



$m_o(t) = \sum_{k=-N}^{N} a_k x(t-kT)$; $\quad a_k$: $S_1, S_2, \ldots S_M \qquad$ BINAREN $a_k: 0, 1$

$x(t)$ - STANDARDEN SIGNAL $\quad (\delta(t)$ ILI ⊓ $)$

$U_o(j\omega) = \sum_{k=-N}^{N} a_k X(j\omega) e^{-j\omega kT} \qquad U_i(j\omega) = H(j\omega) \cdot X(j\omega)$

$$H(j\omega) = H_T(j\omega)\cdot H_L(j\omega)\cdot H_R(j\omega) \qquad Y_i(j\omega) = \sum_{k=-N}^{N} a_k H(j\omega) X(j\omega) e^{-j\omega kT}$$

$$u_i(t) = \sum_{k=-N}^{N} a_k\, y(t-kT) \qquad y(t-kT) = \mathcal{F}^{-1}\{Y(j\omega)\}$$

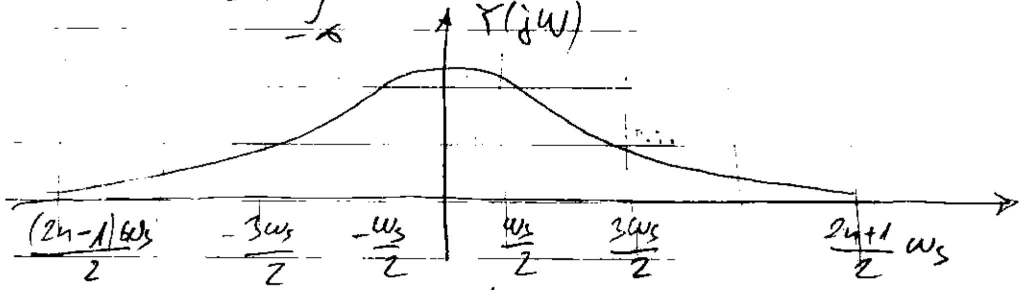
$$\boxed{y(nT) = y_0\,\delta_{n,0}\,, \quad \delta_{i,j} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}}$$

$\delta_{i,j}$ – KRONEKEROVA KONSTANTA (DELTA FUNKCIJA)

$i=j \Rightarrow$ ZA SOPSTVENITE SIGNALIZACIONI INTERV. $\quad i\neq j$ ZA DRUGITE

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)X(j\omega)\, e^{j\omega t}\, d\omega \qquad y(nT) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{H(j\omega)X(j\omega)}_{Y(j\omega)} e^{j\omega nT}\, d\omega$$

$$y(nT) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j\omega)\, e^{j\omega nT}\, d\omega = y_0\,\delta_{n,0}$$



$$y(nT) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\int_{\frac{(2n-1)\omega_s}{2}}^{(2n+1)\omega_s/2} Y(j\omega)\, e^{j\omega nT}\, d\omega = \left|\begin{array}{l} \omega = \gamma + n\omega_s \\ d\omega = d\gamma \\ \omega = (2n+1)\frac{\omega_s}{2} \quad \gamma = n\omega_s + \frac{\omega_s}{2} - n\omega_s \\ \omega = (2n-1)\frac{\omega_s}{2} \quad \gamma = n\omega_s - \frac{\omega_s}{2} - n\omega_s \end{array}\right.$$

$$y(nT) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\int_{-\omega_s/2}^{\omega_s/2} Y(j\gamma + n\omega_s)\, e^{j\gamma\cdot m\cdot T}\cdot e^{jn\cdot m\cdot \omega_s\cdot T}\, d\gamma$$

$$T\cdot\omega_s = T\,\frac{2\pi}{T} = 2\pi \qquad \boxed{T = \frac{2\pi}{\omega_s} \quad \omega_s = \frac{2\pi}{T}}$$

$$y(nT) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\int_{-\omega_s/2}^{\omega_s/2} Y(j\gamma + n\omega_s)\, e^{j\gamma nT}\, \underbrace{e^{j2\pi n m}}_{=1}\, d\gamma$$

$$y(nT) = \frac{1}{2\pi}\int_{-\omega_s/2}^{\omega_s/2} \sum_{n=-\infty}^{\infty} Y(j\gamma + n\omega_s)\, e^{j\gamma n\frac{2\pi}{\omega_s}}\, d\gamma$$

$$y(nT) = \frac{1}{2\pi}\int_{-\omega_s/2}^{\omega_s/2} e^{j\gamma n\frac{2\pi}{\omega_s}} \left(\sum_{n=-\infty}^{\infty} Y(j\gamma + n\omega_s)\right) d\gamma \qquad \gamma = \omega$$

$$\boxed{y(nT) = \frac{1}{2\pi}\int_{-\omega_s/2}^{\omega_s/2} e^{j\omega n\frac{2\pi}{\omega_s}} \left(\sum_{n=-\infty}^{\infty} Y(j\omega + n\omega_s)\right) d\omega = y_0\cdot\delta_{n,0}}$$

$$y(0) = y_0 \qquad \delta_{0,0} = 1 \quad \delta_{1,0} = 0$$

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$$\sum_{n=-\infty}^{\infty} Y\left[j(\omega + n\omega_s)\right] = K = \frac{2\pi}{\omega_s}\cdot Y_0 = T\cdot Y_0 \qquad |\omega| \le \frac{\omega_s}{2}$$

DRAGI 076605167
P. SVET.

DOKAZ: $y(nT) = \frac{1}{2\pi}\int_{-\omega_s/2}^{\omega_s/2} \frac{2\pi}{\omega_s} Y_0\, e^{jn\frac{2\pi}{\omega_s}\omega}\, d\omega$

$$\int_{-\omega_s/2}^{\omega_s/2} e^{jn\omega\frac{2\pi}{\omega_s}}\, d\omega = \frac{e^{jn\omega\frac{2\pi}{\omega_s}}}{jn\frac{2\pi}{\omega_s}}\Bigg|_{-\omega_s/2}^{\omega_s/2} \quad 2\cdot\frac{1}{2}\left(\frac{e^{jn\frac{\omega_s}{2}\frac{2\pi}{\omega_s}} - e^{-jn\frac{\omega_s}{2}\frac{2\pi}{\omega_s}}}{n\frac{2\pi}{\omega_s}}\right)$$

$$= \frac{2}{n\frac{2\pi}{\omega_s}}\cdot \sin n\pi$$

$$y(nT) = \frac{1}{2\pi}\cdot\frac{2\pi}{\omega_s}\cdot Y_0\cdot\frac{2}{n\cdot\frac{2\pi}{\omega_s}}\cdot\sin(n\pi) = Y_0\cdot\frac{\sin n\pi}{n\pi} = Y_0\cdot\delta_{n,0}$$

$n = 0 \quad \boxed{y(nT) = Y_0}$ $\qquad \boxed{\delta_{0,0} = 1 \quad n\neq 0\ \ \delta_{n,0} = 0}$

- Neka: $x(t) = \delta(t)$ $\qquad H(j\omega) = A(\omega)\cdot e^{+j\chi(\omega)} = A(\omega)\left[\cos\chi(\omega) + j\sin\chi(\omega)\right]$

$$\boxed{\sum_{n=-\infty}^{\infty} H\left[j(\omega + n\omega_s)\right] = K \qquad |\omega| \le \frac{\omega_s}{2}}$$

I NYQUIST CRITERIA

1. $\sum_{n=-\infty}^{\infty} A(\omega + n\omega_s)\cos(\chi(\omega) + n\omega_s) = K \qquad |\omega| \le \frac{\omega_s}{2}$

2. $\sum_{n=-\infty}^{\infty} A(\omega + n\omega_s)\sin(\chi(\omega) + n\omega_s) = 0 \qquad |\omega| \le \frac{\omega_s}{2}$

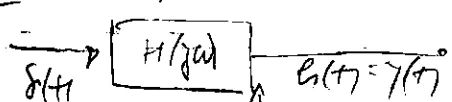
$2\cdot\frac{3\omega_s}{2} = 3\omega_s$ $\qquad x\cdot\pi\cdot 60 = \frac{\pi}{2} \quad x = \frac{1}{120} = \frac{1}{3\cdot 40}$

$x\cdot\pi\cdot\frac{N}{2} = \frac{\pi}{2} \Rightarrow \boxed{x = \frac{1}{N}}$ $\qquad f_s = 2f_m$

$$\boxed{\sum_{n=-\infty}^{\infty} A[\omega + n\omega_s]\cos[\chi(\omega + n\omega_s)] = K \qquad |\omega| \le \frac{\omega_s}{2}}$$

$\left(\omega_{g\,min} = \frac{\omega_s}{2}\right) \rightarrow$ NO TADA BI SE KORISTIO ISP (IDEALAN SISTEM ZA KENJOS)

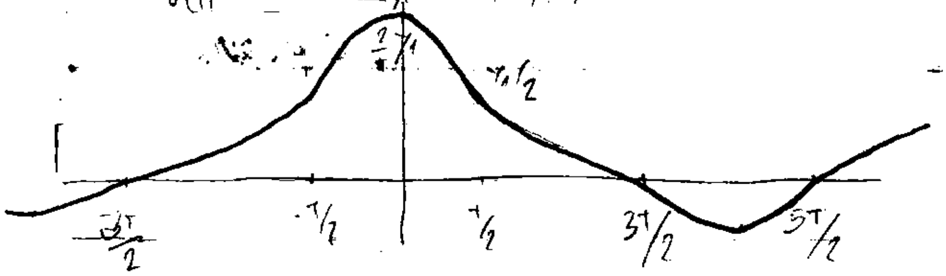
- **II NYQUIST KRITERIUM**



$$h(t) = 2\pi\cdot f_m\,\frac{\sin 2\pi f_s t}{2\pi f_s}$$

$t = \frac{3T}{2}$ $\quad 2\pi f_s\frac{3T}{2} = 2\pi\cdot\frac{1}{T}\cdot\frac{3T}{2} = 3\pi$

$t = \frac{T}{2}$ $\quad 2\pi f_s t = 2\pi\cdot\frac{1}{T}\cdot\frac{T}{2} = \pi$



$$y\left[(2n-1)\frac{T}{2}\right] = \frac{y_1}{2}\left[\delta_{n,0}+\delta_{n,1}\right]$$

$n=0$ $\quad y\left[-\frac{T}{2}\right] = \frac{y_1}{2}\left[\delta_{n,0}+\delta_{n,1}\right] = \frac{y_1}{2}$

$n=1$ $\quad y\left[\frac{T}{2}\right] = \frac{y_1}{2}\left[\delta_{n,0}+\delta_{n,1}\right] = \frac{y_1}{2}$

$n=\pm 2, \pm 3, \ldots \quad y\left[(2n-1)\frac{T}{2}\right] = 0$

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j\omega)e^{j\omega t}d\omega \; ; \quad t=(2n-1)\frac{T}{2}$$

$$y\left[(2n-1)\frac{T}{2}\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j\omega)e^{j\omega(2n-1)\frac{T}{2}}d\omega = \left|\begin{matrix}\omega_s=\frac{2\pi}{T}\\ T=\frac{2\pi}{\omega_s}\\ f_s=\frac{1}{T_s}\end{matrix}\right| =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j\omega)e^{j\omega(2n-1)\frac{2\pi}{2\omega_s}}d\omega \qquad Y(j\omega) = X(j\omega)\cdot H(j\omega)$$

$$y\left[(2n-1)\frac{T}{2}\right] = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{(2n-1)\frac{\omega_s}{2}}^{(2n+1)\omega_s/2} Y(j\omega)e^{j\omega(2n-1)\frac{\pi}{\omega_s}}d\omega = \left|\begin{matrix}\nu+n\omega_s=\omega\\ d\omega=d\nu\\ \omega=n\omega_s+\frac{\omega_s}{2} & \nu=\frac{\omega_s}{2}\\ \omega=n\omega_s-\frac{\omega_s}{2} & \nu=-\frac{\omega_s}{2}\end{matrix}\right|$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{-\omega_s/2}^{\omega_s/2} Y[j(\nu+n\omega_s)]e^{j\nu\frac{\pi}{\omega_s}(2m-1)}\cdot e^{jn\pi(2m-1)}d\nu = \left|\nu=\omega\right|$$

$$j\omega(2m-1)\frac{\pi}{\omega_s} = j(\nu+n\omega_s)(2m-1)\frac{\pi}{\omega_s} = j\nu\frac{\pi}{\omega_s}(2m-1) + jn\omega_s(2m-1)\frac{\pi}{\omega_s}$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{-\omega_s/2}^{\omega_s/2} Y[j(\omega+n\omega_s)]e^{j\pi\frac{2m-1}{\omega_s}\omega}\cdot \underbrace{e^{j2mn\pi}}_{1}\cdot\underbrace{e^{-jn\pi}}_{(-1)^n}d\omega$$

$$y\left[(2m-1)\frac{T}{2}\right] = \frac{1}{2\pi}\int_{-\omega_s/2}^{\omega_s/2} e^{j\pi\frac{2m-1}{\omega_s}\omega}\underbrace{\sum_{n=-\infty}^{\infty}(-1)^n Y[j(\omega+n\omega_s)]}_{\$}d\omega$$

Za da: $y\left[(2m-1)\frac{T}{2}\right] = \frac{y_1}{2}\left[\delta_{m,0}+\delta_{m,1}\right]$ $\$$ treba da bude:

$$\boxed{\sum_{n=-\infty}^{\infty}(-1)^n Y[j(\omega+n\omega_s)] = \frac{2\pi}{\omega_s}y_1\cos\frac{\pi\omega}{\omega_s} = y_1\cdot T\cos\frac{\pi\omega}{\omega_s} \quad |\omega|\le\frac{\omega_s}{2}}$$

II NYQUIST

$$y\left[(2m-1)\frac{T}{2}\right] = \frac{1}{2\pi}\int_{-\omega_s/2}^{\omega_s/2} y_1 T\cos\frac{\pi\omega}{\omega_s}\,e^{j\pi\frac{2m-1}{\omega_s}\omega}d\omega = \frac{y_1}{2}\left[\delta_{m,0}+\delta_{m,1}\right]$$

$x(t) \to \delta(t) \qquad Y(j\omega) = H(j\omega)$

$$\sum_{n=-\infty}^{\infty} (-1)^n H[j(\omega+n\omega_s)] = \gamma_1 T \cos\frac{\pi\omega}{\omega_s} = K\cos\frac{\pi\omega}{\omega_s} \qquad |\omega| \le \frac{\omega_s}{2}$$

1) $\sum_{n=-\infty}^{\infty} (-1)^n A(\omega+n\omega_s)\cdot\cos\chi(\omega+n\omega_s) = K\cos\frac{\pi\omega}{\omega_s} \qquad |\omega|\le\frac{\omega_s}{2}$

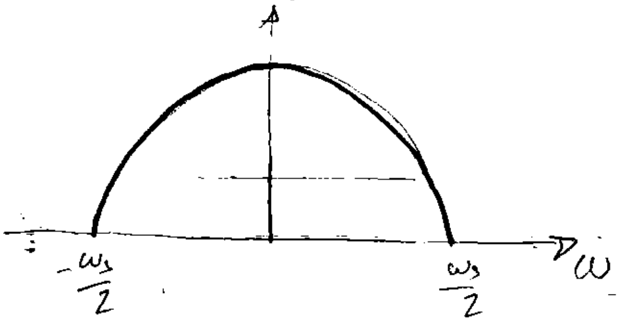
2) $\sum_{n=-\infty}^{\infty} (-1)^n \cdot A(\omega+n\omega_s)\sin\chi(\omega+n\omega_s) = 0 \qquad |\omega|\le\frac{\omega_s}{2}$

• Najmanji moguć opseg za $n=0$:

$A(\omega)\cdot\cos\chi(\omega) = K\cdot\cos\frac{\pi\omega}{\omega_s} \qquad |\omega|\le\frac{\omega_s}{2}$

$A(\omega)\sin\chi(\omega) = 0 \qquad\qquad T = \frac{1}{f_s} = \frac{2\pi}{\omega_s}$

$$A(\omega) = \begin{cases} \gamma_1 T\cdot\cos\frac{\pi\omega}{\omega_s} & |\omega|\le\frac{\omega_s}{2} \\ 0 & |\omega| > \frac{\omega_s}{2}\end{cases}$$



$f_{c\,min} = \frac{\omega_s}{2}$

$$y(t) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} \gamma_1 T\cos\frac{\pi\omega}{\omega_s} e^{j\omega t} d\omega = \frac{2}{\pi}\gamma_1 \frac{\cos\omega_c t}{1-\left(\frac{\omega_s t}{\pi}\right)^2}$$

$\omega_s = 2\omega_c \qquad \omega_c = \frac{1}{2}\omega_s$

$$(*) = \frac{1}{2}\int_{-\omega_c}^{\omega_c}\left(e^{j\frac{\pi\omega}{\omega_s}} + e^{-j\frac{\pi\omega}{\omega_s}}\right)e^{j\omega t}d\omega = \frac{1}{2}\int_{-\omega_c}^{\omega_c} e^{j\left(\frac{\pi}{\omega_s}+t\right)\omega}d\omega + \frac{1}{2}\int_{-\omega_c}^{\omega_c} e^{-j\left(\frac{\pi}{\omega_s}-t\right)\omega}d\omega$$

$$= \frac{1}{2}\,\frac{e^{j\left(\frac{\pi}{\omega_s}+t\right)\omega_c} - e^{-j\left(\frac{\pi}{\omega_s}+t\right)\omega_c}}{j\left(\frac{\pi}{\omega_s}+t\right)} + \frac{1}{2}\,\frac{e^{-j\left(\frac{\pi}{\omega_s}-t\right)\omega_c} - e^{j\left(\frac{\pi}{\omega_s}-t\right)\omega_c}}{-j\left(\frac{\pi}{\omega_c}-t\right)} =$$

$$= \frac{\sin\left(\frac{\pi}{\omega_s}+t\right)\omega_c}{\frac{\pi}{\omega_s}+t} + \frac{\sin\left(\frac{\pi}{\omega_s}-t\right)\omega_c}{\frac{\pi}{\omega_s}-t} = \frac{\left(\frac{\pi}{\omega_s}-t\right)\sin\left(\frac{\pi}{\omega_s}+t\right)\omega_c + \left(\frac{\pi}{\omega_s}+t\right)\sin\left(\frac{\pi}{\omega_s}-t\right)\omega_c}{\left(\frac{\pi}{\omega_s}\right)^2 - t^2}$$

$\nabla\cos(\omega_c t) \qquad \nabla\cos(\omega_c t)$

$$= \frac{\left(\frac{\pi}{2\omega_c}-t\right)\sin\left(\frac{\pi}{2}+\omega_c t\right) + \left(\frac{\pi}{2\omega_c}+t\right)\sin\left(\frac{\pi}{2}-t\omega_c\right)}{\left(\frac{\pi}{\omega_s}\right)^2\left[1-\left(\frac{t\omega_s}{\pi}\right)^2\right]} = \frac{\frac{2\pi}{2\omega_c}\cdot\cos\omega_c t}{\left(\frac{\pi}{2\omega_c}\right)^2\left[1-\left(\frac{2t\omega_c}{\pi}\right)^2\right]}$$

$$= \frac{4\omega_c^2}{\pi^2}\cdot\frac{\pi}{\omega_c}\cdot\frac{\cos\omega_c t}{1-\left(\frac{t\omega_s}{\pi}\right)^2} = \frac{4\omega_c}{\pi}\,\frac{\cos\omega_c t}{1-\left(\frac{t\omega_s}{\pi}\right)^2}$$

U ... uvodimo:

$$y(t) = \frac{1}{2\pi}\cdot\gamma_1 T\cdot\frac{4\omega_c}{\pi}\,\frac{\cos\omega_c t}{1-\left(\frac{t\omega_s}{\pi}\right)^2} = \frac{1}{\pi^2}\cdot\gamma_1\frac{2\pi}{\omega_s}\cdot\frac{2\cdot\omega_s}{2}\,\frac{\cos\omega_c t}{1-\left(\frac{t\omega_s}{\pi}\right)^2} = \frac{2\gamma_1}{\pi}\,\frac{\cos\omega_c t}{1-\left(\frac{t\omega_s}{\pi}\right)^2}$$

$$y(t) = \frac{2}{\pi} y_1 \frac{\cos \omega_c t}{1 - \left(\frac{2\omega_c t}{\pi}\right)^2} \qquad \omega_c = \frac{1}{2}\omega_s$$

$$\omega_c = \frac{\omega_s}{2} = \frac{2\pi}{2T} = \frac{\pi}{T}$$

$$t = \frac{T}{2} \qquad y\left(\frac{T}{2}\right) = \frac{2}{\pi} y_1 \frac{\cos \frac{\pi}{T}\cdot\frac{T}{2}}{1 - \left(\frac{2}{\pi}\cdot\frac{\pi}{T}\cdot\frac{T}{2}\right)^2} = \frac{2}{\pi} y_1 \frac{0}{0}$$

$$\lim_{t \to \frac{T}{2}} \frac{\cos \frac{\pi}{T} t}{1 - \left(\frac{2}{T} t\right)^2} = \lim_{t \to \frac{T}{2}} \frac{-\left(\sin \frac{\pi}{T} \cdot t\right)\frac{\pi}{T}}{-2\left(\frac{2}{T}t\right)\cdot\frac{2}{T}}$$

$$= \lim_{t \to \frac{T}{2}} \frac{\pi \sin\left(\frac{\pi}{T}t\right)}{\frac{8t}{T}} = \frac{\pi \sin \frac{\pi}{2}}{\frac{8}{T}\cdot\frac{T}{2}} = \frac{\pi}{4}$$

$$y\left(\frac{T}{2}\right) = \frac{2}{\pi} \cdot y_1 \cdot \frac{\pi}{4} = \frac{y_1}{2} \qquad \sin \frac{\pi}{T} \cdot \frac{5T}{2} = \frac{5\pi}{2}$$

- S.P. KOJ U ISPOLNUVA I, I, II E PODIGNAT KOSINUS:

$$A(\omega) = K \begin{cases} \frac{1}{2} + \frac{1}{2}\cos\frac{\pi\omega}{\omega_s} = \cos^2 \frac{\pi\omega}{2\omega_s}, & |\omega| \le \omega_s \\ 0 & |\omega| > \omega_s \end{cases}$$

MMV

$$\frac{1}{2} + \frac{1}{2}\left(\cos\left(\frac{\pi\omega}{2\omega_s} + \frac{\pi\omega}{2\omega_s}\right)\right) = \frac{1}{2} + \frac{1}{2}\cos\frac{\pi\omega}{2\omega_s}\cdot\cos\frac{\pi\omega}{2\omega_s} - \frac{1}{2}\sin\frac{\pi\omega}{2\omega_s}\sin\frac{\pi\omega}{2\omega_s}$$

$$= \frac{1}{2}\left(1 - \sin^2\frac{\pi\omega}{2\omega_s}\right) + \frac{1}{2}\cos^2\frac{\pi\omega}{2\omega_s} = \cos^2\frac{\pi\omega}{2\omega_s}$$

$$y = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c}\left(\frac{1}{2} + \frac{1}{2}\cos\frac{\pi\omega}{\omega_s}\right)e^{j\omega t}d\omega = \underbrace{\frac{1}{2\pi}\frac{1}{2}\int_{-\omega_c}^{\omega_c} e^{j\omega t}d\omega}_{①} + \underbrace{\frac{1}{2}\frac{1}{2\pi}\int_{-\omega_c}^{\omega_c}\cos\frac{\pi\omega}{\omega_s}e^{j\omega t}dt}_{②}$$

$$② = \frac{1}{4\pi}\frac{4\omega_c}{\pi}\frac{\cos\omega_c t}{1 - \left(\frac{2t\omega_c}{\pi}\right)^2} \qquad ① = \frac{1}{4\pi}\frac{1}{jt}\left(e^{j\omega_c t} - e^{-j\omega_c t}\right) = \frac{1}{2\pi t}\sin\omega_c t$$

$$② + ① = \frac{\omega_c}{\pi^2}\frac{\cos\omega_c t}{1 - \left(\frac{2t\omega_c}{\pi}\right)^2} + \frac{\sin\omega_c t}{2\pi t} = \frac{\frac{\omega_c}{\pi^2}2\pi t\cos\omega_c t + \sin\omega_c t - \left(\frac{2t\omega_c}{\pi}\right)^2\sin(\omega_c t)}{\left[1 - \left(\frac{2t\omega_c}{\pi}\right)^2\right]2\pi t}$$

$$= \frac{\frac{2\omega_c}{\pi}t\cos\omega_c t + \sin\omega_c t - \left(\frac{2t\omega_c}{\pi}\right)^2\sin(\omega_c t)}{\left[1 - \left(\frac{2T\omega_c}{\pi}\right)^2\right]2\pi t}$$

? NE ZNAM KAKO SE REŠAVA!!!

$$y(t) = \frac{K}{T}\;\frac{1}{1 - \left(\frac{2\omega_c t}{\pi}\right)^2}\;①\quad\frac{\sin(\omega_c t)}{2\omega_c t}\;②$$

① ISPOLNUVA II NC

② ISPOLNUVA I NC

• TRANSVERZALEN FILTER:

$$y(t) = A_l \cdot x(t) + A_{l+1} x(t-T) + \dots + A_0 x(t-lT) + \dots + A_l x(t-2lT)$$

$$y(t) = \sum_{k=-l}^{l} A_k x[t-(k+l)T] = e^{-j\omega lT} \sum_{k=-l}^{l} A_k X(j\omega) e^{-j\omega kT}$$

$$H_k(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = e^{-j\omega lT} \sum_{k=-l}^{l} A_k e^{-j\omega kT}$$

$$f_s = \frac{1}{T_s} = \frac{f_b}{ld M}$$

$$\frac{2\omega_c t}{\pi}\left(\cos\omega_c t - \frac{2\omega_c t}{\pi}\sin\omega_c t\right) + \sin\omega_c t$$

$$\frac{2\omega_c t}{\pi}\cos(2\omega_c t - \omega_c t) = \frac{2\omega_c t}{\pi}[\cos 2\omega_c t \cdot \cos\omega_c t + \sin 2\omega_c t \cdot \sin\omega_c t]$$

$$\sin(2\omega_c t) = 2\sin\omega_c t \cdot \cos\omega_c t$$

$$\frac{2\omega_c t}{\pi} \cdot (1 \cdot \sin\omega_c t) + 2\sin 2\omega_c t - 2\left(\frac{2t\,\omega_c}{\pi}\right)^2 \sin 2\omega_c t$$

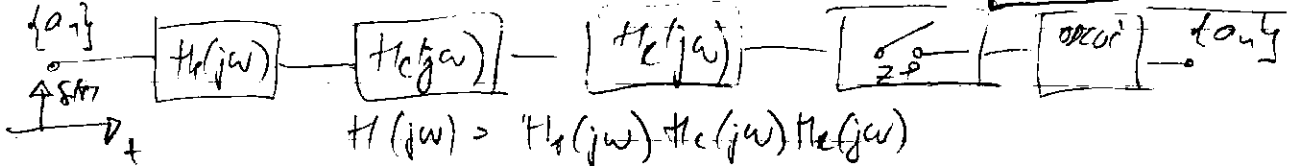
DIGITALNI TK   POVTORUVANJE

• PRENOS VO OSNOVEN OPSEG:

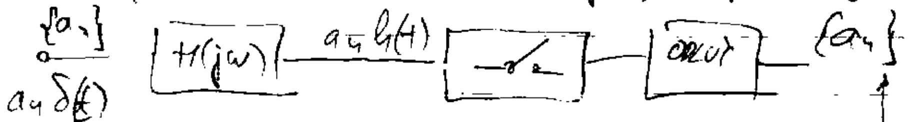
$v_d = \frac{1}{T_d}$ [bd] DIGITALNA BRZINA $1\,bd = \frac{1\,DIGIT}{s}$

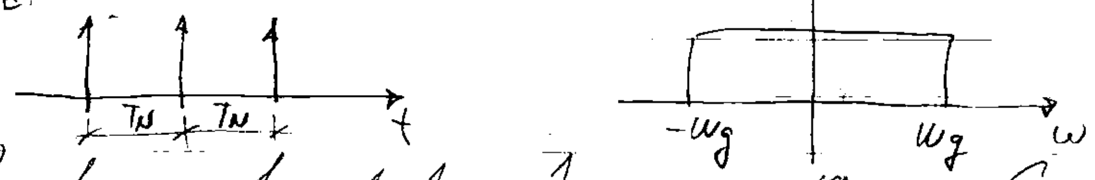
$v_b = v_d \cdot ld M$ $M = 2^n$

- DODAR S/N TREBA MALA $P(e)$ $P(\varepsilon)_{min} = 10^{-7}$



$$H(j\omega) = H_1(j\omega) H_c(j\omega) H_e(j\omega)$$





$2f_g = f_s$ $f_g = \frac{1}{2}\frac{1}{T_N} = \frac{1}{2T_N}$ I NYQIST CRITERIA

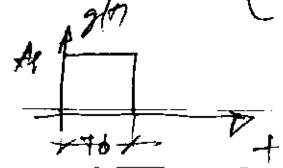
• ~~[illegible]~~ IZOBLIČUVANJA

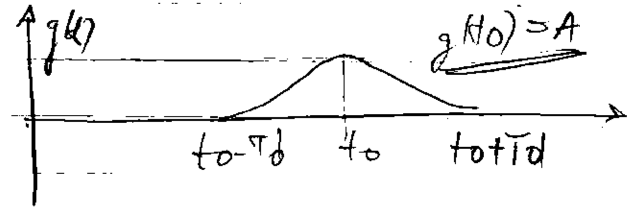
$$g_p(t) = \begin{cases} A_p & |t| \le \tau_0 \\ 0 & NADVOR \end{cases} \Rightarrow \text{TESEN PRAVOAGOLEN IMPULS}$$

• Пред колото за земање примероци овој П • може • да е тесна како:

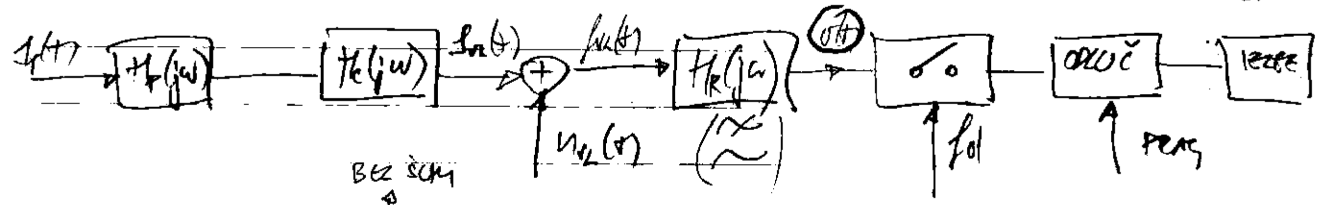
$$g(t) = \begin{cases} \text{произволен облик} & t_0 - T_d \le t \le t_0 + T_d \\ 0 & \text{надвор} \end{cases}$$

$g(t_0) = A$





• Детекција за сигналите пренесени во основен опсег



без шум

$r_{vl}(t) = s_{vl}(t) + n_{vl}(t)$ $v(t) = s(t) + n(t)$ — бел шум но ограничен од влезниот филтер

$s_p(t) = \sum_n a_n g_p(t - nT_d)$ $a_n \in \{0,1\}$ за униполарен бинарен сигнал

$$g_p(t) = \begin{cases} A_p; & 0 \le t \le T_D \\ 0; & \text{надвор} \end{cases}$$

$s(t) = \sum_n a_n g(t - nT_d)$ $g(t) = \begin{cases} \text{произволен} & t_0 - T_d \le t \le t_0 + T_d \\ 0 & \text{надвор} \end{cases}$ $g(t_0) = A$

• $P(E) = ?$ за униполарен бинарен сигнал

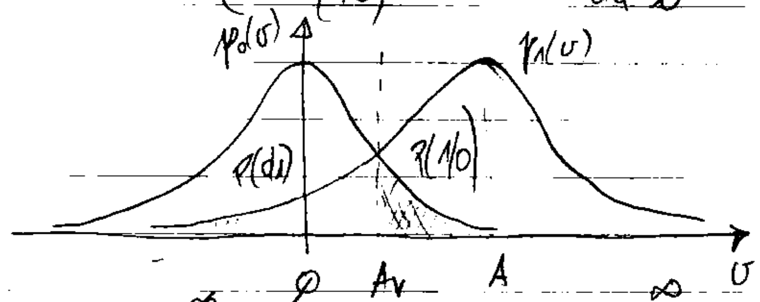
$M = 2$ $a_n \in \{0,1\}$ $s(t_0) = A$, $a_n = 1$ $s(t_0) = 0$, $a_n = 0$

$P(E) = P(0)\,P(1/0) + P(1) \cdot P(0/1)$ $P(x_i/y_i) = \dfrac{P(x_i, y_i)}{P(y_i)}$

$v(t) = s(t) + n(t)$

$$v(t_0) = \begin{cases} A + n(t_0) & a_n = 1 \\ n(t_0) & a_n = 0 \end{cases}$$

$A_r = \dfrac{A}{2}$ $P(0) = P(1)$



$$P(1/0) = \int_{A_r}^{\infty} p_0(v)\,dv = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{A_r}^{\infty} e^{-\frac{v^2}{2\sigma_n^2}}\,dv$$

$t = \dfrac{v}{\sqrt{2\sigma_n^2}}$ $v = A_r$ $t = \dfrac{A_r}{\sqrt{2\sigma_n^2}}$ $dt = \dfrac{dv}{\sqrt{2\sigma_n^2}}$

$$P(1/0) = \frac{\sqrt{2\sigma_n^2}}{\sqrt{2\pi\sigma^2}} \int_{A_r/\sqrt{2\sigma_n^2}}^{\infty} e^{-t^2}\,dt = \frac{1}{\sqrt{\pi}}\left[\int_0^{\infty} e^{-t^2}\,dt - \frac{2}{2}\int_0^{\frac{A_r}{\sqrt{2\sigma_n^2}}} e^{-t^2}\,dt\right]$$

($\int_0^{\infty} e^{-t^2}dt \to \frac{\sqrt{\pi}}{2}$)

$$\underline{P(1/0)} = \frac{1}{2}\left[1 - \frac{2}{\sqrt{\pi}}\int_0^{A_r/\sqrt{2\sigma_N^2}} e^{-t^2} dt\right] = \frac{1}{2}\left[1 - \Phi\left(\frac{A_r}{\sqrt{2\sigma_N^2}}\right)\right]$$

Wavelets and other orthogonal Systems, Walter, Shen

$$P(0/1) = \int_{-\infty}^{A_r} p_1(v)\, dv = \int_{-\infty}^{A_r} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(v-A)^2}{2\sigma_N^2}} dv \qquad \frac{v-A}{\sqrt{2\sigma_N^2}} = t$$

$$\frac{dv}{\sqrt{2\sigma_N^2}} = dt \quad dv = \sqrt{2\sigma_N^2}\, dt$$

$v = -\infty \quad t = -\infty \qquad v = A_r \quad t = \frac{A_r - A}{\sqrt{2\sigma_N^2}}$

$$P(0/1) = \frac{\sqrt{2\sigma_N^2}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\frac{A_r-A}{\sqrt{2\sigma_N^2}}} e^{-t^2} dt = \frac{1}{\sqrt{\pi}}\left[\int_{-\infty}^{0} e^{-t^2} dt + \frac{2}{2}\int_0^{\frac{A_r-A}{\sqrt{2\sigma_N^2}}} e^{-t^2} dt\right]$$

$$\underline{P(0/1)} = \frac{1}{2}\left[1 + \Phi\left(\frac{A_r - A}{\sqrt{2\sigma_N^2}}\right)\right] = \frac{1}{2}\left[1 - \Phi\left(\frac{A - A_r}{\sqrt{2\sigma_N^2}}\right)\right]$$

$$\hat{\rho}^2 = \frac{A^2}{\sigma_N^2} \qquad b_0 = \frac{A_r}{\sigma_N} \qquad \hat{P}_S^2 = A^2 \quad P_N = \sigma_N^2$$

→ srednja snaga na ulazu

→ vršna snaga na korisnom signalu

$$P(\varepsilon) = P(0)\frac{1}{2}\left[1 - \Phi\left(\frac{A_r}{\sqrt{2\sigma_N^2}}\right)\right] + P(1)\frac{1}{2}\left[1 - \Phi\left(\frac{A - A_r}{\sqrt{2\sigma_N^2}}\right)\right]$$

$$P(0) = P(1) = \frac{1}{2} \Rightarrow \boxed{A_r = \frac{A}{2}} \qquad \boxed{P(\varepsilon) = 2\cdot\frac{1}{2}\frac{1}{2}\left[1 - \Phi\left(\frac{A}{2\sqrt{2\sigma_N^2}}\right)\right] = \frac{1}{2}\left[1 - \Phi\left(\frac{A}{2\sqrt{2\sigma_N^2}}\right)\right]}$$

$$\mathrm{erf}(x) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{x} e^{-u^2} du \qquad \mathrm{erfc}(x) = \frac{1}{\sqrt{\pi}}\int_x^{\infty} e^{-u^2} du = 1 - \mathrm{erf}\,x$$

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-u^2} du - \frac{1}{\sqrt{\pi}}\int_{-\infty}^{x} e^{-u^2} du = \frac{1}{\sqrt{\pi}}\int_x^{\infty} e^{-u^2} du$$

$$b_0 = \frac{A_r}{\sigma_N} \quad \hat{\rho}^2 = \frac{A^2}{\sigma_N^2} \quad P(1/0) = \frac{1}{2}\left[1 - \Phi\left(\frac{b_0}{\sqrt{2}}\right)\right]$$

$$P(0/1) = \frac{1}{2}\left[1 - \Phi\left(\frac{\hat{\rho} - b_0}{\sqrt{2}}\right)\right] \qquad \boxed{P(\varepsilon) = \frac{P(0)}{2}\left[1 - \Phi\left(\frac{b_0}{\sqrt{2}}\right)\right] + \frac{P(1)}{2}\left[1 - \Phi\left(\frac{\hat{\rho} - b_0}{\sqrt{2}}\right)\right]}$$

$$\Phi(x) = 2\,\mathrm{erf}(x) - 1 = 2\frac{1}{\sqrt{\pi}}\int_{-\infty}^{x} e^{-u^2} du - \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-u^2} du =$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{x} e^{-u^2} du - \left(\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-u^2} du - \frac{1}{\sqrt{\pi}}\int_{-\infty}^{x} e^{-u^2} du\right) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{x} e^{-u^2} du - \frac{1}{\sqrt{\pi}}\int_x^{\infty} e^{-u^2} du$$

$$\boxed{\Phi(x) = \mathrm{erf}(x) - \mathrm{erfc}(x) = 1 - 2\,\mathrm{erfc}(x) = 2\,\mathrm{erf}(x) - 1}$$

$$P(e) = \frac{1}{2}\left[1 - \phi\left(\frac{A}{2\sqrt{2\sigma_N^2}}\right)\right] = \frac{1}{2}\left[1 - 1 + 2\,\mathrm{erfc}\left(\frac{A}{2\sqrt{2\sigma_N^2}}\right)\right] = \mathrm{erfc}\left(\frac{A}{2\sqrt{2\sigma_N^2}}\right)$$

УНИПОЛАРЕН

- $P(e) = ?$ ЗА БИПОЛАРЕН СИГНАЛ

$$v(t) = s(t) + n(t) \qquad v(t_0) = \begin{cases} \frac{A}{2} + n(t_0) & a_n = 1 \\ -\frac{A}{2} + n(t_0) & a_n = \varnothing \end{cases}$$

$$P(e) = P(0)P(1/0) + P(1)P(0/1) \qquad A_T = \varnothing$$

$$P(1/0) = \frac{1}{\sqrt{2\pi\sigma_N^2}}\int_0^\infty e^{-\frac{(v+\frac{A}{2})^2}{2\sigma^2}}dv \quad \Big/ \; t = \frac{v+\frac{A}{2}}{\sqrt{2\sigma^2}} \quad dt = \frac{dv}{\sqrt{2\sigma^2}} \quad v=0 \; t = \frac{A}{2\sqrt{2\sigma^2}} \quad v=\infty \; t=\infty$$

$$= \frac{1}{\sqrt{\pi}}\int_{A/2\sqrt{2\sigma^2}}^{\infty} e^{-t^2}dt = \frac{1}{\sqrt{\pi}}\underbrace{\int_0^\infty e^{-t^2}dt}_{\frac{\sqrt{\pi}}{2}} - \frac{1}{\sqrt{\pi}}\int_0^{\frac{A}{2\sqrt{2\sigma^2}}} e^{-t^2}dt$$

$$= \frac{1}{2}\left[1 - \frac{2}{\sqrt{\pi}}\int_0^{\frac{A}{2\sqrt{2\sigma^2}}} e^{-t^2}dt\right] = \frac{1}{2}\left[1 - \phi\left(\frac{A}{2\sqrt{2\sigma^2}}\right)\right]$$

$$P(0/1) = P(1/0) = \frac{1}{2}\left[1 - \phi\left(\frac{A}{2\sqrt{2\sigma^2}}\right)\right]$$

$$P(e) = \frac{1}{2}\cdot\frac{1}{2}\cdot 2\left[1 - \phi\left(\frac{-A}{2\sqrt{2\sigma_N^2}}\right)\right] = \mathrm{erfc}\left(\frac{A}{2\sqrt{2\sigma_N^2}}\right) \qquad A_T = \varnothing \quad P(0) = P(1) = \frac{1}{2}$$

БИПОЛАРЕН

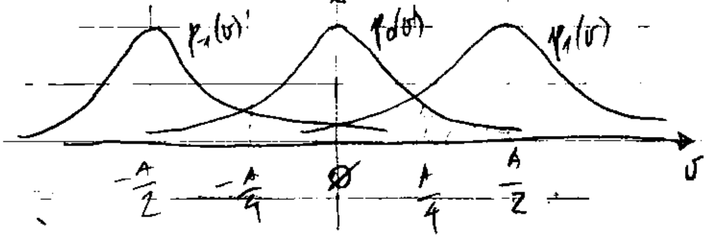
УНИПОЛАРНИОТ И ПОЛАРНИОТ СИГНАЛ ИМААТ ИСТА $P(e)$

$$P_{S,P} = \left(\frac{A}{2}\right)^2 P\left(\frac{A}{2}\right) + \left(-\frac{A}{2}\right)^2 P\left(\frac{A}{-2}\right) = 2\cdot\frac{1}{2}\cdot\frac{A^2}{4} = \frac{A^2}{4}$$

$$P_{SU} = A^2 \cdot P(A) = \frac{A^2}{2}$$

БИПОЛАРНИОТ ИМА 2x ПОМАЛА ПОТРОШУВАЧКА

- ПСЕУДОТЕРНАРЕН (АЛТЕРНАТИВНО БИПОЛАРЕН) СИГНАЛ $P(e) = ?$



$$v(t_0) = s(t_0) + n(t_0)$$

$$v(t_0) = \begin{cases} \frac{A}{2} + n(t_0) & a_n = 1 \\ n(t_0) & a_n = \varnothing \\ -\frac{A}{2} + n(t_0) & a_n = -1 \end{cases}$$

$$s(t) = \sum_{n=-\infty}^{\infty} a_n g(t - nT_d)$$

$$P(e) = P(0)\left[P(1/0) + P(-1/0)\right] + P(1)\left[P(0/1) + \underbrace{P(-1/1)}_{\approx\varnothing}\right] + P(-1)\left[P(0/-1) + \underbrace{P(1/-1)}_{\approx\varnothing}\right]$$

$$P(0) = P(1) = P(-1) = \frac{1}{3} \qquad P(1/0) = P(-1/0) = P(0/1) = P(0/-1)$$

$$P(1/-1) = P(-1/1) \doteq \varnothing$$

$$P(e) = \frac{1}{3}\left[P(1/0) + P(-1/0) + P(0/1) + P(0/-1)\right]$$

$$P(e) = \frac{4}{3}P(1/0) = \frac{4}{3}\int_{A/4}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(v-\frac{A}{2})^2}{2\sigma^2}}dv \qquad t = \frac{v - \frac{A}{2}}{\sqrt{2\sigma_N^2}} \quad v = \frac{A}{4} \; t = \frac{A}{4\sqrt{2\sigma_N^2}} \quad v = \infty \; t = \infty$$

$$t = \frac{\frac{A}{2} - v}{\sqrt{2\sigma_N^2}} \quad (+)$$

$$P(e) = \frac{4}{3}\int_{A/4\sqrt{2\sigma_N^2}}^{\infty} \frac{1}{\sqrt{\pi}} e^{-t^2} dt = \frac{4}{3}\frac{1}{\sqrt{\pi}}\left(\int_0^{\infty} e^{-t^2}dt - \frac{1}{\sqrt{\pi}}\int_0^{A/4\sqrt{2\sigma_N^2}} e^{-t^2}dt\right) =$$

$$= \frac{4}{3}\left(\frac{1}{2} - \frac{1}{2}\frac{2}{\sqrt{\pi}}\int_0^{A/4\sqrt{2\sigma_N^2}} e^{-t^2}dt\right) = \frac{2}{3}\left[1 + \Phi\left(\frac{A}{4\sqrt{2\sigma_N^2}}\right)\right]$$

$$P(e) = \frac{4}{3}P(1/0) = \frac{1}{\sqrt{2\pi\sigma_N^2}}\int^{\infty} e^{-\frac{v^2}{2\sigma_N^2}} dv \qquad \frac{v}{\sqrt{2\sigma_N^2}} = t \qquad v=\infty \quad t=\infty$$

$$v = \frac{A}{4} \quad t = \frac{A}{4\sqrt{2\sigma_N^2}}$$

$$P(e) = \frac{4}{3}\frac{1}{\sqrt{\pi}}\int_{\frac{A}{4\sqrt{2\sigma_N^2}}}^{\infty} e^{-t^2}dt = \frac{1}{\sqrt{\pi}}\left[\underbrace{\int_0^{\infty} e^{-t^2}dt}_{\frac{\sqrt{\pi}}{2}} - \int_0^{\frac{A}{4\sqrt{2\sigma_N^2}}} e^{-t^2}dt\right] =$$

$$P(e) = \frac{1}{2}\frac{4}{3}\left[1 - \Phi\left(\frac{A}{4\sqrt{2\sigma_N^2}}\right)\right]$$

$$\Phi(t) = 1 - 2\,\mathrm{erfc}(t)$$

$$P(\varepsilon) = \frac{1}{2}\left[2\,\mathrm{erfc}(t)\right] = \mathrm{erfc}(t)$$

$$\mathrm{erfc} = \frac{1}{\sqrt{\pi}}\int_x^{\infty} e^{-u^2}du$$

$$P(\varepsilon) = \frac{4}{3}\,\mathrm{erfc}\left(\frac{A}{4\sqrt{2\sigma_N^2}}\right)$$

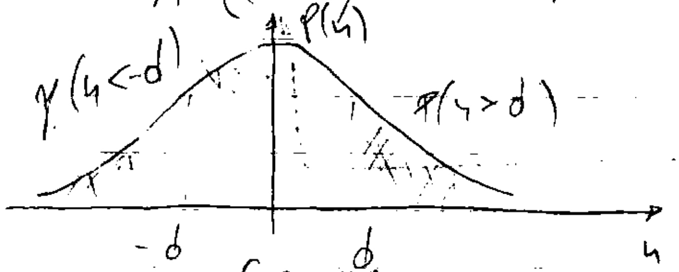
PSEUDOTERNAREN

$A = 5 \quad \sigma = 0{,}235 \quad \Rightarrow \quad P(\varepsilon) = 1{,}13 \cdot 10^{-7}$

- VEROVATNOĆA NA GREŠKA KAD M-EN SIGNAL

$P(u_i) = \frac{1}{M} \qquad u_i = 0, 1, 2, \ldots M$

$$P(e) = \frac{1}{M}\left\{(M-2)\,P(|u|>d) + 2P(u>d)\right\} \qquad \underbrace{P(u>d)}_{\text{NAJNISKOTO NIVO}} = \underbrace{P(u<-d)}_{\text{NAJVISOKO}}$$



$$P(|u|>d) = P(u>d \vee u<-d) = P(u>d) + P(u<-d) = 2P(u>d)$$

$$P(e) = \frac{1}{M}\left\{(M-2)\,P(|u|>d) + P(|u|>d)\right\} = \frac{M-1}{M}P(|u|>d)$$

$$P(|u|>d) = 2P(u>d) = 2\frac{1}{\sqrt{2\pi\sigma_N^2}}\int_d^{\infty} e^{-\frac{v^2}{2\sigma_N^2}}dv \qquad \frac{v}{\sqrt{2\sigma_N^2}} = t \qquad v=d \quad t = \frac{d}{\sqrt{2\sigma_N^2}}$$

$$P(|u|>d) = 2\frac{\sqrt{2\sigma_N^2}}{\sqrt{2\pi\sigma_N^2}}\int_{\frac{d}{\sqrt{2\sigma_N^2}}}^{\infty} e^{-t^2}dt = \frac{2}{\sqrt{\pi}}\left(-\int_0^{d/\sqrt{2\sigma_N^2}} e^{-t^2}dt + \int_0^{\infty} e^{-t^2}dt\right)$$

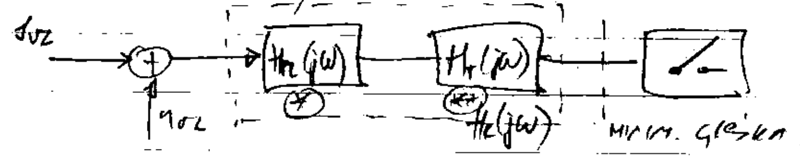
$$= \frac{2}{\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2}\Phi\left(\frac{d}{\sqrt{2\sigma_N^2}}\right)\right] = \left[1 - \Phi\left(\frac{d}{\sqrt{2\sigma_N^2}}\right)\right]$$

$$P(|u|>d) = \frac{2(M-1)}{M}\,\mathrm{erfc}\,\frac{d}{\sqrt{2\sigma_N^2}} \qquad P(|u|>d) = \frac{M-1}{M}\left[1 - \Phi\left(\frac{d}{\sqrt{2\sigma_N^2}}\right)\right]$$

M-AREN

$$P(\varepsilon)\Big|_{M=3} = \frac{2\cdot 2}{3}\,\mathrm{erfc}\,\frac{d}{\sqrt{2\sigma_1^2}} = \frac{4}{3}\,\mathrm{erfc}\,\frac{d}{\sqrt{2\sigma_N^2}} \qquad d=\frac{A}{4}$$

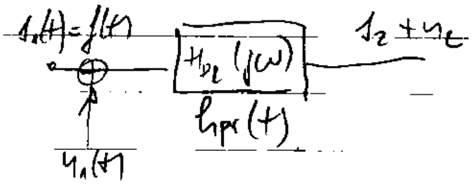
$M \nearrow \quad P(\varepsilon)\nearrow$ ЗАТОА ТРЕБА $\nearrow \frac{S}{N}$ т.е. $\frac{d}{\sqrt{2}\,\sigma_N}$



(*) ↓ ISI

(**) ↓ noise error

- ПРИЛАГОДЕН ФИЛТЕР



$$\frac{|s_2(t)|^2}{\overline{n_2^2(t)}}$$

$$s_2(t_0) = \int_{-\infty}^{\infty} s_1(\tau)\,h(t_0-\tau)\,d\tau$$

$$n_2(t_0) = \int_{-\infty}^{\infty} n_1(\tau)\,h(t_0-\tau)\,d\tau$$

$$s_2(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\,H_R(j\omega)\,e^{+j\omega t}\,d\omega \qquad F(j\omega) = \mathcal{F}\{s(t)\} = \mathcal{F}\{f(t)\}$$

$$s_2(t_0) = A = \frac{1}{2\pi}\left|\int_{-\infty}^{\infty} F(j\omega)\,H_R(j\omega)\,e^{j\omega t_0}\,d\omega\right|$$

$n_1(t)$ – БЕЛ ГАУСОВ ШУМ СО $\Phi_{n_1}(\omega) = n_0$ ТОГАШ ~~СНАГАТА~~ СНАГАТА НА ШУМОТ НА ИЗЛЕЗ ОД ФИЛТЕРОТ ЌЕ БИДЕ

$$R_{n_2}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_{n_2}(\omega)\,d\omega \qquad \Phi_{n_2}(\omega) = |H(j\omega)|^2\cdot\underbrace{\Phi_{n_1}(\omega)}_{n_0}$$

$$\boxed{\overline{n_2^2(t)} = R_{n_2}(0) = \frac{n_0}{2\pi}\int_{-\infty}^{\infty}|H(j\omega)|^2\,d\omega = \sigma_N^2}$$ МАУ

$$\sigma_N^2 = \overline{n_2^2(t)} - \overline{n_2}^2 = \overline{n_2^2(t)} \qquad \overline{n_2} = 0$$

$$\hat{\rho}^2 = \frac{A^2}{\sigma_N^2} = \frac{\frac{1}{(4\pi^2)}\left|\int_{-\infty}^{\infty}F(j\omega)\,H_R(j\omega)\,e^{j\omega t_0}\,d\omega\right|^2}{n_0\,\frac{1}{2\pi}\int_{-\infty}^{\infty}|H_R(j\omega)|^2\,d\omega}$$

00:02:3F:0A:48:19

$$\left|\int_{-\infty}^{\infty}X(j\omega)\,Y(j\omega)\,d\omega\right|^2 \le \int_{-\infty}^{\infty}|X(j\omega)|^2\,d\omega\int_{-\infty}^{\infty}|Y(j\omega)|^2\,d\omega$$ ШВАРЦОВО НЕРАВЕНСТВО

$$R_{ff}(\tau) = \int_{-\infty}^{\infty} f(t)\cdot f(t+\tau)\,dt = \int_{-\infty}^{\infty} f(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}F(j\omega)\,e^{j\omega\tau}\,e^{j\omega t}\,d\omega\right]dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}F(j\omega)\underbrace{\left[\int_{-\infty}^{\infty}f(t)\,e^{j\omega t}\,dt\right]}_{F^*(j\omega)}e^{j\omega\tau}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}F(j\omega)\,F^*(j\omega)\,e^{j\omega\tau}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}|F(j\omega)|^2 e^{j\omega\tau}\,d\omega$$

$$\mathcal{E} = \mathcal{R}_f(0) = \int_{-\infty}^{\infty} f^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega$$

$$\frac{A^2}{\sigma_N^2 \mathcal{E}} = \frac{\frac{1}{4\pi^2}\left|\int_{-\infty}^{\infty} F(j\omega)\cdot H_{PR}(j\omega)\, e^{j\omega t_0} d\omega\right|^2}{\frac{\eta_0}{2\pi}\int_{-\infty}^{\infty}|H_{PR}(j\omega)|^2 d\omega \;\; \frac{1}{2\pi}\int_{-\infty}^{\infty}|F(j\omega)|^2 d\omega}$$

ШВАРЦОВО НЕРАВ:

$$\left|\int_{-\infty}^{\infty} F(j\omega) H_{PR}(j\omega) e^{j\omega t_0} d\omega\right|^2 \le \int_{-\infty}^{\infty}|F(j\omega)|^2 d\omega \int_{-\infty}^{\infty}|H_{PR}(j\omega)|^2 d\omega$$

(1) $Y(j\omega) = k\, X^*(j\omega)$ т.е. $H_{PR}(j\omega) = k\, F^*(j\omega)\, e^{-j\omega t_0}$ → ВО ОВОЈ СЛУЧАЈ ЌЕ ВАЖИ ЕДНАКВОСТ НА ШВАРЦОВОТО НЕРАВЕНСТВО

$$\frac{A^2}{\sigma_N^2 \mathcal{E}} \le \frac{\int_{-\infty}^{\infty}|F(j\omega)|^2 d\omega \int_{-\infty}^{\infty}|H_{PR}(j\omega)|^2 d\omega}{\eta_0 \int_{-\infty}^{\infty}|H_{PR}(j\omega)|^2 d\omega \int_{-\infty}^{\infty}|F(j\omega)|^2 d\omega}$$

$$\frac{A^2}{\sigma_N^2 \mathcal{E}} \le \frac{1}{\eta_0}$$

$$\frac{A^2}{\sigma_N^2} \le \frac{\mathcal{E}}{\eta_0}$$

$\mathcal{E}$ – ЕНЕРГИЈА НА СИГНАЛОТ НА ВЛЕЗ ОД ФИЛТЕРОТ (ВКУПНА ЕНЕРГИЈА СИГНАЛ)

η_0 – СПЕКТРАЛНА ГУСТИНА НА БЕЛИОТ ШУМ НА ВЛЕЗ ОД ФИЛТЕРОТ

АКО Е ИСПОЛНЕТО (1) ТОГАШ ГО ДОБИВАМЕ МАКС. ОДНОС S/N НА ВЛЕЗ ОД ФИЛТЕРОТ

$$\hat{\rho}^2_{max} = \left.\frac{A^2}{\sigma_N^2}\right|_{max} = \frac{\mathcal{E}}{\eta_0} \quad (11)$$

ОПТИМАЛНА ПРЕНОСНА К-КА НА P.F Е ДАДЕНА СО (1)

• ЕНЕРГИЈА НА ВЛЕЗОТ Е:

$$\mathcal{E} = \int_{-\infty}^{\infty} f^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|F^2(j\omega)|^2 d\omega$$

$\mathcal{E}^2 = A^2 \cdot T_d$ ЗА ОДАМЕН ИМПУЛС СО АМП. A И ВРЕМЕ T_d

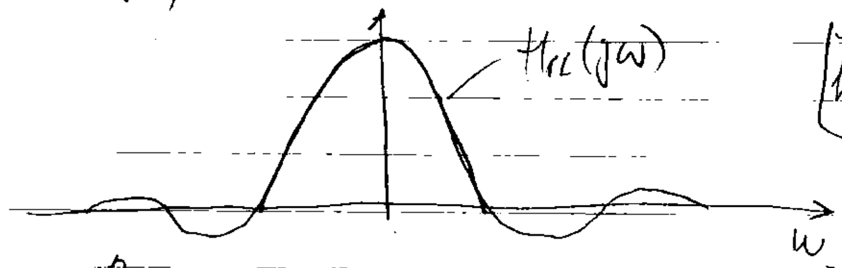
$$P(e) = \operatorname{erfc}\frac{A}{2\sqrt{2}\,\sigma_N} = \operatorname{erfc}\frac{1}{2}\sqrt{\frac{\mathcal{E}}{2\eta_0}} = f\left(\sqrt{\frac{\mathcal{E}}{\eta_0}}\right)$$

$$P(e) = \frac{2(M-1)}{M}\operatorname{erfc}\left(\frac{d}{\sqrt{2}\sigma_N}\right) = \frac{4}{3}\operatorname{erfc}\frac{A}{4\sqrt{2}\,\sigma_N} = \frac{4}{3}\operatorname{erfc}\left(\frac{A}{4}\sqrt{\frac{A}{2\eta_0}}\right)$$

$$P(e) = f\left(\sqrt{\frac{\mathcal{E}}{\eta_0}}\right) \Rightarrow \text{ЗА АМИ}$$

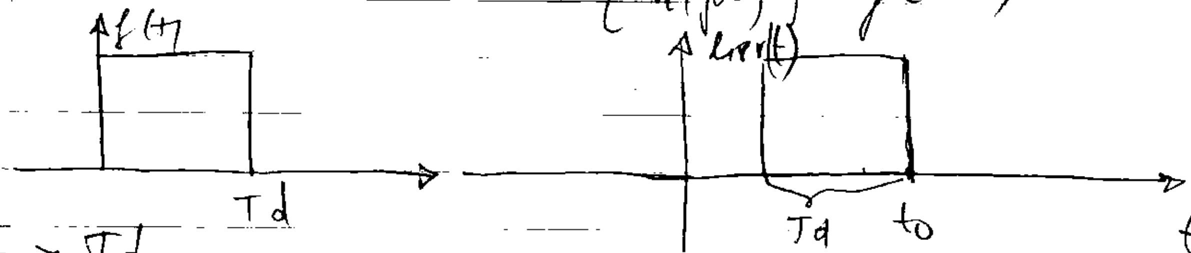
$H_{KR}(j\omega) = K\, F^*(j\omega)\, e^{-j\omega t_0}$ $\qquad \mathcal{F}^{-1}\{H_{KR}(j\omega)\} = h_{KR}(t)$

$$h_{KR}(t) = K \cdot f(t_0 - t)$$



$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(j\omega)\, e^{-j\omega t_0}\, e^{+j\omega t}\, d\omega = \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\, e^{+j\omega(t_0-t)}\, d\omega\right]^* = f^*(t_0-t)$$

$f(t)$ – РЕАЛНА $\rightarrow$ $\mathcal{F}^{-1}\{H_{KR}(j\omega)\} = f(t_0-t)$



$t_0 > T_d$ ЗА ~~М~~ ВИДЕ ЦЕЛИОТ СИГНАЛОТ ЗА ВКЛУЧУВАЊЕ

$$s_2(t_0) = A = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_{KR}(j\omega)\cdot F(j\omega)\, e^{j\omega t_0}\, dt = \frac{K}{2\pi}\int_{-\infty}^{\infty} F^*(j\omega)\, e^{-j\omega t_0} F(j\omega)\, e^{j\omega t_0}\, d\omega$$

$$s_2(t_0) = A = \frac{K}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega = \underline{K \cdot E}$$

A^2 – МАХ МОМЕНТНА СНАГА НА ВЛЕЗ. СИГ. $\cdot E$

$$N = \sigma_N^2 = \frac{N_0}{2\pi}\int_{-\infty}^{\infty} |H_{KR}(j\omega)|^2 d\omega = \frac{N_0 K^2}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega = N_0 K^2 E$$

$$\left.\frac{A}{\sigma_N^2}\right|_{max} = \frac{K \cdot E}{N_0 K^2 E} = \frac{1}{N_0 K}$$

- РАСПРЕДЕЛБА НА АМПЛИТУДАТА НА ИЗЛЕЗ ОД ОПТИМАЛНИОТ ФИЛТЕР

$$p_\rho(r) = \frac{r}{\sigma_N^2}\, e^{-\frac{r^2+A^2}{2\sigma_N^2}}\, I_0\!\left(\frac{Ar}{\sigma_N^2}\right)$$

$$I_0\!\left(\frac{Ar}{\sigma_N^2}\right) = \frac{1}{2\pi}\int_0^{2\pi} e^{\frac{Ar}{\sigma_N^2}\cdot\cos\varphi}\, d\varphi$$

$$p_{r,\varphi}(r,\varphi) = \frac{r}{2\pi\sigma_N^2}\, e^{-\frac{r^2+A^2+2Ar\cos\varphi}{2\sigma_N^2}}$$

$$p_\varphi = \int_0^{A} \frac{r}{2\pi\sigma_N^2}\, e^{-\frac{r^2+A^2+2Ar\cos\varphi}{2\sigma_N^2}}\, dr = \frac{1}{2\pi\sigma_N^2}\int_0^{A} r\, e^{-\frac{r^2}{2\sigma_N^2}}\cdot e^{-\frac{A^2}{2\sigma_N^2}}\cdot e^{-\frac{Ar\cos\varphi}{\sigma_N^2}}\, dr$$

$$p_\varphi = \left(\frac{e^{\frac{A^2\sin^2\varphi}{2\sigma_N^2}}\, e^{-\frac{A^2}{2\sigma_N^2}}}{2\pi\sigma_N^2}\right) = K \int_0^A r \cdot e^{-\frac{r^2+2Ar\cos\varphi + A\sin^2\varphi}{2\sigma_N^2}}\, dr = K\int_0^A r\, e^{-\frac{(r+A\sin\varphi)^2}{2\sigma_N^2}}\, dr$$

$$v = \int_0^A e^{-\frac{(r+A\sin\varphi)^2}{2\sigma_N^2}}\, dr \qquad t = \frac{(r+A\sin\varphi)}{\sqrt{2\sigma_N^2}} \qquad r=A \quad t=\frac{A(1+\sin\varphi)}{\sqrt{2}\sigma_N}$$

$$dt = dr/(\sqrt{2}\sigma_N) \qquad r=0 \quad t=\frac{A\sin\varphi}{\sqrt{2}\sigma_N}$$

$$v=\sqrt{2}\sigma_N\int_{A\sin\varphi/\sqrt{2}\sigma_N}^{A(1+\sin\varphi)/\sqrt{2}\sigma_N} e^{-t^2}\,dt = \frac{\sqrt{2}\sigma_N}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}\,2}{2}\left[\int_0^{t_1} e^{-t^2}\,dt - \int_0^{t_2}e^{-t^2}\,dt\right]$$

$$v = \frac{\sqrt{2\pi\sigma_N^2}}{2}\left[\Phi\left[\frac{A+\sin\varphi}{\sqrt{2\sigma_N^2}}\right] - \Phi\left[\frac{A\sin\varphi}{\sqrt{2\sigma_N^2}}\right]\right]$$

$$p_\varphi = u\cdot v - \int v\,du$$

$$\boxed{\begin{aligned}&\int e^{-x^2}dx = \frac{\sqrt{\pi}}{2}\Phi(x)\\ &\int \Phi(t)\,dt = t\cdot\Phi(t) + \frac{e^{-t^2}}{\sqrt{\pi}}\end{aligned}}$$

$$v=\int e^{-\frac{(r+A\sin\varphi)^2}{2\sigma_N^2}}\,dr = \sqrt{2\sigma_N^2}\int e^{-\frac{(r+A\sin\varphi)^2}{2\sigma_N^2}}\, d\left(\frac{r+A\sin\varphi}{\sqrt{2\sigma_N^2}}\right) =$$

$$= \sqrt{2\sigma_N^2}\cdot\frac{\sqrt{\pi}}{2}\,\Phi\left[\frac{r+A\sin\varphi}{\sqrt{2\sigma_N^2}}\right] = \frac{\sqrt{2\pi\sigma_N^2}}{2}\,\Phi\left[\frac{r+A\sin\varphi}{\sqrt{2\sigma_N^2}}\right]$$

$$p_\varphi = K\frac{\sqrt{2\pi\sigma^2}}{2}\, r\cdot \Phi\left[\frac{r+A\sin\varphi}{\sqrt{2\sigma_N^2}}\right]\Bigg|_0^A - \int_0^A \frac{\sqrt{2\pi\sigma_N^2}}{2}\,\Phi\left[\frac{r+A\sin\varphi}{\sqrt{2\sigma_N^2}}\right]dr =$$

$$= A K \frac{\sqrt{2\pi\sigma^2}}{2}\,\Phi\left(\frac{A+A\sin\varphi}{\sqrt{2\sigma_N^2}}\right) - \frac{\sqrt{2\pi\sigma_N^2}}{2}\,K\sqrt{2\sigma_N^2}\int_0^A \Phi\left[\frac{r+A\sin\varphi}{\sqrt{2\sigma_N^2}}\right] d\left(\frac{r+A\sin\varphi}{\sqrt{2\sigma_N^2}}\right)$$

$$= A\frac{\sqrt{2\pi\sigma^2}}{2}K\,\Phi\left(\frac{A+A\sin\varphi}{\sqrt{2\sigma_N^2}}\right) - K\sigma_N^2\sqrt{\pi}\left\{\frac{r+A\sin\varphi}{\sqrt{2\sigma_N^2}}\cdot\Phi\left[\frac{r+A\sin\varphi}{\sqrt{2\sigma_N^2}}\right]\right\}\Bigg|_0^A - \underbrace{K\sigma_N^2\sqrt{\pi}\,\frac{e^{-\frac{(r+A\sin\varphi)^2}{2\sigma_N^2}}}{\sqrt{\pi}}\Bigg|_0^A}_{K_1} =$$

$$= \frac{A\cdot K\sqrt{2\pi\sigma^2}}{2}\,\Phi\left(\frac{A+A\sin\varphi}{\sqrt{2\sigma_N^2}}\right) - \frac{K\cdot\sqrt{2\sigma_N^2\pi}}{2}\left[(A+A\sin\varphi)\,\Phi\left(\frac{A+A\sin\varphi}{\sqrt{2\sigma_N^2}}\right) - A\sin\varphi\,\Phi\left(\frac{A\sin\varphi}{\sqrt{2\sigma_N^2}}\right)\right]$$

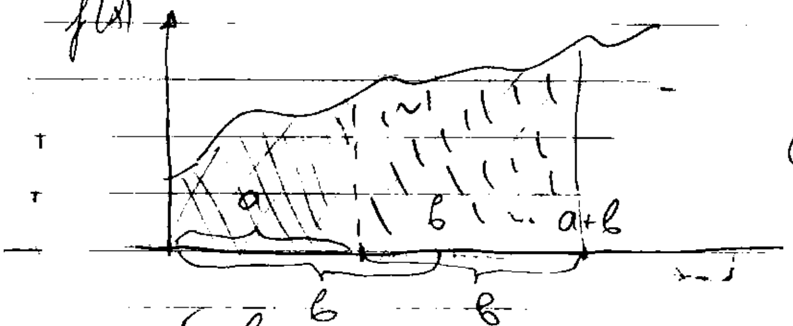
$$= \frac{A\sqrt{2\pi\sigma_N^2}\,K\cdot A\sin\varphi}{2}\,\Phi\left(\frac{A\sin\varphi}{\sqrt{2\sigma_N^2}}\right) - \frac{AK\cdot\sqrt{2\pi\sigma_N^2}}{2}\,A\sin\varphi\,\Phi\left(\frac{A+A\sin\varphi}{\sqrt{2\sigma_N^2}}\right) - K_1$$

$$\Psi_\varphi = \frac{A^2 K \cdot \sqrt{2\pi\sigma_N^2}}{2} \cdot \sin\varphi \left[\Phi\left(\frac{A\sin\varphi}{\sqrt{2\pi\sigma_N^2}}\right) - \Phi\left(\frac{A + A\sin\varphi}{\sqrt{2\pi\sigma_N^2}}\right) \right] - K_1$$

$$K = e^{\frac{A^2\sin^2\varphi}{2\sigma_N^2}} e^{-\frac{A^2}{2\sigma_N^2}} \cdot \frac{1}{2\pi\sigma_N^2}$$

$$\Psi_\varphi = \frac{A^2 e^{\frac{A^2\sin^2\varphi}{2\sigma_N^2}} e^{-\frac{A^2}{2\sigma_N^2}} \sin\varphi}{2\sqrt{2\pi\sigma_N^2}} \left[\Phi\left(\frac{A\sin\varphi}{\sqrt{2\pi\sigma_N^2}}\right) - \Phi\left(\frac{\overset{b+a}{A + A\sin\varphi}}{\sqrt{2\pi\sigma_N^2}}\right) \right] - K_1$$

$$\int_0^{a+b} e^{-x^2}dx = \int_0^b e^{-x^2}dx - \int_0^a e^{-x^2}dx + \int_0^a e^{-x^2}dx$$



$$(\#) = \int_0^a f(x)dx - \int_0^{a+b} f(x)dx =$$

$$= -\int_0^b f(x)\,dx$$

$$(\#) = -\left\{ \int_0^b f(x)dx - \int_0^a f(x)dx \right\} = \int_0^a f(x)dx - \int_0^b f(x)dx$$

$$\Psi_\varphi = \frac{1}{2\sqrt{2\pi\sigma_N^2}} A^2 \overbrace{e^{\frac{A^2\sin^2\varphi}{2\sigma_N^2}} e^{-\frac{A^2}{2\sigma_N^2}}}^{= -\frac{A^2}{2\sigma_N^2}\cos^2\varphi} \sin\varphi \left[\Phi\left(\frac{A\sin\varphi}{\sqrt{2\pi\sigma_N^2}}\right) - \Phi\left(\frac{A}{\sqrt{2\pi\sigma_N^2}}\right) \right] - K_1$$

$$K_1 = K\sigma_N^2 \sqrt{\pi}\, e^{-\left(\frac{x + A\sin\varphi}{\sqrt{2\sigma_N^2}}\right)^2} \Big|_0^A = K\sigma_N^2 \left(e^{-\left(\frac{A + A\sin\varphi}{\sqrt{2\sigma_N^2}}\right)^2} - e^{-\left(\frac{A\sin\varphi}{\sqrt{2\sigma_N^2}}\right)^2} \right)$$

~~$= e^{\frac{A^2\sin^2\varphi}{}} e^{-\frac{A}{2\sigma^2}} \frac{1}{2\pi\sigma_N^2} \sigma_N^2 \left(e^{\frac{A}{\sqrt{2\sigma^2}}} e^{\frac{A\sin\varphi}{\sqrt{2\sigma_N}}} - e^{\frac{A\sin\varphi}{\sqrt{2\sigma_N^2}}} \right)$~~

~~$K_1 = \frac{1}{2\pi} e^{A^2\sin^2\varphi} e^{-\frac{A}{2\sigma^2}} e^{\frac{A\sin\varphi}{\sqrt{2\sigma^2}}} \left(e^{\frac{A}{\sqrt{2\sigma_N}}} \right)$~~

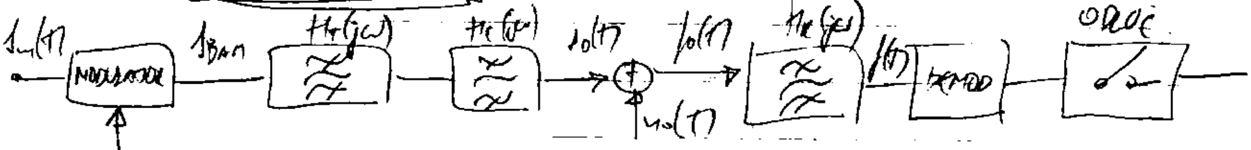
$$K_1 = e^{\frac{A^2\sin^2\varphi}{2\sigma^2}} \cdot e^{-\frac{A^2}{2\sigma^2}} \frac{1}{2\pi\sigma_N^2} \sigma_N^2 \left[e^{-\frac{A^2 + 2A\sin\varphi + A^2\sin^2\varphi}{2\sigma^2}} - e^{-\frac{A^2\sin^2\varphi}{2\sigma_N^2}} \right]$$

$$= e^{-\frac{A^2}{2\sigma^2}} \cdot \frac{1}{2\pi} \left[e^{-\frac{A^2 + 2A\sin\varphi}{2\sigma^2}} - 1 \right] = \frac{1}{2\pi} \left[e^{-\frac{A^2 + A\sin\varphi}{\sigma^2}} - e^{-\frac{A^2}{2\sigma^2}} \right]$$

$$\boxed{\Psi_\varphi = \frac{1}{2\sqrt{2\pi\sigma_N^2}} A^2 e^{-\frac{A^2}{2\sigma_N^2}\cos^2\varphi} \sin\varphi \left[\Phi\left(\frac{A\sin\varphi}{\sqrt{2\pi\sigma}}\right) - \Phi\left(\frac{A}{\sqrt{2\pi\sigma_N^2}}\right) \right] - \frac{1}{2\pi}\left(e^{-\frac{A^2 + A\sin\varphi}{\sigma^2}} - e^{-A^2/2\sigma^2} \right)}$$

• Дигитална амплитудна модулација (DAM)

Гјорѓи 070410610



$$s_n(t) = s_g(t) = \sum_{n=-\infty}^{\infty} a_n g_p(t - nT_d) \qquad g_p = \begin{cases} A_1 & 0 \le t \le T_d \\ 0 & \text{инаку} \end{cases}$$

DAM $a_n \in \{1, 0\}$

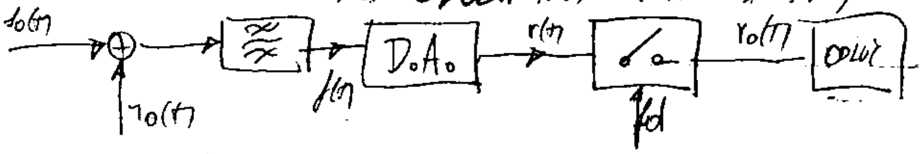
$$s_{DAM}(t) = s_g(t)\cdot\cos(\omega_0 t) = \begin{cases} A_1\cos\omega_0 t & a_n = 1,\ t \in T_d \\ 0 & a_n = 0,\ t \in T_d \end{cases}$$

$$f_0(t) = s_0(t) + n_0(t)$$

$$f(t) = s(t) + n(t) \qquad s(t) = \begin{cases} g(t)\cos(\omega_0 t) & a_n = 1 \\ 0 & a_n = 0 \end{cases}$$

$$f_s = \frac{f_g}{2} \; ; \qquad f_g = 2 f_s = 2\frac{1}{T_d}$$

• Некохерентна демодулација на DAM



$$f(t) = \begin{cases} s(t) + n(t) & a_n = 1 \\ n(t) & a_n = 0 \end{cases}$$

$$f(t) = \begin{cases} g(t)\cos\omega_0 t + n(t) \\ n(t) \end{cases} = \begin{cases} [g(t) + x(t)]\cos\omega_0 t - y(t)\sin\omega_0 t & a_n = 1 \\ x(t)\cos\omega_0 t - y(t)\sin\omega_0 t & a_n = 0 \end{cases}$$

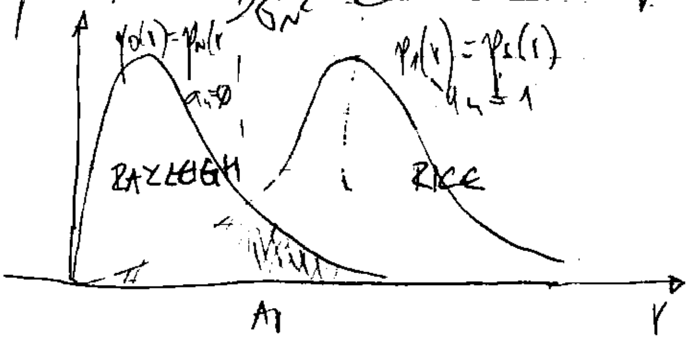
$$r(t) = \begin{cases} \sqrt{[g(t) + x(t)]^2 + y^2(t)} & a_n = 1 \\ \sqrt{x^2(t) + y^2(t)} & a_n = 0 \end{cases}$$

$$g_p(t_0) = A \qquad f(t_0) = [A + x(t)]\cos\omega_0 t - y(t)\sin\omega_0 t \quad a_n = 1$$

Распределба на анвелопата:

$$p_s(r) = \frac{r}{\sigma_n^2} e^{-\frac{(r-A)^2}{2\sigma_n^2}} I_0\left(\frac{A r}{\sigma_n^2}\right) \Rightarrow \text{сигнал}$$

$$p_N(r) = \frac{r}{\sigma_n^2} e^{-\frac{r^2}{2\sigma_n^2}} \Rightarrow \text{шум}$$



$$P(e) = P(0)\cdot P(1/0) + P(1)P(0/1)$$

$$P(0/1) = \int_{-\infty}^{A_1} p_1(r)\,dr$$

$$P(1/0) = \int_{A_1}^{\infty} p_0(r)\,dr$$

$$P(e) = \frac{1}{2}\int_0^{A_r} p_1(r)\,dr = \frac{1}{2}\int_{-\infty}^{A_r} p_0(r)\,dr$$

$$\frac{\partial P(e)}{\partial A_r} = \frac{1}{2}\left[p_1(A_r) - p_0(A_r)\right] = 0$$

$$\frac{A_r}{\sigma_N^2} e^{-\frac{A_r^2 + A^2}{2\sigma_N^2}} I_0\left(\frac{A A_r}{\sigma_N^2}\right) = \frac{A_r}{\sigma_N^2} e^{-\frac{A_r^2}{2\sigma_N^2}}$$

MINIMIZIRANO P(E) JE DODAVA ZA A_r VO PROSTOROT

$$I_0\left(\frac{A A_r}{\sigma_N^2}\right) = e^{-\frac{A_r^2}{2\sigma_N^2} + \frac{A_r^2 + A^2}{2\sigma_N^2}} = e^{\frac{A^2}{2\sigma_N^2}}$$

$$I_0\left(\frac{A A_r}{\sigma_N^2}\right) = \frac{1}{2\pi}\int_0^{2\pi} e^{\frac{A A_r}{\sigma_N^2}\cdot\cos(\varphi)} d\varphi$$

$$A_{ropt} = f\left(\frac{A^2}{\sigma_N^2}\right) = f(\hat{\rho}^2) \qquad \hat{\rho}^2 = \frac{A^2}{\sigma_N^2} \qquad b_0 = \frac{A}{\sigma_N}$$

$$P_s = P(0)\cdot 0^2 + P(1)\cdot A^2 = \frac{A^2}{2} \qquad \rho^2 = \frac{A^2}{2\sigma_N^2} \qquad \rho^2 = \frac{\hat{\rho}^2}{2}$$

$$b_{opt} = \sqrt{2 + \frac{\hat{\rho}^2}{4}}$$

$$P(1/0) = \int_{A_r}^{\infty} \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr = -\int_{A_r}^{\infty} e^{-\frac{r^2}{2\sigma^2}} d\left(\frac{-r^2}{2\sigma^2}\right) = -e^{-\frac{r^2}{2\sigma^2}}\Big|_{A_r}^{\infty}$$

$$P(1/0) = -\left(0 - e^{-\frac{A_r^2}{2\sigma^2}}\right) = e^{-\frac{A_r^2}{2\sigma^2}} = e^{-\frac{b_0^2}{2}}$$

$$P(0/1) = \int_{\infty}^{A_r} \frac{r}{\sigma^2} e^{-\frac{(r^2 + A^2)}{2\sigma^2}} I_0\left(\frac{A\cdot r}{\sigma^2}\right) dr = 1 - P(1/1)$$

$$P(0/1) = 1 - \int_{A_r}^{\infty} \frac{r}{\sigma_N^2} e^{-\frac{(r^2 + A^2)}{2\sigma_N^2}} I_0\left(\frac{A r}{\sigma_N^2}\right) dr = 1 - \int_{b_0}^{\infty} r_0 e^{-\frac{(r_0^2 + \hat{\rho}^2)}{2}} I_0(r_0\cdot\hat{\rho})\,dr_0$$

$$r_0 = \frac{r}{\sigma_N} \qquad b_0 = \frac{A_r}{\sigma_N}$$

- DEFINICIJA NA Q FUNKCIJA

$$Q(a,b) = \int_b^{\infty} x\, e^{-\frac{x^2 + a^2}{2}} I_0(a\cdot x)\,dx$$

$$P(0/1) = 1 - Q\left(\frac{A}{\sigma_N}, \frac{A_r}{\sigma_N}\right) = 1 - Q(\hat{\rho}, b_0)$$

$$P(e) = P(1)\left[1 - Q(\hat{\rho}, b_0)\right] + P(0)\cdot e^{-b_0^2/2}$$

$$P(e) = P(1)\left[1 - Q\left(\hat{\rho}, \sqrt{2 + \frac{\hat{\rho}^2}{4}}\right)\right] + P(0)\cdot e^{-\frac{1}{2}\left(2 + \frac{\hat{\rho}^2}{4}\right)}$$

• Ако $\hat\rho \gg 1$ тогаш $Q\left(\rho, \sqrt{2+\frac{\hat\rho^2}{4}}\right) = Q\left(\rho, \frac{\hat\rho}{2}\right)$

$$Q(\hat\rho, b_0) \doteq \frac{1}{2}\left[1+\Phi\left(\frac{\hat\rho-b_0}{\sqrt{2}}\right)\right] = \mathrm{erf}\left(\frac{\hat\rho-b_0}{\sqrt{2}}\right)$$

$$\frac{1}{2}\left[\frac{2}{\sqrt{\pi}}\int_{-\infty}^{0} e^{-t^2}dt + \frac{2}{\sqrt{\pi}}\int_{0}^{x} e^{-t^2}dt\right] = \frac{1}{2}\frac{2}{\sqrt{\pi}}\int_{-\infty}^{x} e^{-t^2}dt = \mathrm{erf}(x)$$

$$\mathrm{erfc}(x) = \frac{1}{\sqrt{\pi}}\int_{x}^{\infty} e^{-t^2}dt \quad \text{тогаш} \quad \boxed{\mathrm{erfc}(x) \doteq \frac{1}{2x\sqrt{\pi}} e^{-x^2}}$$

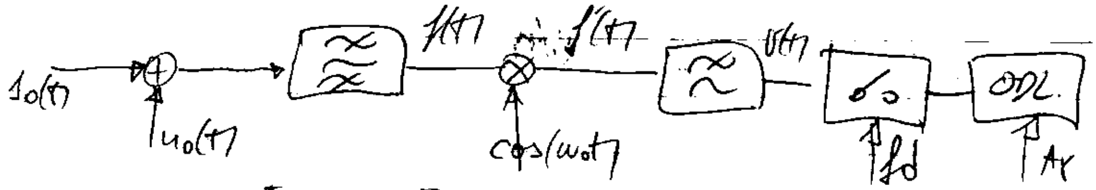
$$Q\left(\hat\rho, \frac{\hat\rho}{2}\right) \overset{x}{=} \mathrm{erf}\frac{\rho}{2\sqrt{2}} \doteq 1 - \frac{\sqrt{2}}{\rho\sqrt{\pi}} e^{-\frac{\hat\rho^2}{8}} \doteq 1 - \frac{1}{\hat\rho}\sqrt{\frac{2}{\pi}}\, e^{-\frac{\hat\rho^2}{8}}$$

• Ако $P(0) = P(1) = \frac{1}{2}$

$$\boxed{P(\varepsilon) = \underbrace{\frac{1}{2}\frac{1}{\hat\rho}\sqrt{\frac{2}{\pi}}\, e^{-\frac{\hat\rho^2}{8}}}_{(1)} + \underbrace{\frac{1}{2} e^{-\frac{\hat\rho^2}{8}}}_{(2)}}$$

Ако: $\hat\rho \gg 1$ (2) $\gg$ (1) $\quad \boxed{P(\varepsilon) \doteq \frac{1}{2} e^{-\frac{\hat\rho^2}{8}}}$

• **Кохерентна демодулација на ДАМ**



$$f(t) = \begin{cases} [g(t)+x(t)]\cos(\omega_0 t) - y(t)\cos(\omega_0 t) & a_k = 1 \\ x(t)\cos(\omega_0 t) - y(t)\cdot\cos(\omega_0 t) & a_k = 0 \end{cases}$$

$$\cos\omega_0 t \cdot \cos\omega_0 t = \frac{1}{2}\left[\cos(\omega_0 t+\omega_0 t) + \cos(\omega_0 t-\omega_0 t)\right] = \frac{1}{2}\left[1 + \cos(2\omega_0 t)\right]$$

$$\sin\omega_0 t \cdot \sin\omega_0 t = \frac{1}{2}\left[\cos(\omega_0 t-\omega_0 t) - \cos(\omega_0 t+\omega_0 t)\right] = \frac{1}{2}\left[1 - \cos(2\omega_0 t)\right]$$

$$\sin\omega_0 t \cdot \cos\omega_0 t = \frac{1}{2}\left[\sin(\omega_0 t+\omega_0 t) + \sin(\omega_0 t-\omega_0 t)\right] = \frac{1}{2}\sin(2\omega_0 t)$$

$$f'(t) = \begin{cases} \frac{1}{2}[g(t)+x(t)](1+\cos(2\omega_0 t)) - \frac{1}{2}y(t)\sin(2\omega_0 t) & a_k = 1 \\ \frac{1}{2}x(t)\cos(\omega_0 t) - \frac{1}{2}y(t)\cdot\sin(2\omega_0 t) & a_k = 0 \end{cases}$$

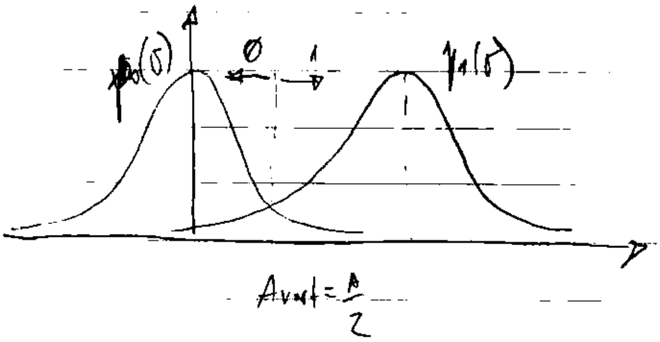
$$v(t) = \begin{cases} k_D[g(t)+x(t)] \\ k_D\cdot x(t) \end{cases}$$

k_D – константа од филтер

$D_p = \frac{k_D^2}{2}$ – константа на демод. по сигнал

$$v(t_0) = \begin{cases} k_D[A+x(t_0)] & a_k = 1 \\ k_D x(t_0) & a_k = 0 \end{cases}$$

$v(t)$ ⇒ Гаусова распределба !!!



$$P(\varepsilon) = P(0)\cdot P(1/0) + P(1)P(0/1)$$

$$A_{vrat} = \frac{A}{2}$$

$$P(1/0) = \operatorname{erfc}\frac{A_r}{\sqrt{2\sigma_n^2}} = \operatorname{erfc}\frac{A}{2\sqrt{2\sigma_n^2}}$$

$$P(0/1) = \operatorname{erfc}\frac{A - A_r}{\sqrt{2\sigma_n^2}} = \operatorname{erfc}\frac{A}{2\sqrt{2\sigma_n^2}}$$

$$P(1/0) = \operatorname{erfc}\frac{\hat{\rho}}{2\sqrt{2}} = P(0/1) = \operatorname{erfc}\frac{\hat{\rho}}{2\sqrt{2}}$$

$$P(\varepsilon) = \frac{1}{2}\cdot 2 \cdot \operatorname{erfc}\frac{\hat{\rho}}{2\sqrt{2}} = \operatorname{erfc}\frac{\hat{\rho}}{2\sqrt{2}} \doteq \left|\hat{\rho} \gg 1\right| \doteq \frac{1}{\hat{\rho}}\sqrt{\frac{2}{\pi}}\, e^{-\frac{\hat{\rho}^2}{8}}$$

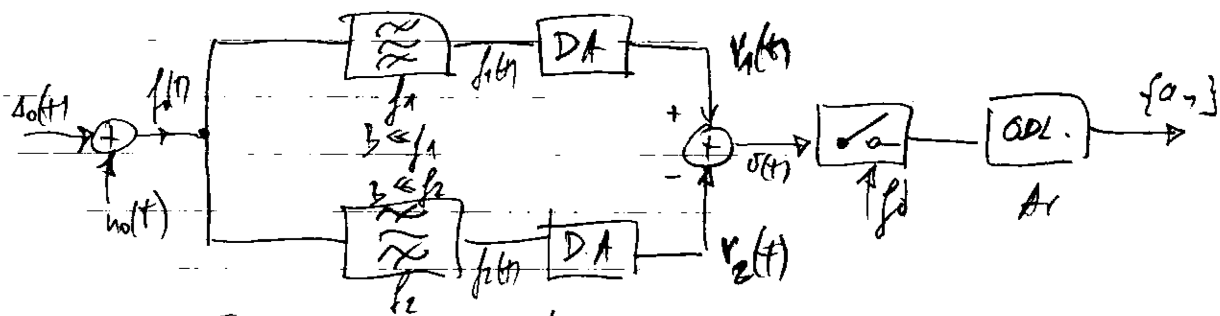
- ДИГИТАЛНА ФРЕК. МОДУЛАЦИЈА

$$s_n(t) = \sum_{n=-\infty}^{\infty} a_n g_T(t - nT_d) \qquad g_T(t) = \begin{cases} A_r; & 0 \le t \le T_d \\ 0; & \text{надвор} \end{cases}$$

$a_n = \{-1, 1\}$ БФМ

$$s_{БФМ} = \begin{cases} s_n(t)\cos\omega_1 t; & a_n = 1 \\ s_n(t)\cdot\cos\omega_2 t; & a_n = -1 \end{cases} = \begin{cases} A_r\cos\omega_1 t & „1“ \\ A_1\cos\omega_2 t & „0“ \end{cases}$$

- БФМ РА-ИК СО НЕКОХЕРЕНТНА ДЕМОДУЛАЦИЈА



$$f_1(t) = \begin{cases} [g(t) + x_1(t)]\cos\omega_0 t - y_1(t)\sin\omega_0 t & a_n = 1 \\ x_1(t)\cos\omega_0 t - y_1(t)\sin\omega_0 t & a_n = -1 \end{cases}$$

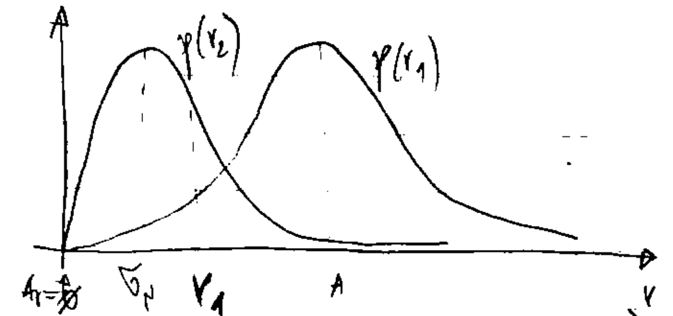
$$f_2(t) = \begin{cases} x_2(t)\cos\omega_0 t - y_2(t)\sin\omega_0 t & a_n = 1 \\ [g(t) + x_2(t)]\cos\omega_0 t - y_2\sin\omega_0 t & a_n = -1 \end{cases}$$

$$v_1(t) = \begin{cases} \sqrt{[g(t)+x_1(t)]^2 + y_1^2} & a_n = 1 \quad \text{РАЈСОВА (RICE)} \\ \sqrt{x_1^2(t) + y_1^2(t)} & a_n = -1 \quad \text{РЕЛЕЕВА (RAYLEIGH)} \end{cases}$$

$$v_2(t) = \begin{cases} \sqrt{x_2^2(t) + y_2^2(t)} & a_n = 1 \\ \sqrt{[g(t) + x_2(t)]^2 + y_2^2(t)} & a_n = -1 \end{cases}$$

$v(t) = v_1(t) - v_2(t)$; $g(t_0) = A$

$$\boxed{\begin{array}{ll} v_1(t) > v_2(t) & a_n = 1 \\ v_1(t) < v_2(t) & a_n = -1 \end{array}}$$



$$r_1(t_0) - r_2(t_0) \lessgtr 0 \quad \begin{matrix} > \text{„1"} \\ < \text{„0"} \end{matrix}$$ (K4)

$P(r_2 > r_1) = ?\quad a_k = 1$

$P(r_1 > r_2) = ?\quad a_k = -1$

$$P(e) = q(1)\cdot P(0/1) + q(0)\, P(1/0)$$

$$r(t_0) = r_1 \qquad P(r_2 > r_1) = P(0/1)_{r_1} = \int_{r_1}^{\infty} p(r_2)\, dr_2$$

(MMV)

$$P(0/1) = \overline{P(0/1)_{r_1}} = \int_0^{\infty} P(0/1)_{r_1}\, p_s(r_1)\, dr_1 \; ; \quad P(0/1)_{r_1} = \frac{1}{\sigma_N^2}\int_{r_1}^{\infty} r_2 e^{-\frac{r_2^2}{2\sigma_N^2}} dr_2$$

$$P(0/1) = -e^{-\frac{r_2^2}{2\sigma_N^2}}\Big|_{r_1}^{\infty} = e^{-\frac{r_1^2}{2\sigma_N^2}}$$

$$P(0/1) = \int_0^{\infty} e^{-\frac{r_1^2}{2\sigma_N^2}} \cdot \frac{r}{\sigma_N^2} \cdot e^{-\frac{r_1^2 - A^2}{2\sigma_N^2}} \cdot I_0\left(\frac{A r_1}{\sigma_N^2}\right) dr_1$$

$$r_1\sqrt{2} = y \qquad \frac{A}{\sqrt{2}} = A' \; ; \quad P(0/1) = \int_0^{\infty} \frac{r_1}{\sigma_N^2} e^{-\frac{(2r_1^2 - A^2)}{2\sigma_N^2}} I_0\left(\frac{A r_1}{\sigma_N^2}\right) dr_1 =$$

$$= \int_0^{\infty} \frac{y}{\sqrt{2}\sigma_N^2} e^{-\frac{y^2 - 2A'^2}{2\sigma_N^2}} I_0\left(\frac{\sqrt{2}A' \cdot \frac{y}{\sqrt{2}}}{\sigma_N^2}\right) \frac{dy}{\sqrt{2}} =$$

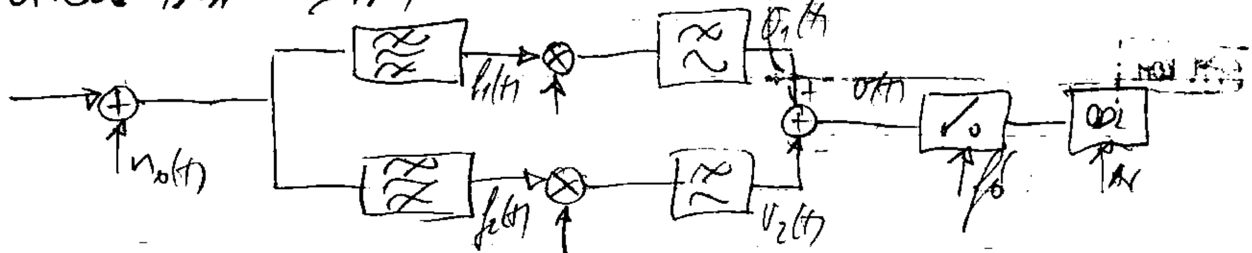
$$= \frac{1}{\sqrt{2}\cdot\sqrt{2}} e^{-\frac{A'^2}{2\sigma_N^2}} \underbrace{\int_0^{\infty} \frac{y}{\sigma_N^2} e^{-\frac{y^2 - A'^2}{2\sigma_N^2}} I_0\left(\frac{A' \cdot y}{\sigma_N^2}\right) dy}_{1} = \frac{1}{2} e^{-\frac{A^2}{4\sigma_N^2}}$$

$$P(1/0)_{r_2} = \int_{r_2}^{\infty} p(r_1)\, dr_1 = P(r_1 > r_2) = \frac{1}{2} e^{-\frac{A^2}{4\sigma_N^2}}$$

$$P(e) = P(0/1) = P(1/0) = \frac{1}{2} e^{-A^2/4\sigma_N^2} = \frac{1}{2} e^{-\frac{\rho'^2}{4}}$$ (BFM)

$P(e)|_{BFM} < P(e)|_{BAM}$

- Koherentna BFM



$$f_1(t) = \begin{cases} [g(t) + x(t)]\cdot\cos\omega_0 t - y_1(t)\cdot\sin\omega_0 t & a_k = 1 \\ x_1(t)\cdot\cos\omega_0 t - y_1(t)\sin\omega_0 t & a_k = -1 \end{cases}$$

$$= \begin{cases} x_2(t)\cdot\cos\omega_0 t - y_2(t)\cdot\sin\omega_0 t & a_n = 1 \\ [g(t) + x_2(t)]\cos\omega_0 t - y_2(t)\sin\omega_0 t & a_n = -1 \end{cases}$$

$$U_1(t) = \begin{cases} K_D[g(t) + x_1(t)] & a_n = 1 \\ K_D\cdot x_1(t) & a_n = -1 \end{cases} \qquad U_2(t) = \begin{cases} K_D x_2(t) & a_n = 1 \\ K_D[g(t) + x_2(t)] & a_n = -1 \end{cases}$$

$$x(t) = x_1 - x_2$$

$$U(t) = U_1 - U_2 = \begin{cases} K_D[x(t) + g(t)] & a_n = 1 \\ K_D[x(t) - g(t)] & a_n = -1 \end{cases}$$

$$g(t_0) = A \qquad U(t) = \begin{cases} K_D[x(t) + A] & a_n = 1 \\ K_D[x(t) - A] & a_n = -1 \end{cases}$$

$$U(t_0) = U_1(t) - U_2(t) \gtrless 0 \qquad \gtrless \begin{matrix} \text{"1"} \\ \text{"0"} \end{matrix}$$

$$p_\xi(t) = \frac{1}{\sqrt{4\pi\sigma_N^2}}\, e^{-\frac{t^2}{4\sigma_N^2}} \qquad \sigma_\xi^2 = \sigma_{x_1}^2 + \sigma_{x_2}^2 = 2\sigma_{x_1}^2 = 2\sigma_N^2$$

$$P(0) = P(1) = \frac{1}{2} \qquad A_T = 0$$

$$P(e) = 2\cdot\frac{1}{2}\,P(1/0) = \int_0^\infty p_{-1}(U)\,dU = \int_0^\infty \frac{1}{\sqrt{4\pi\sigma_N^2}}\, e^{-\frac{(U+A)^2}{4\sigma_N^2}}\,dU$$

$$\frac{U+A}{2\sigma_N} = t \qquad dU = 2\sigma_N\,dt \qquad U = 0 \quad t = \frac{A}{2\sigma_N} \qquad U = \infty \quad t = \infty$$

$$P(e) = \frac{2\sigma_N}{\sqrt{\pi}\,2\sigma_N}\int_{A/2\sigma_N}^{\infty} e^{-t^2}dt = \frac{1}{\sqrt{\pi}}\left[\int_0^\infty e^{-t^2}dt - \frac{2}{2}\int_0^{A/2\sigma_N} e^{-t^2}dt\right] \qquad \frac{\sqrt{\pi}}{2}$$

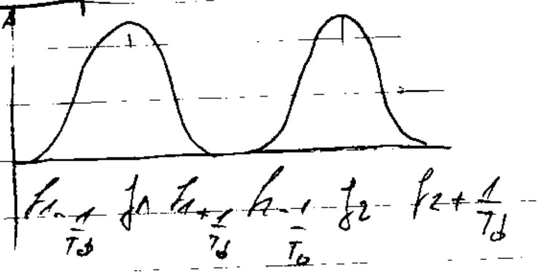
$$P(e) = \frac{1}{2}\left[1 - \Phi\left(\frac{A}{2\sigma_N}\right)\right] = [\Phi(t) = 1 - 2\,\mathrm{erfc}(t)] = \mathrm{erfc}\left(\frac{A}{2\sigma_N}\right)$$

$$P(e) = \mathrm{erfc}\left(\frac{\hat\rho}{2}\right) \qquad \hat\rho \gg 1 \qquad \mathrm{erfc}\,x \doteq \frac{1}{2x\sqrt{\pi}}\,e^{-x^2}$$

$$P(e) = \frac{1}{\hat\rho}\,\frac{1}{\sqrt{\pi}}\,e^{-\frac{\hat\rho^2}{4}}$$

BFM KOH.

BAM $P(e) \doteq \frac{1}{\hat\rho}\,\frac{4}{\sqrt{\pi}}\,e^{-\frac{\hat\rho^2}{8}}$



$$f_1 + \frac{1}{T_d} \leq f_2 - \frac{1}{T_d}$$

$$f_2 - f_1 \geq \frac{2}{T_d} \qquad B_{VK} = \frac{4}{T_d}$$

• DIGITALNA FAZNA MODULACIJA

$$s_{DPM} = Re\left\{\sum_{n=-\infty}^{\infty} g_p(t-nT_d)\, e^{j(\omega_0 t+\psi_n)}\right\}$$

$$s_{DPM} = \cos\omega_0 t \sum_{n=-\infty}^{\infty} \cos\psi_n\, g_p(t-nT_d) + \sin\omega_0 t \sum_{n=-\infty}^{\infty} \sin\psi_n\, g_p(t-nT_d)$$

$a_n = \cos\psi_n$
$b_n = \sin\psi_n$

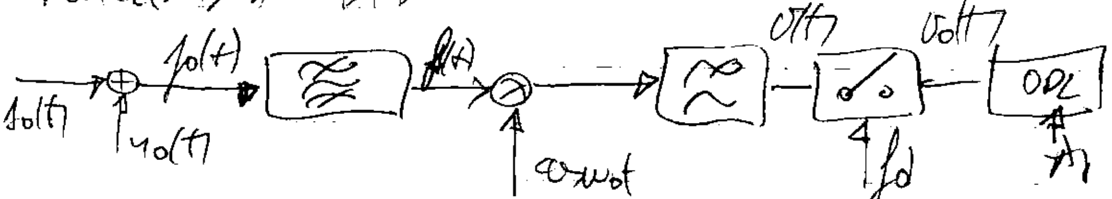
• BPM $\boxed{\psi_n \in (0,\pi)}$ $a_n \in \{-1,1\}$, $b_n \in \{0,0\}$

$$\boxed{s_{BPM}(t) = \cos\omega_0 t \sum_{n=-\infty}^{\infty} a_n\, g_p(t-nT_d) \qquad a_n \in \{-1,1\}}$$

$$g_p = \begin{cases} A_p & 0 \le t \le T_d \\ 0 & \text{inače} \end{cases}$$

$$s_{BPM}(t) = \begin{cases} A_p \cos\omega_0 t, & a_n = 1 \quad \text{za } 0 \le t \le T_d \\ -A_p \cos\omega_0 t, & a_n = -1 \quad -\text{''}- \end{cases}$$

• KOHERENTNA BPD



$$f(t) = \begin{cases} [g(t)+x(t)]\cos\omega_0 t - y(t)\sin\omega_0 t & a_n = 1 \\ [-g(t)+x(t)]\cos\omega_0 t - y(t)\sin\omega_0 t & a_n = -1 \end{cases}$$

$g(t_0) = A$
$$v(t) = \begin{cases} k_D[g(t)+x(t)] & a_n = 1 \\ k_D[-g(t)+x(t)] & a_n = -1 \end{cases}$$

$$v(t_0) = \begin{cases} k_D[A+x(t)] & a_n = 1 \\ k_D[-A+x(t)] & a_n = -1 \end{cases} \qquad \sigma^2 = \sigma_\xi^2 = 2\sigma_N^2$$

$$P(\varepsilon) = P(1/0) = \int_0^{\infty} p_0(v)\,dv = \frac{1}{\sqrt{2\sigma_N^2}} \int_0^{\infty} e^{-\frac{(v+A)^2}{2\sigma_N^2}}\,dv = \mathrm{erfc}\left(\frac{A}{\sqrt{2}\sigma_N}\right)$$

$$P_{FM}(\varepsilon) = \mathrm{erfc}\left(\frac{A}{2\sigma_N}\right) = \left| \sigma_N^2 = \frac{\sigma^2}{2},\ \sigma_N = \frac{\sigma}{\sqrt{2}} \right| = \mathrm{erfc}\left(\frac{A}{\sqrt{2}\sigma}\right)$$

$$P(\varepsilon) = \mathrm{erfc}\left(\frac{\xi}{\sqrt{2}}\right). \quad \mathrm{erfc}(x) = \frac{1}{2x\sqrt{\pi}} e^{-x^2} \quad P(\varepsilon) = \frac{1}{2\xi}\sqrt{\frac{2}{\pi}}\, e^{-\frac{\xi^2}{2}}$$

$)_{BPM} < P(\varepsilon)_{BFM} < P(\varepsilon)_{ZAM}$

ИМЕНА НА ПРИЛАГОДЕН ФИЛТЕР НА СИСТЕМИТЕ СО МОДУЛ.

$H_{PL} = K \cdot S^{*}(j\omega)\, e^{-j\omega t_0}$

$s(t) = g_0(t)\cos(\omega_0 t)$

$H_{PL}(j\omega) = \frac{K}{2}\left[G_0[j(\omega-\omega_0)] + G_0[j(\omega+\omega_0)]\right]^{*} e^{-j\omega t_0}$

$g_0(t) = A \quad -\frac{T_d}{2} \le t \le \frac{T_d}{2} \qquad G(j\omega) = A T_d \frac{\sin\frac{\omega T_d}{2}}{\frac{\omega T_d}{2}}$

$$H_{PL}(j\omega) = \frac{K}{2}\left[A T_d \frac{\sin\frac{(\omega-\omega_0)T_d}{2}}{\frac{(\omega-\omega_0)T_d}{2}} + A T_d \frac{\sin\frac{(\omega+\omega_0)T_d}{2}}{\frac{(\omega+\omega_0)T_d}{2}}\right]^{*} e^{-j\omega t_0}$$

- ОДНОС S/N НА ИЗЛЕЗ ОД ФИЛТЕРОТ

$$S/N = \frac{A^2}{\sigma_r^2} = \frac{\varepsilon}{\eta_0} \qquad \boxed{\rho = \sqrt{\frac{\varepsilon}{\eta_0}}}$$

$$\varepsilon = \int_{-T_d/2}^{T_d/2} A^2\cos^2(\omega_0 t)\,dt = \frac{A^2}{\omega_0}\left(\frac{\omega_0 t}{2} + \frac{1}{4}\sin(2\omega_0 t)\right)\Bigg|_{-T_d/2}^{T_d/2}$$

$$\int \cos^2 x\,dx = \int \cos(x)\,d\sin(x) = \int \sqrt{1-\sin^2(x)}\,d\sin(x) =$$

$$= \int \sqrt{1-u^2}\,du = \left| \begin{matrix} 1-u^2 = t^2 \quad u = \sqrt{t^2+1} \\ -2u\,du = 2t\,dt \\ du = \frac{2t\,dt}{\sqrt{1+t^2}} \end{matrix} \right| =$$

$$= \int \frac{2t^2\,dt}{\sqrt{1+t^2}} \quad ?$$

$$\cos^2 x = \cos x \cdot \cos x = \frac{1}{2}\left[\cos(x+x) + \cos(x-x)\right] = \frac{1}{2}\left[1+\cos(2x)\right]$$

$$\int \frac{1}{2}\left[1+\cos(2x)\right]dx = \frac{x}{2} + \frac{1}{4}\int \cos(2x)\,d(2x) = \frac{x}{2} + \frac{1}{4}\sin(2x)$$

$$\varepsilon = \frac{A^2}{\omega_0}\left\{\left[\frac{\omega_0 T_d}{4} + \frac{1}{4}\sin(\omega_0 T_d)\right] - \left[-\frac{\omega_0 T_d}{4} - \frac{1}{4}\sin\frac{\omega_0 T_d}{2}\right]\right.$$

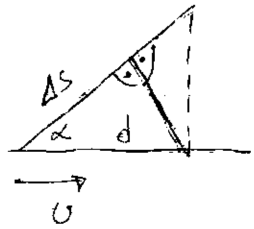
$$= \frac{A^2}{\omega_0}\left[\frac{\omega_0 T_d}{2} + \frac{1}{2}\sin(\omega_0 T_d)\right] \qquad \begin{matrix} T_d = K \cdot T_0 \\ \omega_0 = \frac{2\pi}{T_0} = \frac{2\pi K}{T_d} \end{matrix}$$

$$E = \frac{A^2}{\omega_0}\left[\frac{\omega_0 T_d}{2} + \frac{1}{2}\sin\left(\frac{2\pi k}{T_d} \cdot T_d\right)\right] = \frac{A^2 T_d}{2}$$

• SIMULATION OF FLAT FADING USING METHODS FOR CLASSROOM INSTRUCTIONS;

$$s(t) = \sum_{i=1}^{N} a_i \cos(\omega_i t + \phi_i)$$

$v = \frac{d}{\Delta t}$, $\Delta t = d/v$



$$\varphi_d = \frac{2\pi \Delta s}{\lambda} = \frac{2\pi}{\lambda} d\cos\alpha \qquad \omega_d = \frac{\varphi_d}{\Delta t} = \frac{\varphi_d}{d/v}$$

$$\omega_d = \frac{v}{1\cdot d}\cdot\frac{2\pi}{\lambda}\cdot d\cos\alpha = \frac{2\pi v}{\lambda}\cos\alpha$$

$$f_d = \omega_d/2\pi = \frac{v}{\lambda}\cos\alpha \qquad f_d = f_{dmax}\cos\alpha \qquad f_{dmax} = v/\lambda$$

$$\omega_{di} = \frac{2\pi v}{c/f_c}\cos\psi_i = \frac{\omega_c v}{c}\cos\psi_i$$

$$s(t) = \sum_{i=1}^{N} a_i\cos(\omega_c t + \omega_{di} t + \phi_i)$$

$$s(t) = \cos\omega_c t \underbrace{\sum_{i=1}^{N} a_i \cos(\omega_{di}t+\phi_i)}_{I(t)} - \sin\omega_c t \underbrace{\sum_{i=1}^{N} a_i\sin(\omega_{di}t+\phi_i)}_{Q(t)}$$

$$\boxed{s(t) = I(t)\cos\omega_c t - Q(t)\sin\omega_c t}$$

$$r(t) = \sqrt{I(t)^2 + Q(t)^2} \qquad I(t) = r(t)\cos\varphi \qquad Q(t) = r(t)\sin\varphi \qquad \boxed{\varphi = \arctan\frac{Q(t)}{I(t)}}$$

PDF $f(r) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}$ $r \geq 0$

$$\text{Chi-square}(x) = \sqrt{\frac{[x - \text{expected}(x)]^2}{\text{expected}(x)}}$$

DEGREE OF FREEDOM !

Chi-square DISTRIBUTION DEPENDS ON THE DOF

• MEASURING RESPONSE
- CHAMPION OFFER 900.000 customers
- CHALLENGER " 100.000 "
- 50.000 FROM ORIGINAL POPULATION WOULD HAVE RESPONDED TO CHALLENGER IF THEY RECEIVED IT.
- 5.000 ARE EXPECTED TO RESPOND 10% OF THE POPULATION THAT RECEIVED CHALLENGER OFFER

→ SAMPLE RESPONSE RATE 5%

→ POPULATION RESPONSE RATE 5% $\frac{50.000}{1.000.000} = 0.05 = 5\%$

$\frac{50}{1.000.000} = \frac{5}{10^5} = 0.005 = 0.5\%$ ALL REPORTED

$\frac{995.000}{1.000.000} = 99.5\%$ $\frac{45.000}{50.000} = 90\%$ $\frac{95.000}{100.000} = 95\%$

$P(k,n) = \binom{n}{k} p^k \cdot q^{n-k}$ $q = 1-p$

BINOMIAL

- STANDARD ERROR OF PROPORTION (APPROXIMATING BINOMIAL ~ N)

CHALLENGER

$$SEP = \sqrt{\frac{p*(1-p)}{N}}$$

p - AVERAGE (IN %)
N - SIZE OF POPULATION

$SEP = \sqrt{\frac{5\% \cdot 95\%}{100.000}} = 0.07\%$

$p = 5\%$
68% CONFIDENT INTERVAL IS
$4.93\% \div 5.07\%$

95% CONFIDENT INTERVAL (2σ)
$4.85\% \div 5.14\%$

- CHAMPION
FOR 4.9, 5.0, 5.1 CONFIDENCE BOUNDS OVERLAP ⇒
DESIGNS FOR CHAMPION & CHALLENGER ARE SAME
- DIFFERENCE OF PROPORTIONS
- STANDARD ERROR OF A DIFFERENCE OF PROPORTIONS

$$SEDP = \sqrt{\frac{p_1*(1-p_1)}{N_1} + p_2\frac{(1-p_2)}{N_2}}$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{u^2}{2}} du = \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{-\frac{u^2}{2}} du + \frac{1}{\sqrt{2\pi}}\int_{0}^{x} e^{-\frac{u^2}{2}} du\right] =$$

$$= \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{\pi}} \cdot \sqrt{2}\int_{-\infty}^{0} e^{-t^2} dt + \frac{1}{\sqrt{\pi}}\sqrt{2}\int_{0}^{x/\sqrt{2}} e^{-t^2} dt\right] =$$

$$= \left[\frac{1}{2} + \frac{1}{2}\frac{2}{\sqrt{\pi}}\int_{0}^{x/\sqrt{2}} e^{-t^2} dt\right] = \frac{1}{2}\left[1 + erf\left(\frac{x}{\sqrt{2}}\right)\right]$$

$$\Phi_{\mu\sigma} = \frac{1}{2}\left[1 + erf\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)\right]$$

$$NORMSDIST(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$ excel

$$I = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-z}^{z} e^{-\frac{x^2}{2}} dx \qquad t = \frac{x}{\sqrt{2}} \quad x = z \quad t = \frac{z}{\sqrt{2}}$$

$$dx = \sqrt{2}\,dt \quad x = -z \quad t = -\frac{z}{\sqrt{2}}$$

$$\frac{1}{\sqrt{2\pi\sigma^2}} \left[\sqrt{2} \int_{-z/\sqrt{2}}^{0} e^{-t^2} dt + \sqrt{2} \int_{0}^{z/\sqrt{2}} e^{-t^2} dt \right] = \frac{1}{\sqrt{2\pi\sigma^2}} 2\sqrt{2} \int_{0}^{z/\sqrt{2}} e^{-t^2} dt$$

$$I = \frac{1}{\sigma} \frac{2}{\sqrt{\pi}} \int_{0}^{z/\sqrt{2}} e^{-t^2} dt = erf\left(\frac{z}{\sqrt{2}}\right)$$

$$Chi\text{-}square(x) = \sqrt{\frac{(x - expected(x))^2}{expected(x)}}$$

- RAYLEIGH FADING SIMULATION; $f_c = 900 MHz$

N – NUMBER OF MULTIPATHS $N = 4:40$

- FOR EACH N SIMULATION TIME INTERVAL 1250 WAVELENGTHS

$T_0 = \frac{1}{f_0} = \frac{1}{9 \cdot 10^8} = 1.11$ usec $\quad T_{sim} = 1.3875 \cdot 10^{-6}$ sec $= 1.4\,\mu$sec

SIMULATION IS REPEATED 50 TIMES AND AVERAGED

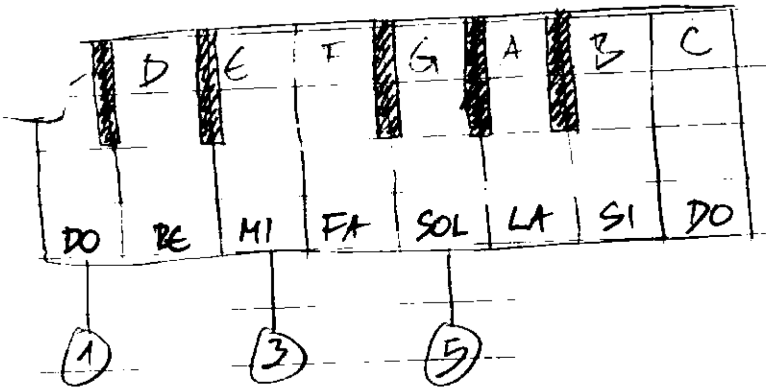
$$s(t) = \sum_{i=1}^{N} a_i \cos(\omega_c t + \phi_i) \qquad (1)$$

$$s(t) = \sum_{i=1}^{N} a_i \cos(\omega_c t + \omega_d t + \phi_i) \qquad (2)$$

- a_i – weibull distributed (weibrnd)
- ϕ_i – UNIFORMLY DISTRIBUTED $[0, 2\pi]$ (unifrnd)
- DEMODULATE WITH „demod" TO GET $I(t)$ & $Q(t)$

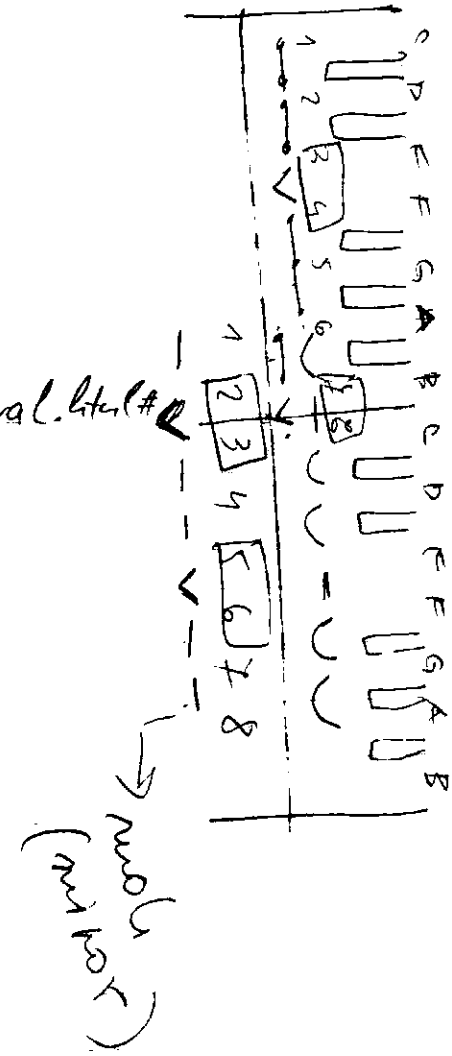
ENVELOPE: $\sqrt{I(t)^2 + Q(t)^2} = r(t)$

- TEST $r(t)$ AGAINST RAYLEIGH DISTRIBUTION USING CHI-SQUARE TEST (APPENDIX II)
- SO MENUVAŠE NA „N" MOŽE DA SE VIDI DEKA $r(t)$ E VO SOGLASNOST SO RAYLEIGH PDF ZA $N \geq 6$



MOL - MINOR (NEZNO TUZNO)
DUR - MAJOR (AGRESIVNO)

www.andromedanet.com/tutorer-general.html#

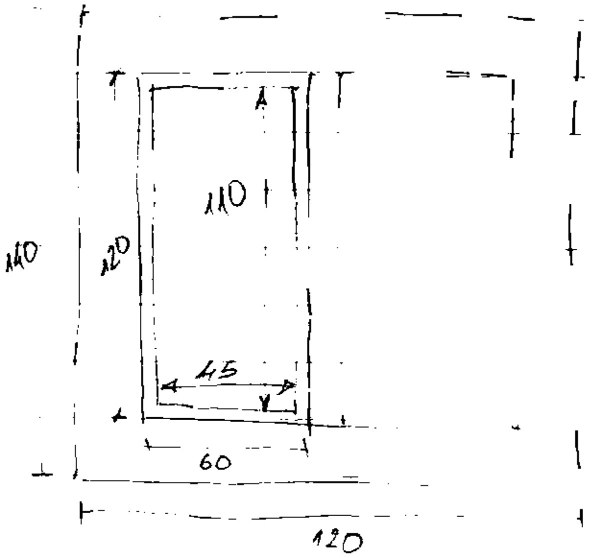


9, 10, 11

- FAKTURA ERICSSON
- DISTRIBUTED PRINCO DVDs (START 27.07.08)

||||

C dur ~ A mol
G dur ~ E mol



16 × 10 16
800

1,4 × 4 = 5,6
1,4 × 2 = 2,8
14,8
≈ 1 m²
25 = 200 €

TERMOPAN STAKLO SO ARGON

1 PROZOR TERMOPAN STAKLO $0,45 \cdot 1,10 = 0,495 \doteq 0,5\ m^2$

RAMKA: $2 \times 120 + 2 \times 140 + 2 \times 120 + 2 \times 60 =$
$= 240 + 280 + 240 + 120 = 480 + 400 = 880$

8,8 m × 10 € = 88 €
\+ TERMOPAN 12,5 €

100 €

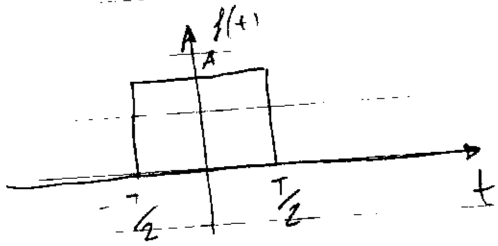
070200127
0800112345

IGOR
046251500
stonkj
jstosik
nimbuzz

90 → 30 >5%
90 → 60 >7%
90 → 8 >10% 12% → 8
7% → 30
8% → 15

TURAN

$$\frac{\sin\left(n\omega_0 \frac{\tau}{2}\right)}{n\omega_0 \frac{\tau}{2}} = \left| \begin{matrix} T = 2\tau \\ \tau = \frac{T}{2} \end{matrix} \right| = \frac{\sin\left(n \frac{2\pi}{T} \cdot \frac{T}{4}\right)}{n \frac{2\pi}{T} \cdot \frac{T}{4}} = \frac{\sin\left(\frac{n\pi}{2}\right)}{\frac{n\pi}{2}}$$



$$F(j\omega) = \int_{-T/2}^{T/2} f(t)\, e^{-j\omega t}\, dt$$

$$F(j\omega) = \left| \begin{matrix} u = -j\omega t & t = -\frac{T}{2},\ u = +j\omega\frac{T}{2} \\ du = -j\omega dt & t = \frac{T}{2},\ u = -j\omega\frac{T}{2} \end{matrix} \right| = \int_{+j\frac{\omega T}{2}}^{-j\frac{\omega T}{2}} A \cdot e^{u} \frac{dt}{-j\omega}$$

$$F(j\omega) = \frac{AT}{-j\omega T} e^{u} \Big|_{j\frac{\omega T}{2}}^{-j\frac{\omega T}{2}} = \frac{2AT}{2j\omega T} e^{u} \Big|_{-j\frac{\omega T}{2}}^{j\frac{\omega T}{2}} = \frac{2AT}{\omega T} \frac{1}{2j} \left(e^{j\frac{\omega T}{2}} - e^{-j\frac{\omega T}{2}}\right) \rightarrow \sin\frac{\omega T}{2}$$

$$F(j\omega) = A \cdot T \frac{\sin\frac{\omega T}{2}}{\frac{\omega T}{2}} = \left| \begin{matrix} T = \frac{1}{f_0} \\ \omega = 2\pi f \end{matrix} \right| = AT \frac{\sin\frac{2\pi f}{2 f_0}}{\frac{2\pi f}{2 f_0}} = AT \frac{\sin\frac{\pi f}{f_0}}{\frac{\pi f}{f_0}} = AT \operatorname{sinc}\left(\frac{f}{f_0}\right)$$

$$2 \times 5800 = 11600$$

$$\begin{array}{r} 11600 \\ 2600 \\ \hline 14200 \end{array}$$

33x21 = 693

30.5x22.5 = 686

MORI DC 2.157

BLADES OF GLORY "

ASTRONAUT FARMER 2.149

ANCHORMAN 2.147

NEXT 2.141

$$\overline{\xi \cdot \eta} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \cdot y\, p_{\xi\eta}(x,y)\, dx\, dy$$

$$p_{\xi\eta}(x,y) = p_\xi(x) \cdot p_\eta(y)$$

$$\overline{\xi \cdot \eta} = \int_{-\infty}^{\infty} x\, p_\xi(x)\, dx \int_{-\infty}^{\infty} y\, p_\eta(y)\, dy = \bar{\xi} \cdot \bar{\eta}$$

$$p_{\xi\eta}(x,y) = p(x) \cdot p(y/x) = p(y) \cdot p(x/y) = \left| \xi \,\&\, \eta \text{ stat nezavisni} \right|$$

$$\Rightarrow \boxed{p_{\xi\eta}(x,y) = p(x) \cdot p(y)}$$

108, 109, 155